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A Generalized Telegraph Process with Resetting to the Origin Driven by Bernoulli Trials

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Abstract

We study a finite-velocity random motion on the real line with alternating velocities $-v$ and c, subject to stochastic resetting to the origin. At each velocity-switching epoch, a Bernoulli trial determines whether the particle is instantaneously reset to the origin with probability θ or it continues its motion with the opposite velocity with probability $1-\theta$. The resulting reset times form a renewal process. We derive the probability distributions of the position-velocity process when the intertimes between consecutive velocity changes have identical (i) exponential distribution, and (ii) Erlang-2 distribution. We also obtain explicit expressions for the moments of the position process and, in case (i), the mean-square distance between the process with resetting and an independent counterpart without resetting. We further characterize the stationary distribution, expressing it in terms of the asymptotic average of the local time of the position process. Finally, we investigate the limiting behavior of the position density under a Kac-type scaling. In both cases (i) and (ii), the position density, conditioned on random initial velocity, converges to that of the Brownian motion with Poisson resetting.

Keywords Finite-velocity random motion · Alternating process · Exponential intertimes · Erlang intertimes · Stationary distribution · Kac's scaling

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1 Introduction and Background

The resetting mechanism of finite-velocity random motions has attracted growing interest in the last few years. It refers to a diffusion process subject to resets, occurring at random times and relocating the process to a given position, which usually coincides with the origin or the starting point. Generally, the random dynamics of the process and the resetting mechanism are taken to be independent of each other, while the reset to a given position occurs according to a Poisson process or a more general counting process. Specifically, the diffusive model is a suitable modification of the so-called (integrated) telegraph process, also known as Goldstein–Kac process. The latter is a continuous-time process that describes the motion of a particle on the real line, where changes in the directions of the two alternating velocities are governed by the Poisson process (see, e.g., Beghin et al. [1]). We assume that the two alternating velocities have different sign and are denoted by $-v < 0$ and $c > 0$. Many extensions and features of the classical model have been proposed and studied recently. For instance, see Cinque and Orsingher [2] for the analysis of the maximum of the asymmetric telegraph process, Martinucci et al. [3] for a generalized telegraph process, and Ricciuti and Toaldo [4] for generalized telegraph processes subject to semi-Markov perturbations.

We highlight some instances of special interest for the present investigation:

(i) *The assumptions of Erlang distributed intertimes between consecutive velocity changes*. This is useful to model velocity changes governed by Poisson-driven collisions, in which only certain fixed collisions cause the effective velocity changes. Results on one-dimensional and multidimensional random motions with velocities alternating at Erlang distributed random times can be found in Di Crescenzo [5], Kolomiiets et al. [6] and Pogorui and Rodríguez-Dagnino [7]. More general instances of generalized telegraph processes based on gamma-distributed intertimes between velocity changes have been studied recently by Martinucci et al. [8].

(ii) *The inclusion of a reset mechanism in finite-velocity random motions*. This allows to describe several kinds of systems subject to abrupt changes. The proposal of reset processes in applications in physics is motivated often by the need of including a stabilizing rule in the dynamics of run-and-tumble particles. See, for instance, Evans and Majumdar [9], and Masoliver [10] where the effects of resetting mechanisms on telegrapher's processes are studied. Reset processes deserve interest also in other applied fields, as mathematical finance, geophysics, biomathematics, genetics. See, also, Evans et al. [11] and Di Crescenzo et al. [12] for motivations and applications in such areas. Moreover, recent contributions can be found in Boyer and Majumdar [13], Can et al. [14], Godrèche and Luck [15]. Stochastic resets have been considered also for the analysis of other types of random processes (see, e.g., Abundo [16], Bonaccorsi et al. [17], Colantoni et al. [18], Ratanov [19]).

(iii) *The adoption of random trials in modeling the dynamics underlying finite-velocity random motions*. This allows to describe certain situations, such as the motion of particles subject to collisions, which can result in changes in direction. Stimulated by applications of urn schemes modeling the preferential attachment principle, Crimaldi et al. [20] studied a telegraph process where each new velocity is determined by the outcome of (two types of) random trials. Also, Singh et al. [21] and Belan [22] treated restart mechanisms of processes based on Bernoulli trials.

(iv) *The analysis of limit results under the Kac's scaling*. Kac [23] noticed that the telegraph process yields the standard Brownian motion when both the velocity and the intensity of its changes tend suitably to $+\infty$. Hence, a typical problem in this field is the identification of the limiting diffusion processes that arise from modified/extended versions of the telegraph process under the so-called Kac's scaling. To place our scaling result in context, Table 1 summarizes some representative Kac-type limits available in the literature. We also recall Section 2.7 of Ratanov and Kolesnik [31], as well as the different scaling and limits obtained for the telegraph process driven by the geometric counting process (cf. Di Crescenzo et al. [32], [12]), and for the Kac-Ornstein-Uhlenbeck processes (cf. Ratanov [33]).

Motivated by the above studies, the objective of this paper is to investigate the telegraph process on the real line subject to resets to the origin driven by random trials. At the end of each (forward or backward) displacement, a Bernoulli trial is performed so that the particle is subject to an instantaneous reset to the origin with probability $\theta \in [0, 1]$, or it begins a new displacement along the other direction with probability $1-\theta$. The reset to the origin is modeled by an instantaneous jump, in the sense that its duration is negligible with respect to the time scale of the motion. The Bernoulli trials are independent of the random displacements, and after each reset the motion restarts with the same initial velocity according to the original dynamics, independently of the past. Therefore, reset times form a renewal process. To the best of our knowledge, telegraph processes with resets driven by Bernoulli trials at velocity-switching epochs have not been studied previously. This model then fills this gap by introducing a richer scheme, in which velocity reversals and resets are determined through a common sequence of Bernoulli decisions, rather than viewing the resetting mechanism as an external process independent of the dynamics. This may be useful in applied contexts, for instance for failures/restarts in communication systems naturally occurring at decision epochs, or for biological agents resetting/deviating after encounters or collisions.

The main contributions of this paper are the following:

(a) *Derivation of the probability law for telegraph processes with Bernoulli-trial resets*. The analysis of the particle's position process is based on the Markov property of the reset times and on the knowledge of the distribution of the intertimes separating velocity changes. We first perform the analysis of the process for general intertimes.

(b) *Explicit characterization of the distribution and moments for exponential and Erlang-2 intertimes*. The choice of such intertimes allows us to obtain tractable expressions for the transition law of the process, with its mean and variance, thanks to the Markov and semi-Markov nature of such distributions, in line with previous results mentioned for example in Table 1.

(c) *Analysis of mean-square distances between processes.* Using the explicit computation of the moments of the position process mentioned above, in the case of exponential intertimes we investigate the mean-square distance between the process subject to resets and an independent version of the same process in absence of resets, by assuming that the two processes have identical initial velocity. We follow the approach adopted in Section 4.1 of [12], for the telegraph process driven by the geometric counting process with Poisson-based resetting. The relevance of this problem in the study of random motions emerges, for instance, in the recent contribution of Barrera and Lukkarinen [34], which provides quantitative bounds for the Wasserstein distance between the telegraph process and the Brownian motion.

(d) *Determination of the stationary distribution.* Typically, finite-velocity random motions on the real line do not possess a stationary distribution unless the intertimes between velocity changes decrease stochastically (see [35]) or a restoring drift force is present (e.g., [36], [37]). In other cases, the restoring force is superimposed by external effects, such has random catastrophes inducing resets. A specific aim of this article is to show how the presence of

Table 1 A synthesis of some limit results based on the Kac's scaling.

	reference	process	limit process
(i)	[24], Lemma 2	telegraph process	Brownian motion
(ii)	[25], Corollary 5.1	asymmetric telegraph process	Brownian motion with drift
(iii)	[8], Theorem 4.2	telegraph process with gamma intertimes	Brownian motion
(iv)	[26], Proposition 4.3 [27], Theorems 4.1 and 4.2	telegraph process with deterministic jumps	mixture of Brownian motions with drift
(v)	[28], Section 7.1	asymmetric circular telegraph process	circular Brownian motion with drift
(vi)	[29], Proposition 2.3	squared telegraph process	squared Brownian motion
(vii)	[30], Remark 4.2	time-changed telegraph process at Brownian times	iterated Brownian motion
(viii)	this paper, Propositions 10 and 18	telegraph process with Bernoulli-trial resets and exponential/Erlang-2 intertimes	Brownian motion subject to Poisson-driven resets

resets driven by Bernoulli trials leads to a stationary distribution, expressible in terms of the asymptotic average of the local time of the process.

(e) *Diffusion limit under a Kac-type scaling.* In the cases of exponential and Erlang-2 intertimes between consecutive velocity changes, we let the velocity and the rate of the velocity changes tend suitably to $+\infty$, while the reset probability θ tends to zero. Under these Kac-type conditions, we prove that the probability density of the particle's position converges to that of the Brownian motion subject to Poisson-driven resets at the origin.

The paper is organized as follows: In Section 2 we introduce the finite-velocity random motion characterized by alternating velocities and the resetting mechanism subject to Bernoulli trials. In Section 3 we discuss the probability density functions (p.d.f.) of the process in the absence of resets and in the specific case of interest. In Section 4 we obtain the probability law of the process when the intertimes are exponentially distributed. We investigate also the moments, various asymptotic results and the mean-square distance between the processes with and without resets at the origin. Then, we perform the limit analysis in two cases of interest: (i) when $t \to +\infty$, leading to the stationary distribution mentioned above, and (ii) under Kac-type scaling conditions. An analogous investigation is performed in Section 5 when the random times separating consecutive velocity reversals have Erlang-2 distribution. In Section 6 we present a discussion focusing on the differences between Evans and Majumdar [9] and the present paper. Finally, in Section 7 we draw our conclusions.

Throughout the paper, we denote the indicator function by $\mathbf{1}_A$, i.e. $\mathbf{1}_A = 1$ if A is true, and $\mathbf{1}_A = 0$ otherwise. Moreover, $\stackrel{d}{=}$ means equality in distribution, $\mathbb{R}$ denotes the set of real numbers, with $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}_0^+ = [0, +\infty)$, whereas $\mathbb{N} = \{1, 2, \ldots\}$.

2 Renewal Times and Intertimes Distributions

Consider the stochastic process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$, having state-space $\mathbb{R} \times \{-v, c\}$ and describing the motion of a particle starting from the origin and moving with alternating velocity $-v$ and c. Here, $X(t)$ and $V(t)$ describe respectively the position and the velocity of the particle at time t. The initial position and initial velocity are fixed, i.e.

$$X(0) = 0, \qquad V(0) = y, \quad y \in \{-v, c\}, \quad -v < 0 < c. \tag{1}$$

The motion alternates randomly in time, so that its direction (forward or backward) is determined at each instant by the sign of the velocity. Moreover, the particle is subject to resets which force it to return to the origin and to restart the motion. The resetting mechanism is based on Bernoulli trials, that are independent of the random displacements. Specifically, at the end of each (forward or backward) displacement a Bernoulli trial is performed, so that the particle is subject to an instantaneous reset to the origin with probability θ, or it begins a new displacement along the other direction with probability $1 - \theta$, for a given $\theta \in [0, 1]$. After each reset, the motion restarts with the same velocity taken at time 0, and then proceeds adopting the same rules, independently of the previous history of the motion. Hence, the sequence of resets constitutes the renewal instants of a renewal process, say $\{K(t), t \in \mathbb{R}_0^+\}$.

At any time $t \in \mathbb{R}^+$ one has $\mathsf{P}(X(t) \in [-vt, ct], V(t) \in \{-v, c\}) = 1$. Figure 1 shows a sample path of the position process that undergoes three resets.

We denote by U_i and D_i the non-negative absolutely continuous random variables (r.v.'s) describing the duration of the i-th interval during which the motion goes forward and backward, respectively. Moreover, we assume that $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ are mutually independent sequences of independent and identically distributed (i.i.d.) r.v.'s.

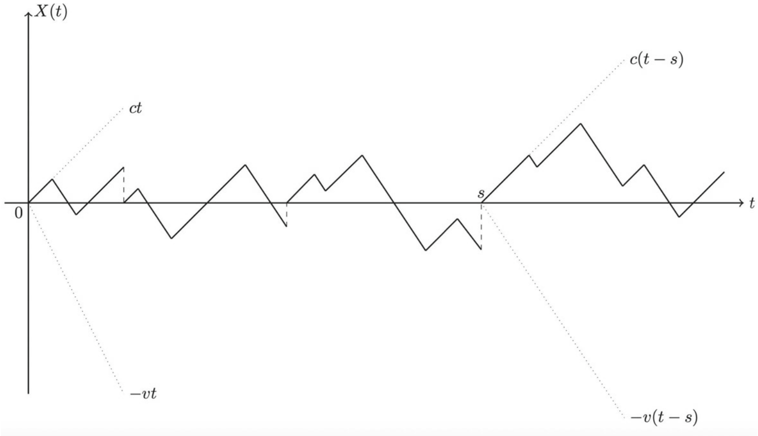

Fig. 1 A sample path of $X(t)$, with $V(0) = c$ and three resets

For the intertime U_i, $i \in \mathbb{N}$, we denote the cumulative distribution function (c.d.f.) by $F_{U_i}(x) = \mathsf{P}(U_i \le x)$, the survival function by $\overline{F}_{U_i}(x) = 1 - F_{U_i}(x)$, and the probability density function (p.d.f.) by $f_{U_i}(x)$, for $x \in \mathbb{R}$. Similarly, for D_i, $i \in \mathbb{N}$, we introduce the functions $F_{D_i}(x) = \mathsf{P}(D_i \le x)$, $\overline{F}_{D_i}(x) = 1 - F_{D_i}(x)$ and $f_{D_i}(x)$. Furthermore, we define the sums of the forward and backward displacements

$$U^{(0)} = D^{(0)} = 0, \qquad U^{(k)} = \sum_{i=1}^{k} U_i, \qquad D^{(k)} = \sum_{i=1}^{k} D_i, \qquad k \in \mathbb{N}. \tag{2}$$

We recall that at the end of each (forward or backward) displacement a Bernoulli trial is performed, leading to a reset to the origin with probability θ, or to a new displacement along the other direction with probability $1 - \theta$. Hence, due to independence, the r.v. describing the number of displacements (trials) before a reset has geometric distribution with parameter θ. Consequently, the inter-renewal times of the renewal process $\{K(t), t \in \mathbb{R}_0^+\}$ can be expressed as suitable mixtures with geometric mixing distribution. The renewal process that counts the number of resets in $(0, t]$ is expressed as

$$K(t) = \sum_{n=1}^{+\infty} \mathbf{1}_{\{R_n \le t\}}, \qquad t \in \mathbb{R}_0^+, \tag{3}$$

where $0 < R_1 < R_2 < \ldots$ are the reset times. The p.d.f. of R_n is denoted by $f_R^{(n)}(t)$, being the n-fold convolution of $f_R^{(1)}(t) \equiv f_R(t)$, and thus $F_R^{(n)}(t) = \int_0^t f_R^{(n)}(s)\,\mathrm{d}s$ is the corresponding c.d.f. Due to independence, the inter-occurrence times $R_{n+1} - R_n$, for $n \in \mathbb{N}$, are identically distributed as R_1.

In order to describe the resetting mechanism, we define the mean

$$M(t) := \mathsf{E}[K(t)] = \sum_{n=1}^{+\infty} F_R^{(n)}(t), \qquad t \in \mathbb{R}_0^+, \tag{4}$$

also known as the renewal function of $\{K(t), t \in \mathbb{R}_0^+\}$. The corresponding renewal intensity is denoted as

$$\rho(t) := \frac{\mathrm{d}M(t)}{\mathrm{d}t}, \qquad t \in \mathbb{R}_0^+. \tag{5}$$

Since the particle can perform a reset at the end of each displacement w.p. θ, and recalling Eq. (2), if $V(0) = c$ then

$$R_1 = \begin{cases} U^{(k)} + D^{(k-1)}, & \text{w.p. } p_{2k-1} \text{ (if the reset occurs immediately after the } k\text{-th forward displacement, } k \in \mathbb{N}), \\ U^{(k)} + D^{(k)}, & \text{w.p. } p_{2k} \text{ (if the reset occurs immediately after the } k\text{-th backward displacement, } k \in \mathbb{N}). \end{cases} \tag{6}$$

where

$$p_j = \theta(1-\theta)^{j-1}, \qquad j \in \mathbb{N} \tag{7}$$

denotes the probability that the first reset occurs immediately after the j-th particle's displacement. Therefore, in this case the c.d.f. of R_1 can be expressed as

$$\begin{aligned} F_R(t) &:= F_R^{(1)}(t) = \mathsf{P}(R_1 \le t) \\ &= \sum_{k=1}^{\infty} p_{2k-1} \mathsf{P}\left[U^{(k)} + D^{(k-1)} \le t\right] + \sum_{k=1}^{\infty} p_{2k} \mathsf{P}\left[U^{(k)} + D^{(k)} \le t\right], \qquad t \in \mathbb{R}_0^+. \end{aligned} \tag{8}$$

The case when $V(0) = -v$ can be treated similarly, by symmetry.

3 The Stochastic Process and Its Probability Law

In this section we study the probability law of the position-velocity process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$, with state-space $\mathbb{R} \times \{-v, c\}$ and initial conditions (1). In general, the analysis of finite-velocity random motions related to the telegraph process is based on the resolution of the PDE's for the relevant densities. However, for the process under investigation such an approach is not convenient, since the nature of the problem allows us to use a more effective strategy based on the analysis of the intertimes between consecutive velocity changes and the renewal times of the resets.

We first analyze the process in the absence of resets and then in the general case.

Remark 1 Denoting by W_1 the first running time, we have $W_1 \stackrel{d}{=} D_1$ if $y = -v$, and $W_1 \stackrel{d}{=} U_1$ if $y = c$. Hence, its survival function is given by

$$\overline{F}_{W_1}(t) = \begin{cases} \overline{F}_{D_1}(t), & \text{if } y = -v \\ \overline{F}_{U_1}(t), & \text{if } y = c, \end{cases} \qquad t \in \mathbb{R}. \tag{9}$$

3.1 Analysis in the Absence of Resets

Let us first investigate the process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ when no resets occur in $(0, t]$, i.e. when $R_1 > t$. At every instant $t \in \mathbb{R}^+$ the position $X(t)$ is confined in $[-vt, ct]$. Hence, if the particle does not change velocity in $(0, t]$, then it occupies one of the extremes of $[-vt, ct]$ depending on the initial velocity $V(0) = y \in \{-v, c\}$, so that

$$\mathsf{P}[X(t) = yt, V(t) = y, R_1 > t \mid X(0) = 0, V(0) = y] = \overline{F}_{W_1}(t), \tag{10}$$

where $\overline{F}_{W_1}(t)$ is given in (9). Otherwise, if the particle changes velocity or it is subject to reset at least once in $(0, t]$, then it is located in a state belonging to $(-vt, ct)$. The p.d.f. of the particle's position at time $t \in \mathbb{R}^+$ when no reset has occurred before, is denoted by

$$p_0(x, t \mid y) = \mathsf{P}[X(t) \in \mathrm{d}x, R_1 > t \mid X(0) = 0, V(0) = y]/\mathrm{d}x, \tag{11}$$

for $y \in \{-v, c\}$ and $x \in (-vt, ct)$. Moreover,

$$
\begin{aligned}
f_0(x,t\,|\,y) &= \mathsf{P}[X(t) \in \mathrm{d}x,\, V(t) = c,\, R_1 > t \,|\, X(0) = 0,\, V(0) = y]/\mathrm{d}x, \\
b_0(x,t\,|\,y) &= \mathsf{P}[X(t) \in \mathrm{d}x,\, V(t) = -v,\, R_1 > t \,|\, X(0) = 0,\, V(0) = y]/\mathrm{d}x
\end{aligned}
\tag{12}
$$

denote the sub-densities concerning the forward and backward motion at time t, respectively. Recalling (7), we can now introduce the probability

$$
\overline{P}_j := (1-\theta)^j, \qquad j \in \mathbb{N}_0 \tag{13}
$$

that no resets occur immediately after any of the first j particle's displacements. Moreover, due to positions (2), hereafter $f_U^{(k)}(x)$ and $f_D^{(k)}(x)$ will denote the p.d.f.'s of $U^{(k)}$ and $D^{(k)}$, respectively, with $f_U^{(0)}(x)$ and $f_D^{(0)}(x)$ being the Dirac-delta function.

In the following proposition we obtain the general expressions of the p.d.f.'s introduced in Eq. (12). The proof is similar to that of Theorem 2.1 of [5], and is based on a renewal procedure for the positions occupied at random instants of velocity reversals.

Proposition 1 *For the stochastic process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$, the p.d.f.'s (11) and (12), for all $t \in \mathbb{R}^+$, $x \in (-vt, ct)$, and $y \in \{-v, c\}$, satisfy*

$$
p_0(x,t\,|\,y) = f_0(x,t\,|\,y) + b_0(x,t\,|\,y), \tag{14}
$$

where

$$
f_0(x,t\,|\,c) = \frac{1}{c+v}\sum_{k=1}^{+\infty}\overline{P}_{2k}\, f_D^{(k)}(t-\tau^*)\int_{t-\tau^*}^{t} f_U^{(k)}(s-t+\tau^*)\overline{F}_{U_{k+1}}(t-s)\,\mathrm{d}s, \tag{15}
$$

$$
b_0(x,t\,|\,c) = \frac{1}{c+v}\sum_{k=0}^{+\infty}\overline{P}_{2k+1}\, f_U^{(k+1)}(\tau^*)\int_{\tau^*}^{t} f_D^{(k)}(s-\tau^*)\overline{F}_{D_{k+1}}(t-s)\,\mathrm{d}s,
$$

and

$$
f_0(x,t\,|-v) = \frac{1}{c+v}\sum_{k=0}^{+\infty}\overline{P}_{2k+1}\, f_D^{(k+1)}(t-\tau^*)\int_{t-\tau^*}^{t} f_U^{(k)}(s-t+\tau^*)\overline{F}_{U_{k+1}}(t-s)\,\mathrm{d}s,
$$

$$
b_0(x,t\,|-v) = \frac{1}{c+v}\sum_{k=1}^{+\infty}\overline{P}_{2k}\, f_U^{(k)}(\tau^*)\int_{\tau^*}^{t} f_D^{(k)}(s-\tau^*)\overline{F}_{D_{k+1}}(t-s)\,\mathrm{d}s,
$$

with $\tau^* = \tau^*(x,t) = (vt+x)/(c+v)$ and where $\overline{P}_j$ is defined in (13).

Proof For $V(0) = c$, imposing that no resets occur in $(0, t]$, conditioning on the number k of velocity reversals from $-v$ to c in $(0, t]$, and on the last instant s preceding t in which the particle changes velocity from $-v$ to c, and recalling the first of (12), the forward density is expressed as

$$
\begin{aligned}
f_0(x,t\,|\,c)\,\mathrm{d}x = \sum_{k=1}^{+\infty}\overline{P}_{2k}\int_0^t &\mathsf{P}\{U^{(k)} + D^{(k)} \in \mathrm{d}s,\, cU^{(k)} - vD^{(k)} \\
&+ c(t-s) \in \mathrm{d}x,\, U_{k+1} > t-s\}.
\end{aligned}
$$

By performing a transformation as in the proof of Theorem 2.1 of [5], and taking into account that $X(s) \geq -vs$ implies $s \geq \frac{ct-x}{c+v} \equiv t-\tau^*$, we have

$$f_0(x,t\,|\,c) = \frac{1}{c+v} \sum_{k=1}^{+\infty} \overline{P}_{2k}\, f_D^{(k)}\left(\frac{ct-x}{c+v}\right) \int_{t-\tau^*}^{t} f_U^{(k)}\left(s - \frac{ct-x}{c+v}\right) \overline{F}_{U_{k+1}}(t-s)\,\mathrm{d}s.$$

The Eq. (15) thus follows. The remaining p.d.f.'s can be obtained in a similar way. □

Remark 2 (i) The densities given in Proposition 1 satisfy suitable symmetry conditions, as f_0 can be obtained from b_0 by interchanging U_k with D_k, c with v, x with $-x$, and vice-versa.

(ii) From Eqs. (8)÷(11) the following normalization condition holds, for all $t \in \mathbb{R}^+$:

$$\overline{F}_{W_1}(t) + \int_{-vt}^{ct} p_0(x,t\,|\,y)\,\mathrm{d}x + F_R(t) = 1.$$

Remark 3 The densities defined in (12) allow us to obtain also the so-called flow function

$$w_0(x,t\,|\,y) = f_0(x,t\,|\,y) - b_0(x,t\,|\,y), \qquad x \in (-vt, ct), \quad t \in \mathbb{R}^+, \quad y \in \{-v, c\}.$$

According to Orsingher [24], in a large ensemble of particles moving according to the above mentioned rules, the function $w_0(x,t\,|\,y)$ represents the excess of forward moving particles with respect to backward moving ones near state x at time t, when no resets occur up to time t. Hence, if the initial velocity is chosen at random with equal probability, then the related flow function is given by the following mixture:

$$w_0(x,t) := \frac{1}{2}\left[w_0(x,t\,|\,-v) + w_0(x,t\,|\,c)\right], \qquad x \in (-vt, ct), \quad t \in \mathbb{R}^+. \tag{16}$$

3.2 Analysis in the General Case

Let us now focus on the process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$, having state-space $\mathbb{R} \times \{-v, c\}$ and subject to the initial condition specified in Eq. (1). The restarting velocity of the particle after any reset is identical to the initial velocity $V(0) = y \in \{-v, c\}$. Hence, by taking into account the effect of instantaneous resets, the conditional probability law of $\{(X(t), V(t)),$ $t \in \mathbb{R}^+\}$ consists of two components:

(i) A discrete component:

$$\mathsf{P}[X(t) = yt, V(t) = y\,|\,X(0) = 0, V(0) = y], \qquad y \in \{-v, c\},$$

for the probability that the particle does not change velocity in $(0, t]$ and then it occupies one of the extremes of $[-vt, ct]$ at time $t \in \mathbb{R}^+$.

(ii) An absolutely continuous component, described by the following p.d.f.:

$$p(x,t\,|\,y) = \mathsf{P}[X(t) \in \mathrm{d}x\,|\,X(0) = 0, V(0) = y]/\mathrm{d}x, \qquad x \in (-vt, ct), \tag{17}$$

that accounts for the position x occupied at time $t \in \mathbb{R}^+$ when the particle changes velocity or has a reset at least once in $(0, t]$.

Let us now provide a general expression for the probability law described above. It is based on renewal arguments, similar to those exploited in Eq. (23) of Evans and Majumdar [9]. We recall that the resets to the origin occur according to the renewal process $\{K(t), t \in \mathbb{R}_0^+\}$, whose renewal function $M(t)$ and renewal intensity $\rho(t)$ are given respectively in (4) and (5).

Theorem 2 *For the stochastic process* $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ *conditional on the initial state* $X(0) = 0$, $V(0) = y \in \{-v, c\}$, *for* $t \in \mathbb{R}^+$ *one has*

$$\mathsf{P}[X(t) = yt, V(t) = y \mid X(0) = 0, V(0) = y] = \overline{F}_{W_1}(t), \tag{18}$$

where $\overline{F}_{W_1}(t)$ *is given in (9), and, for* $x \in (-vt, ct)$,

$$\begin{aligned} p(x, t \mid y) = p_0(x, t \mid y) &+ \int_0^{t-m(x)} \rho(s)\, p_0(x, t - s \mid y)\, \mathrm{d}s \\ &+ \frac{1}{|y|}\, \rho\Big(t - \frac{x}{y}\Big)\, \overline{F}_{W_1}\Big(\frac{x}{y}\Big)\, \mathbf{1}_{\{0 < \frac{x}{y} < t\}}, \end{aligned} \tag{19}$$

where $p_0(x, t \mid y)$ *is the p.d.f. obtained in Proposition 1,*

$$m(x) := \max\Big\{ -\frac{x}{v}, \frac{x}{c} \Big\}, \qquad x \in \mathbb{R}, \tag{20}$$

and $\rho(s)$ *is the renewal intensity defined in (5).*

Proof Eq. (18) immediately follows noting that the left-hand-side is equal to $\mathsf{P}[D_1 > t]$ if $y = -v$, and $\mathsf{P}[U_1 > t]$ if $y = c$. For the p.d.f. defined in (17), the decomposition in the right-hand-side of (19) holds upon the following remarks, by recalling that after any reset (that is a renewal event) the motion proceeds with the same initial velocity y and is governed by the same probabilistic laws independently of the past.

(i) Recalling Eq. (11), the first term on the right-hand side of (19) is the p.d.f. of the motion of the particle without undergoing resetting in $(0, t]$.

(ii) The second term on the right-hand side of (19) concerns the case in which at least a reset occurs in $(0, t]$, the last reset occurring at time $s \in (0, t]$ with renewal intensity $\rho(s)$, and then the motion proceeds with at least a velocity change and no resets in $(s, t]$. Hence, the integral term in Eq. (19) follows from applying the strong Markov property to the process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$. Indeed, after a reset the process restarts from state 0, with the same initial velocity $V(0) = y$, and the subsequent evolution is independent of the past. Therefore, conditioning on the last reset time $s < t$ and recalling that $x = c(t - s)$ if $y = c$, and $x = -v(t - s)$ if $y = -v$, then two cases arise: $s < t - \frac{x}{c}$ for $x > 0$, and $s < t + \frac{x}{v}$ for $x < 0$.

(iii) Through the third term on the right-hand side of (19) we account for at least a reset in $(0, t]$, with the last reset at time $s \in (0, t]$, and with no velocity changes (and thus no resets) in $(s, t]$. So at time t the particle occupies the state $x = -v(t - s)$ if $y = -v$, or the state $x = c(t - s)$ if $y = c$. Formally, this term is expressed as

$$\int_0^t \rho(s)\, \delta(x - y(t - s))\, \overline{F}_{W_1}(t - s)\, \mathrm{d}s,$$

where $\delta(\cdot)$ denotes the Dirac-delta function. Hence, the thesis follows by its scaling property $\delta(yx) = \frac{1}{|y|}\delta(x)$. □

Hereafter we show the normalization condition for $X(t)$.

Proposition 3 *For all* $t \in \mathbb{R}^+$ *and* $y \in \{-v, c\}$ *we have*

$$\mathsf{P}[X(t) \in [-vt, ct] \mid X(0) = 0, V(0) = y] = 1.$$

Proof Integrating both sides of Eq. (19) for $x \in (-vt, ct)$, we have that

$$\mathsf{P}[X(t) \in (-vt, ct) \mid X(0) = 0, V(0) = y]$$

is expressed as the sum of three terms. Due to (11) and Remark 1, the first term is given by

$$\mathcal{I}_1 = \int_{-vt}^{ct} p_0(x, t \mid y)\, \mathrm{d}x = \mathsf{P}[X(t) \in (-vt, ct),\, R_1 > t,\, W_1 \leq t \mid X(0) = 0,\, V(0) = y]. \tag{21}$$

The second term, thanks to the Fubini-Tonelli theorem, Eq. (20) and the independence of the renewals, is expressed as

$$\begin{aligned}\mathcal{I}_2 &= \int_{-vt}^{ct} \mathrm{d}x \int_0^{t-m(x)} \rho(s)\, p_0(x, t-s \mid y)\, \mathrm{d}s = \int_0^t \mathrm{d}s \int_{-v(t-s)}^{c(t-s)} \rho(s)\, p_0(x, t-s \mid y)\, \mathrm{d}x \\ &= \int_0^t \rho(s)\, \mathsf{P}[X(t) \in (-v(t-s), c(t-s)),\, R_1^{(s)} > t-s \mid X(s) = 0,\, V(s) = y]\, \mathrm{d}s,\end{aligned} \tag{22}$$

where $R_1^{(s)}$ indicates the length of the time interval between the reset instant s and the first subsequent reset instant. For the third term we have

$$\begin{aligned}\mathcal{I}_3 &= \frac{1}{|y|} \int_{-vt}^{ct} \rho\Big(t - \frac{x}{y}\Big)\, \overline{F}_{W_1}\Big(\frac{x}{y}\Big)\, \mathbf{1}_{\{0<\frac{x}{y}<t\}}\, \mathrm{d}x = \int_0^t \rho(s)\, \overline{F}_{W_1}(t-s)\, \mathrm{d}s \\ &= \int_0^t \rho(s)\, \mathsf{P}[X(t) = y(t-s),\, W_1 > t-s \mid X(s) = 0,\, V(s) = y]\, \mathrm{d}s.\end{aligned} \tag{23}$$

The term (22) is equal to the probability that at least a reset occurs in $(0, t)$, with the last one at time s, and at least a velocity change occurs in (s, t). The term (23) is equal to the probability that at least a reset occurs in $(0, t)$, with the last one at time s, and that no velocity changes occur in (s, t). Hence, the sum $\mathcal{I}_2 + \mathcal{I}_3$ yields

$$\mathsf{P}[X(t) \in (-vt, ct),\, R_1 \leq t \mid X(0) = 0,\, V(0) = y].$$

By summing up the last term to the probability in (21), and noting that $\{R_1 \leq t\} \subseteq \{W_1 \leq t\}$, one obtains $\mathsf{P}(W_1 \leq t)$. Hence, the thesis follows by Eq. (18) and recalling that the support of $X(t)$ at time t is $[-vt, ct]$. □

In the following, we focus on the probability law of $\{(X(t), V(t)),\, t \in \mathbb{R}_0^+\}$ in some special cases when the distribution of the intertimes $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ are known, i.e. when they have (i) exponential distribution, and (ii) Erlang-2 distribution.

4 Exponentially Distributed Intertimes

Let us analyze the random motion when the times separating velocity changes of the particle have identical exponential distribution. Here, the intertimes $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ are assumed to be exponentially distributed with parameter $\lambda \in \mathbb{R}^+$. Hence, with reference to the notation introduced in Section 2, for any $i \in \mathbb{N}$ we have

$$\overline{F}_{U_i}(t) = \overline{F}_{D_i}(t) = e^{-\lambda t}, \qquad t \in \mathbb{R}^+ \tag{24}$$

and the random sums introduced in (2) have Erlang p.d.f.'s

$$f_U^{(k)}(t) = f_D^{(k)}(t) = \frac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}, \qquad t \in \mathbb{R}^+, \quad k \in \mathbb{N}. \tag{25}$$

These assumptions give that the counting process describing the occurrence of velocity changes is a Poisson process with intensity λ, as for the classical telegraph process.

Remark 4 If the intertimes $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ of $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ are exponentially distributed with parameter $\lambda \in \mathbb{R}^+$ then it is well known that, for a given $\theta \in [0, 1]$, the reset time R_1 is exponentially distributed with parameter $\theta\lambda$, being a geometric mixture of Erlang distributed random times due to (6). In this case, the renewal intensity (5) is constant:

$$\rho(t) = \theta\lambda, \qquad t \in \mathbb{R}_0^+. \tag{26}$$

To determine the probability law of the process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$, we recall the modified Bessel function of first kind, for ν real,

$$I_\nu(x) = \sum_{k=0}^{+\infty} \frac{1}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}, \qquad x \in \mathbb{R}^+. \tag{27}$$

We first analyze the motion when no resets occur. Indeed, making use of Proposition 1, we are now able to determine the p.d.f.'s defined in Eqs. (11) and (12).

Proposition 4 *If the intertimes $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ of $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ are exponentially distributed with parameter $\lambda \in \mathbb{R}^+$ then the sub-densities concerning the forward and the backward motion at time $t \in \mathbb{R}^+$ when no resets occur up to time t, for $x \in (-vt, ct)$, are given by*

$$\begin{aligned} f_0(x, t \mid c) &= \frac{(1-\theta)\lambda\, e^{-\lambda t}}{c+v} \sqrt{\frac{\tau^*}{t-\tau^*}}\, I_1\Big[2(1-\theta)\lambda\sqrt{\tau^*(t-\tau^*)}\Big], \\ b_0(x, t \mid c) &= \frac{(1-\theta)\lambda\, e^{-\lambda t}}{c+v}\, I_0\Big[2(1-\theta)\lambda\sqrt{\tau^*(t-\tau^*)}\Big], \end{aligned} \tag{28}$$

and

$$\begin{aligned} f_0(x, t \mid -v) &= \frac{(1-\theta)\lambda\, e^{-\lambda t}}{c+v}\, I_0\Big[2(1-\theta)\lambda\sqrt{\tau^*(t-\tau^*)}\Big], \\ b_0(x, t \mid -v) &= \frac{(1-\theta)\lambda\, e^{-\lambda t}}{c+v} \sqrt{\frac{t-\tau^*}{\tau^*}}\, I_1\Big[2(1-\theta)\lambda\sqrt{\tau^*(t-\tau^*)}\Big], \end{aligned} \tag{29}$$

with $\tau^* = (x+vt)/(c+v)$, and $I_\nu(x)$ defined in (27). Hence, the conditional p.d.f. of $X(t)$ with no resets until time t can be expressed as

$$\begin{aligned} p_0(x, t \mid c) &= \frac{(1-\theta)\lambda\, e^{-\lambda t}}{c+v} \sum_{k=0}^{1} \left(\frac{\tau^*}{t-\tau^*}\right)^{k/2} I_k\Big[2(1-\theta)\lambda\sqrt{\tau^*(t-\tau^*)}\Big], \\ p_0(x, t \mid -v) &= \frac{(1-\theta)\lambda\, e^{-\lambda t}}{c+v} \sum_{k=0}^{1} \left(\frac{t-\tau^*}{\tau^*}\right)^{k/2} I_k\Big[2(1-\theta)\lambda\sqrt{\tau^*(t-\tau^*)}\Big]. \end{aligned} \tag{30}$$

The expressions provided in Proposition 4 can be obtained by straightforward calculations, making use of Proposition 1 and Eqs. (13), (24) and (25). Equivalently, they follow from the analogous formulas for the classical telegraph process, by taking $(1-\theta)\lambda$ as intensity of the underlying Poisson process, the latter counting the velocity changes when no resets occur. It is useful to note the characteristic, also found in other finite-velocity random motions, that the argument of the Bessel functions in Eqs. (28), (29) and (30), vanishes precisely at the extremes of the diffusion interval $[-vt, ct]$.

Remark 5 The densities shown in Proposition 4 allow to obtain the flow function $w_0(x, t \mid y)$ given in Remark 3 and, in particular, the flow function when the initial velocity is chosen at random w.p. $\frac{1}{2}$, defined in Eq. (16). The expression is omitted since it is obtained by direct

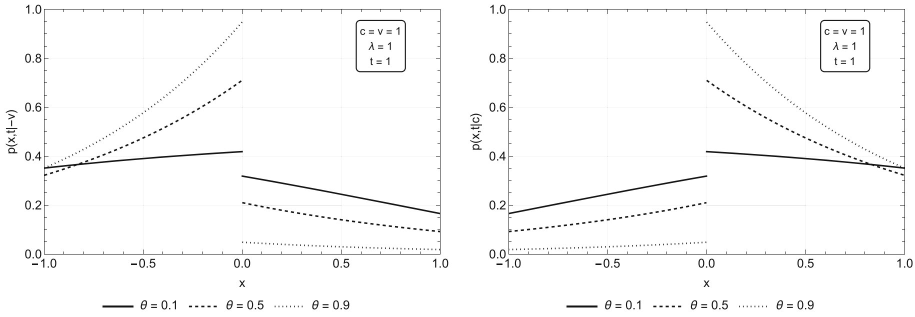

Fig. 2 Plots of $p(x, t \mid y)$, for $y = -v$ (left) and $y = c$ (right), given in (32), for various choices of θ.

calculations. In particular, by taking $\theta = 0$ it identifies with the flow functions given in Eq. (13) of Theorem 1 of Orsingher [24] (for $\lambda_1 = \lambda_2$ and $v = c$) and in Eq. (4.3) of Theorem 4.1 of Beghin et al. [1] when no drift is assumed and the asymmetry is due only to different rates (for $\lambda_1 \neq \lambda_2$ and $v = c$).

Now, resorting to Theorem 2 we are able to provide the probability law of $X(t)$ in the case of exponential intertimes.

Theorem 5 *If the intertimes* $\{U_i, i \in \mathbb{N}\}$ *and* $\{D_i, i \in \mathbb{N}\}$ *of* $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ *are exponentially distributed with parameter* $\lambda \in \mathbb{R}^+$ *then, for all* $t \in \mathbb{R}^+$ *and* $y \in \{-v, c\}$*, we have*

$$\mathsf{P}[X(t) = yt, V(t) = y \mid X(0) = 0, V(0) = y] = e^{-\lambda t}. \tag{31}$$

Moreover, for $x \in (-vt, ct)$*, one has*

$$p(x, t \mid y) = p_0(x, t \mid y) + \theta\lambda \int_0^{t-m(x)} p_0(x, t-s \mid y)\, \mathrm{d}s + \frac{\theta\lambda}{|y|} e^{-\lambda \frac{x}{y}} \mathbf{1}_{\{0 < \frac{x}{y} < t\}}, \tag{32}$$

where $p_0(x, t \mid y)$ *is the probability p.d.f. obtained in Proposition 4, and* $m(x)$ *is given in (20).*

Proof Eq. (31) follows from (18). The right-hand-side of Eq. (32) is obtained making use of (19) and recalling (26). □

Unfortunately, the integral in the right-hand-side of (32) is not available in closed form, since it involves expressions as

$$\int_0^{t-x/c} e^{-\lambda(t-s)} \left(\frac{x + v(t-s)}{c(t-s) - x}\right)^{k/2} I_k\left[\frac{2(1-\theta)\lambda}{c+v}\sqrt{[x + v(t-s)][c(t-s) - x]}\right] \mathrm{d}s,$$

for $k \in \{0, 1\}$, when $y = c$ and $x \in (0, ct)$, for instance.

Remark 6 Under the assumptions of Proposition 4 and Theorem 5 the following symmetry properties hold for all $t \in \mathbb{R}^+$ and $x \in (-vt, ct)$:

$$p_0^{(-v,c)}(x, t \mid c) = p_0^{(-c,v)}(-x, t \mid -v), \qquad p^{(-v,c)}(x, t \mid c) = p^{(-c,v)}(-x, t \mid -v),$$

where $p_0^{(c_1,c_2)}$ and $p^{(c_1,c_2)}$ denote the relevant p.d.f.'s when the backward and forward velocities are c_1 and c_2, respectively.

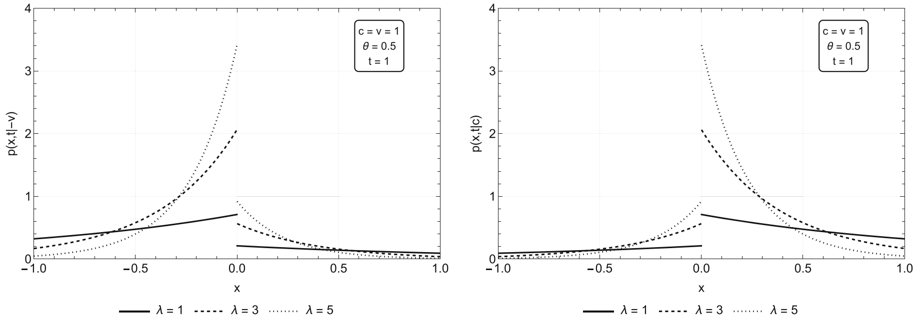

Fig. 3 Plots of $p(x, t \mid y)$, for $y = -v$ (left) and $y = c$ (right), given in (32), for various choices of λ

Figures 2 and 3 show some plots of $p(x, t \mid y)$ obtained by means of numerical calculations. They confirm the symmetry pinpointed in Remark 6. Clearly, the existence of a discontinuity of $p(x, t \mid y)$ at $x = 0$ is much more evident for larger θ, and for larger λ, i.e. for larger values of the renewal intensity (26). This is typical for processes subject to reset in the origin (see, e.g., Fig. 2 of Di Crescenzo et al. [12] for the telegraph process driven by the geometric counting process with Poisson-based resetting).

It is now essential to show that the probability mass of $X(t)$ is unity over $[-vt, ct]$.

Remark 7 Under the assumptions of Theorem 5 it is not hard determining explicitly the terms used in Proposition 3 to show that, for $y \in \{-v, c\}$,

$$\mathsf{P}[-vt \le X(t) \le ct \mid X(0) = 0, V(0) = y] = 1, \qquad t \in \mathbb{R}^+. \tag{33}$$

Indeed, in this case from Eqs. (21), (22) and (23), recalling (14), and making use of (27) and (28), for $y = c$ one has, for $t \in \mathbb{R}^+$:

$$\begin{aligned}
\mathcal{I}_1 &= \int_{-vt}^{ct} p_0(x, t \mid c)\,\mathrm{d}x \\
&= e^{-\lambda t}\Bigg\{\sum_{k=1}^{\infty}\left(\frac{(1-\theta)\lambda}{c+v}\right)^{2k}\frac{1}{k!(k-1)!}\int_{-vt}^{ct}(x+vt)^k(ct-x)^{k-1}\,\mathrm{d}x \\
&+ \sum_{k=0}^{\infty}\left(\frac{(1-\theta)\lambda}{c+v}\right)^{2k+1}\frac{1}{(k!)^2}\int_{-vt}^{ct}(x+vt)^k(ct-x)^k\,\mathrm{d}x\Bigg\} \\
&= e^{-\lambda t}\big[\cosh((1-\theta)\lambda t) + \sinh((1-\theta)\lambda t) - 1\big] \\
&= e^{-\theta\lambda t} - e^{-\lambda t},
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}_2 &= \int_0^t \mathrm{d}s \int_{-v(t-s)}^{c(t-s)} \rho(s)\,p_0(x, t-s \mid c)\mathrm{d}x \\
&= \theta\lambda \int_0^t e^{-\lambda(t-s)}\Big[\cosh((1-\theta)\lambda(t-s)) + \sinh((1-\theta)\lambda(t-s)) - 1\Big]\mathrm{d}s \\
&= \theta\lambda \int_0^t e^{-\lambda(t-s)}\left[e^{\lambda(1-\theta)(t-s)} - 1\right]\mathrm{d}s \\
&= 1 - e^{-\theta\lambda t} - \theta(1 - e^{-\lambda t}),
\end{aligned}$$

and

$$\mathcal{I}_3 = \int_0^t \rho(s)\,\overline{F}_{W_1}(t-s)\,\mathrm{d}s = \int_0^t \theta\lambda\, e^{-\lambda(t-s)}\,\mathrm{d}s \\ = \theta\,(1-e^{-\lambda t}).$$

Therefore, the sum $\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3$ gives $1 - e^{\lambda t}$, which added to the right-hand side of (31) gives the desired result. The case $y = -v$ can be treated similarly.

Remark 8 We analyze the limiting behavior of the p.d.f. (32) when x tends to the extremes of the diffusion interval $(-vt, ct)$. Under the assumptions of Theorem 5, making use of Eqs. (27) and (30), and due to the symmetry stated in Remark 6, for any $t \in \mathbb{R}^+$ and $y \in \{-v, c\}$ we have:

$$\lim_{x\to yt} p(x,t\,|\,y) = \frac{(1-\theta)(1+(1-\theta)\lambda t)\lambda e^{-\lambda t}}{c+v} + \frac{\theta\lambda e^{-\lambda t}}{|y|},$$

$$\lim_{x\to y^* t} p(x,t\,|\,y) = \frac{(1-\theta)\lambda e^{-\lambda t}}{c+v},$$

where

$$y^* = \begin{cases} c, & \text{if } y = -v \\ -v, & \text{if } y = c. \end{cases} \tag{34}$$

It is easy to see that the second limit above is decreasing in θ, whereas the first one in general is not monotonic in θ, as also confirmed in Figure 2.

Remark 9 Let us consider the limiting behavior of the p.d.f. (32) when the reset probability θ tends to the limit cases. Under the assumptions of Theorem 5, for any $t \in \mathbb{R}^+$ we have:

$$\lim_{\theta\to 0} p(x,t\,|\,c) = \frac{\lambda\, e^{-\lambda t}}{c+v} \sum_{k=0}^{1} \left(\frac{\tau^*}{t-\tau^*}\right)^{k/2} I_k\Big[2\lambda\sqrt{\tau^*(t-\tau^*)}\Big],$$

$$\lim_{\theta\to 0} p(x,t\,|-v) = \frac{\lambda\, e^{-\lambda t}}{c+v} \sum_{k=0}^{1} \left(\frac{t-\tau^*}{\tau^*}\right)^{k/2} I_k\Big[2\lambda\sqrt{\tau^*(t-\tau^*)}\Big],$$

these results being in agreement with the limits of a classical telegraph process. Moreover, for $y \in \{-v, c\}$ one has

$$\lim_{\theta\to 1} p(x,t\,|\,y) = \frac{\lambda}{|y|} e^{-\lambda\frac{x}{y}} \mathbf{1}_{\{0<\frac{x}{y}<t\}},$$

i.e. the exponential p.d.f. that arises when at every intertime there is a reset to the origin.

4.1 Moments of the Position Process with Exponential Intertimes

In this section we focus on the moments of $X(t)$. In particular, we determine the first and the second moment of $X(t)$ subject to resets when the intertimes are exponentially distributed. To this aim, we denote by $\mathsf{E}_y[X^k(t)]$ the k-th conditional moment of $X(t)$ under the initial condition $\{X(0) = 0, V(0) = y\}$, for $y \in \{-v, c\}$.

Theorem 6 *Under the assumptions of Theorem 5, for all $t \in \mathbb{R}^+$ and $y \in \{-v, c\}$, we have*

$$\mathsf{E}_y[X(t)] = \frac{1}{2\lambda}\left[\frac{c-v}{\theta}\big(1-e^{-\theta\lambda t}\big) + \operatorname{sgn}(y)\,\frac{c+v}{2-\theta}\big(1-e^{-(2-\theta)\lambda t}\big)\right] \tag{35}$$

and

$$
\begin{aligned}
\mathsf{E}_y[X^2(t)] &= \left(\frac{c-v}{\sqrt{2}\theta\lambda}\right)^2\big(1-e^{-\theta\lambda t}(1+\theta\lambda t)\big) - \left(\frac{c+v}{\sqrt{2}(2-\theta)\lambda}\right)^2 \\
&\quad + \operatorname{sgn}(y)\left(\frac{2\,y\,(c+v)}{(2-\theta)^2\theta\lambda^2} - \frac{(c^2-v^2)t\,e^{-(2-\theta)\lambda t}}{2(2-\theta)\lambda}\right) - \frac{c+v}{4(2-\theta)^2(1-\theta)\theta\lambda^2}e^{-\lambda t} \\
&\quad \times \Bigg[(c+v)[2(1-\theta)^2\cosh((1-\theta)\lambda t) + (2-\theta)^2\sinh((1-\theta)\lambda t)] \\
&\quad + \operatorname{sgn}(y)(3y+y^*) \\
&\quad [2(1-\theta)\cosh((1-\theta)\lambda t) + (2-(2-\theta)\theta)\sinh((1-\theta)\lambda t)]\Bigg],
\end{aligned}
\tag{36}
$$

where y^* is defined in (34).

The proof is given in Appendix A.1.

Remark 10 We can study the monotonicity of $\mathsf{E}_y[X(t)]$, for $y \in \{-v, c\}$. From Eq. (35) one has

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}_y[X(t)] = \frac{1}{2}\left[(c-v)e^{-\theta\lambda t} + \operatorname{sgn}(y)(c+v)e^{-(2-\theta)\lambda t}\right], \qquad t \in \mathbb{R}^+.
$$

Hence, by setting

$$
t_M := \frac{1}{2(1-\theta)\lambda}\log\left(\frac{c+v}{|c-v|}\right),
$$

we get:

(i) If $c < v$, then $\mathsf{E}_c[X(t)]$ is strictly increasing for $0 < t < t_M$, has a maximum for $t = t_M$, and it is strictly decreasing for $t > t_M$.
(ii) If $c \geq v$, then $\mathsf{E}_c[X(t)]$ is strictly increasing for all $t > 0$.

By symmetry, we have also:

(i) If $v < c$, then $\mathsf{E}_{-v}[X(t)]$ is strictly decreasing for $0 < t < t_M$, has a minimum for $t = t_M$, and it is strictly increasing for $t > t_M$.
(ii) If $v \geq c$, then $\mathsf{E}_{-v}[X(t)]$ is strictly decreasing for all $t > 0$.

Making use of Theorem 6, now we can show that the moments $\mathsf{E}_y[X^k(t)]$, $k = 1, 2$, admit finite limits as $t \to +\infty$.

Corollary 1 *Under the assumptions of Theorem 6, for $y \in \{-v, c\}$ and y^* given in (34), one has:*

$$
\begin{aligned}
\lim_{t\to+\infty}\mathsf{E}_y[X(t)] &= \frac{c-v-py^*}{(2-\theta)\theta\lambda}, \\
\lim_{t\to+\infty}\mathsf{E}_y[X^2(t)] &= \frac{2\big[y^2 - cv(2-\theta)(1-\theta) + (y^*)^2(1-\theta)\big]}{\big[(2-\theta)\theta\lambda\big]^2}.
\end{aligned}
$$

The behavior described in Remark 10 and in Corollary 1 is confirmed in Figure 4, which shows some instances of $\mathsf{E}_y[X(t)]$, $y \in \{-v, c\}$, for different values of c and v. The plots confirm that the mean of $X(t)$ has a finite limit for $t \to +\infty$. They also show that, for fixed t, $\mathsf{E}_y[X(t)]$ is increasing in $c - v$, which represents the net velocity of the motion in a time unit in absence of resets.

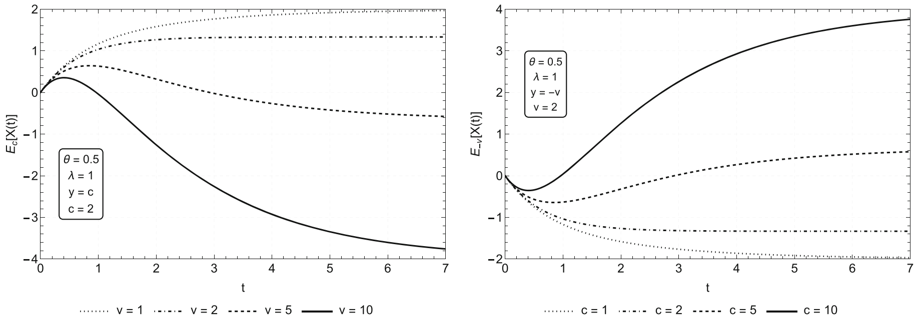

Fig. 4 Plots of $\mathsf{E}_y[X(t)]$, given in (35), for $y = c$ (left) and $y = -v$ (right), for various choices of the non-initial velocity

Remark 11 Under the assumptions of Theorem 5, from Eq. (35) one has the symmetry relation

$$\mathsf{E}_c[X(t)] + \mathsf{E}_{-v}[X(t)] = (c - v)\frac{1 - e^{-\theta\lambda t}}{\theta\lambda}, \qquad t \in \mathbb{R}_0^+. \tag{37}$$

The term in the right-hand side of (37) tends to $(c - v)t$ as $\theta \to 0$, similarly as noted in Remark 3.1 of [35].

Let us now analyze the behavior of the conditional mean $\mathsf{E}_y[X(t)]$ for the limit cases of θ, i.e., when (i) $\theta \to 0$ (no resets), and (ii) $\theta \to 1$ (resets at all velocity changes).

Remark 12 Under the assumptions of Theorem 5, for all $t \in \mathbb{R}^+$ and $y \in \{-v, c\}$, we obtain the following limits, which are in agreement with the case in absence of resets:

$$\begin{aligned}\lim_{\theta\to 0} \mathsf{E}_y[X(t)] &= \frac{1}{2}\left[(c - v)t + \operatorname{sgn}(y)\frac{c + v}{2\lambda}\left(1 - e^{-2\lambda t}\right)\right],\\ \lim_{\theta\to 0} \mathsf{E}_y[X^2(t)] &= \frac{1}{8\lambda^2}\Big[(c + v)(c + v - \operatorname{sgn}(y)2(c - v)\lambda t)e^{-2\lambda t}\\ &\quad + 4\lambda t(y^2 + cv) + 2[\lambda t(c - v)]^2 - (c + v)^2\Big].\end{aligned} \tag{38}$$

Moreover, we have

$$\begin{aligned}\lim_{\theta\to 1} \mathsf{E}_y[X(t)] &= \frac{y}{\lambda}\left(1 - e^{-\lambda t}\right),\\ \lim_{\theta\to 1} \mathsf{E}_y[X^2(t)] &= 2\left(\frac{y}{\lambda}\right)^2\left[1 - (1 + \lambda t)e^{-\lambda t}\right].\end{aligned}$$

These limits depend only on the initial velocity y since now, after each interarrival/reset, the velocity restart as y and the motion does not take the opposite velocity y^*.

Remark 13 In agreement with Remark 6, under the assumptions of Theorem 6, due to (35) the following symmetry property holds:

$$\mathsf{E}_c^{(-v,c)}[X(t)] = -\mathsf{E}_{-v}^{(-c,v)}[X(t)], \qquad t \in \mathbb{R}_0^+,$$

where $\mathsf{E}_y^{(c_1,c_2)}[X(t)]$, $y \in \{-v, c\}$, denotes the mean of $X(t)$ when the backward and forward velocities are c_1 and c_2, respectively.

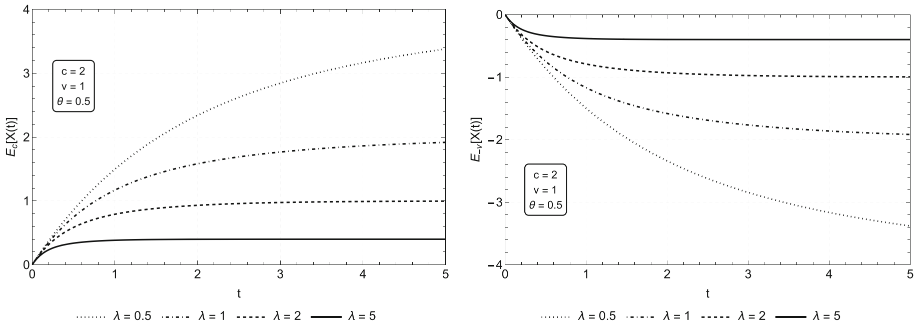

Fig. 5 Plots of $\mathsf{E}_y[X(t)]$, for $y = c$ (left) and $y = -v$ (right), for various choices of λ

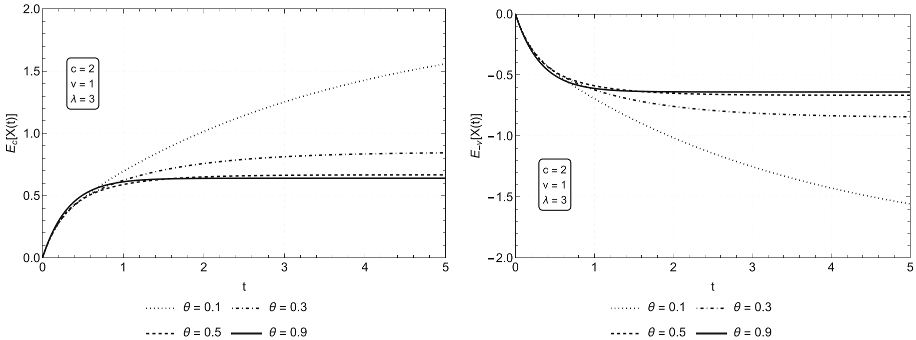

Fig. 6 Plots of $\mathsf{E}_y[X(t)]$, for $y = c$ (left) and $y = -v$ (right), for various choices of θ

The symmetry behavior described in Remark 13 is confirmed in Figures 5 and 6, where plots of the mean of $X(t)$ are shown for various choices of λ and θ, respectively. We can see that $\mathsf{E}_y[X(t)]$, for fixed t, approaches the origin for larger θ and for larger λ, i.e. for larger values of the renewal intensity (26). This is in agreement with the behavior exhibited by $p(x, t \mid y)$ near $x = 0$ in Figures 2 and 3.

Let us denote by $\mathsf{Var}_y[X(t)]$ the variance of the process $X(t)$ under the initial condition $\{X(0) = 0, V(0) = y\}$, for $y \in \{-v, c\}$. Thanks to Eqs. (35) and (36) the following result can be obtained after some calculations.

Corollary 2 *Under the assumptions of Theorem 5, the variance of $X(t)$ can be expressed as follows, for all $t \in \mathbb{R}^+$ and $y \in \{-v, c\}$:*

$$
\begin{aligned}
&\mathsf{Var}_y[X(t)] \\
&= \frac{c+v}{2(2-\theta)^2\lambda^2}\left[\frac{|y|}{\theta}\left(4 - \frac{(2-\theta)^2}{1-\theta}e^{-\theta\lambda t}\right) - (c+v) + \left(|y^*| + \frac{|y|}{1-\theta}\right)e^{-(2-\theta)\lambda t}\right] \\
&\quad + \left(\frac{c-v}{\sqrt{2}\,\theta\lambda}\right)^2 (1 - (1+\theta\lambda t)e^{-\theta\lambda t}) - \operatorname{sgn}(y)\,\frac{(c^2-v^2)t\,e^{-(2-\theta)\lambda t}}{2(2-\theta)\lambda} \\
&\quad - \left[\frac{(c-v)(1-e^{-\theta\lambda t})}{2\theta\lambda} + \operatorname{sgn}(y)\,\frac{(c+v)(1-e^{-(2-\theta)\lambda t})}{2(2-\theta)\lambda}\right]^2,
\end{aligned}
$$

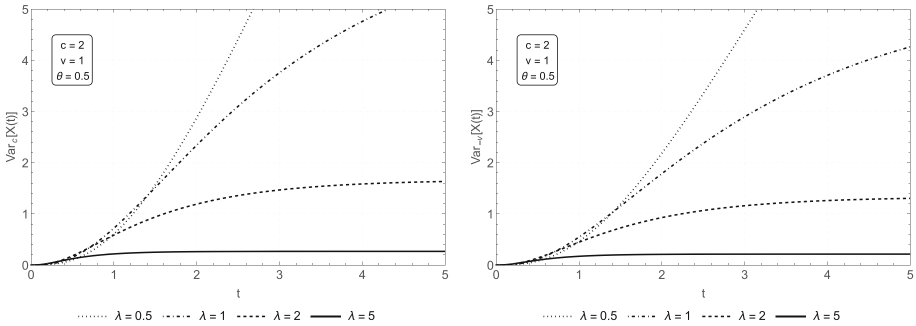

Fig. 7 Plot of $\mathsf{Var}_y[X(t)]$ for $y = c$ (left) and $y = -v$ (right), for various choices of λ

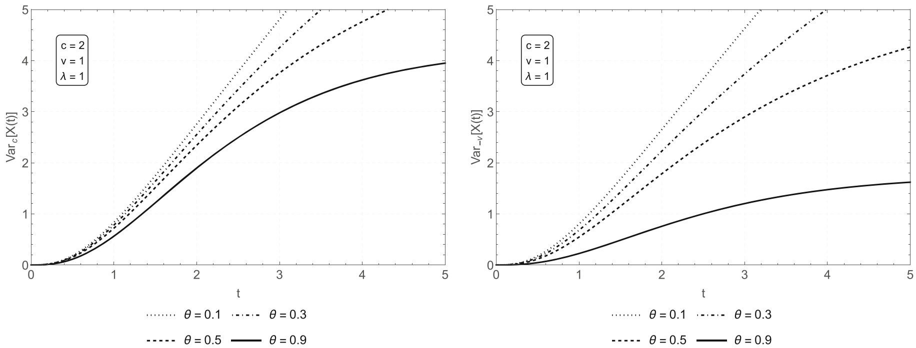

Fig. 8 Plot of $\mathsf{Var}_y[X(t)]$ for $y = c$ (left) and $y = -v$ (right), for various choices of θ

where y^* is defined in (34). Moreover, the following limit holds:

$$\lim_{t\to+\infty} \mathsf{Var}_y[X(t)] = \frac{y^2 - 2cv(1-\theta)^2 + (y^*)^2(1-\theta^2)}{\left[(2-\theta)\theta\lambda\right]^2}. \tag{39}$$

Some plots of $\mathsf{Var}_y[X(t)]$, $y \in \{-v, c\}$, are shown in Figures 7 and 8 as a function of t for different values of λ and θ, respectively. They show that the variance, for fixed t, is decreasing in λ and in θ. Indeed, to increase the renewal intensity (26) amplifies the concentration near the origin, and consequently decreases the variance.

Due to random changes in the direction and instantaneous resets, the initial velocity $V(0) = y$ becomes irrelevant early. Over time, the system's behavior is primarily determined by the reset probability θ and the rate of directional changes λ, rather than by the initial velocity. The plots show that the variance first increases with time before stabilizing at a constant value, as given in Eq. (39). Moreover, as $t \to +\infty$ the effect of the initial conditions diminishes, and the variance reaches a steady value determined by the process parameters. In Figure 7, we see that as λ increases, the variance grows more quickly at first. This suggests that higher frequency of directional changes speeds up the diffusion. In Figure 8, we observe that as θ increases, the variance decreases. This reinforces the idea that resets prevent the process from straying too far from the origin, thereby limiting the dispersion.

Remark 14 Under the assumptions of Theorem 5, for all $t \in \mathbb{R}^+$ and $y \in \{-v, c\}$, we have

$$\lim_{\theta\to 0} \mathsf{Var}_y[X(t)] = \left(\frac{c+v}{4\lambda}\right)^2 \left[1 + 4\lambda t - e^{-4\lambda t}\big(1 - 2e^{2\lambda t}\big)^2\right]; \tag{40}$$

$$\lim_{\theta\to 1} \mathsf{Var}_y[X(t)] = \left(\frac{y}{\lambda}\right)^2 \big(1 - 2\lambda t\, e^{-\lambda t} - e^{-2\lambda t}\big).$$

If $\theta \to 0$, i.e. when the reset mechanism vanishes, in the long run the process loses memory of its initial velocity because the directions are continually reshuffled by velocity reversals. Clearly, the right-hand-side of Eq. (40) is in agreement with the variance of the classical telegraph process. Indeed, if $c = v$ and the initial velocity is random as in (44), then it is not hard to see that our results lead to the variance given in Eq. (28) of Orsingher [24].

If $\theta \to 1$ then the process is reset to the origin at every occurrence of velocity changes. This implies that the initial velocity is continuously reset. In this case, the position of the process remains conditioned by its initial velocity, and for this reason it persists in the limiting expression of the variance.

4.2 Asymptotic Results Under Exponential Intertimes

In this section we focus on limit results for the p.d.f. of $X(t)$ in two different cases.

(i) Stationary density. The results given in Corollary 1 suggest the existence of a stationary regime. With reference to Theorem 5, since the probability (31) tends to 0, hereafter we investigate the stationary p.d.f. of $X(t)$.

Proposition 7 *Let the intertimes $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ of the process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ be exponentially distributed with parameter $\lambda \in \mathbb{R}^+$. Then, for $y \in \{-v, c\}$ one has:*

$$w(x) := \lim_{t\to+\infty} p(x, t \mid y) = \theta\lambda\, L_0^\infty(x \mid y) + \frac{\theta\lambda}{|y|} e^{-\lambda\frac{x}{y}} \mathbf{1}_{\{\frac{x}{y}>0\}}, \qquad x \in \mathbb{R}. \tag{41}$$

where

$$L_0^\infty(x \mid y) := \int_{m(x)}^{+\infty} p_0(x, u \mid y)\, \mathrm{d}u \qquad x \in \mathbb{R}. \tag{42}$$

Proof Due to Proposition 4, we have

$$p_0(x, t \mid y) \sim C\, e^{-\lambda t}\left[I_0(A) + R^{1/2} I_1(A)\right],$$

where $C = const$, $R = R(t) \sim const$, $A = A(t) \le (1-\theta)\lambda t$ for large t, and $I_k(A) \sim e^A/\sqrt{2\pi A}$, so that $p_0(x, t \mid y) = O(e^{-\theta\lambda t})$ as $t \to +\infty$. Hence, by setting $u = t - s$ in the integral in the right-hand-side of Eq. (32) and by taking $t \to +\infty$ in Eq. (32) we obtain (41). □

Remark 15 *(i)* The properties considered in the proof of Proposition 7 and the local integrability of $p_0(x, u \mid y)$ in a neighborhood of $m(x)$ ensure that $L_0^\infty(x \mid y)$ is well defined for all $x \in \mathbb{R}$ and $y \in \{-v, c\}$.

(ii) Since the essential support of $p_0(x, t \mid y)$ can be expressed as $x \in \mathbb{R}$, $t > m(x)$, the function $L_0^\infty(x \mid y)$ represents the asymptotic average of the local time that the process $X(t)$, in the absence of resets and originating at y, spends in the state x.

(iii) In the right-hand-side of Eq. (41), the first term accounts for the sample-paths whose most recent reset occurs at some finite time and is followed by at least one velocity-switching

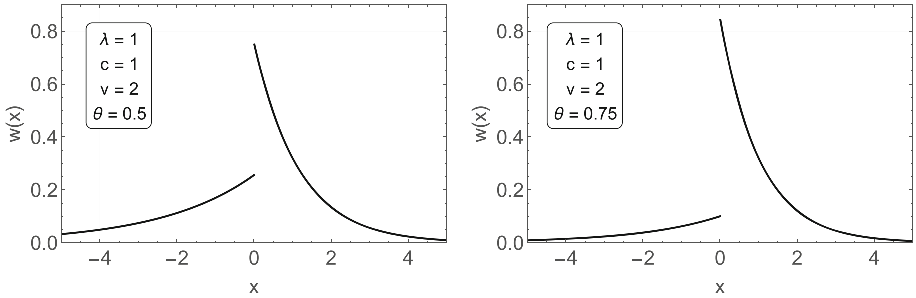

Fig. 9 Plot of $w(x)$, obtained from Eq. (41), with $y = c$

event, whereas the second term corresponds to the sample-paths that experience a reset and then move from the origin with no subsequent velocity change up to the observation time. In conclusion, the existence of the stationary density is a consequence of the resetting mechanism, since the term $p_0(x, t \mid y)$ vanishes as $t \to +\infty$.

Even if the integral in (42) seems not available in closed form (see, for instance, Ratanov and Turov [38]), it can be evaluated numerically. Figure 9 shows two examples of $w(x)$ when $y = c$ (the case $y = -v$ is symmetric), where the last term in Eq. (41) has an exponential trend with support $\mathbb{R}^+$, whereas $L_0^\infty(x \mid y)$ is a continuous function for $x \in \mathbb{R}$, with a cusp at $x = 0$. This yields a jump at $x = 0$, which is typical for processes with resets (see, e.g., Fig. 2 of [12]).

(ii) Kac's scaling. We recall that for the classical telegraph process with velocity changes driven by the Poisson process, it is well known that an asymptotic limit holds under the Kac's scaling, which involve both intensity and velocity, leading to the Gaussian transition function of the Brownian motion (as reported in case (i) of Table 1). Hereafter, in analogy with Lemma 2 of Orsingher [24], we aim to investigate the limit behavior of the probability law of $X(t)$ under the following Kac-type conditions:

$$\lambda \to +\infty, \qquad \frac{c^2}{\lambda} \to \sigma^2 \in \mathbb{R}^+, \qquad \theta\lambda \to \kappa \in \mathbb{R}^+. \tag{43}$$

We focus on the process conditional on the random initial velocity, i.e.

$$C_0 := \{X(0) = 0,\, V(0) \stackrel{d}{=} V_0\}, \qquad \text{with } \mathsf{P}(V_0 = -v) = \mathsf{P}(V_0 = c) = \frac{1}{2}. \tag{44}$$

Clearly, recalling Eq. (11), in this case the p.d.f. of the particle position at time t, when no reset has occurred before, with obvious notation, is expressed as

$$p_0(x, t) = \frac{1}{2}\big(p_0(x, t \mid -v) + p_0(x, t \mid c)\big). \tag{45}$$

In the special case when $v = c$, recalling identity $\frac{\mathrm{d}}{\mathrm{d}x} I_0(x) = I_1(x)$, and making use of Eqs. (27), (30), (45), we obtain the following explicit form, for $t \in \mathbb{R}^+$ and $x \in (-ct, ct)$:

$$\begin{aligned} p_0(x, t) &= \frac{e^{-\lambda t}}{2c}\left[(1-\theta)\lambda\, I_0\left(\frac{(1-\theta)\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \frac{\partial}{\partial t} I_0\left(\frac{(1-\theta)\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right] \\ &= \frac{e^{-\lambda t}}{2c}\left[(1-\theta)\lambda\, I_0\left(\frac{(1-\theta)\lambda}{c}\sqrt{c^2t^2 - x^2}\right) + \frac{(1-\theta)\lambda\, c\, t}{\sqrt{c^2t^2 - x^2}}\, I_1\left(\frac{(1-\theta)\lambda}{c}\sqrt{c^2t^2 - x^2}\right)\right]. \end{aligned} \tag{46}$$

Note that, for $\theta = 0$ (i.e., for no resets), Eq. (46) yields the p.d.f. of the classical telegraph process given, for instance, in Theorem 1 of Orsingher [24].

For the forthcoming analysis it is essential the asymptotic estimate of the modified Bessel function (27), i.e. (cf. Eq. (9.7.1) of Abramowitz and Stegun [39])

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \qquad \text{as} \quad z \to +\infty. \tag{47}$$

We begin by considering the density $p_0(x, t)$ in the limit conditions specified in (43).

Proposition 8 *Let the intertimes $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ of the process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ be exponentially distributed with parameter $\lambda \in \mathbb{R}^+$. Then, the following limit holds for all $t \in \mathbb{R}^+$:*

$$\lim_{\substack{\lambda\to+\infty\\ c^2/\lambda\to\sigma^2\\ \theta\lambda\to\kappa}} p_0(x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{x^2}{2\sigma^2 t}-\kappa t}, \qquad x \in \mathbb{R}. \tag{48}$$

The proof is given in Appendix A.2.

Remark 16 It is worth noting the analogy with the result given in the Lemma 2 of Orsingher [24]. Indeed, as expected, the right-hand side of Eq. (48) corresponds to the Gaussian transition function of the Brownian motion at t, times the probability that 0 events occur in $[0, t]$ for a Poisson process having intensity κ.

In analogy with Eq. (17), and recalling Eq. (44), for $t \in \mathbb{R}^+$ and $x \in (-ct, ct)$

$$p(x, t) = \mathsf{P}[X(t) \in \mathrm{d}x \mid C_0]/\mathrm{d}x = \frac{1}{2}\big(p(x, t \mid -c) + p(x, t \mid c)\big) \tag{49}$$

denotes the p.d.f. of the absolutely continuous component of the probability law of $X(t)$ when the initial velocity is random (with equal probabilities), with $v = c$. In this case, due to (32) and (49), the following relation holds for $t \in \mathbb{R}^+$ and $x \in (-ct, ct)$:

$$p(x, t) = p_0(x, t) + \theta\lambda \int_0^{t-m(x)} p_0(x, t-s)\,\mathrm{d}s + \frac{\theta\lambda}{2c}\left(e^{\lambda\frac{x}{c}}\mathbf{1}_{\{-ct<x<0\}} + e^{-\lambda\frac{x}{c}}\mathbf{1}_{\{0\le x<ct\}}\right). \tag{50}$$

To analyze the behavior of $p(x, t)$ under the limit conditions (43), we first provide the following result for the Brownian motion subject to Poisson-driven resets at the origin, expressed in terms of the complementary error function

$$\mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2}\mathrm{d}t, \qquad x \in \mathbb{R}. \tag{51}$$

Proposition 9 *The probability density of the Brownian motion process with mean 0 and infinitesimal variance σ^2, with $\sigma \in \mathbb{R}^+$, subject to instantaneous resets at 0 occurring with rate $\kappa \in \mathbb{R}^+$, for $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$ is given by*

$$\begin{aligned}\varphi(x, t; \kappa) := {} & \frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{x^2}{2\sigma^2 t}-\kappa t}\\ & + \frac{\sqrt{2\kappa}}{4\sigma}\left\{e^{-\sqrt{2\kappa}\frac{|x|}{\sigma}}\mathrm{Erfc}\left(\frac{|x|}{\sqrt{2\sigma^2 t}} - \sqrt{\kappa t}\right) - e^{\sqrt{2\kappa}\frac{|x|}{\sigma}}\mathrm{Erfc}\left(\frac{|x|}{\sqrt{2\sigma^2 t}} + \sqrt{\kappa t}\right)\right\}.\end{aligned} \tag{52}$$

Proof Due to the Markov property, the p.d.f. of interest at time $t \in \mathbb{R}^+$ can be decomposed as (see, for instance, the approach used by Giorno et al. [40])

$$\varphi(x, t; \kappa) = e^{-\kappa t}\phi(x, t) + \kappa \int_0^t e^{-\kappa\tau}\phi(x, \tau)\,\mathrm{d}\tau, \qquad x \in \mathbb{R},$$

where $\phi(x, t)$ is the Gaussian density at x, with mean 0 and variance $\sigma^2 t$. Then, the expression (52) follows from some calculations. □

We are now able to investigate the limit behavior of $p(x, t)$.

Proposition 10 *Let the intertimes $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ of the process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ be exponentially distributed with parameter $\lambda \in \mathbb{R}^+$. If the initial velocity is random as specified in Eq. (44) and if $v = c$, then the following limit holds for all $t \in \mathbb{R}^+$:*

$$\lim_{\substack{\lambda\to+\infty \\ c^2/\lambda\to\sigma^2 \\ \theta\lambda\to\kappa}} \mathsf{P}[X(t) = yt, V(t) = y \mid C_0] = 0, \qquad y \in \{-c, c\}, \tag{53}$$

and, for $x \in \mathbb{R}$,

$$\lim_{\substack{\lambda\to+\infty \\ c^2/\lambda\to\sigma^2 \\ \theta\lambda\to\kappa}} p(x, t) = \varphi(x, t; \kappa), \tag{54}$$

where $\varphi(x, t; \kappa)$ is given in (52).

The proof is given in Appendix A.3.

Remark 17 Due to Eq. (52), it is straightforward to verify that $\varphi(x, t; \kappa)$ is an even function in $x \in \mathbb{R}$ and it exhibits a cusp behavior at $x = 0$, such that

$$\varphi(0, t; \kappa) = \frac{e^{-\kappa t}}{\sqrt{2\pi\sigma^2 t}} + \frac{\sqrt{2\kappa}}{2\sigma}\,\mathrm{Erf}\big(\sqrt{\kappa t}\big), \qquad \lim_{x\to 0^\pm}\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x, t; \kappa) = \mp\frac{\kappa}{\sigma^2},$$

where $\mathrm{Erf}(x) = 1 - \mathrm{Erfc}(x)$. Furthermore, it is a *bona-fide* p.d.f., i.e., $\int_{\mathbb{R}} \varphi(x, t; \kappa)\,dx = 1$ for $t \in \mathbb{R}^+$.

For various choices of the parameters, Figure 10 shows some plots of density $p(x, t)$, evaluated applying numerical methods to Eq. (50), and of its approximation $\varphi(x, t; \kappa)$ given in Eq. (52), with c^2/λ and $\lambda\theta$ fixed. The plots confirm that the approximation between the two densities improves as θ approaches 0 and when λ increases.

4.3 Mean-Square Distance

In this section we investigate the mean-square distance between the process subject to resets, $X(t)$, and an independent copy of the process in absence of resets, henceforth denoted by $X_0(t)$. Specifically, we define the distance

$$\begin{aligned}\mathsf{D}_y^\lambda(p, t) &:= \mathsf{E}_y[|X(t) - X_0(t)|^2] \\ &= \mathsf{E}_y[X^2(t)] + \mathsf{E}_y[X_0^2(t)] - 2\mathsf{E}_y[X(t)]\mathsf{E}_y[X_0(t)], \qquad t \in \mathbb{R}_0^+.\end{aligned} \tag{55}$$

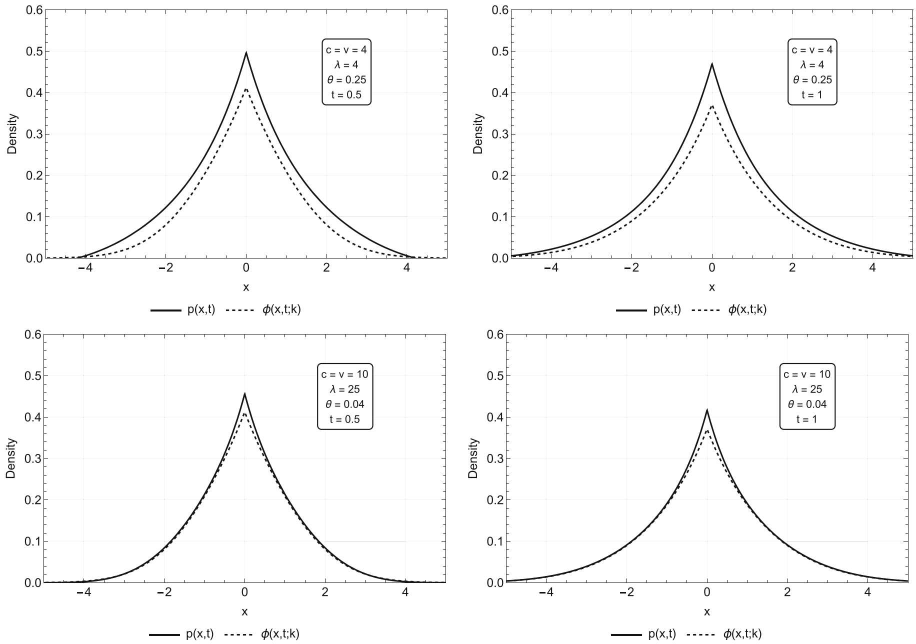

Fig. 10 Plot of $p(x,t)$, obtained from Eq. (50) and its approximation $\varphi(x,t;\kappa)$, from Eq. (52), with $\kappa=\theta\lambda$ and $\sigma=c/\sqrt{\lambda}$, for various choices of the parameters

Corollary 3 *Under the assumptions of Theorem 6, the mean-square distance defined in Eq. (55), for $\lambda\in\mathbb{R}^+$, $y\in\{-v,c\}$, $\theta\in[0,1]$ and $t\in\mathbb{R}_0^+$ is given by*

$$
\begin{aligned}
&\mathsf{D}_y^{\lambda}(p,t)\\
&=\frac{1}{8\lambda^2}\Bigg\{4(c+v)\Bigg[|y|\left(\lambda t+\frac{4}{(2-\theta)^2 p}+\frac{e^{-\lambda t(2-\theta)}}{(2-\theta)^2}\right)-\mathrm{sgn}(y)\lambda t(c-v)\\
&\times\left(\frac{e^{-2\lambda t}}{2}+\frac{e^{-\lambda t(2-\theta)}}{2-\theta}\right)\Bigg]-(c+v)^2\left((1-e^{-2\lambda t})+\mathrm{sgn}(y)\frac{4}{(2-\theta)^2}\right)+(c-v)^2\\
&\times\left[2\lambda^2t^2+\left(\frac{2}{\theta}\right)^2\left(1-e^{-\theta\lambda t}(1+\theta\lambda t)\right)\right]-4\lambda\left[(c-v)t+\mathrm{sgn}(y)\frac{(1-e^{-2\lambda t})(c+v)}{2\lambda}\right]\\
&\times\left[\frac{(1-e^{-\theta\lambda t})(c-v)}{\theta}+\mathrm{sgn}(y)\frac{(1-e^{-\lambda t(2-\theta)})(c+v)}{2-\theta}\right]-\frac{4y^*e^{-\lambda t}(c+v)}{(2-\theta)^2(1-\theta)p}\\
&\times\Big[(4-p)(1-\theta)\cosh\big((1-\theta)\lambda t\big)\Big]+\big(4-(3-p)p\big)\sinh\big((1-\theta)\lambda t\big)\Bigg\}.
\end{aligned}
\tag{56}
$$

Proof By using Eqs. (35), (36) and (55), after some calculations we obtain Eq. (56). □

From Eq. (56), with $\mathsf{D}_y^{\lambda}(p_0,t):=\lim_{\theta\to p_0}\mathsf{D}_y^{\lambda}(p,t)$, for $y\in\{-v,c\}$ and $t\in\mathbb{R}_0^+$ one has

$$\mathsf{D}_y^{\lambda}(0,t)=2\,\mathsf{Var}_y[X_0(t)],$$

and

$$\mathsf{D}_y^{\lambda}(1,t)=\frac{1}{8\lambda^2}\bigg[e^{-2\lambda t}(c+v)\Big(4y^*(1-e^{-\lambda t})+(c+v)+\operatorname{sgn}(y)2\lambda t(c-v)\Big)$$
$$+4y^*e^{-\lambda t}\Big(y+3y^*-\operatorname{sgn}(y)2\lambda t(c+v)\Big)+y+3y^*$$
$$-\operatorname{sgn}(y)2(c^2-v^2)+2\lambda t\Big(-2y^*(y^*+3y)+\lambda t(c-v)^2\Big)\bigg],$$

where $\mathsf{Var}_y[X_0(t)]$ is given in the right-hand-side of (40), and y^* is defined in (34).

Remark 18 From Corollary 3, we have that the distance $\mathsf{D}_y^{\lambda}(p,t)$ is asymptotically quadratic as t diverges. In fact, due to (55), for any $y\in\{-v,c\}$ we have

$$\lim_{t\to+\infty}\frac{\mathsf{D}_y^{\lambda}(p,t)}{t^2}=\frac{1}{4}(c-v)^2. \tag{57}$$

Moreover, in the special case $v=c$ it is asymptotically linear in t, with

$$\lim_{t\to+\infty}\frac{\mathsf{D}_y^{\lambda}(p,t)}{t}=\frac{c^2}{\lambda}. \tag{58}$$

The result in Eq. (58) is in agreement with Remark 3 of Orsingher [24], this suggesting that the reset does not impact the long-term diffusivity of the process, which is governed by c^2/λ.

Figures 11 and 12 show some plots of $\mathsf{D}_y^{\lambda}(p,t)$, given in Eq. (56), for various choices of λ and θ, respectively, for $y\in\{-v,c\}$. In all cases, $\mathsf{D}_y^{\lambda}(p,t)$ is monotonically decreasing in λ and θ, for large t. This result is explained by noting that a high frequency of velocity reversals (large λ) or a high reset probability θ reduces the growth of the mean-square distance between the two processes. Specifically, if λ is large then the process without reset oscillates more frequently, this reducing the difference compared to the process with reset. Further, for large θ, the two processes behave similarly, and their distance is reduced over time. This aligns with the results in Remark 18:

(i) If $c=v$ in Eq. (58), then the distance grows quadratically with time, with the growth amplitude determined by the difference $c-v$. A larger velocity difference leads to a greater distance over time.

(ii) If $c\neq v$ in Eq. (57), then the distance grows linearly with time and it is inversely proportional to λ, which means that the growth of the distance slows as λ increases.

5 Erlang-2 Distributed Intertimes

In this section we analyze a different scheme, in which the intertimes $\{U_i, i\in\mathbb{N}\}$ and $\{D_i, i\in\mathbb{N}\}$ have Erlang distribution with equal parameters $(\lambda,2)$. Hence, for any $i\in\mathbb{N}$ and $\lambda\in\mathbb{R}^+$, the survival functions of U_i and D_i are

$$\overline{F}_{U_i}(t)=\overline{F}_{D_i}(t)=e^{-\lambda t}(1+\lambda t),\qquad t\in\mathbb{R}^+. \tag{59}$$

Then, in this case the random sums given in Eq. (2) have $(\lambda,2k)$-Erlang p.d.f.'s, for $k\in\mathbb{N}$,

$$f_U^{(k)}(t)=f_D^{(k)}(t)=\frac{\lambda^{2k}\,t^{2k-1}\,e^{-\lambda t}}{(2k-1)!},\qquad t\in\mathbb{R}^+. \tag{60}$$

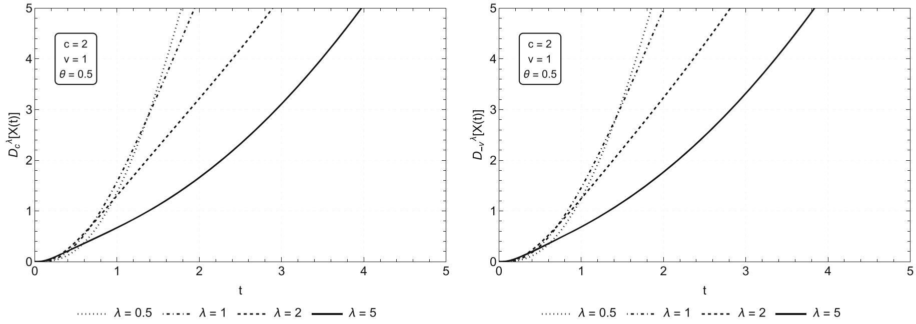

Fig. 11 Plot of $\mathrm{D}_y^{\lambda}(p,t)$ for $y=c$ (left) and $y=-v$ (right), and various choices of λ

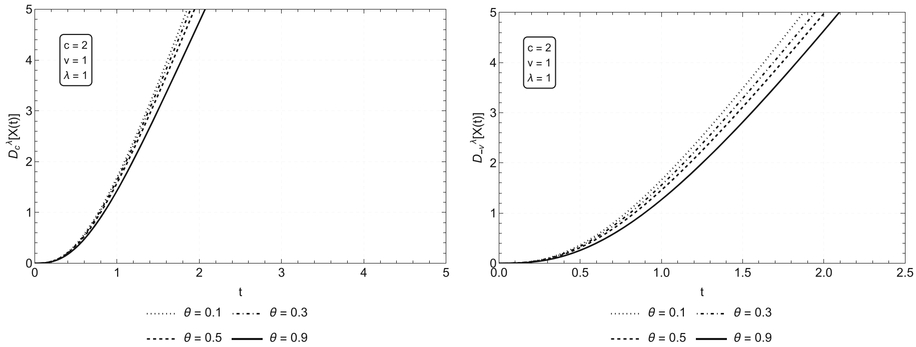

Fig. 12 Plot of $\mathrm{D}_y^{\lambda}(p,t)$ for $y=c$ (left) and $y=-v$ (right), and various choices of θ

Proposition 11 *If the intertimes* $\{U_i, i\in\mathbb{N}\}$ *and* $\{D_i, i\in\mathbb{N}\}$ *of* $\{(X(t),V(t)), t\in\mathbb{R}_0^+\}$ *are Erlang distributed with parameters* $(\lambda,2)$*, for* $\lambda\in\mathbb{R}^+$*, then for* $\theta\in[0,1]$*, the reset time* R_1 *has p.d.f.*

$$f_R^{(1)}(t)=\frac{\theta\lambda\, e^{-\lambda t}}{\sqrt{1-\theta}}\ \sinh\left(\sqrt{1-\theta}\,\lambda t\right),\qquad t\in\mathbb{R}^+. \tag{61}$$

Moreover, the renewal intensity is given by

$$\rho(t)=\frac{\theta\lambda}{2}(1-e^{-2\lambda t}),\qquad t\in\mathbb{R}_0^+. \tag{62}$$

Proof Eq. (61) is obtained by differentiating Eq. (8) and making use of Eq. (60). Consequently, the renewal intensity (62) is derived by Eqs. (4) and (5), and adopting an inverse Laplace transform-based approach. □

Let us now analyze the process $\{(X(t),V(t)), t\in\mathbb{R}_0^+\}$ before the occurrence of the first reset. To determine the probability densities defined in Eqs. (11) and (12), we recall the two-index pseudo-Bessel function (cf. Di Crescenzo [5] for its properties):

$$S_{i,j}^{(n,r)}(\alpha,\beta)=\sum_{k=0}^{+\infty}\frac{\alpha^{nk+i}\beta^{rk+j}}{(nk+i)!(rk+j)!},\qquad \alpha,\beta\in\mathbb{R}, \tag{63}$$

defined for $n,r\in\mathbb{N}$ and $i,j\in\mathbb{N}_0$. Recalling Proposition 1, the following result holds.

Proposition 12 *If the intertimes* $\{U_i, i \in \mathbb{N}\}$ *and* $\{D_i, i \in \mathbb{N}\}$ *of* $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ *have Erlang distribution with equal parameters* $(\lambda, 2)$*, with* $\lambda \in \mathbb{R}^+$*, then the sub-densities of forward and backward motion when no reset occurs up to time* $t \in \mathbb{R}^+$*, for* $x \in (-vt, ct)$*, are given by*

$$
\begin{aligned}
f_0(x, t \mid c) &= \frac{(1-\theta)\lambda e^{-\lambda t}}{c+v} \sum_{j=0}^{1} S_{2+j,1}^{(2,2)}\big(\lambda\tau^*, (1-\theta)\lambda(t-\tau^*)\big), \\
b_0(x, t \mid c) &= \frac{\lambda e^{-\lambda t}}{c+v} \sum_{j=0}^{1} S_{1,j}^{(2,2)}\big((1-\theta)\lambda\tau^*, \lambda(t-\tau^*)\big),
\end{aligned}
\tag{64}
$$

and

$$
\begin{aligned}
f_0(x, t \mid -v) &= \frac{\lambda e^{-\lambda t}}{c+v} \sum_{j=0}^{1} S_{1,j}^{(2,2)}\big((1-\theta)\lambda(t-\tau^*), \lambda\tau^*\big), \\
b_0(x, t \mid -v) &= \frac{\lambda(1-\theta) e^{-\lambda t}}{c+v} \sum_{j=0}^{1} S_{2+j,1}^{(2,2)}\big(\lambda(t-\tau^*), (1-\theta)\lambda\tau^*\big),
\end{aligned}
\tag{65}
$$

with $\tau^* = (x+vt)/(c+v)$*, and with* $S_{i,j}^{(n,r)}$ *defined in (63).*

Proof Making use of (59) and (60), then Eq. (15) gives

$$
f_0(x, t \mid c) = \frac{(1-\theta)\lambda e^{-\lambda t}}{c+v} \sum_{k=1}^{+\infty} \frac{[(1-\theta)\lambda(t-\tau^*)]^{2k-1}}{[(2k-1)!]^2}
\sum_{j=0}^{1} \int_{t-\tau*}^{t} (s-t+\tau^*)^{2k-1} \frac{\lambda^j}{j!} (t-s)^j \mathrm{d}s.
$$

Since the latter integral is equal to

$$
\frac{(2k-1)!}{(2k+j)!} \lambda^j (\tau^*)^{2k+j}
$$

and recalling the function (63), then the first of Eq. (64) follows after some calculations. The other p.d.f.'s in Eqs. (64) and (65) can be obtained in a similar way. □

The following theorem gives the probability law of the process in the general case.

Theorem 13 *Let the intertimes* $\{U_i, i \in \mathbb{N}\}$ *and* $\{D_i, i \in \mathbb{N}\}$ *of* $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ *have Erlang distribution with equal parameters* $(\lambda, 2)$*, with* $\lambda \in \mathbb{R}^+$*. Then, for all* $t \in \mathbb{R}^+$ *and* $y \in \{-v, c\}$*, we have*

$$
\mathsf{P}[X(t) = yt, V(t) = c \mid X(0) = 0, V(0) = y] = e^{-\lambda t}(1+\lambda t). \tag{66}
$$

Moreover, for $x \in (-vt, ct)$*, one has*

$$
\begin{aligned}
p(x, t \mid y) &= p_0(x, t \mid y) + \frac{\theta\lambda}{2} \int_0^{t-m(x)} \left(1 - e^{-2\lambda s}\right) p_0(x, t-s \mid y) \mathrm{d}s \\
&\quad + \frac{\theta\lambda}{2|y|} \left[1 - e^{-2\lambda\left(t - \frac{x}{y}\right)}\right] e^{-\lambda\frac{x}{y}} \left(1 + \lambda\frac{x}{y}\right) \mathbf{1}_{\{0 < \frac{x}{y} < t\}},
\end{aligned}
\tag{67}
$$

where $p_0(x, t \mid y) = f_0(x, t \mid y) + b_0(x, t \mid y)$*, with p.d.f.'s* f_0 *and* b_0 *obtained in Proposition 12, and* $m(x)$ *given in Eq. (20).*

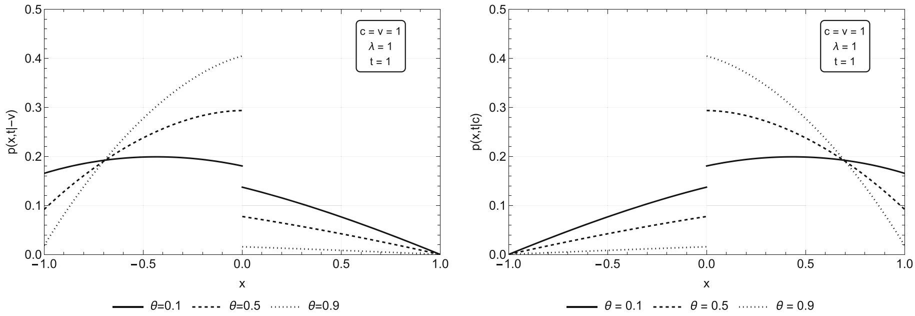

Fig. 13 Plots of $p(x, t \mid y)$, for $y = -v$ (left) and $y = c$ (right), for various choices of θ

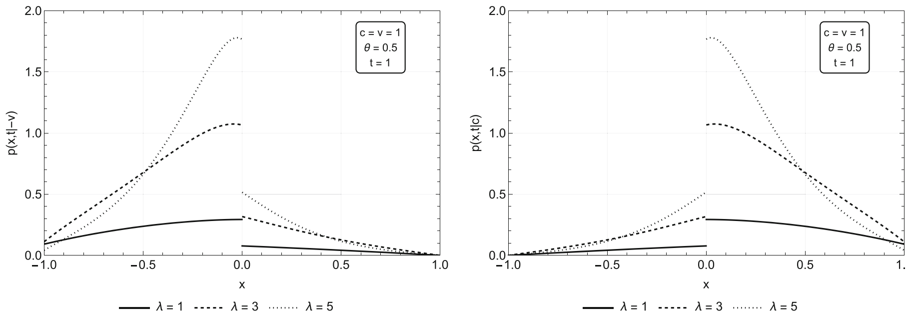

Fig. 14 Plots of $p(x, t \mid y)$, for $y = -v$ (left) and $y = c$ (right), for various choices of λ

Proof Eq. (66) follows from (18) and recalling (59). The right-hand-side of Eq. (67) is determined making use of (19) and recalling (62). □

In the case of exponential intertimes, the normalization condition (33) has been proved formally (cf. Remark 7). Instead, in the case of Erlang-2 intertimes, it can be shown by means of numerical computations.

Remark 19 The symmetry properties expressed in Remark 6 hold also under the assumptions of Proposition 12 and Theorem 13.

Figures 13 and 14 show plots of $p(x, t \mid y)$ get through numerical calculations. They exhibit the symmetry stated in Remark 19. Even in this case a discontinuity of the p.d.f. arises at $x = 0$, as already seen in Section 4 for exponential intertimes. Moreover, as for Figures 2 and 3 the discontinuity of $p(x, t \mid y)$ at $x = 0$ is much more evident for larger θ and λ. This is in agreement with the property that the renewal intensity (62) is increasing both in θ and λ.

Remark 20 Similarly as in Remark 8, we can study the limiting behavior of the p.d.f. given in Theorem 13. As a consequence of Eqs. (63), (64), (65) and (67), under the assumptions of Theorem 13, for $t \in \mathbb{R}^+$ and $y \in \{-v, c\}$, with y^* defined in (34), we have:

$$\lim_{x \to yt} p(x, t \mid y) = \frac{(1-\theta)\lambda^2 t e^{-\lambda t}}{c+v} + \frac{\theta\lambda(1+\lambda t)e^{-\lambda t}}{2|y|}, \qquad \lim_{x \to y^* t} p(x, t \mid y) = 0.$$

Remark 21 Let us now consider the limiting behavior of the p.d.f. (67) when θ tends to the endpoints of [0, 1]. Under the assumptions of Theorem 13 we have:

$$\begin{aligned}
\lim_{\theta\to 0} p(x,t\,|\,c) &= \frac{\lambda e^{-\lambda t}}{c+v}\sum_{j=0}^{1}\left[S_{2+j,1}^{(2,2)}\big(\lambda\tau^*,\lambda(t-\tau^*)\big)+S_{1,j}^{(2,2)}\big(\lambda\tau^*,\lambda(t-\tau^*)\big)\right],\\
\lim_{\theta\to 0} p(x,t\,|-v) &= \frac{\lambda e^{-\lambda t}}{c+v}\sum_{j=0}^{1}\left[S_{1,j}^{(2,2)}\big(\lambda(t-\tau^*),\lambda\tau^*\big)+S_{2+j,1}^{(2,2)}\big(\lambda(t-\tau^*),\lambda\tau^*\big)\right],
\end{aligned} \tag{68}$$

where $\tau^*=(x+vt)/(c+v)$. The results in Eqs. (68) agree with the densities in Theorem 3.1 of Di Crescenzo [5] having Erlang intertimes with parameters $(\lambda, 2)$. Moreover, we have

$$\lim_{\theta\to 1} p(x,t\,|\,y)=\frac{\lambda}{2|y|}\left[1-e^{-2\lambda\left(t-\frac{x}{y}\right)}\right]e^{-\lambda\frac{x}{y}}\left(1+\lambda\frac{x}{y}\right)\mathbf{1}_{\{0<\frac{x}{y}<t\}}. \tag{69}$$

When $\theta\to 1$ there is a high probability of finding the particle in intervals close to the origin due to the effect of the high frequency of resets. Moreover, by integrating the density on the right-hand side of (69) we get $1-e^{-\lambda t}(1+\lambda t)$, this being in agreement with Eq. (66).

5.1 Moments of the Position Process with Erlang-2 Intertimes

Let $\mathsf{E}_y[X(t)]$ and $\mathsf{Var}_y[X(t)]$ denote respectively the mean and the variance conditional on $\{X(0)=0, V(0)=y\}$, with $y\in\{-v,c\}$. Making use of Eq. (67), in the following theorem we obtain the first and second conditional moments of $X(t)$.

Theorem 14 *Under the assumptions of Theorem 13, for all $t\in\mathbb{R}^+$ and $y\in\{-v,c\}$, we have*

$$\begin{aligned}
&\mathsf{E}_y[X(t)]\\
&=y\,t\,e^{-\lambda t}(1+\lambda t)+\frac{\theta\,y}{2\lambda}(3-2e^{-\lambda t}\big(2+\lambda t(1+\lambda t)\big)+e^{-2\lambda t})\\
&\quad+\sum_{k=0}^{+\infty}\sum_{j=0}^{1}\frac{(1-\theta)^{2k+1}}{\lambda}\left\{\frac{\big(2(c-v)(k+1)+y(j+1)\big)(1-\theta)}{(4(k+1)+j+1)!}\times\left[e^{-\lambda t}(\lambda t)^{4(k+1)+j+1}\right.\right.\\
&+\frac{\theta}{2}\left.\left(\Gamma[4(k+1)+j+2,0,\lambda t]+(-1)^j\,e^{-2\lambda t}\Gamma[4(k+1+j+2,-\lambda t,0)]\right)\right]\\
&\quad+\frac{\big(2(c-v)(k+1)+y^*(j-1)\big)}{(2(2k+1)+j+1)!}\left[e^{-\lambda t}(\lambda t)^{2(2k+1)+j+1}\right.\\
&\quad\left.\left.+\frac{\theta}{2}\left(\Gamma[2(2k+1)+j+2,0,\lambda t]+(-1)^j\,e^{-2\lambda t}\Gamma[2(2k+1)+j+2,-\lambda t,0)]\right)\right]\right\},
\end{aligned} \tag{70}$$

and

$$
\begin{aligned}
&\mathsf{E}_y[X^2(t)]\\
&= (y\,t)^2\, e^{-\lambda t}(1+\lambda t) + \frac{\theta\, y^2}{\lambda^2}\left(4 - 2e^{-\lambda t}\big(2+\lambda t(6+\lambda t(1+\lambda t))\big) + e^{-2\lambda t}\right)\\
&\quad + \sum_{k=0}^{+\infty}\sum_{j=0}^{1} \frac{(1-\theta)^{2k+1}}{\lambda^2}\left\{ \frac{g_{j,k}^{(1)}\,(1-\theta)}{(4(k+1)+j+2)!}\bigg[e^{-\lambda t}(\lambda t)^{4(k+1)+j+2} \right.\\
&\quad + \frac{\theta}{2}\bigg(\Gamma[4(k+1)+j+3, 0, \lambda t] + (-1)^j\, e^{-2\lambda t}\,\Gamma[4(k+1)+j+3, -\lambda t, 0]\bigg)\bigg]\\
&\quad + \frac{g_{j,k}^{(2)}}{(2(2k+1)+j+2)!}\bigg[e^{-\lambda t}(\lambda t)^{2(2k+1)+j+2}\\
&\quad \left. + \frac{\theta}{2}\bigg(\Gamma[2(2k+1)+j+3, 0, \lambda t] + (-1)^j\, e^{-2\lambda t}\,\Gamma[2(2k+1)+j+2, -\lambda t, 0]\bigg)\bigg]\right\},
\end{aligned}
\tag{71}
$$

with y^* defined in (34), $\Gamma[a, z_0, z_1]$ the generalized incomplete gamma function, and

$$
\begin{aligned}
g_{j,k}^{(1)} &= y^2(2k+j+4)(2k+j+3) + 4cv(2k+j+3)(k+1) + 2(y^*)^2(2k+3)(k+1),\\
g_{j,k}^{(2)} &= 2y^2(2k+3)(k+1) + 4cv(2k+j+1)(k+1) + (y^*)^2(2k+j+2)(2k+j+1).
\end{aligned}
\tag{72}
$$

Proof For all $t \in \mathbb{R}^+$ and $y \in \{-v, c\}$, making use of the Eqs. (66) and (67) we have

$$
\begin{aligned}
\mathsf{E}_y[X(t)] &= e^{-\lambda t}(1+\lambda t) + \int_{-vt}^{ct} x\, p_0(x, t \mid y)\, \mathrm{d}x + \frac{\theta\lambda}{2}\int_0^t \mathrm{d}s \int_{-v(t-s)}^{c(t-s)} x\, p_0(x, t-s \mid y)\, \mathrm{d}x\\
&\quad + \frac{\theta\lambda}{2|y|}\int_{-vt}^{ct} x\left[1 - e^{-2\lambda\left(t-\frac{x}{y}\right)}\right] e^{-\lambda\frac{x}{y}}\left(1+\lambda\frac{x}{y}\right)\mathbf{1}_{\{0<\frac{x}{y}<t\}}\,\mathrm{d}x.
\end{aligned}
$$

After some calculations we get (70). Then, Eq. (71) is obtained similarly. □

The following result, whose proof is given in Appendix A.4, shows that the mean and the variance of $X(t)$ both tend to a constant limit when t goes to infinity.

Corollary 4 *Under the assumptions of Theorem 14, the following limits hold for all $t \in \mathbb{R}^+$, for y^* defined in (34),*

$$
\lim_{t\to+\infty} \mathsf{E}_y[X(t)] = \frac{3py + (1-\theta)[4(c-v) - py^*]}{2\lambda(2-\theta)p}, \tag{73}
$$

$$
\lim_{t\to+\infty} \mathsf{E}_y[X^2(t)] = \frac{2\big[(4-(3-p)p)\big[y^2 + 2((y^*)^2 + 2cv)(1-\theta)\big] - 2cv(1-\theta)p^2\big]}{((2-\theta)\theta\lambda)^2}. \tag{74}
$$

Some plots of $\mathsf{E}_y[X(t)]$, for $y \in \{-v, c\}$, are given in Figures 15 and 16 for various choices of λ and θ, respectively. They show that the mean of $X(t)$, for fixed t, approaches the origin for larger θ and for larger λ, in analogy with the same behavior exhibited in Figures 5 and 6 in the case of exponential intertimes. Further, the symmetry behavior is also confirmed, i.e.,

$$
\mathsf{E}_c^{(-v,c)}[X(t)] = -\mathsf{E}_{-v}^{(-c,v)}[X(t)], \qquad t \in \mathbb{R}_0^+,
$$

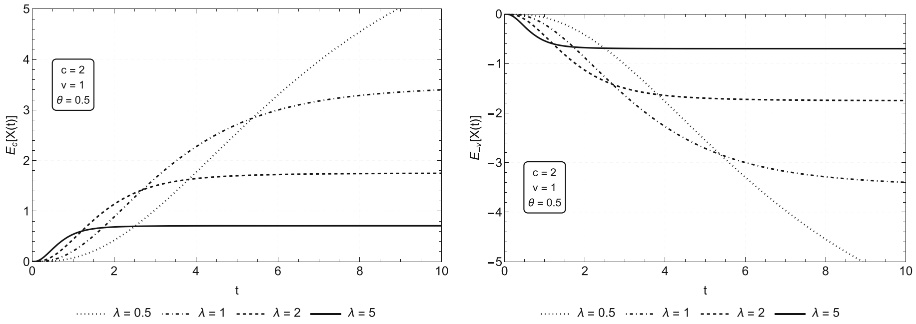

Fig. 15 Plot of the mean (70) for $y = c$ (left) and $y = -v$ (right) for various choices of λ

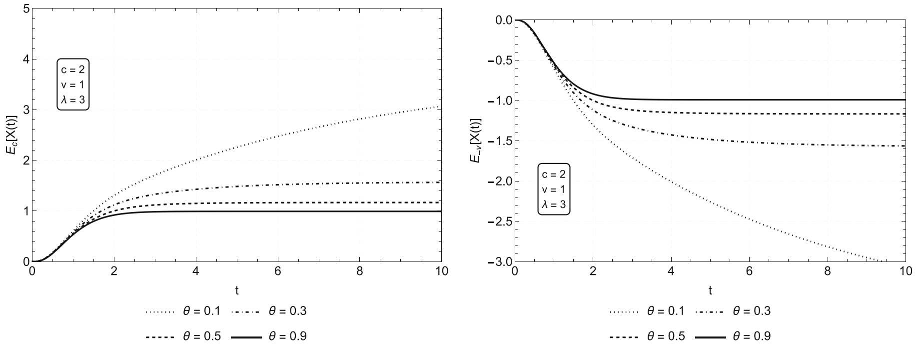

Fig. 16 Plot of the mean (70) for $y = c$ (left) and $y = -v$ (right) for various choices of θ

where $\mathsf{E}_y^{(c_1,c_2)}[X(t)]$, for $y \in \{-v, c\}$, indicates the mean when the backward and forward velocities are c_1 and c_2, respectively.

We avoid to give the expression of $\mathsf{Var}_y[X(t)]$ since it is quite cumbersome, and thus provide just the asymptotic result for $t \to +\infty$.

Remark 22 Under the assumptions of Theorem 14, for all $t \in \mathbb{R}^+$ and $y \in \{-v, c\}$, one has:

$$\lim_{t\to+\infty} \mathsf{Var}_y[X(t)]$$
$$= \frac{7(py)^2 + (1-\theta)\left[\left(10cv - (y^*)^2(1-\theta)\right)p^2 + \left(4(c+v)\right)^2 + 64cv(1-\theta)\right]}{(2(2-\theta)\theta\lambda)^2}.$$

5.2 Asymptotic results under Erlang-2 intertimes

In analogy with Section 4.2, let us now analyze the limit results for the p.d.f. of the process $X(t)$ driven by Erlang intertimes with parameters $(\lambda, 2)$, i.e., with survival functions (59).

(i) Stationary density. As for the case of exponential intertimes, Corollary 4 suggests the existence of a stationary regime. Hence, we refer to Theorem 13 to investigate the stationary p.d.f. of $X(t)$, by noting that the probability (66) tends to 0.

Proposition 15 *Let the intertimes $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ of the process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ have Erlang distribution with parameters $(\lambda, 2)$, for $\lambda \in \mathbb{R}^+$. Then, for $y \in \{-v, c\}$ we*

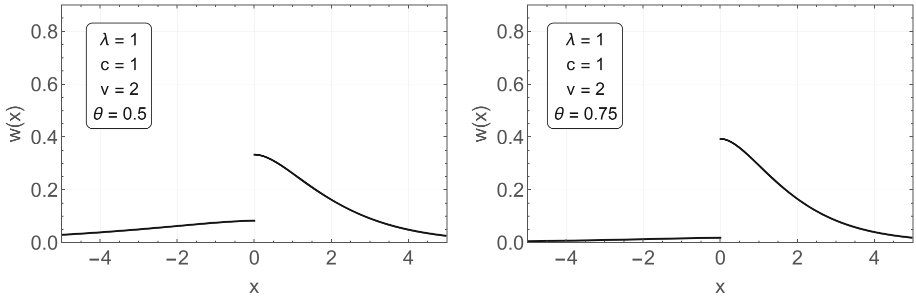

Fig. 17 Plot of $w(x)$, obtained from Eq. (41), with $y = c$

have:

$$w(x) := \lim_{t\to+\infty} p(x,t\,|\,y) = \frac{\theta\lambda}{2}\, L_0^{\infty}(x\,|\,y) + \frac{\theta\lambda}{2|y|}\, e^{-\lambda\frac{x}{y}}\left(1+\lambda\frac{x}{y}\right)\mathbf{1}_{\{\frac{x}{y}>0\}}, \qquad x\in\mathbb{R}. \tag{75}$$

where $L_0^{\infty}(x\,|\,y)$ is defined as in Eq. (42).

Proof Recalling Proposition 12 and Eq. (63), the density $p_0(x,t\,|\,y)$ is a linear combination of terms as

$$g(x,t) \sim C\, e^{-\lambda t} S_{i,j}^{(2,2)}(\alpha,\beta), \qquad \text{where} \quad \alpha\,\beta = (1-\theta)\lambda^2 \frac{(x+vt)(ct-x)}{(c+v)^2},$$

with $C = const$. Since the growth of $S_{i,j}^{(2,2)}(\alpha,\beta)$ is governed by $e^{2\sqrt{\alpha\,\beta}}$, for $t\to+\infty$ we have

$$p_0(x,t\,|\,y) = O(e^{-\delta t}), \qquad \delta = \lambda\left(1-\frac{2\sqrt{c\,v}}{c+v}\sqrt{1-\theta}\right) > 0.$$

Moreover, by setting $u = t-s$ in the integral in the right-hand-side of Eq. (67) it becomes

$$\int_{m(x)}^{t}\left[1-e^{-2\lambda(t-u)}\right] p_0(x,u\,|\,y)\mathrm{d}u.$$

Since the term in square brackets is confined in $[0,1]$ and the p.d.f. $p_0(x,u\,|\,y)$ decays exponentially fast, the dominated convergence theorem yields the first term in the right-hand-side of Eq. (75) for $t\to+\infty$. The second terms also follows from Eq. (67). $\square$

The analogies between the Propositions 7 and 15 lead to the same comments given in Remark 15 also in the case of Erlang distributed intertimes. Essentially, the main diversities stems from the different forms of underlying densities p_0 and renewal intensities (26) and (62).

In Figure 17 we show two examples of $w(x)$ evaluated numerically when $y = c$ (the case $y = -v$ is symmetric), where the last term in Eq. (41) has support $\mathbb{R}^+$, whereas $L_0^{\infty}(x\,|\,y)$ is continuous for $x\in\mathbb{R}$. The shape is similar to the exponential case treated in Figure 9, but in the present case the distribution is more spread out since the intertimes are stochastically larger.

(ii) Kac's scaling. Let us now investigate the asymptotic limit under Kac's scaling for the process having Erlang intertimes, and conditional on the same random initial velocity specified in (44). Recalling Eq. (11), even in this case the p.d.f. $p_0(x,t)$ of the particle position

at time t, when no reset has occurred before, is given by Eq. (45), where

$$p_0(x,t\,|\,y)=f_0(x,t\,|\,y)+b_0(x,t\,|\,y),\qquad y\in\{-v,c\}. \tag{76}$$

Hereafter we see that, despite that the expression of $p_0(x,t)$ involves the two-index pseudo-Bessel function (63), in the special case $v=c$ it is worth a more compact form that involves the Bessel functions of the first kind, with ν real, i.e.

$$J_\nu(x)=\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu},\qquad x\in\mathbb{R}^+. \tag{77}$$

Proposition 16 *If the intertimes $\{U_i, i\in\mathbb{N}\}$ and $\{D_i, i\in\mathbb{N}\}$ of $\{(X(t),V(t)), t\in\mathbb{R}_0^+\}$ have Erlang distribution with equal parameters $(\lambda,2)$, with $\lambda\in\mathbb{R}^+$, and if $v=c$, then, for all $t\in\mathbb{R}^+$ and $x\in(-ct,ct)$, the p.d.f. expressed by Eq. (45) is given by*

$$\begin{aligned}
p_0(x,t)=\frac{e^{-\lambda t}}{2c}\Bigg\{&\frac{\lambda\,c^2t^2}{c^2t^2-x^2}\,I_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right)\\
&+\left(1-\frac{c^2t^2+x^2}{(1-\theta)\lambda\,t\,(c^2t^2-x^2)}\right)\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right)\\
&+\frac{\lambda\,x^2}{c^2t^2-x^2}\,J_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right)\\
&+\left(\frac{c^2t^2+x^2}{(1-\theta)\lambda\,t\,(c^2t^2-x^2)}\right)\frac{\partial}{\partial t}J_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right)\Bigg\},
\end{aligned} \tag{78}$$

where $I_0(x)$ and $J_0(x)$ are given in (27) and (77), respectively.

Proof Making use of Eqs. (64) and (65) in (45), and recalling the identity given in (76), the density (78) follows after some calculations, for $v=c$, for all $t\in\mathbb{R}^+$ and $x\in(-ct,ct)$. $\square$

We remark that $p_0(x,t)$ can be computed by means of Eq. (78) and noting that

$$\begin{aligned}
\frac{\partial}{\partial t}I_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right)&=\frac{\sqrt{1-\theta}\,\lambda\,c\,t}{\sqrt{c^2t^2-x^2}}\,I_1\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right),\\
\frac{\partial}{\partial t}J_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right)&=-\frac{\sqrt{1-\theta}\,\lambda\,c\,t}{\sqrt{c^2t^2-x^2}}\,J_1\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right),
\end{aligned} \tag{79}$$

due to the series forms (27) and (77).

We now provide the limit density $p_0(x,t)$ under the Kac-type conditions (43).

Proposition 17 *Let the intertimes $\{U_i, i\in\mathbb{N}\}$ and $\{D_i, i\in\mathbb{N}\}$ of the process $\{(X(t),V(t)), t\in\mathbb{R}_0^+\}$ have Erlang distribution with equal parameters $(\lambda,2)$, with $\lambda\in\mathbb{R}^+$. Then the following limit holds for all $t\in\mathbb{R}^+$:*

$$\lim_{\substack{\lambda\to+\infty\\ c^2/\lambda\to\sigma^2\\ \theta\lambda\to\kappa}} p_0(x,t)=\frac{1}{\sqrt{2\pi\sigma^2t}}\,e^{-\frac{x^2}{2\sigma^2t}-\frac{\kappa t}{2}},\qquad x\in\mathbb{R}. \tag{80}$$

The proof is given in Appendix A.5.

Remark 23 The right-hand side of Eq. (80) is identical to the Gaussian transition function of the Brownian motion at time t, times the probability that 0 events occur in $[0,t]$ for a Poisson process having rate $\frac{\kappa}{2}$. The difference with the case treated in Proposition 8, where the last term refers to rate κ, arises from the different arguments of the Bessel functions in Eqs. (46) and (78).

Due to Eqs. (19) and (59), and recalling the renewal intensity (62), in analogy with Eq. (50), in the present case the density (49), for $t \in \mathbb{R}^+$ and $x \in (-ct, ct)$, for $v = c$, is expressed as

$$\begin{aligned} p(x,t) &= p_0(x,t) + \frac{\theta\lambda}{2}\int_0^{t-m(x)} \left(1-e^{-2\lambda s}\right) p_0(x,t-s)\,\mathrm{d}s \\ &\quad + \frac{\theta\lambda}{2c}\left\{\left[1-e^{-2\lambda\left(t+\frac{x}{c}\right)}\right] e^{\lambda\frac{x}{c}}\left(1-\lambda\frac{x}{c}\right)\mathbf{1}_{\{-ct<x<0\}}\right. \\ &\quad \left. + \left[1-e^{-2\lambda\left(t-\frac{x}{c}\right)}\right] e^{-\lambda\frac{x}{c}}\left(1+\lambda\frac{x}{c}\right)\mathbf{1}_{\{0\le x<ct\}}\right\}. \end{aligned} \tag{81}$$

Let us now obtain the limit of $p(x,t)$ under the asymptotic Kac-type conditions.

Proposition 18 *Let the intertimes $\{U_i, i \in \mathbb{N}\}$ and $\{D_i, i \in \mathbb{N}\}$ of the process $\{(X(t), V(t)), t \in \mathbb{R}_0^+\}$ have Erlang distribution with equal parameters $(\lambda, 2)$, with $\lambda \in \mathbb{R}^+$. If the initial velocity is random as specified in Eq. (44) and if $v = c$, then the following limit behavior holds for all $t \in \mathbb{R}^+$:*

$$\lim_{\substack{\lambda\to+\infty \\ c^2/\lambda\to\sigma^2 \\ \theta\lambda\to\kappa}} \mathsf{P}[X(t)=yt, V(t)=y \,|\, C_0] = 0, \qquad y \in \{-c, c\}, \tag{82}$$

and, for $x \in \mathbb{R}$,

$$\lim_{\substack{\lambda\to+\infty \\ c^2/\lambda\to\sigma^2 \\ \theta\lambda\to\kappa}} p(x,t) = \varphi(x,t;\kappa/2), \tag{83}$$

where the function φ is defined in Eq. (52).

Proof The result (82) is immediate from (66) and the initial condition (44). For the first term on the right side of (67), under the given assumptions the corresponding limit is (cf. (80))

$$\frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{x^2}{2\sigma^2 t}-\frac{\kappa t}{2}}.$$

Then, we use the transformation $s = u\big(t-m(x)\big)$ in the second term on the right-hand-side of (81), so that the conditions (43) give:

$$\begin{aligned} \lim_{\substack{\lambda\to+\infty \\ c^2/\lambda\to\sigma^2 \\ \theta\lambda\to\kappa}} \frac{\theta\lambda}{2}\int_0^{t-m(x)} \left(1-e^{-2\lambda s}\right) p_0(x,t-s)\mathrm{d}s &= \frac{\kappa t}{2\sqrt{2\pi\sigma^2}}\int_0^1 \frac{e^{-\frac{x^2}{2\sigma^2 t(1-u)}-\frac{\kappa t(1-u)}{2}}}{\sqrt{t(1-u)}}\mathrm{d}u \\ &= \frac{\kappa\,|x|}{\sqrt{2\pi\sigma^2}}\int_0^{\frac{\sqrt{t}}{|x|}} e^{-\frac{1}{2\sigma^2 y^2}-\frac{\kappa x^2 y^2}{2}}\,\mathrm{d}y, \end{aligned}$$

where the transformation $\sqrt{t(1-u)} = y\,|x|$ has been used in the last equality. Making use of some calculations and of Eq. 7.4.33 of Abramowitz and Stegun [39], the second term on

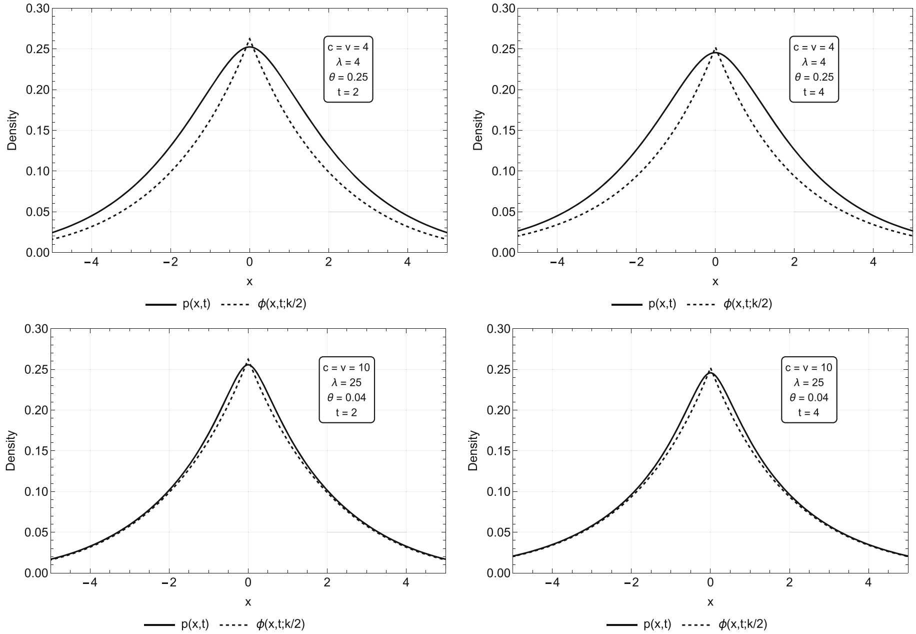

Fig. 18 Plot of $p(x,t)$, given in Eq. (81), and its approximation $\varphi(x,t;\kappa/2)$, with $\kappa=\theta\lambda$ and $\sigma=c/\sqrt{\lambda}$, for various choices of the parameters

the right side of (67) yields

$$\frac{\sqrt{\kappa}}{4\sigma}\left\{e^{-\sqrt{\kappa}\frac{|x|}{\sigma}}\operatorname{Erfc}\left(\frac{|x|}{\sqrt{2\sigma^2 t}}-\sqrt{\frac{\kappa t}{2}}\right)-e^{\sqrt{\kappa}\frac{|x|}{\sigma}}\operatorname{Erfc}\left(\frac{|x|}{\sqrt{2\sigma^2 t}}+\sqrt{\frac{\kappa t}{2}}\right)\right\}.$$

Moreover, under the limit conditions (43), the third term on the right side of (67) tends to 0. Thus, summing the two non-vanishing contributions, the result in Eq. (83) finally follows. □

It is worth to point out that the density given in Eq. (83) is identical to the limit result obtained in Eq. (54) in the case of exponentially distributed intertimes, but replacing the parameter κ with $\kappa/2$. Hence, the comments given in Remark 17 apply also in the present case. The difference between the parameters κ and $\kappa/2$ reflects the different distributions of the intertimes.

The densities of interest are shown in Figure 18 for some choices of the parameters, with c^2/λ and $\lambda\theta$ fixed; the plots of $p(x,t)$ are obtained numerically, and the approximations $\varphi(x,t;\kappa/2)$ are due to Proposition 18. As for the exponential case treated in Section 4.2, even for Erlang-2 intertimes the plots confirm that the goodness of the approximation improves when θ tends to 0 and when λ increases.

Finally, we note that the p.d.f. (52) tends to the density of the Brownian motion as $\kappa\to 0$. Hence, the limiting results given in Propositions 10 and 18, reported as case (viii) of Table 1, agree with the analogous results presented as cases (i) and (iii) in the Table 1 for the classical telegraph process and the telegraph process with gamma intertimes, respectively.

6 Discussion

Our results can be compared with those obtained by Evans and Majumdar [9], who investigated a run-and-tumble particle subject to stochastic resetting. In both studies, a particle moves along a line with finite velocity, changes direction at random times, and is instantaneously returned to the origin whenever a reset occurs.

The main difference concerns the reset mechanism. In Evans and Majumdar [9], resets are generated by an external Poisson process, and therefore occur independently of the particle motion. In the present paper, instead, the reset mechanism is internal and is governed by Bernoulli trials, so that resets to the origin occur as an alternative to velocity reversals.

Another difference concerns the velocity immediately after a reset. Evans and Majumdar [9] consider both the case in which the velocity is preserved and the case in which it is randomly selected after resetting. Here, instead, after each reset the particle always restarts from the origin with the same initial velocity. Furthermore, while the paper [9] assumes exponentially distributed durations of the random displacements, our analysis also cover the case of Erlang-2 distributed intertimes between consecutive velocity changes.

The objectives of the two investigations are quite different. Evans and Majumdar [9] focus mainly on first-passage properties, especially the mean first-passage time to the origin. In contrast, the present study emphasizes the explicit characterization of the probability law of the process, its moments, the effect of resetting on the mean-square distance from the corresponding process without resets, and the diffusion limits. Nevertheless, under suitable Kac-type scaling conditions, both models converge to Brownian motion with resets, the only difference being the reset intensity.

7 Concluding Remarks

In this paper, we analyzed a generalized finite-velocity random motion of a particle running with two alternating negative and positive given velocities. The motion is subject to possible instantaneous resets to the origin, governed by Bernoulli trials, so that the sequence of resets forms a renewal process. Under these assumptions, we derived the probability laws of the position process in two scenarios, i.e. when the intertimes between velocity changes follow (i) an exponential distribution with parameter λ, and (ii) an Erlang distribution with parameters $(\lambda, 2)$. We also discussed various results about the moments of the process, that allowed to study, in case (i), the mean-square distance between the process with resets and an independent copy without resets. The analysis of the mean-square distance under Erlang-2 intertimes involves very cumbersome expressions, and thus has been omitted. Moreover, for both cases, we investigated the asymptotic p.d.f. of the process, conditioned on a random initial velocity, under limit conditions of Kac-type. In analogy with similar cases studied previously, we showed that the limiting density coincides with the p.d.f. of Brownian motion subject to Poissonian resets to the origin. This result holds for both (i) exponential and (ii) Erlang-2 inter-time distributions. The only difference lies in the intensity of the Poisson process governing the resets, which, in case (ii), is half that of case (i).

The investigation on the convergence of the limit density to the analogous p.d.f. of the Brownian motion subject to resets leaves open the question of weak convergence at the level of sample paths. This is left as a possible direction for future research, together with applications in engineering, financial and actuarial sciences. Various applications in biomathematics can be envisaged as well. Indeed, stochastic resetting models have recently proved useful in the study of viral drug resistance dynamics (see Ramoso et al. [41]) and search processes motivated by DNA lesion repair (cf. Calvert and Evans [42]). This suggests potential application areas for the framework developed in this paper.

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Data Availability All data generated or analysed during this study are included in this article.

Declarations

Competing Interest All authors contributed equally to this paper. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

A Some Proofs

A.1 Proof of Theorem 6

For all $t \in \mathbb{R}^+$ and $y \in \{-v, c\}$, making use of the Eqs. (31) and (32) we have

$$\begin{aligned}\mathsf{E}_y[X(t)] &= yt\, e^{-\lambda t} + \int_{-vt}^{ct} x\, p_0(x, t \mid y)\, \mathrm{d}x \\ &\quad + \theta\lambda \int_0^t \mathrm{d}s \int_{-v(t-s)}^{c(t-s)} x\, p_0(x, t-s \mid y)\, \mathrm{d}x + \frac{\theta\lambda}{|y|} \int_{-vt}^{ct} x\, e^{-\lambda \frac{x}{y}} \mathbf{1}_{\{0<\frac{x}{y}<t\}}\, \mathrm{d}x.\end{aligned} \tag{84}$$

Using (30), we note that:

$$\int_{-vt}^{ct} x\, p_0(x, t \mid y)\, \mathrm{d}x = -yt\, e^{-\lambda t} + \frac{e^{-\theta\lambda t}}{2}(c-v)t + \mathrm{sgn}(y) \frac{e^{-\lambda t}}{2(1-\theta)\lambda}(c+v) \sinh((1-\theta)\lambda t), \tag{85}$$

and

$$\begin{aligned}&\theta\lambda \int_0^t \mathrm{d}s \int_{-v(t-s)}^{c(t-s)} x\, p_0(x, t-s \mid y)\, \mathrm{d}x \\ &= \frac{y\,\theta(1+\lambda t)e^{-\lambda t}}{\lambda} - \frac{y\,\theta}{\lambda} + \frac{c-v}{2\theta\lambda} - \frac{e^{-\theta\lambda t}}{2\theta\lambda}(c-v)(1+\theta\lambda t) \\ &+ \mathrm{sgn}(y) \frac{c+v}{2(2-\theta)\lambda} \left[1 - e^{-\lambda t} \left(\cosh((1-\theta)\lambda t) + \frac{1}{1-\theta} \sinh((1-\theta)\lambda t) \right) \right],\end{aligned} \tag{86}$$

and

$$\frac{\theta\lambda}{|y|} \int_{-vt}^{ct} x\, e^{-\lambda \frac{x}{y}} \mathbf{1}_{\{0<\frac{x}{y}<t\}}\, \mathrm{d}x = -\frac{y\,\theta(1+\lambda t)e^{-\lambda t}}{\lambda} + \frac{y\,\theta}{\lambda}. \tag{87}$$

Hence, substituting Eqs. (85), (86) and (87) into (84), after straightforward calculations we obtain Eq. (35). The derivation of Eq. (36) proceeds similarly.

A.2 Proof of Proposition 8

Under the conditions (43), for t $\in \mathbb{R}^+$ one has

$$\frac{(1-\theta)\lambda}{c}\sqrt{c^2t^2-x^2} \sim (1-\theta)\lambda\left(t-\frac{x^2}{2c^2t}\right),$$

and thus, by virtue of (47), the following asymptotic behaviors hold:

$$\frac{e^{-\lambda t}}{2c}(1-\theta)\lambda\, I_0\left(\frac{(1-\theta)\lambda}{c}\sqrt{c^2t^2-x^2}\right) \sim \frac{e^{-(1-\theta)\lambda\frac{x^2}{2c^2t}-\theta\lambda t}}{2\sqrt{\frac{2\pi}{1-\theta}\frac{c^2}{\lambda}\left(t-\frac{x^2}{2c^2t}\right)}}, \tag{88}$$

$$\frac{e^{-\lambda t}}{2c}\frac{(1-\theta)\lambda\, c\, t}{\sqrt{c^2t^2-x^2}}\, I_1\left(\frac{(1-\theta)\lambda}{c}\sqrt{c^2t^2-x^2}\right) \sim \frac{1}{1-\frac{x^2}{2c^2t^2}}\cdot\frac{e^{-(1-\theta)\lambda\frac{x^2}{2c^2t}-\theta\lambda t}}{2\sqrt{\frac{2\pi}{1-\theta}\frac{c^2}{\lambda}\left(t-\frac{x^2}{2c^2t}\right)}}. \tag{89}$$

Due to (43), both terms in the right-hand-side of Eqs. (88) and (89) tend to

$$\frac{1}{2\sqrt{2\pi\sigma^2t}}\, e^{-\frac{x^2}{2\sigma^2t}-\kappa t}.$$

Hence, from (46) we finally obtain the thesis (48).

A.3 Proof of Proposition 10

The limit (53) follows immediately from (31) and (44). Let us now analyze the asymptotic behavior of the right-hand side of (50). For the first term, the limit is, see Eq. (48),

$$\frac{1}{\sqrt{2\pi\sigma^2t}}\, e^{-\frac{x^2}{2\sigma^2t}-\kappa t}. \tag{90}$$

About the second term, by making use of the transformation $s = u\,(t-m(x))$ and applying the limits we have:

$$\begin{aligned}\lim_{\substack{\lambda\to+\infty\\ c^2/\lambda\to\sigma^2\\ \theta\lambda\to\kappa}} \theta\lambda\int_0^{t-m(x)} p_0(x,t-s)\,\mathrm{d}s &= \frac{\kappa t}{\sqrt{2\pi\sigma^2}}\int_0^1 \frac{e^{-\frac{x^2}{2\sigma^2t(1-u)}-\kappa t(1-u)}}{\sqrt{t(1-u)}}\,\mathrm{d}u\\ &= \frac{2\,\kappa\,|x|}{\sqrt{2\pi\sigma^2}}\int_0^{\frac{\sqrt{t}}{|x|}} e^{-\frac{1}{2\sigma^2y^2}-\kappa x^2y^2}\,\mathrm{d}y,\end{aligned} \tag{91}$$

where the transformation $\sqrt{t(1-u)} = y\,|x|$ is used in the last equality. Recalling the definition of the complementary error function (51), after some calculations and using Eq. 7.4.33 of Abramowitz and Stegun [39], the last term of Eq. (91) becomes

$$\frac{\sqrt{2\kappa}}{4\sigma}\left\{e^{-\sqrt{2\kappa}\frac{|x|}{\sigma}}\,\mathrm{Erfc}\left(\frac{|x|}{\sqrt{2\sigma^2t}}-\sqrt{\kappa t}\right)-e^{\sqrt{2\kappa}\frac{|x|}{\sigma}}\,\mathrm{Erfc}\left(\frac{|x|}{\sqrt{2\sigma^2t}}+\sqrt{\kappa t}\right)\right\}. \tag{92}$$

Finally, under the limit conditions (43) the third term on the right-hand side of Eq. (50) tends to 0. Hence, the result in Eq. (54) holds by summing the contributions given in Eqs. (90) and (92).

A.4 Proof of Corollary 4

Using Eq. (70), one has:

$$\lim_{t\to+\infty} \mathsf{E}_y[X(t)] = \sum_{k=0}^{+\infty}\sum_{j=0}^{1} \frac{(1-\theta)^{2k+1}p}{2\lambda}\Big\{\big(2(c-v)(k+1)+y(j+1)\big)(1-\theta) + 2(c-v)(k+1)+y^*(j-1)\Big\} + \frac{3py}{2\lambda}. \tag{93}$$

Computing the sum in (93) on the index j we obtain

$$\lim_{t\to+\infty} \mathsf{E}_y[X(t)] = \frac{3py}{2\lambda} + \frac{(1-\theta)p}{2\lambda}\sum_{k=0}^{+\infty}(1-\theta)^{2k}\Big\{4(c-v)(2-\theta)(k+1) - (y^*+3y(1-\theta))\Big\}. \tag{94}$$

Recalling the geometric-type series and using (94) one has:

$$\lim_{t\to+\infty} \mathsf{E}_y[X(t)] = \frac{1}{2\lambda}\Big\{3py + \frac{4(c-v)(1-\theta)}{\theta(2-\theta)} - \frac{(1-\theta)}{2-\theta}(y^*+3y(1-\theta))\Big\}. \tag{95}$$

From Eq. (95), we obtain Eq. (73). Using Eq. (71), we get:

$$\lim_{t\to+\infty} \mathsf{E}_y[X^2(t)] = \frac{4py^2}{\lambda^2} + \sum_{k=0}^{+\infty}\sum_{j=0}^{1}\frac{(1-\theta)^{2k+1}p}{2\lambda^2}\{g_{j,k}^{(1)}(1-\theta)+g_{j,k}^{(2)}\}, \tag{96}$$

with $g_{j,k}^{(1)}$ and $g_{j,k}^{(2)}$ given in (72). Computing the sum in (96) on the index j we obtain

$$\lim_{t\to+\infty} \mathsf{E}_y[X^2(t)] = \frac{4py^2}{\lambda^2} + \frac{2(1-\theta)p}{\lambda^2}\sum_{k=0}^{+\infty}(1-\theta)^{2k}\{y^2(2(2-\theta)k^2 - 8p(k+1)+13k+11) + ((y^*)^2+2cv)(k+1)(5+2(2-\theta)k-3p) - cvp(k+1)\}.$$

Finally, computing the sum on k, Eq. (74) is obtained.

A.5 Proof of Proposition 17

We first recall the asymptotic estimate of the modified Bessel function (77), i.e. (cf. Eq. (9.2.1) of Abramowitz and Stegun [39])

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \qquad \text{as} \quad z \to +\infty. \tag{97}$$

Under the limit conditions (43), for $t \in \mathbb{R}^+$ one has

$$\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)} \sim \left(1-\frac{\theta}{2}\right)\lambda\left(t - \frac{x^2}{2c^2t}\right).$$

Hence, due to (47) and (97) we get, respectively,

$$I_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right) \sim \frac{e^{\left(1-\frac{\theta}{2}\right)\lambda\left(t-\frac{x^2}{2c^2t}\right)}}{\sqrt{2\pi\left(1-\frac{\theta}{2}\right)\lambda\left(t-\frac{x^2}{2c^2t}\right)}},$$

$$J_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right) \sim \sqrt{\frac{2}{\pi\left(1-\frac{\theta}{2}\right)\lambda\left(t-\frac{x^2}{2c^2t}\right)}}\cos\left(\left(1-\frac{\theta}{2}\right)\lambda\left(t-\frac{x^2}{2c^2t}\right)-\frac{\pi}{4}\right).$$

For the first and the third term in the right-hand-side of (78) one has, respectively,

$$\frac{e^{-\lambda t}}{2c}\left(\frac{\lambda\, c^2t^2}{c^2t^2-x^2}\right)I_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right) \sim \frac{c^2t^2}{c^2t^2-x^2}\cdot\frac{e^{-\frac{\lambda}{c^2}\left(1-\frac{\theta}{2}\right)\frac{x^2}{2t}-\frac{\theta}{2}\lambda t}}{2\sqrt{2\pi\left(1-\frac{\theta}{2}\right)\left(\frac{c^2t}{\lambda}-\frac{x^2}{2\lambda t}\right)}}, \tag{98}$$

and

$$\frac{e^{-\lambda t}}{2c}\left(\frac{\lambda\, x^2}{c^2t^2-x^2}\right)J_0\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right) \\ \sim \frac{x^2}{c^2t^2-x^2}\cdot\frac{e^{-\lambda t}}{\sqrt{2\pi\left(1-\frac{\theta}{2}\right)\left(\frac{c^2t}{\lambda}-\frac{x^2}{2\lambda t}\right)}}\cos\left(\left(1-\frac{\theta}{2}\right)\lambda\left(t-\frac{x^2}{2c^2t}\right)-\frac{\pi}{4}\right). \tag{99}$$

Similarly, recalling the identities (79), for the second and the forth term in the right-hand-side of (78), one has, respectively,

$$\frac{e^{-\lambda t}}{2c}\left(1-\frac{c^2t^2+x^2}{(1-\theta)\lambda\, t\,(c^2t^2-x^2)}\right)\left(\frac{\sqrt{1-\theta}\,\lambda\, c\, t}{\sqrt{c^2t^2-x^2}}\right)I_1\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right) \\ \sim \left(1-\frac{c^2t^2+x^2}{(1-\theta)\lambda\, t\,(c^2t^2-x^2)}\right)\left(\frac{1-\frac{\theta}{2}}{1-\frac{x^2}{2c^2t^2}}\right)\frac{e^{-\frac{\lambda}{c^2}\left(1-\frac{\theta}{2}\right)\frac{x^2}{2t}-\frac{\theta}{2}\lambda t}}{2\sqrt{2\pi\left(1-\frac{\theta}{2}\right)\left(\frac{c^2t}{\lambda}-\frac{x^2}{2\lambda t}\right)}} \tag{100}$$

and

$$-\frac{e^{-\lambda t}}{2c}\left(\frac{c^2t^2+x^2}{(1-\theta)\lambda\, t\,(c^2t^2-x^2)}\right)\left(\frac{\sqrt{1-\theta}\,\lambda\, c\, t}{\sqrt{c^2t^2-x^2}}\right)J_1\left(\frac{\lambda}{c}\sqrt{(1-\theta)(c^2t^2-x^2)}\right) \\ \sim -\left(\frac{c^2t^2+x^2}{(1-\theta)\lambda\, t\,(c^2t^2-x^2)}\right)\left(\frac{1-\frac{\theta}{2}}{1-\frac{x^2}{2c^2t^2}}\right)\frac{e^{-\lambda t}}{\sqrt{2\pi\left(1-\frac{\theta}{2}\right)\left(\frac{c^2t}{\lambda}-\frac{x^2}{2\lambda t}\right)}} \\ \times\cos\left(\left(1-\frac{\theta}{2}\right)\lambda\left(t-\frac{x^2}{2c^2t}\right)-\frac{3\pi}{4}\right). \tag{101}$$

It is not hard to see that, under the limit conditions (43), both the terms in the right-hand side of Eqs. (98) and (100) tend to

$$\frac{1}{2\sqrt{2\pi\sigma^2t}}\,e^{-\frac{x^2}{2\sigma^2t}-\frac{\kappa t}{2}},$$

whereas the terms given in Eqs. (99) and (101) tend to 0. This finally yields Eq. (80), due to Eqs. (78) and (79).

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