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On a bilateral birth-death process with alternating rates

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Abstract We consider a bilateral birth-death process characterized by a constant transition rate λ from even states and a possibly different transition rate μ from odd states. We determine the probability generating functions of the even and odd states, the transition probabilities, mean and variance of the process for arbitrary initial state. Some features of the birth-death process confined to the non-negative integers by a reflecting boundary in the zero-state are also analyzed. In particular, making use of a Laplace transform approach we obtain a series form of the transition probability from state 1 to the zero-state.

Keywords Birth-death processes · Alternating rates · Probability generating functions · Transition probabilities · Symmetry

Mathematics Subject Classification (2000) 60J80 · 60J85

1 Introduction

Birth-death processes were introduced to describe random growth (see, for instance, Ricciardi [17] for an accurate description of birth-death processes in the context of

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This paper is dedicated to the memory of Professor Luigi Maria Ricciardi, who passed away in Naples on May 7, 2011.

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population dynamics). Furthermore, they arise as natural descriptors of time-varying phenomena in several applied fields such as queueing, epidemiology, epidemics, optics, neurophysiology, etc. An extensive survey has been provided in Parthasarathy and Lenin [15]. In particular, in Section 9 of that paper certain birth-death processes are used to describe the time changes in the concentrations of the components of a chemical reaction, and their role in the study of diatomic molecular chains is emphasized.

Moreover, Stockmayer et al. [19] gave an example of application of stochastic processes in the study of chain molecular diffusion, by modeling a molecule as a freely-joined chain of two regularly alternating kinds of atoms. The two kinds of atoms have alternating jump rates, and these rates are reversed for odd labeled beads. By invoking the master equations for even and odd numbered bonds, the authors obtained the exact time-dependent average length of bond vectors.

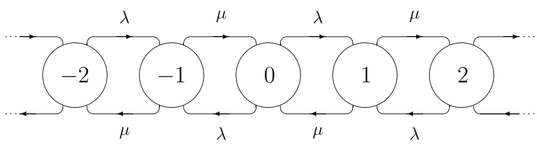
Inspired by this work, Conolly et al. [4] studied an infinitely long chain of atoms joined by links of equal length. The links are assumed to be subject to random shocks, that force the atoms to move and the molecule to diffuse. The shock mechanism is different according to whether the atom occupies an odd or an even position on the chain. The originating stochastic model is a randomized random walk on the integers with an unusual exponential pattern for the inter-step time intervals. The authors analyze some features of this process and investigate also its queue counterpart, where the walk is confined to the non negative integers. Various results concerning this queueing system with "chemical" rules (the so-called "chemical queue") were obtained also by Tarabia and El-Baz [20,21] and more recently by Tarabia et al. [22].

Another example arising in a chemical context where the role of parity is crucial is provided in Lente [14], where the probability of a more stable enantiomer is different according on whether the number of chiral molecules is even or odd.

Stimulated by the above researches, in this paper we consider a birth-death process $N(t)$ on the integers with a transition rate λ from even states and a possibly different rate μ from odd states. This model arises by suitably modifying the death rates of the process considered in the above papers. A detailed description of the model is performed in Sect. 2, where the probability generating functions of even and odd states and the transition probabilities of the process are obtained for arbitrary initial state. Certain symmetry properties of the transition probabilities are also given. In Sect. 3 we study the birth-death process obtained by superimposing a reflecting boundary in the zero-state. In particular, by making use of a Laplace transform approach, we obtain the probability of a transition from state 1 to the zero-state. Formulas for mean and variance of both processes are also provided. We remark that some preliminary results on the process under investigation are given in Iuliano and Martinucci [13] for the case of zero initial state.

It should be mentioned that closed-form results on bilateral birth-death processes have been obtained in the past only in few solvable cases, such as those in the above mentioned papers, and those given in Di Crescenzo [5], Di Crescenzo and Martinucci [9], Pollett [16].

Fig. 1 Transition rate diagram of $N(t)$



2 Transient distribution

We consider a birth-death process $\{N(t);\ t \geq 0\}$ with state-space $\mathbb{Z}$, and denote by

$$p_{k,n}(t) = P\{N(t) = n \mid N(0) = k\}, \quad t \geq 0, \quad n \in \mathbb{Z}$$

its transition probabilities, where $k \in \mathbb{Z}$ is the initial state. We assume that $N(t)$ is characterized by a transition rate λ from any even state to the two neighboring states, and by a possibly different transition rate μ from any odd state to the neighboring states. In other terms, denoting by

$$\nu_{j,n} = \lim_{h \to 0} \frac{1}{h} P\{N(t+h) = n \mid N(t) = j\}$$

the time-homogeneous transition rates of $N(t)$ from state j to state n, we assume that the allowed transitions are characterized by the following rates:

$$\nu_{2n,2n\pm1} = \lambda, \quad \nu_{2n\pm1,2n} = \mu, \quad \forall n \in \mathbb{Z}, \tag{1}$$

with $\lambda, \mu > 0$. The associated transition rate diagram of this process is given in Fig. 1. We note that rates (1) are different from those of the birth-death model considered in Conolly et al. [4] and Tarabia et al. [22], where $\nu_{2n,2n+1} = \nu_{2n+1,2n} = \lambda$ and $\nu_{2n-1,2n} = \nu_{2n,2n-1} = \mu$ for any $n \in \mathbb{Z}$.

Due to assumptions (1), the transition probabilities of $N(t)$ satisfy the following system of differential-difference equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} p_{k,2n}(t) = \mu\, p_{k,2n-1}(t) - 2\lambda\, p_{k,2n}(t) + \mu\, p_{k,2n+1}(t), \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t} p_{k,2n+1}(t) = \lambda\, p_{k,2n}(t) - 2\mu\, p_{k,2n+1}(t) + \lambda\, p_{k,2n+2}(t), \end{cases} \tag{2}$$

for any $t \geq 0$, $n \in \mathbb{Z}$ and for any initial state $k \in \mathbb{Z}$. The initial condition is expressed by:

$$p_{k,n}(0) = \delta_{n,k}, \tag{3}$$

where $\delta_{n,k}$ is the Kronecker's delta. We notice that in the special case when $\lambda = \mu$ process $N(t)$ identifies with the so-called "randomized random walk" (see Conolly [3]).

In order to obtain the state probabilities of $N(t)$, hereafter we develop a probability generating function-based approach. We recall that this method has been used in the past to determine probabilities of interest in several stochastic models (see, for instance, Giorno and Nobile [10] and Ricciardi and Sato [18] for the distribution of the range of one-dimensional random walks). Let us define the probability generating functions of the sets of even and odd states of $N(t)$, respectively:

$$F_k(z,t) := \sum_{j=-\infty}^{+\infty} z^{2j} p_{k,2j}(t), \quad G_k(z,t) := \sum_{j=-\infty}^{+\infty} z^{2j+1} p_{k,2j+1}(t), \tag{4}$$

with $z \in \mathbb{Z}$. Note that, due to (3), the following initial conditions hold:

$$F_k(z,0) = \begin{cases} z^k & k \text{ even} \\ 0 & k \text{ odd}, \end{cases} \qquad G_k(z,0) = \begin{cases} 0 & k \text{ even} \\ z^k & k \text{ odd}. \end{cases} \tag{5}$$

From system (2) we have that the generating functions (4) satisfy the following differential system:

$$\begin{cases} \dfrac{\partial}{\partial t} F_k(z,t) = \mu\, z G_k(z,t) - 2\lambda\, F_k(z,t) + \dfrac{\mu}{z}\, G_k(z,t), \\[2ex] \dfrac{\partial}{\partial t} G_k(z,t) = \lambda\, z F_k(z,t) - 2\mu\, G_k(z,t) + \dfrac{\lambda}{z}\, F_k(z,t), \end{cases}$$

so that

$$\frac{\partial}{\partial t}\begin{pmatrix} F_k(z,t) \\ G_k(z,t) \end{pmatrix} = A \cdot \begin{pmatrix} F_k(z,t) \\ G_k(z,t) \end{pmatrix}, \quad A := \begin{pmatrix} -2\lambda & \mu \dfrac{z^2+1}{z} \\[2ex] \lambda \dfrac{z^2+1}{z} & -2\mu \end{pmatrix}.$$

Hence, by use of standard methods, due to conditions (5) we come to

$$\begin{pmatrix} F_{2k}(z,t) \\ G_{2k}(z,t) \end{pmatrix} = \mathrm{e}^{At} \cdot \begin{pmatrix} z^{2k} \\ 0 \end{pmatrix}, \tag{6}$$

and

$$\begin{pmatrix} F_{2k+1}(z,t) \\ G_{2k+1}(z,t) \end{pmatrix} = \mathrm{e}^{At} \cdot \begin{pmatrix} 0 \\ z^{2k+1} \end{pmatrix}, \tag{7}$$

where

$$\mathrm{e}^{At} = \exp\left\{ \begin{pmatrix} -2\lambda & \mu \dfrac{z^2+1}{z} \\[2ex] \lambda \dfrac{z^2+1}{z} & -2\mu \end{pmatrix} t \right\}.$$

By straightforward calculations we have $A = S \cdot V \cdot S^{-1}$, where

$$S = \begin{pmatrix} \mu - \lambda - \dfrac{h(z)}{z} & \mu - \lambda + \dfrac{h(z)}{z} \\ \lambda \dfrac{z^2+1}{z} & \lambda \dfrac{z^2+1}{z} \end{pmatrix}, \qquad V = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}, \tag{8}$$

for $v_1 = -(\lambda+\mu) - h(z)/z$ and $v_2 = -(\lambda+\mu) + h(z)/z$, and where

$$S^{-1} = -\frac{z}{2\lambda(z^2+1)h(z)} \begin{pmatrix} \lambda(z^2+1) & z(\lambda-\mu) - h(z) \\ -\lambda(z^2+1) & z(\mu-\lambda) - h(z) \end{pmatrix}, \tag{9}$$

with

$$h(z) := \sqrt{(\mu z^2 + \lambda)(\lambda z^2 + \mu)}.$$

If the initial state is even $(2k)$, Eqs. (6) and (8)–(9) give

$$\mathrm{e}^{At} \cdot \begin{pmatrix} z^{2k} \\ 0 \end{pmatrix} = \begin{pmatrix} \mu - \lambda - \dfrac{h(z)}{z} & \mu - \lambda + \dfrac{h(z)}{z} \\ \lambda \dfrac{(z^2+1)}{z} & \lambda \dfrac{(z^2+1)}{z} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{v_1 t} & 0 \\ 0 & \mathrm{e}^{v_2 t} \end{pmatrix} \begin{pmatrix} -\dfrac{z^{2k+1}}{2h(z)} \\ \dfrac{z^{2k+1}}{2h(z)} \end{pmatrix},$$

and then

$$\mathrm{e}^{At} \cdot \begin{pmatrix} z^{2k} \\ 0 \end{pmatrix} = \mathrm{e}^{-(\lambda+\mu)t} \cdot \frac{z^{2k}}{h(z)} \begin{pmatrix} h(z) \cosh[t\,\dfrac{h(z)}{z}] + z(\mu-\lambda) \sinh[t\,\dfrac{h(z)}{z}], \\ \lambda(z^2+1) \sinh[t\,\dfrac{h(z)}{z}] \end{pmatrix}. \tag{10}$$

Hence, from Eqs. (6) and (10) we obtain the explicit expression of the probability generating functions when the initial state is even:

$$F_{2k}(z,t) = \mathrm{e}^{-(\lambda+\mu)t} \frac{z^{2k}}{h(z)} \left\{ h(z) \cosh\left[\frac{t\,h(z)}{z}\right] + z(\mu-\lambda) \sinh\left[\frac{t\,h(z)}{z}\right] \right\}, \tag{11}$$

$$G_{2k}(z,t) = \mathrm{e}^{-(\lambda+\mu)t} \frac{z^{2k}}{h(z)} \lambda \left(z^2+1\right) \sinh\left[\frac{t\,h(z)}{z}\right]. \tag{12}$$

Similarly, if the initial state is odd $(2k+1)$ the explicit expression of the probability generating functions is:

$$F_{2k+1}(z,t) = \mathrm{e}^{-(\lambda+\mu)t}\,\frac{z^{2k+1}}{h(z)}\,\mu\left(z^2+1\right)\sinh\left[\frac{t\,h(z)}{z}\right], \tag{13}$$

$$G_{2k+1}(z,t) = \mathrm{e}^{-(\lambda+\mu)t}\,\frac{z^{2k+1}}{h(z)}\left\{h(z)\cosh\left[\frac{t\,h(z)}{z}\right] + z(\lambda-\mu)\sinh\left[\frac{t\,h(z)}{z}\right]\right\}. \tag{14}$$

We are now able to provide the state probabilities.

Proposition 1 *For all $l, r \in \mathbb{Z}$ and $t \geq 0$ the transition probabilities of $N(t)$ are:*

$$\begin{aligned} p_{2l,2r}(t) &= \mathrm{e}^{-(\lambda+\mu)t}\sum_{n=|r-l|}^{+\infty}\left[\frac{(\lambda t)^{2n}}{(2n)!} + \left(\frac{\mu-\lambda}{\lambda}\right)\frac{(\lambda t)^{2n+1}}{(2n+1)!}\right] \\ &\quad\times\sum_{k=0}^{n-|r-l|}\binom{n}{k}\binom{n}{k+|r-l|}\left(\frac{\lambda}{\mu}\right)^{-2k-|r-l|}, \end{aligned} \tag{15}$$

$$\begin{aligned} p_{2l,2r+1}(t) &= \mathrm{e}^{-(\lambda+\mu)t}\left\{\sum_{n=|r-l|}^{+\infty}\frac{(\lambda t)^{2n+1}}{(2n+1)!}\sum_{k=0}^{n-|r-l|}\binom{n}{k}\binom{n}{k+|r-l|}\left(\frac{\lambda}{\mu}\right)^{-2k-|r-l|}\right. \\ &\quad\left.+\sum_{n=|r-l+1|}^{+\infty}\frac{(\lambda t)^{2n+1}}{(2n+1)!}\sum_{k=0}^{n-|r-l+1|}\binom{n}{k}\binom{n}{k+|r-l+1|}\left(\frac{\lambda}{\mu}\right)^{-2k-|r-l+1|}\right\}. \end{aligned} \tag{16}$$

$$\begin{aligned} p_{2l+1,2r}(t) &= \mathrm{e}^{-(\lambda+\mu)t}\left\{\sum_{n=|r-l-1|}^{+\infty}\frac{(\mu t)^{2n+1}}{(2n+1)!}\sum_{k=0}^{n-|r-l-1|}\binom{n}{k}\binom{n}{k+|r-l-1|}\left(\frac{\mu}{\lambda}\right)^{-2k-|r-l-1|}\right. \\ &\quad\left.+\sum_{n=|r-l|}^{+\infty}\frac{(\mu t)^{2n+1}}{(2n+1)!}\sum_{k=0}^{n-|r-l|}\binom{n}{k}\binom{n}{k+|r-l|}\left(\frac{\mu}{\lambda}\right)^{-2k-|r-l|}\right\}. \end{aligned} \tag{17}$$

$$\begin{aligned} p_{2l+1,2r+1}(t) &= \mathrm{e}^{-(\lambda+\mu)t}\sum_{n=|r-l|}^{+\infty}\left[\frac{(\mu t)^{2n}}{(2n)!} + \left(\frac{\lambda-\mu}{\lambda}\right)\frac{(\mu t)^{2n+1}}{(2n+1)!}\right] \\ &\quad\times\sum_{k=0}^{n-|r-l|}\binom{n}{k}\binom{n}{k+|r-l|}\left(\frac{\mu}{\lambda}\right)^{-2k-|r-l|}. \end{aligned} \tag{18}$$

Proof It follows by extracting the coefficients of x^{2r} and x^{2r+1} in (11)–(14), respectively. □

Figure 2 shows some plots of transition probabilities given in Proposition 1.

2.1 Symmetry properties

The relevance of symmetry properties of transition functions of birth-death processes has been emphasized in Anderson and McDunnough [1] and in Di Crescenzo [6], for instance. We stress that the role of symmetry is closely connected to the analysis of the

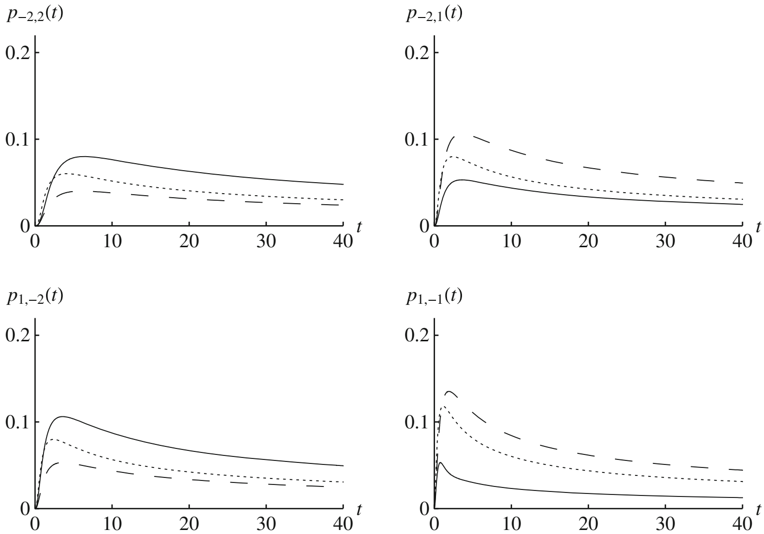


Fig. 2 Plots of some transition probabilities for $(\lambda, \mu) = (1, 2)$ (*solid line*), $(\lambda, \mu) = (2, 2)$ (*dotted line*), $(\lambda, \mu) = (2, 1)$ (*dashed line*)

first-passage-time problem in Markov process. See, for instance, the contributions of Giorno et al. [11,12] and Di Crescenzo et al. [7,8], where some relations involving the transition probability density functions and the first-passage-time density functions of symmetric diffusion processes in the presence of suitable time-varying boundaries.

Hereafter we analyze some symmetry properties of the transition probabilities obtained in Proposition 1. The proof is omitted, since it follows from direct analysis of the probabilities (15)–(18). When necessary we emphasize the dependence on the parameters by writing $p_{k,n}(t;\lambda,\mu)$ instead of $p_{k,n}(t)$.

Proposition 2 *For every $t \geq 0$ and $n, k \in \mathbb{Z}$ the following symmetry relations hold:*

$$\begin{array}{lll} (i) & p_{N-k,N-n}(t) = p_{k,n}(t), & \textit{if } N \textit{ is even} \\ (ii) & p_{N-k,N-n}(t;\lambda,\mu) = p_{k,n}(t;\mu,\lambda), & \textit{if } N \textit{ is odd}; \\ (iii) & p_{n,k}(t;\lambda,\mu) = p_{k,n}(t;\mu,\lambda); & \\ (iv) & p_{N+k,N+n}(t) = p_{k,n}(t), & \textit{if } N \textit{ is even} \\ (v) & p_{N+k,N+n}(t;\lambda,\mu) = p_{k,n}(t;\mu,\lambda), & \textit{if } N \textit{ is odd}. \end{array}$$

In Fig. 2 the plots of $p_{-2,1}(t)$ and $p_{1,-2}(t)$ illustrate a case in which property (ii) of Proposition 2 holds.

2.2 Moments

Hereafter we obtain in closed form the mean and the variance of $N(t)$. We shall obtain that the mean is equal to the initial state. This result is intuitively justified by the symmetry of the Markov chain. Indeed, by choosing $N = 2k$ and $n = k - r$ in identity (i) of Proposition 2 we have $p_{k,k+r}(t) = p_{k,k-r}(t)$ $\forall k, r \in \mathbb{Z}$, and $t \geq 0$.

Proposition 3 *For $t \geq 0$ and $k \in \mathbb{Z}$ we have*

$$E[N(t)|N(0)=k]=k, \tag{19}$$

$$Var[N(t)|N(0)=k]=\frac{4\lambda\mu}{\lambda+\mu}\,t+(-1)^k\frac{\lambda(\lambda-\mu)}{(\lambda+\mu)^2}\left[1-\mathrm{e}^{-2(\lambda+\mu)t}\right], \tag{20}$$

Proof The mean (19) easily follows from Eqs. (2) and (3). Moreover, by setting $\psi_k(t) := E[N^2(t)|N(0)=k]$ from system (2) we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi_k(t) &= 2\mu\sum_{n=-\infty}^{+\infty} p_{k,2n+1}(t)+2\lambda\sum_{n=-\infty}^{+\infty} p_{k,2n}(t)\\ &= 2\mu\, G_k(1,t)+2\lambda\, F_k(1,t), \quad t \geq 0,\end{aligned}$$

where F_k and G_k have been defined in (4). Hence, recalling Eqs. (11)–(14), after some calculations we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_k(t)=\begin{cases}\dfrac{4\lambda\mu}{\lambda+\mu}+\dfrac{2\lambda(\lambda-\mu)}{\lambda+\mu}\,\mathrm{e}^{-2(\lambda+\mu)t}, & k \text{ even}\\ \dfrac{4\lambda\mu}{\lambda+\mu}+\dfrac{2\mu(\mu-\lambda)}{\lambda+\mu}\mathrm{e}^{-2(\lambda+\mu)t}, & k \text{ odd}\end{cases}$$

with $\psi_k(0)=k^2$. Finally, Eq. (20) follows. □

3 A reflecting boundary

In this section we consider the case in which the state-space is reduced to the set of non-negative integers. We shall denote by $\{R(t);\ t \geq 0\}$ the birth-death process having state-space $\{0, 1, 2, \ldots\}$, with 0 reflecting, whose rates are identical to those of $N(t)$. This describes, for instance, the number of customers in a queueing system with alternating rates. For $n = 0, 1, 2, \ldots$, let us introduce the transition probabilities

$$q_{k,n}(t) = P\{R(t)=n \mid R(0)=k\}, \quad t \geq 0.$$

The related differential-difference equations are, for $n = 1, 2, \ldots,$

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}q_{k,0}(t)=\mu\, q_{k,1}(t)-\lambda\, q_{k,0}(t),\\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}q_{k,2n}(t)=\mu\, q_{k,2n-1}(t)-2\lambda\, q_{k,2n}(t)+\mu\, q_{k,2n+1}(t),\\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}q_{k,2n-1}(t)=\lambda\, q_{k,2n}(t)-2\mu\, q_{k,2n-1}(t)+\lambda\, q_{k,2n-2}(t),\end{cases} \tag{21}$$

with

$$q_{k,n}(0) = \delta_{n,k}. \tag{22}$$

We point out that the steady-state distribution of $R(t)$ does not exist. Indeed, from system (21) it is not hard to see that $\lim_{t\to\infty} q_{k,n}(t) = 0 \; \forall k, n \in \mathbb{Z}$.

3.1 Moments

Let us now set, for $k \in \mathbb{Z}$,

$$P_k(t) = P\{R(t) \text{ even} \mid R(0) = k\} = \sum_{n=0}^{+\infty} q_{k,2n}(t), \quad t \ge 0. \tag{23}$$

Now mean and variance of $R(t)$ will be formally expressed in terms of (23).

Proposition 4 *For $t \ge 0$ we have*

$$E[R(t)|R(0) = k] = \lambda \int_0^t q_{k,0}(\tau)d\tau + k, \tag{24}$$

$$\begin{aligned} Var[R(t)|R(0) = k] = 2(\lambda-\mu)\int_0^t P_k(\tau)d\tau - \lambda(2k+1)\int_0^t q_{k,0}(\tau)d\tau \\ -\lambda^2\left[\int_0^t q_{k,0}(\tau)d\tau\right]^2 + 2\mu t, \end{aligned} \tag{25}$$

where

$$P_k(t) = \frac{2\mu}{\lambda+\mu} + \frac{\lambda-\mu}{\lambda+\mu}\,\mathrm{e}^{-2(\lambda+\mu)t} + \lambda\int_0^t \mathrm{e}^{-2(\lambda+\mu)(t-\tau)} q_{k,0}(\tau)d\tau. \tag{26}$$

Proof The mean (24) easily follows from system (21) and condition (22). Moreover, from Eqs. (21) we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E[R^2(t)|R(0)=k] &= 2\mu\sum_{n=1}^{+\infty} q_{k,2n+1}(t) + 2\mu q_{k,1}(t) + \lambda q_{k,0}(t) + 2\lambda\sum_{n=1}^{+\infty} q_{k,2n}(t) \\ &= 2\mu[1 - P_k(t) - q_{k,1}(t)] + 2\mu q_{k,1}(t) \\ &\quad +\lambda q_{k,0}(t) + 2\lambda\,[P_k(t) - q_{k,0}(t)] \\ &= 2(\lambda-\mu)P_k(t) + \lambda q_{k,0}(t) + 2\mu, \end{aligned}$$

where $P_k(t)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P_k(t) = -2(\mu+\lambda)\, P_k(t) + \lambda\, q_{k,0}(t) + 2\mu. \tag{27}$$

Since the solution of (27) is Eq. (26), the conditional variance (25) easily follows. □

3.2 Probabilities

We note that when $k = n = 0$ the transition probability is given by (see Sect. 3 of Iuliano and Martinucci [13])

$$q_{0,0}(t) = \frac{\mathrm{e}^{-at}}{a+b} \sum_{k=0}^{+\infty} \frac{(t/2)^{2k}}{k!^2} \left\{ \left(a^{2k+1} + b^{2k+1}\right) {}_1F_2\left(-\frac{1}{2}, k+\frac{1}{2}, k+1, \frac{b^2t^2}{4}\right) + \frac{t\,(a^{2k+2} - b^{2k+2})}{2(k+1)}\, {}_1F_2\left(-\frac{1}{2}, k+1, k+\frac{3}{2}, \frac{b^2t^2}{4}\right)\right\}, \quad t \geq 0,$$

where

$$a = \lambda + \mu, \quad b = \lambda - \mu.$$

Now we analyse the case in which the initial state is $k = 1$. Denoting by

$$\pi_{k,n}(s) := \mathcal{L}_s[q_{k,n}(t)] = \int_0^\infty \mathrm{e}^{-st}\, q_{k,n}(t)\,\mathrm{d}t, \quad s > 0,$$

the Laplace transform of the transition probabilities of $R(t)$, from Eqs. (21) we have:

$$\begin{cases} (\lambda+s)\,\pi_{1,0}(s) = \mu\pi_{1,1}(s) \\ (2\mu+s)\,\pi_{1,1}(s) = 1 + \lambda\pi_{1,2}(s) + \lambda\pi_{1,0}(s) \\ (2\lambda+s)\,\pi_{1,2n}(s) = \mu\pi_{1,2n-1}(s) + \mu\pi_{1,2n+1}(s), \quad n \geq 1 \\ (2\mu+s)\,\pi_{1,2n-1}(s) = \lambda\pi_{1,2n}(s) + \lambda\pi_{1,2n-2}(s), \quad n \geq 2. \end{cases} \tag{28}$$

The solution of system (28) involves the roots of the biquadratic equation

$$\lambda\mu\, x^4 - \left[(\lambda+\mu+s)^2 - \lambda^2 - \mu^2\right] x^2 + \lambda\mu = 0,$$

which are given by

$$\psi_1^2(s) = \frac{(A+B)^2}{a^2-b^2}, \qquad \psi_2^2(s) = \frac{(A-B)^2}{a^2-b^2},$$

with

$$A^2 = (a+s)^2 - a^2, \qquad B^2 = (a+s)^2 - b^2.$$

Since $\psi_1^2(s) > 1$ and $0 < \psi_2^2(s) < 1$, from system (28) we finally obtain:

$$\pi_{1,2n}(s) = \frac{(2\mu + s)(\lambda + s)\left[\psi_2^2(s)\right]^{n+1}}{\lambda^2\left[\mu(1 - \psi_2^2(s)) - s\,\psi_2^2(s)\right]}, \qquad n \geq 1, \tag{29}$$

and, similarly,

$$\pi_{1,2n-1}(s) = \frac{(\lambda + s)\left[\psi_2^2(s)\right]^n\left[1 + \psi_2^2(s)\right]}{\lambda\left[\mu(1 - \psi_2^2(s)) - s\,\psi_2^2(s)\right]}, \qquad n \geq 1. \tag{30}$$

By making use of Eqs. (29) and (30) and substituting in (28), we have

$$\pi_{1,0}(s) = \frac{(2\lambda + s)(2\mu + s) - AB}{\lambda\,\left[s(2\mu + s) + AB\right]}. \tag{31}$$

By inversion of (31) after some calculations we obtain

$$\begin{aligned} q_{1,0}(t) = {} & \frac{\mathrm{e}^{-at}}{2\lambda(a+b)}\int_0^t\left[-b^2\,\frac{I_1(b(t-s))}{b(t-s)} + a^2\,\frac{I_1(a(t-s))}{a(t-s)}\right]h(s)\mathrm{d}s \\ & + \frac{\mathrm{e}^{-at}}{2\lambda(a+b)}(a^2 - b^2)\int_0^t \frac{b(t-s)}{2}\,{}_1F_2\left(\frac{1}{2}, \frac{3}{2}, 2, \frac{b^2(t-s)^2}{4}\right)h(s)\mathrm{d}s, \end{aligned} \tag{32}$$

where

$$h(x) := a\left[I_0(ax) + I_1(ax)\right] + b\left[I_0(bx) - I_1(bx)\right],$$

with $I_n(\cdot)$ denoting the modified Bessel function of the first kind. The evaluation of the integrals in Eq. (32) finally gives the following result.

Proposition 5 *For $t \geq 0$, we have*

$$\begin{aligned} q_{1,0}(t) = \frac{\mathrm{e}^{-at}}{\lambda(a+b)}\Bigg\{ & \sum_{n=0}^{+\infty}\frac{t^{2n}}{n!(n+1)!}\left[\left(\frac{a}{2}\right)^{2n+2} - \left(\frac{b}{2}\right)^{2n+2}\right]\xi\left(\frac{1}{2}, 1, a, b\right) \\ & + \sum_{n=0}^{+\infty}\frac{t^{2n+1}}{n!(n+1)!(2n+1)}\left[\frac{a^2 - b^2}{2}\left(\frac{b}{2}\right)^{2n+1}\right]\xi\left(1, \frac{3}{2}, a, b\right) \\ & + \sum_{n=0}^{+\infty}\frac{t^{2n+1}}{n!(n+1)!(2n+1)}\left[\left(\frac{a}{2}\right)^{2n+2} - \left(\frac{b}{2}\right)^{2n+2}\right]\eta\left(1, \frac{3}{2}, a, b\right) \\ & + \sum_{n=0}^{+\infty}\frac{t^{2n+2}}{n!(n+1)!(2n+1)(2n+2)}\left[\frac{a^2 - b^2}{2}\left(\frac{b}{2}\right)^{2n+1}\right]\eta\left(\frac{3}{2}, 2, a, b\right)\Bigg\}, \end{aligned}$$

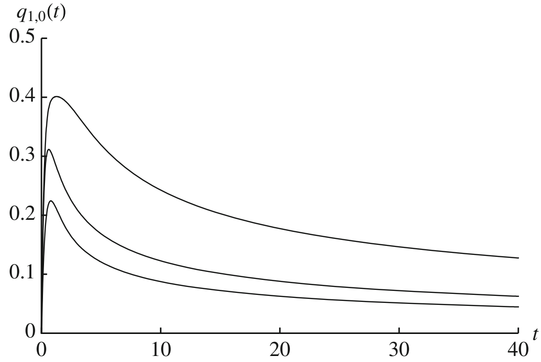


Fig. 3 Plots of $q_{1,0}(t)$ for $(\lambda,\mu)=(1,2),(2,2),(2,1)$, from *top* to *bottom*

where

$$\xi(u,v,a,b) = {}_1F_2\left(\frac{1}{2}; n+u, n+v; \frac{a^2t^2}{4}\right) - {}_1F_2\left(\frac{1}{2}; n+v, n+u; \frac{b^2t^2}{4}\right),$$

$$\eta(u,v,a,b)=a\ {}_1F_2\left(\frac{1}{2}; n+u, n+v; \frac{a^2t^2}{4}\right)+b\ {}_1F_2\left(\frac{1}{2}; n+v, n+u; \frac{b^2t^2}{4}\right).$$

In conclusion, some illustrative plots of $q_{1,0}(t)$ are shown in Fig. 3.

4 Concluding remarks

Stimulated by some previous works on the applications of stochastic processes to the study of chain molecular diffusion, in this paper we have analyzed a birth-death process on $\mathbb{Z}$ characterized by alternating transition rates. The probability generating functions of even and odd states and the transition probabilities of the bilateral process have been obtained when the initial state is arbitrary. A preliminary investigation on the transient behavior of the birth-death process obtained by superimposing a reflecting boundary in the zero-state has also been performed.

In conclusion, the results given in this paper deserve also special interest in the fields of chemical queueing processes and two-periodic random walks, according to the lines traced in various papers, such as Conolly et al. [4] and Böhm and Hornik [2], for instance.

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