

A Nyström method for second-kind Volterra integral equations on the square

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ABSTRACT. In this paper, we propose a Nyström-type method for the numerical approximation of second-kind bivariate Volterra integral equations on the unit square. The method is based on bivariate Generalized Bernstein (GB) polynomials and on a cubature formula defined on equally spaced nodes, allowing the integral operator to be discretized directly without requiring any change of variables. The use of uniform grids makes the proposed approach particularly suitable for applications in which data are naturally sampled at equidistant points.

The approximation framework relies on tensor-product GB polynomials whose degrees may differ with respect to each variable. This flexibility allows different approximation orders to be prescribed according to the regularity of the target function in each variable. As a result, the computational cost can be reduced while preserving high level of accuracy. Compared with classical Bernstein operators, the proposed GB operators achieve higher approximation orders depending on the smoothness of the approximated functions.

The resulting Nyström scheme leads to a well-conditioned linear system and is proved to be stable and convergent. Error estimates are derived in Sobolev and Zygmund spaces. Several numerical experiments are presented to validate the theoretical results and to illustrate the effectiveness of the method for kernels and right-hand side functions with different smoothness properties.

Keywords: Volterra integral equations, Nyström method, generalized Bernstein polynomials, cubature rules.

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1. INTRODUCTION

This paper deals with the numerical approximation of second-kind Volterra integral equations on the square $S := [0, 1] \times [0, 1]$, of the form

$$(1.1) \quad f(\mathbf{y}) + \nu \int_{\mathcal{Y}} h(\mathbf{x}, \mathbf{y}) f(\mathbf{x}), d\mathbf{x} = g(\mathbf{y}), \quad \mathbf{y} := (y_1, y_2), \quad \mathbf{x} := (x_1, x_2) \in S,$$

where $\mathcal{Y} = \{\mathbf{x} \mid 0 \leq x_1 \leq y_1, \ 0 \leq x_2 \leq y_2, \ (y_1, y_2) \in S\}$. Here, f denotes the unknown solution, ν is a nonzero real parameter, while h and g are given functions defined on $S \times S$ and S , respectively.

Two-dimensional Volterra integral equations arise in several areas of applied science and engineering. Applications include electromagnetic theory [6], mathematical models for asexual reproduction processes [24], molecular physics and electrostatics [14], contact problems [15], and heat transfer [12]. These applications motivate the development of efficient and reliable numerical methods.

Over the years, several numerical approaches have been proposed for the approximation of such equations, including collocation and iterated methods [4, 5], Euler-type methods [17], and

methods based on Chebyshev polynomials [2]. More recently, techniques involving geometric series and Schauder fixed point theory have been investigated in [3], together with extrapolation methods [23] and Taylor collocation methods [13].

In this paper, we propose a Nyström-type method based on bivariate Generalized Bernstein (GB) polynomials [20], defined on equally spaced grids of $\mathcal{S}$. This aspect makes such polynomials particularly suitable for applications in which data are naturally sampled on uniform grids.

GB polynomials have been investigated in the numerical treatment of functional equations in the univariate setting for Fredholm and Volterra integral equations [10, 19, 21], and in the bivariate setting for second-kind Fredholm integral equations [20]. The present paper extends this framework to second-kind bivariate Volterra integral equations.

More precisely, we consider GB polynomials

$$B_{\mathbf{m}}^{\ell}(f; x_1, x_2), \quad \mathbf{m} = (m_1, m_2), \quad \ell = (\ell_1, \ell_2),$$

where $m_i, \ell_i \in \mathbb{N}$, $i = 1, 2$. The polynomial degrees may differ with respect to the two variables, namely degree m_1 in x_1 and degree m_2 in x_2 . Similarly to classical bivariate Bernstein polynomials $B_{\mathbf{m}}(f)$, GB polynomials require the values of f on an equidistant grid of $(m_1 + 1) \times (m_2 + 1)$ points. However, owing to the presence of the parameters ℓ_1 and ℓ_2 , they achieve higher approximation orders than the classical Bernstein polynomial sequence. In particular, as $m_1, m_2 \rightarrow \infty$, the convergence rate depends on the smoothness of the target function: the smoother the function, the faster the convergence.

These approximation properties are inherited by the Nyström method we propose here to approximate the solution of equation (1.1). The resulting numerical scheme leads to a well-conditioned linear system and is shown to be stable and convergent. Moreover, error estimates are derived in Sobolev and Zygmund spaces.

The paper is organized as follows. In Section 2, we introduce the functional spaces and the notation used throughout the paper. Section 3 reviews the GB polynomials and presents the cubature formula employed to approximate the integral operator in (1.1). In Section 4, we describe the Nyström method, while Section 5 is devoted to computational details and numerical experiments. Finally, conclusions are drawn in Section 6.

2. NOTATIONS AND FUNCTION SPACES

In the sequel, $\mathcal{C}$ will denote any positive constant that may differ in different formulas. Writing $\mathcal{C} \neq \mathcal{C}(a, b, c, \dots)$ means that the positive constant $\mathcal{C}$ is independent of the parameters $a, b, c, \dots$. Conversely, if we write $\mathcal{C} = \mathcal{C}(a, b, c, \dots)$ this highlights the dependence of $\mathcal{C}$ on $a, b, c, \dots$. The notation $A \sim B$, where A and B are positive quantities depending on some parameters, it will be used if and only if $(A/B)^{\pm 1} \leq \mathcal{C}$, where $\mathcal{C}$ is a positive constant independent of the above parameters. $\mathbb{P}_m$ will denote the space of one-variable algebraic polynomials of degree at most m , whereas $\mathbb{P}_{\mathbf{m}}$, $\mathbf{m} := (m_1, m_2)$, $m_i \in \mathbb{N}$, will indicate the set of all bivariate polynomials whose degree is at most m_1 in the variable x_1 and not exceeding m_2 in the variable x_2 . In what follows, we set $N_0^n := \{0, 1, \dots, n\}$, $n \in \mathbb{N}$.

Finally, if $f(x_1, x_2)$ is a bivariate function, f_{x_i} will denote the univariate functions obtained by fixing the variable x_i , $i = 1, 2$. Similarly, if $k(\mathbf{x}, \mathbf{y})$ is a four-variables function, the notation $k_{\mathbf{y}}$ will mean the bivariate function obtained by fixing the variables $\mathbf{y} := (y_1, y_2)$ and similarly, $k_{\mathbf{x}}$ will denote the bivariate function obtained by fixing the variables $\mathbf{x} := (x_1, x_2)$.

2.1. Univariate spaces. Let $C([0, 1])$ be the set of all continuous functions on the interval $[0, 1]$ endowed with the uniform norm

$$\|F\|_\infty := \max_{x \in [0, 1]} |F(x)|.$$

The error of best approximation of $F \in C([0, 1])$

$$E_n(F) = \min_{P \in \mathbb{P}_n} \|F - P\|, \quad F \in C([0, 1])$$

characterizes the space $C([0, 1])$, i.e.

$$F \in C([0, 1]) \iff \lim_{n \rightarrow \infty} E_n(F) = 0.$$

Setting $\varphi(x) = \sqrt{x(1-x)}$, for any $F \in C([0, 1])$, the φ -modulus of smoothness by Ditzian & Totik of order $k \in \mathbb{N}$ is defined as [8, p. 10]

$$\omega_\varphi^k(F, t) = \sup_{h \leq t} \|\Delta_{h\varphi}^k F\|_\infty,$$

where

$$\Delta_{h\varphi}^k F(x) = \sum_{i=0}^k (-1)^{i-1} \binom{k}{i} F\left(x + \frac{h}{2} \varphi(x) (k - 2i)\right).$$

We recall that ([8], Theorem 2.1.1)

$$(2.2) \quad \omega_\varphi^k(f, t) \sim K_{k,\varphi}(F, t^k) := \inf\{\|F - g\| + t^k \|g^{(k)} \varphi^k\| : g^{(k-1)} \in \mathcal{AC}\},$$

where $\mathcal{AC}$ denotes the space of the absolutely continuous functions on $[0, 1]$.

Using the above modulus of smoothness, it is possible to define the Hölder–Zygmund space of order $\lambda \in \mathbb{R}^+$ as

$$Z_\lambda = \left\{ F \in C([0, 1]) : \sup_{t>0} \frac{\omega_\varphi^k(F, t)}{t^\lambda} < \infty, \quad k > \lambda > 0 \right\},$$

with the norm

$$\|F\|_{Z_\lambda} = \|F\|_\infty + \sup_{t>0} \frac{\omega_\varphi^k(F, t)}{t^\lambda}, \quad k > \lambda.$$

Zygmund spaces can be characterized in terms of $E_n(F)$, [8, Theorem 8.2.1]

$$F \in Z_\lambda \Leftrightarrow E_n(F) = \mathcal{O}(n^{-\lambda}), \quad \forall \lambda \in \mathbb{R}^+,$$

and ([8], Theorem 4.2.1)

$$(2.3) \quad \omega_\varphi^k(F, t) \leq Ct^\lambda \|F\|_{Z_\lambda}, \quad C \neq \mathcal{C}(F, t), \quad k > \lambda.$$

Now, let us consider the following Sobolev–type spaces of integer order $r \geq 1$

$$W_r = \left\{ F \in C([0, 1]), F^{(r-1)} \in \mathcal{AC}(0, 1) : \|F^{(r)} \varphi^r\| < \infty \right\},$$

endowed with the norm

$$\|F\|_{W_r} = \|F\|_\infty + \|F^{(r)} \varphi^r\|_\infty.$$

Functions in W_r can be characterized in terms of $E_n(F)$ [8, Theorem 4.2.1]

$$F \in W_r \Leftrightarrow E_n(F) = \mathcal{O}(n^{-r}), \quad \forall r \in \mathbb{N},$$

and ([8], Theorem 4.2.1)

$$(2.4) \quad \omega_\varphi^k(F, t) \leq Ct^r \|F\|_{W_r}, \quad C \neq \mathcal{C}(F, t), \quad k > r.$$

2.2. Bivariate spaces. Let $C(\mathcal{S})$ be the space of bivariate continuous functions on the square $\mathcal{S}$, equipped with the uniform norm

$$\|f\|_\infty = \max_{\mathbf{x} \in \mathcal{S}} |f(\mathbf{x})|.$$

For smoother functions, we recall [20] the following Sobolev-type subspaces of $C(\mathcal{S})$ of multi-index $\mathbf{r} := (r_1, r_2)$, $r_i \in \mathbb{N}$, $i = 1, 2$ as

$$W_{\mathbf{r}} := \{f \in C(\mathcal{S}) : M_{\mathbf{r}}(f) < \infty\},$$

where

$$(2.5) \quad M_{\mathbf{r}}(f) := \max \left\{ \sup_{x_2 \in [0,1]} \left\| \frac{\partial^{r_1} f_{x_2}}{\partial x_1^{r_1}} \varphi_1^{r_1} \right\|_\infty, \sup_{x_1 \in [0,1]} \left\| \frac{\partial^{r_2} f_{x_1}}{\partial x_2^{r_2}} \varphi_2^{r_2} \right\|_\infty \right\}.$$

Here and in the sequel, the superscript (r_j) denotes the r_j -th derivative of the one-dimensional function f_{x_1} or f_{x_2} and $\varphi_1(x_1) = \sqrt{x_1(1-x_1)}$, $\varphi_2(x_2) = \sqrt{x_2(1-x_2)}$. $W_{\mathbf{r}}$ is endowed with the norm

$$\|f\|_{W_{\mathbf{r}}} = \|f\|_\infty + M_{\mathbf{r}}(f).$$

To deal with more refined spaces of functions, we consider the following Hölder-Zygmund space of multi-index $\lambda := (\lambda_1, \lambda_2)$, $\lambda_i \in \mathbb{R}^+$

$$Z_{\lambda} := \{f \in C(\mathcal{S}) : N_{\lambda}(f) < \infty\},$$

where

$$(2.6) \quad N_{\lambda}(f) := \max \left\{ \sup_{x_2 \in [0,1]} \sup_{t>0} \frac{\omega_{\varphi_1}^{k_1}(f_{x_2}, t)}{t^{\lambda_1}}, \sup_{x_1 \in [0,1]} \sup_{t>0} \frac{\omega_{\varphi_2}^{k_2}(f_{x_1}, t)}{t^{\lambda_2}} \right\},$$

with $k_1 > \lambda_1 > 0$ and $k_2 > \lambda_2 > 0$. Z_{λ} is equipped with the norm

$$\|f\|_{Z_{\lambda}} = \|f\|_\infty + N_{\lambda}(f).$$

3. APPROXIMATION TOOLS

In this section, we summarize the key properties and definitions of univariate GB polynomials introduced and studied in [9, 16, 18]. We then move on to the bivariate case, recalling the 2D GB polynomials investigated in [21, 20]. Thus, we apply these polynomials to construct a cubature rule, successively employed to construct the Nyström method introduced in Section 4.

3.1. Univariate GB polynomials. Let $F \in C([0, 1])$ and denote by $B_n(F)$ the n -th Bernstein polynomial of F

$$B_n(F, z) = \sum_{k=0}^n p_{n,k}(z) F(t_{n,k}), \quad z \in [0, 1],$$

where

$$p_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}, \quad t_{n,k} := \frac{k}{n}, \quad j \in N_0^n.$$

Denoted by I the identity operator in $C([0, 1])$, the GB operator of order $s \in \mathbb{N}$, is defined as [16, 18]

$$B_{n,s} = I - (I - B_n)^s, \quad \text{with} \quad \begin{cases} B_n^0 := I \\ B_n^i := B_n(B_n^{i-1}), \end{cases} \quad i = 1, 2, \dots.$$

The map $B_{n,s} : F \in C([0, 1]) \rightarrow \mathbb{P}_n$ is linear and not always positive. Moreover, as $s \rightarrow \infty$ the polynomial sequence $B_{n,s}(F)$ continuously links Bernstein polynomials $B_n(F)$ to Lagrange polynomials $L_n(F)$ interpolating F at the nodes $t_{n,i}$, i.e. $L_n(F, t_{n,i}) = F(t_{n,i})$, $i \in N_0^n$.

For all n, s , and for any $z \in [0, 1]$, the polynomial $B_{n,s}(F, z)$ can be represented in the basis $\{p_{n,k}(z)\}_{k=0}^n$. Indeed, introducing the matrix $\mathbf{A}_n \in \mathbb{R}^{(n+1) \times (n+1)}$ defined as

$$(\mathbf{A}_n)_{k,j} = p_{n,j}(t_{n,k}),$$

and setting $\mathbf{C}_n^s \in \mathbb{R}^{(n+1) \times (n+1)}$,

$$(3.7) \quad \begin{cases} \mathbf{C}_n^1 = \mathbf{I} \\ \mathbf{C}_n^s = \mathbf{I} + (\mathbf{I} - \mathbf{A}_n) + \dots + (\mathbf{I} - \mathbf{A}_n)^{s-1}, \quad s \geq 2, \end{cases}$$

where $\mathbf{I}$ is the identity matrix of order $n+1$, the following relation holds true [22]

$$(3.8) \quad p_{n,j}^{(s)}(z) = \sum_{k=0}^n p_{n,k}(z) c_{k,j}^s, \quad j \in N_0^n,$$

where $c_{k,j}^s$ denotes the entries of the matrix $\mathbf{C}_n^s$.

We recall that the following recurrence relation holds [22]

$$\mathbf{C}_{n,2^p} = \mathbf{C}_{n,2^{p-1}} + (\mathbf{I} - \mathbf{A})^{2^{p-1}} \mathbf{C}_{n,2^{p-1}},$$

which allows a fast construction of the subsequence $\{B_{n,2^p}\}_{p=1,2,\dots}$.

Now we recall a result proved by Gonska and Zhou in [11], attesting the uniform convergence of the sequence $\{B_{n,s}(F)\}_n$ to any $F \in C([0, 1])$ for s fixed, and for all $n \in \mathbb{N}$,

$$\|F - B_{n,s}(F)\| \leq C \left\{ \omega_\varphi^{2s} \left(F, \frac{1}{\sqrt{n}} \right) + \frac{\|F\|_\infty}{n^s} \right\}, \quad F \in C([0, 1]),$$

with $C = C(s)$, $C \neq C(F, n)$.

3.2. GB polynomials on $\mathcal{S}$. We recall the definition of GB polynomials on the square $\mathcal{S}$ (see, e.g. [20, 21]).

For any bivariate function $f \in C(\mathcal{S})$, with $\mathbf{m} = (m_1, m_2)$ the $\mathbf{m}$ -th Bernstein polynomial is defined as

$$B_{\mathbf{m}} := B_{m_1} \otimes B_{m_2} : C(\mathcal{S}) \rightarrow \mathbb{P}_{\mathbf{m}},$$

and for a given $\ell := (\ell_1, \ell_2)$, let be

$$B_{\mathbf{m}}^\ell := B_{m_1, \ell_1} \otimes B_{m_2, \ell_2} : C(\mathcal{S}) \rightarrow \mathbb{P}_{\mathbf{m}},$$

as introduced in [21] (see also [20] in the special case $m_1 = m_2$ and $\ell_1 = \ell_2$).

For any $f \in C(\mathcal{S})$ the polynomial $B_{\mathbf{m}}^\ell(f, \mathbf{x})$ can be represented as

$$B_{\mathbf{m}}^\ell(f, \mathbf{x}) = \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} p_{m_1, j_1}^{(\ell_1)}(x_1) p_{m_2, j_2}^{(\ell_2)}(x_2) f(\mathbf{t}_{j_1, j_2}^{\mathbf{m}}), \quad \mathbf{t}_{j_1, j_2}^{\mathbf{m}} := (t_{m_1, j_1}, t_{m_2, j_2}).$$

We point out that $B_{\mathbf{m}}^\ell$ preserves polynomials of degree 1 in each variable, i.e.,

$$B_{\mathbf{m}}^\ell(q, \mathbf{x}) = q(\mathbf{x}), \quad \forall q \in \mathbb{P}_1,$$

and, denoting by $\mathcal{L}_{\mathbf{m}}(f, \mathbf{x}) \in \mathbb{P}_{\mathbf{m}}$ the 2D Lagrange polynomial interpolating $\mathbf{f}$ at the grid $\{(t_{m_1, i}, t_{m_2, j})\}_{(i, j) \in N_0^{m_1} \times N_0^{m_2}}$, by [20], for any $f \in C(\mathcal{S})$, one has

$$\lim_{\ell \rightarrow +\infty} B_{\mathbf{m}}^\ell(f, \mathbf{x}) := \lim_{\ell_1, \ell_2 \rightarrow +\infty} B_{\mathbf{m}}^{(\ell_1, \ell_2)}(f, \mathbf{x}) = \mathcal{L}_{\mathbf{m}}(f, \mathbf{x}),$$

holding uniformly in $\mathbf{x} \in \mathcal{S}$. Polynomials $B_{\mathbf{m}}^{\ell}(f)$ have different degrees in each variable and also different parameters ℓ_1, ℓ_2 , and these different choices make them more flexible in approximating functions that exhibit different regularities with respect to each variable.

In addition, the polynomial $B_{\mathbf{m}}^{\ell}(f, \mathbf{x})$ can be conveniently represented in vector form. Indeed, in view of (3.8), one can easily deduce

$$B_{\mathbf{m}}^{\ell}(f, \mathbf{x}) = \mathbf{p}_{m_1}(x_1)^T \mathbf{C}_{m_1}^{\ell_1} \mathbf{F}_{m_1, m_2} \mathbf{C}_{m_2}^{\ell_2} \mathbf{p}_{m_2}(x_2),$$

where

$$\mathbf{p}_{m_i}(x_i) = [p_{m_i, 0}(x_i), \dots, p_{m_i, m_i}(x_i)]^T,$$

$$(3.9) \quad \mathbf{F}_{m_1, m_2}(i, j) = f(\mathbf{t}_{i, j}^{\mathbf{m}}), \quad \mathbf{t}_{i, j}^{\mathbf{m}} := (t_{m_1, i}, t_{m_2, j}), \quad (i, j) \in N_0^{m_1} \times N_0^{m_2}.$$

and the matrices $\mathbf{C}_{m_i}^{\ell_i} \in \mathbb{R}^{(m_i+1) \times (m_i+1)}$ defined as in (3.7) with m_i instead of n and ℓ_i instead of s , for $i = 1, 2$ in both cases.

Regarding the uniform convergence of the sequence $B_{\mathbf{m}}^{\ell}(f)$, the following result holds.

Theorem 3.1. [21] *Let $f \in C(\mathcal{S})$. For any $\mathbf{m} = (m_1, m_2)$, fixing ℓ , one has*

$$\|B_{\mathbf{m}}^{\ell}(f)\|_{\infty} \leq 2^{\ell_1 + \ell_2} \|f\|_{\infty}.$$

Moreover, for m_1, m_2 sufficiently large (say $m_1, m_2 > m_0$), the following estimate holds

$$(3.10) \quad \|f - B_{\mathbf{m}}^{\ell}(f)\|_{\infty} \leq C \max \left\{ \sup_{x_2 \in [0, 1]} \omega_{\varphi_1}^{2\ell_1} \left(f_{x_2}, \frac{1}{\sqrt{m_1}} \right), \sup_{x_1 \in [0, 1]} \omega_{\varphi_2}^{2\ell_2} \left(f_{x_1}, \frac{1}{\sqrt{m_2}} \right) \right\} \\ + C \|f\|_{\infty} \left(\frac{1}{m_1^{\ell_1}} + \frac{1}{m_2^{\ell_2}} \right),$$

where $C = C(\ell)$ and $C \neq C(f, \mathbf{m})$.

Remark 3.1. By (3.10), for any $f \in W_{\mathbf{r}}$ in view of (2.4) one can easily obtain

$$\|f - B_{\mathbf{m}}^{\ell}(f)\|_{\infty} \leq C \left[\|f\|_{W_{\mathbf{r}}} \max \left(\frac{1}{\sqrt{m_1^{r_1}}}, \frac{1}{\sqrt{m_2^{r_2}}} \right) + \|f\|_{\infty} \left(\frac{1}{m_1^{\ell_1}} + \frac{1}{m_2^{\ell_2}} \right) \right],$$

and for any $f \in Z_{\lambda}$, by (2.3)

$$\|f - B_{\mathbf{m}}^{\ell}(f)\|_{\infty} \leq C \left[\|f\|_{Z_{\lambda}} \max \left(\frac{1}{\sqrt{m_1^{\lambda_1}}}, \frac{1}{\sqrt{m_2^{\lambda_2}}} \right) + \|f\|_{\infty} \left(\frac{1}{m_1^{\ell_1}} + \frac{1}{m_2^{\ell_2}} \right) \right],$$

where in both cases $C \neq C(\mathbf{m}, f)$.

Finally, selecting $\ell_1 \geq \frac{r_1}{2}$, and $\ell_2 \geq \frac{r_2}{2}$, we have

$$(3.11) \quad \|f - B_{\mathbf{m}}^{\ell}(f)\|_{\infty} \leq C \begin{cases} \|f\|_{W_{\mathbf{r}}} \max \left(\frac{1}{\sqrt{m_1^{r_1}}}, \frac{1}{\sqrt{m_2^{r_2}}} \right), & f \in W_{\mathbf{r}} \\ \|f\|_{Z_{\lambda}} \max \left(\frac{1}{\sqrt{m_1^{\lambda_1}}}, \frac{1}{\sqrt{m_2^{\lambda_2}}} \right), & f \in Z_{\lambda}. \end{cases}$$

3.3. The GB cubature rule on $\mathcal{Y}$. Consider the integral

$$I(\psi, \mathbf{y}) = \int_{\mathcal{Y}} \psi(\mathbf{x}) d\mathbf{x},$$

where

$$\mathcal{Y} = \{\mathbf{x} \mid 0 \leq x_1 \leq y_1, \quad 0 \leq x_2 \leq y_2, \quad (y_1, y_2) \in \mathcal{S}\},$$

and $\psi \in C(\mathcal{S})$. By approximating the function ψ by the polynomial $B_{\mathbf{m}}^{\ell}(\psi)$, we obtain the following cubature rule

$$(3.12) \quad I(\psi, \mathbf{y}) = Q_{\mathbf{m}}^{\ell}(\psi, \mathbf{y}) + R_{\mathbf{m}, \ell}(\psi, \mathbf{y}),$$

where

$$(3.13) \quad Q_{\mathbf{m}}^{\ell}(\psi, \mathbf{y}) = \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \Lambda_{j_1, j_2}^{\ell}(\mathbf{y}) \psi(\mathbf{t}_{j_1, j_2}^{\mathbf{m}}), \quad \mathbf{t}_{j_1, j_2}^{\mathbf{m}} := (t_{m_1, j_1}, t_{m_2, j_2}),$$

$$(3.14) \quad \Lambda_{j_1, j_2}^{\ell}(\mathbf{y}) = \left(\int_0^{y_1} p_{m_1, j_1}^{(\ell_1)}(x_1) dx_1 \right) \left(\int_0^{y_2} p_{m_2, j_2}^{(\ell_2)}(x_2) dx_2 \right) =: \lambda_{m_1, j_1}^{\ell_1}(y_1) \lambda_{m_2, j_2}^{\ell_2}(y_2),$$

and $R_{\mathbf{m}, \ell}(\psi, \mathbf{y})$ is the remainder term.

About the computation of $\Lambda_{j_1, j_2}^{\ell}(\mathbf{y})$, in view of (3.8) we have

$$p_{m_1, j_1}^{(\ell_1)}(x_1) = \sum_{k_1}^{m_1} p_{m_1, k_1}(x_1) c_{k_1, j_1}^{\ell_1}$$

and hence

$$(3.15) \quad \lambda_{m_1, j_1}^{\ell_1}(y_1) = \sum_{k_1=0}^{m_1} \mathcal{I}_{m_1, k_1}(y_1) c_{k_1, j_1}^{\ell_1},$$

with

$$\mathcal{I}_{m_1, k_1}(y_1) = \int_0^{y_1} p_{m_1, k_1}(x_1) dx_1 = \binom{m_1}{k_1} B_{y_1}(k_1 + 1, m_1 - k_1 + 1),$$

where B_{y_1} is the incomplete Beta function defined as

$$B_{y_1}(a, b) = \int_0^{y_1} x_1^{a-1} (1 - x_1)^{b-1} dx_1.$$

Similarly, we have

$$(3.16) \quad \lambda_{m_2, j_2}^{\ell_2}(y_2) = \sum_{k_2=0}^{m_2} \binom{m_2}{k_2} B_{y_2}(k_2 + 1, m_2 - k_2 + 1) c_{k_2, j_2}^{\ell_2}.$$

By (3.15)–(3.16) combined with (3.14) and (3.13), we have in the end

$$\begin{aligned} Q_{\mathbf{m}}^{\ell}(\psi, \mathbf{y}) &= \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \left(\sum_{k_1=0}^{m_1} \binom{m_1}{k_1} B_{y_1}(k_1 + 1, m_1 - k_1 + 1) c_{k_1, j_1}^{\ell_1} \right) \\ &\quad \times \left(\sum_{k_2=0}^{m_2} \binom{m_2}{k_2} B_{y_2}(k_2 + 1, m_2 - k_2 + 1) c_{k_2, j_2}^{\ell_2} \right) \psi(\mathbf{t}_{j_1, j_2}^{\mathbf{m}}). \end{aligned}$$

Regarding the stability and convergence of the rule, the following result holds.

Theorem 3.2. *For any $\psi \in C(S)$, the cubature formula is numerically stable*

$$\sup_{m_1, m_2} \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} |\Lambda_{j_1, j_2}^{\ell}(\mathbf{y})| < \infty,$$

and convergent

$$(3.17) \quad |R_{\mathbf{m}, \ell}(\psi, \mathbf{y})| \leq C y_1 y_2 \left(\max \left\{ \sup_{x_2 \in [0,1]} \omega_{\varphi_1}^{2\ell_1} \left(\psi_{x_2}, \frac{1}{\sqrt{m_1}} \right), \sup_{x_1 \in [0,1]} \omega_{\varphi_2}^{2\ell_2} \left(\psi_{x_1}, \frac{1}{\sqrt{m_2}} \right) \right\} \right. \\ \left. + \|\psi\|_{\infty} \left(\frac{1}{m_1^{\ell_1}} + \frac{1}{m_2^{\ell_2}} \right) \right).$$

Remark 3.2. By (3.17), by (3.11) if $\psi \in W_{\mathbf{r}}$ and we set $\ell_1 \geq \frac{r_1}{2}$ and $\ell_2 \geq \frac{r_2}{2}$, then we have

$$(3.18) \quad |R_{\mathbf{m}, \ell}(\psi, \mathbf{y})| \leq C y_1 y_2 \|\psi\|_{W_{\mathbf{r}}} \max \left(\frac{1}{\sqrt{m_1^{r_1}}}, \frac{1}{\sqrt{m_2^{r_2}}} \right).$$

Similarly, if $\psi \in Z_{\lambda}$ and $\ell_1 \geq \frac{\lambda_1}{2}$, and $\ell_2 \geq \frac{\lambda_2}{2}$, it is

$$|R_{\mathbf{m}, \ell}(\psi, \mathbf{y})| \leq C y_1 y_2 \|\psi\|_{Z_{\lambda}} \max \left(\frac{1}{\sqrt{m_1^{\lambda_1}}}, \frac{1}{\sqrt{m_2^{\lambda_2}}} \right).$$

4. THE NYSTRÖM METHOD BASED ON THE GB RULE

Let us consider equation (1.1), which can be rewritten in operator form as

$$(4.19) \quad (\mathcal{I} + \mathcal{V})f = g,$$

where $\mathcal{I}$ is the identity operator and

$$\mathcal{V}f(\mathbf{y}) := \nu \int_{\mathbf{Y}} h(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) d\mathbf{x}.$$

First of all, we remark that using standard arguments, if $h(\mathbf{x}, \mathbf{y}) \in C(S \times S)$, then $\mathcal{V} : C(S) \rightarrow C(S)$ is a compact operator and consequently, if $\ker(\mathcal{I} + \mathcal{V}) = \{0\}$, the Fredholm alternative theorem ensures that for any function $g \in C(S)$ the equation (1.1) admits a unique solution in $C(S)$. Moreover, if for some $\mathbf{r} \in \mathbb{N} \times \mathbb{N}$

$$(4.20) \quad \sup_{\mathbf{x} \in S} \|h_{\mathbf{x}}\|_{W_{\mathbf{r}}} < +\infty$$

then $\mathcal{V}f \in W_{\mathbf{r}}$ for any $f \in C(S)$. Similarly, if for some $\lambda \in \mathbb{R}^+ \times \mathbb{R}^+$

$$(4.21) \quad \sup_{\mathbf{x} \in S} \|h_{\mathbf{x}}\|_{Z_{\lambda}} < +\infty$$

then $\mathcal{V}f \in Z_{\lambda}$ for any $f \in C(S)$.

Consequently, if $g \in W_{\mathbf{r}}$ ($g \in Z_{\lambda}$), the solution f of (4.19) belongs to $W_{\mathbf{r}}$ (Z_{λ}).

Now we introduce the Nyström method based on the cubature rule (3.12), to approximate the solution of equation (1.1) in $C(S)$.

Through (3.12), we define the discrete operator

$$(\mathcal{V}_{\mathbf{m}}^{\ell} f)(\mathbf{y}) = \nu \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \Lambda_{j_1, j_2}^{\ell_1, \ell_2}(\mathbf{y}) h(\mathbf{t}_{j_1, j_2}^{\mathbf{m}}, \mathbf{y}) f(\mathbf{t}_{j_1, j_2}^{\mathbf{m}}),$$

with $\Lambda_{j_1, j_2}^{\ell_1, \ell_2}(\mathbf{y})$ given in (3.14), and consider the following discrete equation

$$(4.22) \quad (\mathcal{I} + \mathcal{V}_{\mathbf{m}}^{\ell}) f_{\mathbf{m}}^{\ell} = g,$$

in the unknown $f_{\mathbf{m}}^{\ell}$. To compute it, we evaluate (4.22) at the cubature points, getting the following system in the unknowns $a_{j_1, j_2}^{\ell} = f_{\mathbf{m}}^{\ell}(\mathbf{t}_{j_1, j_2}^{\mathbf{m}})$, $(j_1, j_2) \in N_0^{m_1} \times N_0^{m_2}$

$$(4.23) \quad \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} [\delta_{j_1, i_1} \delta_{j_2, i_2} + \nu \Lambda_{j_1, j_2}^{\ell}(\mathbf{t}_{i_1, i_2}^{\mathbf{m}}) h(\mathbf{t}_{j_1, j_2}^{\mathbf{m}}, \mathbf{t}_{i_1, i_2}^{\mathbf{m}})] a_{j_1, j_2}^{\ell} = g(\mathbf{t}_{i_1, i_2}^{\mathbf{m}}), \quad (i_1, i_2) \in N_0^{m_1} \times N_0^{m_2}.$$

Once the solution $\mathbf{a}^* = \{a_{j_1, j_2}^*, (j_1, j_2) \in N_0^{m_1} \times N_0^{m_2}\}$ of the system (4.23) is computed, we can construct the following Nyström interpolant, which approximates the solution f of equation (1.1) in $C(S)$

$$(4.24) \quad f_{\mathbf{m}}^{\ell}(\mathbf{y}) = g(\mathbf{y}) - \nu \sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \Lambda_{j_1, j_2}^{\ell}(\mathbf{y}) h(\mathbf{t}_{j_1, j_2}^{\mathbf{m}}, \mathbf{y}) a_{j_1, j_2}^*.$$

Denoting by $\mathcal{Q}_{\mathbf{m}, \ell}$ the coefficient matrix of system (4.23) and by

$$\text{cond}(\mathcal{Q}_{\mathbf{m}, \ell}) := \|\mathcal{Q}_{\mathbf{m}, \ell}\|_{\infty} \|\mathcal{Q}_{\mathbf{m}, \ell}^{-1}\|_{\infty}$$

the condition number of the system in infinity norm, the following result holds.

Theorem 4.3. *Assume that $h \in C(S \times S)$ and for a given $g \in C(S)$, let f be the unique solution of (1.1) in $C(S)$. Then the method is stable and convergent in $C(S)$. If in addition, for some $\mathbf{r}$, the kernel h satisfies (4.20) and*

$$(4.25) \quad \sup_{\mathbf{y} \in S} \|h_{\mathbf{y}}\|_{W_{\mathbf{r}}} < +\infty,$$

then, for any $g \in W_{\mathbf{r}}$, the system (4.23) is uniquely solvable and well conditioned too, i.e.

$$\text{cond}(\mathcal{Q}_{\mathbf{m}, \ell}) \leq C, \quad C \neq C(\mathbf{m}), \quad C = C(\ell).$$

Moreover, choosing $\ell_1 \geq \frac{r_1}{2}$ and $\ell_2 \geq \frac{r_2}{2}$, for $\mathbf{m}$ sufficiently large, one has

$$\|f - f_{\mathbf{m}}^{\ell}\|_{\infty} \leq C \|f\|_{W_{\mathbf{r}}} \max \left(\frac{1}{\sqrt{m_1^{r_1}}}, \frac{1}{\sqrt{m_2^{r_2}}} \right),$$

where $C \neq C(\mathbf{m}, f)$ and $C = C(\ell)$.

Proof. Since the Nyström method is based on a cubature rule which is convergent for any continuous function, using standard arguments (see [1]), it follows that the method is stable and convergent in $C(S)$, and the associated linear system is well conditioned. Moreover, since [1, Th.4.1.2]

$$\|f - f_{\mathbf{m}}^{\ell}\|_{\infty} \sim \|\mathcal{V}f - \mathcal{V}_{\mathbf{m}}^{\ell}f\|_{\infty},$$

where the constants in $\sim$ are independent of $\mathbf{m}$ and f , it follows that in S it is for any fixed ℓ

$$\lim_m \|f - f_{\mathbf{m}}^{\ell}\|_{\infty} = 0.$$

On the other hand, under the assumptions $\ell_1 \geq r_1/2$ and $\ell_2 \geq r_2/2$, by (3.18), one has

$$(4.26) \quad |\mathcal{V}f(\mathbf{y}) - \mathcal{V}_{\mathbf{m}}^{\ell}f(\mathbf{y})| \leq C y_1 y_2 \|fh_{\mathbf{y}}\|_{W_{\mathbf{r}}} \max \left(\frac{1}{\sqrt{m_1^{r_1}}}, \frac{1}{\sqrt{m_2^{r_2}}} \right).$$

At first assume that

$$\begin{aligned} M_{\mathbf{r}}(f) &:= \max \left\{ \sup_{x_2 \in [0,1]} \left\| \frac{\partial^{r_1} f_{x_2}}{\partial x_1^{r_1}} \varphi_1^{r_1} \right\|_{\infty}, \sup_{x_1 \in [0,1]} \left\| \frac{\partial^{r_2} f_{x_1}}{\partial x_2^{r_2}} \varphi_2^{r_2} \right\|_{\infty} \right\} \\ &= \sup_{x_2 \in [0,1]} \left\| \frac{\partial^{r_1}}{\partial x_1^{r_1}} (f h_{\mathbf{y}})_{x_2} \varphi_1^{r_1} \right\|_{\infty}. \end{aligned}$$

Then, one has

$$\begin{aligned} \left| \frac{\partial^{r_1}}{\partial x_1^{r_1}} (f h_{\mathbf{y}})(\mathbf{x}) \varphi_1^{r_1}(\mathbf{x}) \right| &\leq \sum_{j=0}^{r_1} \binom{r_1}{j} \left| \frac{\partial^{r_1-j}}{\partial x_1^{r_1-j}} h_{\mathbf{y}}(\mathbf{x}) \varphi_1^{r_1-j}(\mathbf{x}) \right| \left| \frac{\partial^j}{\partial x_1^j} f(\mathbf{x}) \varphi_1^j(\mathbf{x}) \right| \\ &\leq \sum_{j=0}^{r_1} \binom{r_1}{j} \left\| \frac{\partial^{r_1-j} h_{\mathbf{y}, x_2}}{\partial x_1^{r_1-j}} \varphi_1^{r_1-j} \right\|_{\infty} \|f_{x_2}^{(j)} \varphi_1^j\|_{\infty} \\ &\leq \mathcal{C} \|h_{\mathbf{y}, x_2}\|_{W_{r_1}} \|f_{x_2}\|_{W_{r_1}}, \end{aligned}$$

where in the last bound we have taken into account (4.25) and used [7, p. 310, Lemma 2.1]

$$\sum_{j=0}^r \|f_{x_2}^{(j)} \varphi_1^j\|_{\infty} \leq \mathcal{C} \left(\|f\|_{\infty} + \|f_{x_2}^{(r_1)} \varphi_1^{r_1}\|_{\infty} \right).$$

Hence,

$$(4.27) \quad \sup_{x_2 \in [0,1]} \left\| \frac{\partial^{r_1}}{\partial x_1^{r_1}} (f h_{\mathbf{y}})_{x_2} \varphi_1^{r_1} \right\|_{\infty} \leq \mathcal{C} \sup_{x_2 \in [0,1]} \|h_{\mathbf{y}, x_2}\|_{W_{r_1}} \|f_{x_2}\|_{W_{r_1}}.$$

Similarly, proceeding in the case of

$$M_{\mathbf{r}}(f h_{\mathbf{y}}) = \sup_{x_1 \in [0,1]} \left\| \frac{\partial^{r_2}}{\partial x_2^{r_2}} (f h_{\mathbf{y}})_{x_1} \varphi_2^{r_2} \right\|_{\infty},$$

one has

$$(4.28) \quad \sup_{x_1 \in [0,1]} \left\| \frac{\partial^{r_2}}{\partial x_2^{r_2}} (f h_{\mathbf{y}})_{x_1} \varphi_2^{r_2} \right\|_{\infty} \leq \mathcal{C} \sup_{x_1 \in [0,1]} \|h_{\mathbf{y}, x_1}\|_{W_{r_2}} \|f_{x_1}\|_{W_{r_2}}.$$

Combining (4.27) and (4.28) in (4.26), we get

$$\begin{aligned} |\mathcal{V}f(\mathbf{y}) - \mathcal{V}_{\mathbf{m}}^{\ell} f(\mathbf{y})| &\leq \mathcal{C} \max \left(\sup_{x_2 \in [0,1]} \|h_{\mathbf{y}, x_2}\|_{W_{r_1}} \|f_{x_2}\|_{W_{r_1}}, \sup_{x_1 \in [0,1]} \|h_{\mathbf{y}, x_1}\|_{W_{r_2}} \|f_{x_1}\|_{W_{r_2}} \right) \\ &\quad \times \max \left(\frac{1}{\sqrt{m_1^{r_1}}}, \frac{1}{\sqrt{m_2^{r_2}}} \right) \end{aligned}$$

and under assumption (4.20), we have at last

$$|\mathcal{V}f(\mathbf{y}) - \mathcal{V}_{\mathbf{m}}^{\ell} f(\mathbf{y})| \leq \mathcal{C} \|f\|_{W_{\mathbf{r}}} \max \left(\frac{1}{\sqrt{m_1^{r_1}}}, \frac{1}{\sqrt{m_2^{r_2}}} \right),$$

and the theorem is fully proved. $\square$

Theorem 4.4. Assume that $h \in C(\mathcal{S} \times \mathcal{S})$ and for a given $g \in C(\mathcal{S})$, let f be the unique solution of (1.1) in $C(\mathcal{S})$. Then the method is stable and convergent in $C(\mathcal{S})$. If in addition, for some λ , the kernel h satisfies (4.21), the function $g \in Z_\lambda$, and

$$\sup_{\mathbf{y} \in \mathcal{S}} \|h_{\mathbf{y}}\|_{Z_\lambda} < +\infty,$$

then, for $\mathbf{m}$ sufficiently large, the system (4.23) is uniquely solvable and well conditioned too, since

$$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell}) \leq C, \quad C \neq C(\mathbf{m}), \quad C = C(\ell).$$

Moreover, choosing $\ell_1 \geq \frac{\lambda_1}{2}$ and $\ell_2 \geq \frac{\lambda_2}{2}$, one has

$$\|f - f_{\mathbf{m}}^\ell\|_\infty \leq C \|f\|_{Z_\lambda} \max \left(\frac{1}{\sqrt{m_1^{\lambda_1}}}, \frac{1}{\sqrt{m_2^{\lambda_2}}} \right),$$

where $C \neq C(\mathbf{m}, f)$ and $C = C(\ell)$.

We omit the proof because it follows from arguments similar to those used in the proof of Theorem 4.3.

Remark 4.3. By Theorems 4.3 and 4.4, it follows that the convergence rate of the Nyström method behaves as the square root of the best approximation error order of the spaces functions where the solution f lives. Furthermore, the order of convergence of the sequence $\{f_{\mathbf{m}}^\ell\}_{\mathbf{m},\ell}$ increases as the smoothness of the involved functions increases. This remarkable behavior reflects the speed of convergence of the operator on which the quadrature formula is based, and represents, in some sense, a substantial improvement over methods based on equidistant points. In fact, in general, piecewise methods implemented at equidistant points perform significantly worse when the functions involved are smooth.

5. COMPUTATIONAL ASPECTS AND NUMERICAL TESTS

In this section, we provide some tests to show the performance of our method and confirm the theoretical estimates established in Section 4.

But first, we give some details about the linear system (4.23). Setting $N := (m_1 + 1)(m_2 + 1)$ the square system in (4.23) has order N and takes the following matrix form

$$(5.29) \quad \begin{aligned} \mathcal{Q}_{\mathbf{m},\ell} \mathbf{a} &= \mathbf{g}, \\ \mathcal{Q}_{\mathbf{m},\ell} &\in \mathbb{R}^{N \times N}, \quad \mathbf{a} \in \mathbb{R}^N, \quad \mathbf{g} \in \mathbb{R}^N. \end{aligned}$$

The matrix $\mathcal{Q}_{\mathbf{m},\ell}$ can be represented as a block-matrix of order $(m_1 + 1)$

$$(5.30) \quad \mathcal{Q}_{\mathbf{m},\ell} = \begin{bmatrix} Q^{(0,0)} & Q^{(0,1)} & \dots & Q^{(0,m_1)} \\ Q^{(1,0)} & Q^{(1,1)} & \dots & Q^{(1,m_1)} \\ \vdots & \vdots & \ddots & \vdots \\ Q^{(m_1,0)} & Q^{(m_1,1)} & \dots & Q^{(m_1,m_1)} \end{bmatrix},$$

whose blocks $Q^{(i_1,j_1)} \in \mathbb{R}^{(m_2+1) \times (m_2+1)}$ are defined as

$$Q^{(i_1,j_1)} = \delta_{j_1,i_1} \mathbf{I}_{m_2+1} + \nu \mathbf{V}^{(i_1,j_1)}, \quad (i_1, j_1) \in N_0^{m_1} \times N_0^{m_1}.$$

Here, $\mathbf{I}_{m_2+1}$ denotes the identity matrix of order $m_2 + 1$ and $\mathbf{V}^{(i_1,j_1)} \in \mathbb{R}^{(m_2+1) \times (m_2+1)}$ are the entries of the block matrix of order $(m_1 + 1)$

$$\mathbf{V}_{\mathbf{m},\ell} = \mathbf{A}_{\mathbf{m},\ell} \odot \mathbf{H}_{\mathbf{m}} \in \mathbb{R}^{N \times N},$$

$$\mathbf{V}_{\mathbf{m},\ell} = \begin{bmatrix} \mathbf{V}^{(0,0)} & \mathbf{V}^{(0,1)} & \dots & \mathbf{V}^{(0,m_1)} \\ \mathbf{V}^{(1,0)} & \mathbf{V}^{(1,1)} & \dots & \mathbf{V}^{(1,m_1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{V}^{(m_1,0)} & \mathbf{V}^{(m_1,1)} & \dots & \mathbf{V}^{(0,m_1)} \end{bmatrix},$$

$$\mathbf{\Lambda}_{\mathbf{m},\ell} = \begin{bmatrix} \mathbf{\Lambda}^{(0,0)} & \mathbf{\Lambda}^{(0,1)} & \dots & \mathbf{\Lambda}^{(0,m_1)} \\ \mathbf{\Lambda}^{(1,0)} & \mathbf{\Lambda}^{(1,1)} & \dots & \mathbf{\Lambda}^{(1,m_1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Lambda}^{(m_1,0)} & \mathbf{\Lambda}^{(m_1,1)} & \dots & \mathbf{\Lambda}^{(0,m_1)} \end{bmatrix}, \quad \mathbf{H}_{\mathbf{m}} = \begin{bmatrix} \mathbf{H}^{(0,0)} & \mathbf{H}^{(0,1)} & \dots & \mathbf{H}^{(0,m_1)} \\ \mathbf{H}^{(1,0)} & \mathbf{H}^{(1,1)} & \dots & \mathbf{H}^{(1,m_1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{H}^{(m_1,0)} & \mathbf{H}^{(m_1,1)} & \dots & \mathbf{H}^{(0,m_1)} \end{bmatrix},$$

where $\odot$ is the Hadamard product between matrices and with

$$\mathbf{\Lambda}^{(i_1,j_1)}, \mathbf{H}^{(i_1,j_1)} \in \mathbb{R}^{(m_2+1) \times (m_2+1)}, \quad (i_1, j_1) \in N_0^{m_1} \times N_0^{m_1},$$

$$\mathbf{\Lambda}^{(i_1,j_1)}(i_2, j_2) = \Lambda_{j_1, j_2}^{\ell_1, \ell_2}(\mathbf{t}_{i_1, i_2}^{\mathbf{m}}), \quad \mathbf{H}^{(i_1,j_1)}(i_2, j_2) = h(\mathbf{t}_{j_1, j_2}^{\mathbf{m}}, \mathbf{t}_{i_1, i_2}^{\mathbf{m}}), \quad (i_2, j_2) \in N_0^{m_2} \times N_0^{m_2}.$$

Globally, the matrix $\mathcal{Q}_{\mathbf{m},\ell}$ can also be viewed as follows

$$\mathcal{Q}_{\mathbf{m},\ell} = \mathbf{I}_N + \nu \mathbf{V}_{\mathbf{m},\ell},$$

where $\mathbf{I}_N$ is the identity matrix of order N .

Finally, $\mathbf{a}, \mathbf{g}$ are (m_2+1) -block arrays, whose components are vectors of length (m_1+1) , namely

$$\mathbf{a} = [\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_{m_1}]^T \in \mathbb{R}^N, \quad \mathbf{a}_{j_1} = [a_{j_1,0}, a_{j_1,1}, \dots, a_{j_1,m_2}]^T \in \mathbb{R}^{m_2+1}, \quad \forall j_1 \in N_0^{m_1},$$

$$\mathbf{g} = [\mathbf{g}_0, \mathbf{g}_1, \dots, \mathbf{g}_{m_1}]^T \in \mathbb{R}^N, \quad \mathbf{g}_{i_1} = [g(\mathbf{t}_{i_1,0}^{\mathbf{m}}), g(\mathbf{t}_{i_1,1}^{\mathbf{m}}), \dots, g(\mathbf{t}_{i_1,m_2}^{\mathbf{m}})]^T \in \mathbb{R}^{m_2+1}, \quad \forall i_1 \in N_0^{m_1}.$$

Once the system has been solved and the unknown vector $\mathbf{a}$ has been determined, we employ it to compute the Nyström interpolant (4.24).

To show the accuracy of our method, we have selected five tests from the numerous experiments we have carried out. Denoting by $\mathbf{Y}$ a sufficiently dense grid of equally spaced points in the unit square $\mathcal{S}$, in all tables we report the maximum absolute error

$$e_{\mathbf{m}}^{\ell}(f) := \max_{\mathbf{y} \in \mathbf{Y}} |e_{\mathbf{m}}^{\ell}(f, \mathbf{y})|.$$

When the exact solution f is known, we define

$$|e_{\mathbf{m}}^{\ell}(f, \mathbf{y})| := |f(\mathbf{y}) - f_{\mathbf{m}}^{\ell}(\mathbf{y})|.$$

In the case f is unknown, we use the same notation to denote the maximum absolute error computed between successive instances of the procedure for fixed values of the parameter ℓ , namely

$$|e_{\mathbf{m}}^{\ell}(f, \mathbf{y})| := |f_{\mathbf{m}+\mathbf{n}}^{\ell}(\mathbf{y}) - f_{\mathbf{m}}^{\ell}(\mathbf{y})|,$$

where $\mathbf{n} \in \{(0,5), (5,0), (5,5)\}$ depends on the numerical test under consideration. For example, in test cases where the kernel h is smooth with respect to the inner variable x_1 and less regular with respect to the variable x_2 , we construct a sequence of approximations of f by increasing the value of m_2 , while keeping m_1 fixed, i.e., $\mathbf{n} = (0,5)$.

Moreover, in all tests we assess the good conditioning of the linear system (5.29) by displaying the condition number in infinity norm of the matrix $\mathcal{Q}_{\mathbf{m},\ell}$ defined in (5.30).

All computations are performed in double precision using MATLAB R2025b on a 2020 M1 MacBook Pro (16 GB RAM) running macOS.

Example 5.1. Let us consider the following equation

$$f(y_1, y_2) + \frac{1}{5} \int_{\mathcal{Y}} f(x_1, x_2) [y_1 e^{x_1} + x_2 \sqrt{y_2}] dx_1 dx_2 = g(y_1, y_2),$$

with the right-hand side term

$$g(y_1, y_2) = \cos(y_1 + y_2) + \frac{1}{10} \left[y_2^{\frac{3}{2}} (\cos y_2 - \cos(y_1 + y_2)) - \sqrt{y_2} \sin y_1 - y_1 (1 + e^{y_1} (\sin y_1 - \cos y_1)) \right] \\ + \frac{1}{10} [\sqrt{y_2} (\sin(y_1 + y_2) - \sin y_2) + y_1 (\cos y_2 - \sin y_2 + e^{y_1} (\sin(y_1 + y_2) - \cos(y_1 + y_2)))] .$$

Here the exact solution is $f(y_1, y_2) = \cos(y_1 + y_2)$. In this case the right-hand side term g and the kernel h_x belong to Z_λ , $\forall \lambda_1 > 0$, $\lambda_2 = 1$, and therefore the solution f lives at least in that space. In view of Theorem 4.4, the expected rate of the error is very poor. Table 1 reports the results obtained by implementing the method for increasing values of $\mathbf{m}$ and different choices of the ℓ parameter. Note that by increasing ℓ , the number of nodes needed to reach machine precision is reduced. However, beyond a certain threshold - in this case $\ell \geq (2^7, 2^7)$ - no further improvement in accuracy is observed. Furthermore, also the well-conditioning of the system is confirmed.

	$\ell = (2^2, 2^2)$		$\ell = (2^3, 2^3)$		$\ell = (2^4, 2^4)$	
$\mathbf{m}$	$e_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$	$e_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$	$e_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$
(5, 5)	1.40e-04	2.02	1.23e-05	2.02	1.64e-06	2.02
(10, 10)	1.09e-05	2.03	2.09e-07	2.03	1.11e-08	2.03
(15, 15)	2.39e-06	2.03	1.55e-08	2.03	4.34e-10	2.03
(20, 20)	8.00e-07	2.03	2.61e-09	2.03	2.44e-11	2.03
(25, 25)	3.40e-07	2.03	6.23e-10	2.03	2.39e-12	2.03
(30, 30)	1.68e-07	2.03	1.87e-10	2.03	3.24e-13	2.03
(35, 35)	9.24e-08	2.03	6.59e-11	2.03	5.62e-14	2.03
(40, 40)	5.49e-08	2.03	2.63e-11	2.03	1.12e-14	2.03
(45, 45)	3.47e-08	2.03	1.15e-11	2.03	4.22e-15	2.03
(50, 50)	2.29e-08	2.03	5.47e-12	2.03		
(55, 55)	1.58e-08	2.03	2.77e-12	2.03		

	$\ell = (2^5, 2^5)$		$\ell = (2^6, 2^6)$		$\ell = (2^7, 2^7)$	
$\mathbf{m}$	$e_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$	$e_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$	$e_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$
(5, 5)	3.28e-07	2.02	3.48e-07	2.02	3.68e-07	2.02
(10, 10)	1.56e-09	2.03	1.96e-10	2.03	5.48e-11	2.03
(15, 15)	9.18e-12	2.03	2.00e-13	2.03	7.02e-14	2.03
(20, 20)	1.68e-13	2.03	6.38e-15	2.03	2.39e-15	2.03
(25, 25)	4.75e-15	2.03	2.50e-16	2.03		
(30, 30)	1.67e-15	2.03				

TABLE 1. Numerical results of Example 5.1.

Example 5.2. Let us consider the following equation

$$f(y_1, y_2) - \int_{\mathcal{Y}} [y_1 x_1^2 + \cos x_2] f(x_1, x_2) dx_1 dx_2 = y_1 \sin y_2 + \frac{y_1^5}{4} (\cos y_2 - 1) - \frac{y_1^2}{4} \sin^2 y_2,$$

whose exact solution is $f(y_1, y_2) = y_1 \sin y_2$. We compare the results obtained by our method, reported in Table 2, with those of the Taylor collocation method proposed by Laib et al. in [13, Example 4.1, Tables 2 and 3]. Our Nyström method proves to be more accurate and faster than the Taylor collocation method, achieving machine precision for $\mathbf{m} = (20, 20)$ in 1 second.

$\mathbf{m}$	$e_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$
(5,5)	4.78e-06	5.54
(10,10)	1.39e-10	5.57
(15,15)	8.52e-14	5.57
(20,20)	3.22e-15	5.57

TABLE 2. Numerical results of Example 5.2 with $\ell = (2^8, 2^8)$.

Example 5.3. Let us consider the following equation

$$f(y_1, y_2) + 15 \int_{\mathcal{Y}} f(x_1, x_2)(y_1 - x_1)^{\frac{11}{2}} \cos(x_2 + y_2) dx_1 dx_2 = g(y_1, y_2),$$

with the right-hand side function

$$g(y_1, y_2) = y_1 + 2y_2 - \frac{4}{13} y_1^{\frac{13}{2}} [15 \cos y_2 - 15 \cos 2y_2 + y_1 \sin y_2 - (y_1 + 15y_2) \sin 2y_2]$$

chosen so that the exact solution is $f(y_1, y_2) = y_1 + 2y_2$. In this test, the kernel h is smooth with respect to the variables x_2 and y_2 , while it is less regular with respect to x_1 and y_1 , namely $h \in Z_{\lambda} \times Z_{\lambda}$, with $\lambda_1 = \frac{11}{2}, \forall \lambda_2 > 0$. For this reason, fixing $\ell = (2^8, 2^8)$, we compare the performance of the method in two settings: first, by increasing both components of $\mathbf{m}$, and second, by increasing only m_1 , while keeping $m_2 = 10$ fixed. Table 3 shows that the same maximum absolute errors are achieved in both cases. However, in the first case we solve a linear system of order $56^2 = 3136$, requiring 120 seconds of CPU time, whereas in the second case the linear system has a smaller order, $56 * 11 = 616$, and the algorithm requires only 5 seconds. Note that, in the latter case, the conditioning of the coefficient matrix $\mathcal{Q}_{\mathbf{m},\ell}$ remains essentially unchanged.

$\mathbf{m}$	$e_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$	$\mathbf{m}$	$e_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$
(10, 10)	3.88e-06	3.68	(10, 10)	3.88e-06	3.68
(15, 15)	4.51e-07	3.44	(15, 10)	4.51e-07	3.68
(20, 20)	1.05e-07	3.46	(20, 10)	1.05e-07	3.68
(25, 25)	2.98e-08	3.46	(25, 10)	2.98e-08	3.68
(30, 30)	1.09e-08	3.46	(30, 10)	1.09e-08	3.68
(35, 35)	5.74e-09	3.45	(35, 10)	5.74e-09	3.68
(40, 40)	2.89e-09	3.45	(40, 10)	2.89e-09	3.68
(45, 45)	1.31e-09	3.45	(45, 10)	1.31e-09	3.68
(50, 50)	8.21e-10	3.45	(50, 10)	8.21e-10	3.68
(55, 55)	5.11e-10	3.45	(55, 10)	5.11e-10	3.68

TABLE 3. Numerical results of Example 5.3 with $\ell = (2^8, 2^8)$.

Example 5.4. Let us consider the following equation

$$f(y_1, y_2) - \frac{1}{3} \int_{\mathcal{Y}} f(x_1, x_2) (y_1 - x_1)^{\frac{13}{2}} (y_2 - x_2)^{\frac{9}{2}} dx_1 dx_2 = e^{y_1 + y_2},$$

for which the exact solution is unknown. The right-hand side function g is smooth with respect to both variables, whereas the kernel h belongs to the Zygmund space $Z_{\lambda} \times Z_{\lambda}$ with $\lambda = (\frac{13}{2}, \frac{9}{2})$. Since the kernel is less smooth with respect to the inner variable x_2 , we perform a benchmark analysis to investigate the influence of the parameter $\ell = (\ell_1, \ell_2)$ on convergence, fixing ℓ_1 while increasing ℓ_2 . Table 4 shows that this strategy yields an improved convergence rate, providing one additional correct digit compared with the case $\ell_1 = \ell_2$.

$\ell = (2^4, 2^4)$			$\ell = (2^4, 2^6)$		
m	$\epsilon_m^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{m,\ell})$	m	$\epsilon_m^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{m,\ell})$
(5, 5)	4.87e-05	1.02	(5, 5)	4.87e-05	1.02
(10, 10)	6.36e-07	1.02	(10, 10)	6.36e-07	1.02
(15, 15)	4.19e-08	1.02	(15, 15)	4.19e-08	1.02
(20, 20)	7.18e-09	1.02	(20, 20)	6.46e-09	1.02
(25, 25)	2.28e-09	1.02	(25, 25)	1.75e-09	1.02
(30, 30)	1.18e-09	1.02	(30, 30)	7.22e-10	1.02
(35, 35)	6.39e-10	1.02	(35, 35)	3.23e-10	1.02
(40, 40)	3.60e-10	1.02	(40, 40)	1.41e-10	1.02
(45, 45)	2.14e-10	1.02	(45, 45)	7.75e-11	1.02
(50, 50)	1.35e-10	1.02	(50, 50)	4.78e-11	1.02

TABLE 4. Numerical results of Example 5.4.

Example 5.5. Let us consider the following equation

$$f(y_1, y_2) - \frac{1}{\pi} \int_{\mathcal{Y}} f(x_1, x_2) \log(x_1 y_1 + x_2 y_2 + 6) dx_1 dx_2 = (\sin(y_1 y_2))^{\frac{19}{3}},$$

where the exact solution is unknown. In this case, the kernel h is smooth, whereas the right-hand side belongs to the Zygmund space Z_{λ} with $\lambda = (12.\bar{6}, 12.\bar{6})$. The numerical results reported in Table 5 exhibit a faster convergence than theoretically predicted, whose order is $\mathcal{O}(m^{-6.\bar{3}})$ with $m = \min\{m_1, m_2\}$. This behavior is not unexpected: since our method is of Nyström type, it is less sensitive to reduced smoothness of the right-hand side than to singularities in the kernel h . This is also evident from the definition of the Nyström interpolant (4.24), whose accuracy is primarily determined by the accuracy of the cubature rule (3.12).

6. CONCLUSIONS

In this paper, we introduced a Nyström method for the numerical treatment of bivariate Volterra integral equations of the second kind on the unit square $\mathcal{S}$. The method is based on bivariate Generalized Bernstein polynomials defined as the tensor product of their univariate counterparts. The use of GB polynomials allows the construction of the method by exploiting samples of the involved functions on equally spaced grids of points in $\mathcal{S}$. This makes the approach well suited for real-life applications in science and engineering, where data are often obtained through measurements at fixed, equally spaced locations.

Moreover, we considered the possibility that the degree of the GB polynomial may differ with respect to each variable. This leads to a significant reduction in computational time, as

$\mathbf{m}$	$\epsilon_{\mathbf{m}}^{\ell}(f)$	$\text{cond}(\mathcal{Q}_{\mathbf{m},\ell})$
(5,5)	4.40e-05	2.77
(10,10)	3.54e-08	2.78
(15,15)	2.02e-10	2.78
(20,20)	4.51e-12	2.78
(25,25)	5.84e-13	2.78
(30,30)	9.38e-14	2.78
(35,35)	2.64e-14	2.78
(40,40)	8.72e-15	2.78
(45,45)	3.39e-15	2.78
(50,50)	1.50e-15	2.78

TABLE 5. Numerical results of Example 5.5 with $\ell = (2^8, 2^8)$.

it decreases the size of the linear system associated with the method. In particular, when the involved functions exhibit low regularity with respect to one variable and higher smoothness with respect to the other, one can fix a lower degree for the smoother component and increase the degree for the less regular one.

As for the discrete Volterra operator used in the construction of the method, it is defined through a cubature rule on equispaced points that does not require any change of variables, thus allowing the original Volterra operator to be treated directly. We proved the stability and convergence of the proposed Nyström method and demonstrated its effectiveness through several numerical experiments. These confirm the theoretical results for kernels and right-hand side terms with different smoothness, and include a comparison with another method for 2D VIEs introduced in [13].

Motivated by these promising results, future work will focus on the development of numerical methods for 2D Volterra-Fredholm integral equations, incorporating the cubature rule introduced here for the discretization of the Volterra operator.

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