Analysis and Mathematical Physics (2026) 16:47
https://doi.org/10.1007/s13324-026-01178-7

A nonlinear version of Kantorovich operators with *p*-averages: convergence results and asymptotic formula

Mirella Cappelletti Montano[1] · Rosario Corso[2] · Vita Leonessa[3]

Received: 28 November 2025 / Revised: 28 November 2025 / Accepted: 18 February 2026

Abstract
In this paper, we modify the classical Kantorovich operators, very well known in Approximation Theory, by considering p-averages (whose expressions are of the form of L^p (quasi-)norms, $p > 0$). We establish convergence results, an asymptotic formula covering the general setting; moreover, we show that, under suitable assumptions, our operators perform better than the classical Kantorovich ones in approximating functions. Because of the nature of the p-averages, the proposed operators are nonlinear, so their study turns out to be more challenging.

Keywords Kantorovich operators · p-averages · Nonlinear operators · Asymptotic formula

Mathematics Subject Classification 41A10 · 41A35 · 41A60 · 47H99

1 Introduction

The mathematical field of Approximation Theory aims to study how a given function can be replaced, up to a prescribed precision, by a simpler one, typically belonging to a specific class of functions (such as polynomials) or having a particular structure.

✉ Rosario Corso
rosario.corso02@unipa.it

Mirella Cappelletti Montano
mirella.cappellettimontano@uniba.it

Vita Leonessa
vita.leonessa@unibas.it

1 Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Via E. Orabona 4, 70125 Bari, Italy

2 Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, Via Archirafi 34, 90123 Palermo, Italy

3 Dipartimento di Ingegneria, Università degli Studi della Basilicata, Via dell'Ateneo Lucano 10, 85100 Potenza, Italy

Published online: 03 April 2026

The term *approximation* can be interpreted in several ways; for instance, it may refer to pointwise convergence or convergence with respect to a norm (such as the uniform or L^p-norm, with $p \geq 1$). The areas and applications of Approximation Theory are numerous, ranging from Signal Analysis [10], to Computer-Aided Geometric Design [21], Image Processing [6], and even Neural Networks [16].

Classical examples of approximation processes for functions on the interval [0, 1] are the Kantorovich operators [23], defined for integrable functions by

$$(K_n f)(x) = \sum_{k=0}^{n} \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,\mathrm{d}t \right) B_{k,n}(x), \qquad x \in [0,1], n \geq 1.$$

They act as polynomials expressed in the Bernstein basis

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad 0 \leq k \leq n,\ x \in [0,1],$$

with coefficients given by the integral averages

$$(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,\mathrm{d}t. \tag{1.1}$$

These averages can be interpreted as approximations of $f(s)$ for any $s \in \left[\frac{k}{n+1}, \frac{k+1}{n+1}\right]$, and in particular of the values $f\left(\frac{k}{n}\right)$. In fact, the operators K_n were introduced, with the aim of approximating integrable functions on [0, 1], as a modification of the Bernstein operators [26], preserving the same structure but replacing the coefficients $f\left(\frac{k}{n}\right)$ with the above integral averages.

Several studies about Kantorovich operators and their generalizations have been tackled over the decades; here we mention [3, 17, 29, 30], among many others. In particular, Kantorovich operators have been modified by replacing the averages (1.1) with weighted averages [18, 20, 31], averages over mobile intervals [11], or pointwise evaluations only at the endpoints [1].

In this paper, we aim to explore a sequence of Kantorovich-type operators obtained by replacing the averages (1.1) with the power averages of exponent p, in short p-*averages*,

$$\left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p\,\mathrm{d}t \right)^{\frac{1}{p}}, \qquad p > 0.$$

Namely, for every $n \geq 1$, we will consider the operators given by

$$(K_{p,n} f)(x) = \sum_{k=0}^{n} \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p\,\mathrm{d}t \right)^{\frac{1}{p}} B_{k,n}(x), \qquad x \in [0,1],$$

for all non-negative functions $f \in L^p(0,1)$, and by

$$K_{p,n}f = K_{p,n}(f_+) - K_{p,n}(f_-)$$

for a function $f \in L^p(0,1)$ with arbitrary sign, where f_+ and f_- are the non-negative and non-positive parts of f. When $p = 1$, Kantorovich operators are recovered. We stress that we consider $p > 0$ and not only $p \geq 1$, as more usually done when dealing with L^p-spaces. We remark also that L^p-spaces have been objects of interest even in the case $0 < p < 1$ (for instance, in [5, 19, 25]), as well as this case has been successfully considered for ℓ_p-minimization problems (see, for example, [27]).

Our study is motivated by the idea of finding better approximations of functions than those provided by Kantorovich operators. More precisely, let $f : [a,b] \to \mathbb{R}$ be a non-negative bounded measurable function on an interval $[a,b]$ and let $s \in [a,b]$. All the averages

$$A_p(f) := \left(\frac{1}{b-a}\int_a^b f(t)^p \,\mathrm{d}t\right)^{\frac{1}{p}}, \qquad p > 0, \tag{1.2}$$

are different from each other, unless f is constant a.e. in $[a,b]$. Thus, a natural question arises: for which values of p, does $A_p(f)$ gives an approximation of $f(s)$ better than $A_1(f)$, i.e. $|f(s) - A_p(f)| < |f(s) - A_1(f)|$? A simple answer makes use of the Hölder inequality, which states that $A_p(f) < A_q(f)$ if $p < q$. More precisely, assuming that f is not constant a.e. in $[a,b]$,

(i) if $f(s) = \max_{t\in[a,b]} f(t)$, then $A_p(f)$ approximates $f(s)$ better than $A_1(f)$ for every $p > 1$, since $A_1(f) < A_p(f) < f(s)$ for every $p > 1$;
(ii) if $f(s) = \min_{t\in[a,b]} f(t)$, then $A_p(f)$ approximates $f(s)$ better than $A_1(f)$ for every $0 < p < 1$, since $f(s) < A_p(f) < A_1(f)$ for every $0 < p < 1$.

More generally,

(iii) if $A_1(f) < f(s)$, then there is a range of values of $p > 1$ for which $A_p(f)$ approximates $f(s)$ better than $A_1(f)$;
(iv) if $A_1(f) > f(s)$, then there is a range of values of $0 < p < 1$ for which $A_p(f)$ approximates $f(s)$ better than $A_1(f)$.

In conclusion, values of $p \neq 1$ may provide for a better approximation in different situations and it is clear that this argument about the p-averages affects the approximation of f by the operators $K_{p,n}$. Indeed, as it is shown in Section 6, under some assumptions on f and $x \in [0,1]$, $K_{p,n}f$ approximates f better than $K_n f$ at x.

Now, after having discussed the motivation, we point out one of the most peculiar characteristics of the operators $K_{p,n}$: for $p \neq 1$ they are nonlinear operators. This makes the study much more challenging than the linear case, namely the classical Kantorovich operators. Nonlinear operators have been arising an increasing interest in Approximation Theory in recent years. For instance, [28] and many other works, e.g. [4, 7, 14, 15, 24, 32], studied operators expressed in terms of nonlinear kernels which are suitably estimated; [8] considered nonlinear operators based on different algebraic structures and introduced, in particular, the max-product and max-min approximations (modifications of these operators have been studied, e.g., in [9, 13, 22]). To the best of our knowledge, nonlinear operators based on averages of type (1.2) were not previously studied.

With more details, in this paper we investigate basic properties of the operators $K_{p,n}$ like boundedness, continuity (Section 3), convergence at a point, uniformly on [0, 1] or in L^p-spaces as $n \to +\infty$ (Section 4). In addition, we establish an asymptotic formula of Voronovskaya type, that is an explicit expression of $\lim_{n\to+\infty} n((K_{p,n} f)(x) - f(x))$ (related then to the approximation error) when f is twice differentiable at x, in the general setting. We would like to point out that the sum of several terms (specifically, those arising from the Taylor expansion of f centered at x) must be handled in the proof of such a formula. In the linear case, these terms can be treated separately, while for nonlinear operators this separation is not straightforward and requires a more elaborated processing. To get an idea of the problem complexity, most of this paper (the entire Section 5) is devoted to the proof of the asymptotic formula. As far as we are aware, previous studies on nonlinear operators in Approximation Theory have mainly provided asymptotic estimates (see, e.g., [7, 14, 24]), whereas explicit asymptotic formulas like the one proved here seem to be new.
The asymptotic formula (Theorem 5.1) for the operators $K_{p,n}$ has some curious properties, that are discussed after the statement. Here, we mention that it has a differential expression on f and one of its coefficients is, under some assumption on f, discontinuous at the endpoints. Differential equations with discontinuous coefficients are of great interest for the modeling of heterogeneous processes [12, 33].

To conclude, nonlinearity plays a crucial role in many fields of Mathematics or Science. For instance, many real phenomena are better modeled by nonlinear (rather than linear) differential equations; in Image Processing, the so-called median filter performs better than the linear mean filter in the reduction of particular noises; talking about Machine Learning, classifiers based on nonlinear functions or operations often provide improved results. In the same way, we believe that nonlinear operators in Approximation Theory may offer better properties compared to linear operators and this paper gives a contribution in this direction.

2 Preliminaries and notations

For every $p > 0$, we denote by $L^p(a, b)$ the space of all measurable functions $f : [a, b] \to \mathbb{R}$ such that $\|f\|_p := \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} < \infty$, by $L^\infty(a, b)$ the space of all (classes of) essentially bounded measurable functions $f : [a, b] \to \mathbb{R}$ and by $\|f\|_\infty := \sup_{x\in[a,b]} |f(x)| < \infty$. In this paper, the symbols sup and inf stand for the essential supremum and infimum, respectively. Finally, $C(a, b)$ is the space of all continuous functions on $[a, b]$.

We write $f \geq 0$, as usual, to indicate that a function $f : [a, b] \to \mathbb{R}$ is non-negative. Any function $f \in L^p(a, b)$ with real values can be written as difference of two unique non-negative functions in $L^p(a,b)$, i.e. $f = f_+ - f_-$. The functions f_+, f_- are called the *non-negative* and the *non-positive part* of f, respectively.

As known, when the usual identification of measurable functions is made, $\|\cdot\|_p$ is a norm if $p \geq 1$ and a quasi-norm if $0 < p < 1$. In particular, in the latter case the triangle inequality is lost. However, for any $0 < p \leq 1$ and $f, g \in L^p(a,b)$, it results that

$$\|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p, \tag{2.1}$$

and Jensen's inequality in concave version entails the flipped triangle inequality for non-negative functions

$$\|f\|_p + \|g\|_p \leq \|f+g\|_p, \quad f, g \geq 0. \tag{2.2}$$

The properties of $\|\cdot\|_p$ and Holder's inequality imply that the p-averages A_p, defined by (1.2), verify the following statements:

(i) For any $p > 0$, A_p is positively homogeneous, i.e. $A_p(\lambda f) = \lambda A_p(f)$, for all $\lambda \geq 0$ and $f \in L^p(a,b)$, $f \geq 0$.
(ii) For any $p \geq 1$, A_p is subadditive, i.e. $A_p(f+g) \leq A_p(f) + A_p(g)$ for all $f, g \in L^p(a,b)$, $f, g \geq 0$.
(iii) For any $0 < p \leq 1$, A_p is superadditive, i.e. $A_p(f) + A_p(g) \leq A_p(f+g)$ for all $f, g \in L^p(a,b)$, $f, g \geq 0$.
(iv) For any $p > 0$, A_p is monotone, i.e. $A_p(f) \leq A_p(g)$ for all $f, g \in L^p(a,b)$, $0 \leq f \leq g$.
(v) For any $f \in L^\infty(a,b)$ such that $f \geq 0$, $A_p(f)$ is increasing w.r.t. the parameter, i.e. $A_p(f) \leq A_q(f)$ for all $0 < p < q$.

Finally, we recall some properties of the Bernstein basis that we will apply throughout the paper [26]: for every $x \in [0,1]$ and $n \geq 1$, we have

$$\sum_{k=0}^{n} B_{k,n}(x) = 1, \tag{2.3}$$

$$\sum_{k=0}^{n} k B_{k,n}(x) = nx, \qquad \sum_{k=0}^{n} k^2 B_{k,n}(x) = nx(1-x+nx), \tag{2.4}$$

$$(n+1)\int_0^1 B_{k,n}(x)\,\mathrm{d}x = 1. \tag{2.5}$$

3 Definition and basic properties

In this section we define the nonlinear version of the Kantorovich operators, quoted in the introduction, and give some elementary properties.

Definition 3.1 Let $p > 0$ and $n \geq 1$. For a non-negative function $f \in L^p(0,1)$, we define

$$(K_{p,n}f)(x) = \sum_{k=0}^{n} \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p \, \mathrm{d}t \right)^{\frac{1}{p}} B_{k,n}(x), \qquad x \in [0,1]. \tag{3.1}$$

If $f \in L^p(0,1)$ has an arbitrary sign, then we set

$$K_{p,n}f = K_{p,n}(f_+) - K_{p,n}(f_-). \tag{3.2}$$

We will refer to $K_{p,n}$ as a Kantorovich operator with p-averages.

Explicitly, in (3.2) we have

$$(K_{p,n}f)(x) = \sum_{k=0}^{n} (n+1)^{\frac{1}{p}} \left[\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_+(t)^p \, \mathrm{d}t \right)^{\frac{1}{p}} - \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_-(t)^p \, \mathrm{d}t \right)^{\frac{1}{p}} \right] B_{k,n}(x),$$

for $x \in [0,1]$. Obviously, when $p = 1$ we recover the Kantorovich operator [23]

$$(K_n f)(x) = \sum_{k=0}^{n} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t) \, \mathrm{d}t \, B_{k,n}(x), \qquad x \in [0,1],$$

for any $f \in L^1(0,1)$. Other remarks that are immediate consequences of the definition of $K_{p,n}$ and the properties of the p-averages listed in Section 2 are stated below (and hold for any $n \geq 1$).

Remarks 3.2
- (i) For any $p > 0$ and $p \neq 1$, $K_{p,n}$ is a nonlinear operator.
- (ii) For any $p > 0$, $K_{p,n}$ is a positive operator, i.e. it maps non-negative functions into non-negative functions.
- (iii) For any $p > 0$, $K_{p,n}$ preserves the constant functions.
- (iv) For any $p > 0$, $K_{p,n}$ is a homogeneous operator, i.e. $K_{p,n}(\lambda f) = \lambda K_{p,n} f$, for all $\lambda \in \mathbb{R}$ (not necessarily positive) and $f \in L^p(0,1)$.
- (v) For any $p \geq 1$, $K_{p,n}$ is subadditive on the subspace of non-negative functions, i.e. for every $f, g \in L^p(0,1)$ such that $f, g \geq 0$ we have $K_{p,n}(f+g) \leq K_{p,n}f + K_{p,n}g$.
- (vi) For any $0 < p \leq 1$, $K_{p,n}$ is superadditive on the subspace of non-negative functions, i.e. for every $f, g \in L^p(0,1)$ such that $f, g \geq 0$ we have $K_{p,n}f + K_{p,n}g \leq K_{p,n}(f+g)$.
- (vii) For any $p > 0$, $K_{p,n}$ is monotone on the subspace of non-negative functions, i.e. for every $f, g \in L^p(0,1)$ such that $0 \leq f \leq g$ we have $K_{p,n}f \leq K_{p,n}g$.
- (viii) For any $p > 0$, $f \in L^\infty(a,b)$ such that $f \geq 0$, $K_{p,n}f$ is increasing w.r.t. to the parameter, i.e. $K_{p,n}f \leq K_{q,n}f$, for every $0 < p < q$.

In the next proposition we give some bounds for $K_{p,n}f$.

Proposition 3.3 *The following statements hold for any* $p > 0$ *and* $n \geq 1$.

(i) $K_{p,n} : L^\infty(0,1) \to L^\infty(0,1)$ *is a well-defined bounded operator. In particular, for any* $f \in L^\infty(0,1)$*, we have that*

$$\inf_{[0,1]} f \leq K_{p,n} f(x) \leq \sup_{[0,1]} f$$

for all $x \in [0,1]$ *and* $\|K_{p,n} f\|_\infty \leq \|f\|_\infty$.

(ii) $K_{p,n} : L^p(0,1) \to L^p(0,1)$ *is a well-defined bounded operator. In particular,*

(a) *if* $p \geq 1$*, then for any* $f \in L^p(0,1)$ *we have that*

$$\|K_{p,n} f\|_p \leq \|f\|_p;$$

(b) *if* $0 < p < 1$*, then for any* $f \in L^p(0,1)$ *such that* $f \geq 0$*, we have that*

$$\|K_{p,n} f\|_p \geq \|f\|_p.$$

Proof (i) The statements are a direct consequence of the fact that

$$\inf_{[\frac{k}{n+1}, \frac{k+1}{n+1}]} f \leq (n+1)^{\frac{1}{p}} \left[\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_+(t)^p \,\mathrm{d}t \right)^{\frac{1}{p}} - \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_-(t)^p \,\mathrm{d}t \right)^{\frac{1}{p}} \right] \leq \sup_{[\frac{k}{n+1}, \frac{k+1}{n+1}]} f,$$

for all $f \in L^\infty(0,1)$, $n \geq 1$ and $0 \leq k \leq n$.

(ii) We start with the case $p \geq 1$. By using the convexity of the function $y \mapsto y^p$, equations (2.3) and (2.5), the reverse triangle inequality, we obtain, for $f \in L^p(0,1)$,

$$\|K_{p,n} f\|_p^p \leq \int_0^1 \sum_{k=0}^n (n+1) \left| \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_+(t)^p \,\mathrm{d}t \right)^{\frac{1}{p}} - \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_-(t)^p \,\mathrm{d}t \right)^{\frac{1}{p}} \right|^p B_{k,n}(x) \,\mathrm{d}x$$

$$\leq \int_0^1 \sum_{k=0}^n (n+1) \left| \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f_+(t) - f_-(t)|^p \,\mathrm{d}t \right)^{\frac{1}{p}} \right|^p B_{k,n}(x) \,\mathrm{d}x$$

$$\leq \int_0^1 \sum_{k=0}^n (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p \,\mathrm{d}t \, B_{k,n}(x) \,\mathrm{d}x = \sum_{k=0}^n \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t)|^p \,\mathrm{d}t = \|f\|_p^p,$$

which proves the boundedness of the operator and (a).

Now, let $0 < p < 1$. Thanks to (2.1) we get, for $f \in L^p(0,1)$,

$$\|K_{p,n} f\|_p^p \leq \int_0^1 \sum_{k=0}^n (n+1) \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_+(t)^p \,\mathrm{d}t \right) B_{k,n}(x)^p \,\mathrm{d}x$$

$$+ \int_0^1 \sum_{k=0}^n (n+1) \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_-(t)^p \,\mathrm{d}t \right) B_{k,n}(x)^p \,\mathrm{d}x$$

$$\leq (n+1)\left(\sum_{k=0}^{n}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left(f_+(t)^p + f_-(t)^p\right)\,\mathrm{d}t\right)\max_{k=0,\dots,n}\int_0^1 B_{k,n}(x)^p\,\mathrm{d}x$$

$$= \|f\|_p^p\,(n+1)\max_{k=0,\dots,n}\int_0^1 B_{k,n}(x)^p\,\mathrm{d}x,$$

so the operator is bounded. The inequality in (b) can be proved with similar steps as for (a) and taking into account that the function $y \mapsto y^p$ is concave. □

From numerical experiments we observed that the inequality in Proposition 3.3(ii.b) does not necessarily hold when f has a variable sign.

Remarks 3.4 As in the proof of Proposition 3.3(ii) it follows that, for any $p \geq 1$ and $f, g \in L^p(0,1)$, $f, g \geq 0$,

$$\|K_{p,n}f - K_{p,n}g\|_p \leq \|f - g\|_p. \tag{3.3}$$

Indeed,

$$\|K_{p,n}f - K_{p,n}g\|_p^p \leq \int_0^1 \sum_{k=0}^{n}(n+1)\left|\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p\,\mathrm{d}t\right)^{\frac{1}{p}} - \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} g(t)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\right|^p B_{k,n}(x)\,\mathrm{d}x$$

$$\leq \int_0^1 \sum_{k=0}^{n}(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |f(t) - g(t)|^p\,\mathrm{d}t\,B_{k,n}(x)\,\mathrm{d}x = \|f - g\|_p^p.$$

In general, if $f, g \in L^p(0,1)$, $p \geq 1$, then, by using the properties $|f_+ - g_+| \leq |f - g|$ and $|f_- - g_-| \leq |f - g|$, which hold true in every Riesz space, we get that

$$\|K_{p,n}f - K_{p,n}g\|_p \leq 2\|f - g\|_p,$$

so that $K_{p,n} : L^p(0,1) \to L^p(0,1)$ is a continuous operator.
Analogously, for any $p > 0$, $f, g \in L^\infty(0,1)$

$$\|K_{p,n}f - K_{p,n}g\|_\infty \leq M\|f - g\|_\infty,$$

where $M = 1$ if $f, g \geq 0$ and $M = 2$ otherwise.

4 Convergence theorems

In this section we study the convergence of the operators $K_{p,n}$ as $n \to +\infty$, in the sense of pointwise/uniform convergence or in L^p-norm, as it was classically done for Kantorovich operators [26]. We begin the section with some asymptotic estimates that will be needed throughout the paper.

Lemma 4.1 *For any real number $j \geq 1$, we have*

$$\sum_{k=0}^{n} \max\left(\left|\frac{k}{n+1} - x\right|^{j}, \left|\frac{k+1}{n+1} - x\right|^{j}\right) B_{k,n}(x) = \mathcal{O}(n^{-\frac{j}{2}}) \tag{4.1}$$

as $n \to +\infty$ and uniformly w.r.t. $x \in [0, 1]$.

Proof The left hand side in (4.1) can be bounded by

$$2^{j-1} \sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^{j} B_{k,n}(x) + \frac{2^{j-1}}{(n+1)^{j}},$$

in view of the triangular and Jensen's inequalities and of (2.3). Hence, the conclusion follows by the fact that $\sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^{j} B_{k,n}(x) = \mathcal{O}(n^{-\frac{j}{2}})$ as $n \to +\infty$ (see [2, Theorem 1]). □

In what follows, we present a result about the pointwise and uniform convergence for the operators $K_{p,n}$.

Theorem 4.2 *Let $f \in L^{\infty}(0, 1)$ and $p > 0$. Then, $\lim_{n\to+\infty} (K_{p,n} f)(x) = f(x)$ at any point x of continuity of f. Moreover, if $f \in C(0, 1)$, then $\lim_{n\to+\infty} \|K_{p,n} f - f\|_{\infty} = 0$.*

Proof We first consider f to be non-negative. Let $\epsilon > 0$. There exists $\delta > 0$ such that $|f(t) - f(x)| < \epsilon$ for all $t \in [0, 1]$ with $|t - x| < \delta$. Then

$$|(K_{p,n} f)(x) - f(x)| = \left|\sum_{k=0}^{n} \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^{p}\, \mathrm{d}t\right)^{\frac{1}{p}} B_{k,n}(x) - f(x)\right|$$

$$\leq \sum_{k=0}^{n} \left|\left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^{p}\, \mathrm{d}t\right)^{\frac{1}{p}} - f(x)\right| B_{k,n}(x) \leq \sum_{J_1} + \sum_{J_2},$$

where

$$J_1 = \left\{k = 0, \ldots, n : \max\left(\left|\frac{k}{n+1} - x\right|, \left|\frac{k+1}{n+1} - x\right|\right) < \delta\right\}$$

and

$$J_2 = \left\{k = 0, \ldots, n : \max\left(\left|\frac{k}{n+1} - x\right|, \left|\frac{k+1}{n+1} - x\right|\right) \geq \delta\right\}.$$

Let us estimate the first sum. Since f is essentially bounded and continuous at x we obtain

$$\sum_{J_1} \left|\left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^{p}\, \mathrm{d}t\right)^{\frac{1}{p}} - f(x)\right| B_{k,n}(x)$$

$$\leq \sum_{J_1} \max(|i_{k,n}(f) - f(x)|, |s_{k,n}(f) - f(x)|) B_{k,n}(x) \leq \epsilon$$

where, in particular, we set $i_{k,n}(f) = \inf_{[\frac{k}{n+1}, \frac{k+1}{n+1}]} f$ and $s_{k,n}(f) = \sup_{[\frac{k}{n+1}, \frac{k+1}{n+1}]} f$.

Regarding the second sum, by Lemma 4.1, there exists C independent of n, x such that

$$\sum_{J_2} \left| \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p \, \mathrm{d}t \right)^{\frac{1}{p}} - f(x) \right| B_{k,n}(x)$$
$$\leq \sum_{J_2} \left(\left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p \, \mathrm{d}t \right)^{\frac{1}{p}} + f(x) \right) B_{k,n}(x) \leq 2\|f\|_\infty \sum_{J_2} B_{k,n}(x)$$
$$\leq 2\|f\|_\infty \frac{1}{\delta^2} \sum_{J_2} \max \left(\left| \frac{k}{n+1} - x \right|^2, \left| \frac{k+1}{n+1} - x \right|^2 \right) B_{k,n}(x) \leq C \frac{\|f\|_\infty}{\delta^2 n}.$$

This completes the proof of the case where f is non-negative. The general case follows by the inequality

$$|(K_{p,n} f)(x) - f(x)| \leq |(K_{p,n}(f_+))(x) - f_+(x)| + |(K_{p,n}(f_-))(x) - f_-(x)|.$$

If $f \in C(0, 1)$, reasoning in a similar way we get the second statement. □

About the convergence in L^p-spaces, we have the following theorem.

Theorem 4.3 *Let $n \geq 1$ and suppose that one of the following conditions hold*

(i) $p \geq 1$ and $f \in L^p(0, 1)$,
(ii) $p > 0$ and $f \in C(0, 1)$.

Then $\lim_{n \to +\infty} \|K_{p,n} f - f\|_p = 0$.

Proof (i) We start by considering $f \in L^p(0, 1)$, $f \geq 0$, and $\varepsilon > 0$; then, there exists $g \in C(0, 1)$, $g \geq 0$, such that $\|f - g\|_p < \varepsilon/3$.

From Theorem 4.2, there exists $\overline{n} \in \mathbb{N}$ such that, for any $n \in \mathbb{N}$, $n \geq \overline{n}$,

$$\|K_{p,n} g - g\|_p \leq \|K_{p,n} g - g\|_\infty < \varepsilon/3. \tag{4.2}$$

Hence, for $n \geq \overline{n}$, taking (4.2) and (3.3) into account,

$$\|K_{p,n} f - f\|_p \leq \|K_{p,n} f - K_{p,n} g\|_p + \|K_{p,n} g - g\|_p + \|f - g\|_p < \varepsilon.$$

Now let assume $f \in L^p(0, 1)$. By what proved in the first part and by the inequality

$$\|K_{p,n} f - f\|_p \leq \|K_{p,n}(f_+) - K_{p,n}(f_-) - (f_+ - f_-)\|_p$$
$$\leq \|K_{p,n}(f_+) - f_+\|_p + \|K_{p,n}(f_-) - f_-\|_p,$$

the proof is complete.

(ii) In this case the statement follows by the inequality $\|K_{p,n}f - f\|_p \leq \|K_{p,n}f - f\|_\infty$ and from Theorem 4.2. □

5 Asymptotic formula

In this section we will state and prove an asymptotic formula of Voronovskaya type for the operators $K_{p,n}$ covering any case of $p > 0$, $x \in [0, 1]$ and sign of $f(x)$. The result is the following.

Theorem 5.1 *Let $f \in L^\infty(0, 1)$, $x \in [0, 1]$ such that f is two times differentiable at x, and $p > 0$. Then*

$$Vf(x) := \lim_{n \to +\infty} n((K_{p,n}f)(x) - f(x)) \tag{5.1}$$

exists. Moreover,

(i) if $0 < x < 1$, or if $x = 0, 1$ and $f(x) \neq 0$, then

$$Vf(x) = f'(x)\frac{1-2x}{2} + f''(x)\frac{x(1-x)}{2}; \tag{5.2}$$

(ii) if $x = 0$ and $f(0) = 0$, then

$$Vf(0) = \frac{f'(0)}{(p+1)^{\frac{1}{p}}}; \tag{5.3}$$

(iii) if $x = 1$ and $f(1) = 0$, then

$$Vf(1) = -\frac{f'(1)}{(p+1)^{\frac{1}{p}}}. \tag{5.4}$$

Before attacking the proof of Theorem 5.1, we want to highlight some curious and unexpected features of the above mentioned asymptotic formula for the operators $K_{p,n}$.

Let f be two times differentiable at $x \in [0, 1]$. First, restricting on the interval $]0, 1[$, Vf does not depend on p (in other words, (5.1) coincides with the corresponding formula of the Kantorovich operator). In particular, V is linear on f on the interval $]0, 1[$, even though the operator $K_{p,n}$ is nonlinear for $p \neq 1$.

Asymptotically, the nonlinearity is preserved at $x = 0$ and $x = 1$. Indeed, if for instance $f, g : [0, 1] \to \mathbb{R}$ are given by $f(x) = 2x + 1$ and $g(x) = -x - 1$ for $x \in [0, 1]$, then $f(0) \neq 0$, $g(0) \neq 0$, $(f + g)(0) = 0$ and $(f + g)'(0) \neq 0$, so

$$V(f + g)(0) \neq Vf(0) + Vg(0).$$

Moreover, V is linear on the space of two times differentiable functions on $[0, 1]$ which vanish at the endpoints.

Finally, when f is two times differentiable in a neighborhood of $x_0 = 0$ or 1, $f(x_0) = 0$ and $f'(x_0) \neq 0$, then the function Vf is discontinuous at x_0 and exactly

$$\lim_{x \to x_0} Vf(x) = \frac{2}{(p+1)^{\frac{1}{p}}} Vf(x_0).$$

This implies that the convergence in (5.1) is not necessarily uniform with respect to x. Furthermore, as it will be shown in Corollary 6.5, the discontinuity has some indirect positive implications about values of p for better approximations. More generally, we will discuss some consequences of Theorem 5.1 in Section 6.

As one can see, the statement of Theorem 5.1 is very simple. Nevertheless, the non-linearity of the operators $K_{p,n}$ makes the proof decisively more difficult in comparison to the case of the linear operators K_n. This is why we formulate some preparatory lemmas and we organize the section into different parts, to facilitate the understanding of the proof, which is in Subsection 5.3.

5.1 A recurrent Taylor's formula and the remainder estimate

In many parts of the proof we will use the following Taylor's formula and the corresponding estimate of the remainder, in order to approximate a recurrent function with a polynomial of a certain degree. As usual, we denote by

$$\binom{\alpha}{i} = \begin{cases} 1 & i = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!} & i > 0, \end{cases}$$

for any $\alpha \in]0, +\infty[$ and any non-negative integer i.

Lemma 5.2 *Let* $\alpha \in]0, +\infty[$ *and* $q \in \mathbb{N}$ *such that* $\beta \geq \max(\alpha, q+1)$. *Then*

$$(1+y)^\alpha = \sum_{i=0}^{q} \binom{\alpha}{i} y^i + R_q(y), \qquad \forall y \geq -1, \tag{5.5}$$

where

$$|R_q(y)| \leq r(|y|^\beta + |y|^{q+1}), \qquad \forall y \geq -1, \tag{5.6}$$

and r *depends only of* α, β, *and* q.

Proof Let $\alpha \in]0, +\infty[$ and $q \in \mathbb{N}$. By Taylor's formula centered at 0, it is possible to write (5.5) with Lagrange form remainder

$$R_q(y) = \binom{\alpha}{q+1} (1 + z(y))^{\alpha-q-1} y^{q+1}$$

for any $y > -1$, $z(y)$ being a number satisfying

$$y < z(y) < 0, \text{ if } -1 < y < 0, \qquad z(0) = 0, \qquad 0 < z(y) < y, \text{ if } y > 0. \quad (5.7)$$

Taking (5.7) into account, in the case where $\alpha > q + 1$, we have

$$(1 + z(y))^{\alpha-q-1} \leq \begin{cases} (1 + y)^{\alpha-q-1} & \text{if } y \geq 0 \\ 1 & \text{if } -1 \leq y < 0. \end{cases}$$

Now, for any $\beta \geq \max(\alpha, q + 1)$ we can further write by Jensen's inequality

$$(1 + y)^{\alpha-q-1} \leq (1 + y)^{\beta-q-1} \leq 2^{\beta-q-2}(1 + y^{\beta-q-1}), \qquad \forall y \geq 0,$$

so the estimate

$$|(1 + z(y))^{\alpha-q-1} y^{q+1}| \leq 2^{\beta-q-2}|y|^{\beta} + 2^{\beta-q-2}|y|^{q+1}$$

holds for all $y > -1$, which implies (5.6). Moreover, the statement is true also for $y = -1$ by continuity.

Finally, in the case that $0 < \alpha \leq q + 1$, the function $y \mapsto (1 + z(y))^{\alpha-q-1}$ is bounded in the interval $]-1, 0]$ since the function $y \mapsto (1 + y)^{\alpha}$ is so, and it is bounded in the interval $[0, +\infty[$ because $(1 + z(y))^{\alpha-q-1} \leq 1$ by (5.7). Therefore, (5.6) is true for $y > -1$ and by continuity also for $y = -1$, even in the case where $0 < \alpha \leq q + 1$. □

Remarks 5.3 Actually, a careful reading of the proof of Lemma 5.2 shows that a stronger estimate than (5.6) holds, i.e. there exist r_1, r_2 depending only on α, q such that

$$|R_q(y)| \leq \begin{cases} r_1(|y|^{\alpha} + |y|^{q+1}), & \text{if } \alpha > q + 1 \\ r_2|y|^{q+1}, & \text{if } 0 < \alpha \leq q + 1 \end{cases}$$

for all $y \geq -1$. However, we prefer to state Lemma 5.2 with (5.6) to uniform the proof of Theorem 5.1, with respect to any $p > 0$.

5.2 Asymptotic behavior of suitable moments

In the proof of Theorem 5.1 we will manage the asymptotic behavior of certain functions, which are combinations of particular coefficients (in form of integral of suitable powers of $\psi_x(t) = t - x$, where x is fixed) and of the Bernstein basis. Those functions are sometimes (but not always) moments or absolute moments of the operators $K_{p,n}$, so we generally call them, by an abuse of terminology, "moments". The type of moments required for the proof is determined according to the fact the approximating function and some of its derivatives vanish or not at a certain point x. In what follows, $n \geq 1$ and $k = 0, \ldots, n$, as usual.

First of all, in the case $f(x) \neq 0$, the terms

$$a_{k,n}(x) = (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)\,\mathrm{d}t = \frac{2k+1}{2(n+1)} - x, \tag{5.8}$$

$$b_{k,n}(x) = (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^2\,\mathrm{d}t = \frac{3k^2+3k+1}{3(n+1)^2} - \frac{2k+1}{n+1}x + x^2 \tag{5.9}$$

are needed, while in the case $f(x) = f'(x) = 0$ and $f''(x) \neq 0$ only the coefficient

$$d_{p,k,n}(x) = \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |t-x|^{2p}\,\mathrm{d}t\right)^{\frac{1}{p}} \tag{5.10}$$

is involved. The following lemma gives the expressions of the moment limit with these coefficients.

Lemma 5.4 *Uniformly w.r.t* $x \in [0,1]$, *it holds that*

$$\lim_{n\to+\infty} n\left(\sum_{k=0}^{n} a_{k,n}(x)B_{k,n}(x)\right) = \frac{1-2x}{2}, \tag{5.11}$$

$$\lim_{n\to+\infty} n\left(\sum_{k=0}^{n} b_{k,n}(x)B_{k,n}(x)\right) = x(1-x), \tag{5.12}$$

$$\lim_{n\to+\infty} n\left(\sum_{k=0}^{n} a_{k,n}(x)^2 B_{k,n}(x)\right) = x(1-x), \tag{5.13}$$

$$\lim_{n\to+\infty} n\left(\sum_{k=0}^{n} d_{p,k,n}(x)B_{k,n}(x)\right) = x(1-x). \tag{5.14}$$

Proof Equations (5.11) and (5.12) are well-known [26], while (5.13) follows from

$$a_{k,n}(x)^2 = \frac{4k^2+4k+1}{4(n+1)^2} - \frac{2k+1}{n+1}x + x^2$$

and the identities (2.4).

To prove (5.14), we observe that, by mean value theorem for integrals, for every $k = 0, \ldots, n$ there exists $\xi_{k,n} \in \left]\frac{k}{n+1}, \frac{k+1}{n+1}\right[$ such that

$$d_{p,k,n}(x) = \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |t-x|^{2p}\,\mathrm{d}t\right)^{\frac{1}{p}} = (\xi_{k,n} - x)^2.$$

Hence,

$$\left|d_{p,k,n}(x)-\left(\frac{k}{n}-x\right)^2\right| = \left|\xi_{k,n}-\frac{k}{n}\right|\left|\xi_{k,n}+\frac{k}{n}-2x\right| = \left|\xi_{k,n}-\frac{k}{n}\right|\left|\xi_{k,n}-\frac{k}{n}+2\frac{k}{n}-2x\right|$$
$$\leq \left|\xi_{k,n}-\frac{k}{n}\right|^2 + 2\left|\xi_{k,n}-\frac{k}{n}\right|\left|\frac{k}{n}-x\right| \leq \frac{1}{(n+1)^2}+\frac{1}{n+1}\left|\frac{k}{n}-x\right|.$$

Therefore,

$$\left|\sum_{k=0}^{n} d_{p,k,n}(x)B_{k,n}(x)-\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 B_{k,n}(x)\right| \leq \frac{1}{(n+1)^2}+\frac{1}{n+1}\sum_{k=0}^{n}\left|\frac{k}{n}-x\right| B_{k,n}(x).$$

Since $\lim_{n\to+\infty}\sum_{k=0}^{n}\left|\frac{k}{n}-x\right| B_{k,n}(x) = 0$ and $\lim_{n\to+\infty} n\left(\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2 B_{k,n}(x)\right) = x(1-x)$ both uniformly w.r.t. to x (see, for instance [26, Sect. 1.5]), the last inequality implies (5.14). □

We remark that $d_{1,k,n} = b_{k,n}$ for all n, k, and

$$\sum_{k=0}^{n} a_{k,n}(x)B_{k,n}(x) = K_{1,n}(\psi_x), \qquad \sum_{k=0}^{n} b_{k,n}(x)B_{k,n}(x) = K_{1,n}(\psi_x^2),$$

$$\sum_{k=0}^{n} d_{p,k,n}(x)B_{k,n}(x) = K_{p,n}(\psi_x^2).$$

Further moments will appear in the proof of Theorem 5.1 when $f(x) = 0$ and $f'(x) \neq 0$ for $0 < x < 1$, namely

$$\sum_{k=0}^{n} a_{p,k,n}(x)B_{k,n}(x), \qquad \sum_{k=0}^{n} b_{p,k,n}(x)B_{k,n}(x), \tag{5.15}$$

where

$$a_{p,k,n}(x) = \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)_+{}^p\,\mathrm{d}t\right)^{\frac{1}{p}} - \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)_-{}^p\,\mathrm{d}t\right)^{\frac{1}{p}}, \tag{5.16}$$

and

$$b_{p,k,n}(x) = \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)_+{}^p \,\mathrm{d}t\right)^{\frac{1}{p}-1} (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)_+{}^{p+1}\,\mathrm{d}t$$
$$+\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)_-{}^p \,\mathrm{d}t\right)^{\frac{1}{p}-1} (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)_-{}^{p+1}\,\mathrm{d}t. \tag{5.17}$$

Note that

$$\sum_{k=0}^{n} a_{p,k,n}(x)B_{k,n}(x) = K_{p,n}(\psi_x),$$

and in (5.17) we mean that $\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)_+{}^p \,\mathrm{d}t\right)^{\frac{1}{p}-1} = 0$ if $x \geq \frac{k+1}{n+1}$, and similarly $\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)_-{}^p \,\mathrm{d}t\right)^{\frac{1}{p}-1} = 0$ if $x \leq \frac{k}{n+1}$, even when $p \geq 1$ (situations where the standard meanings are lost). Furthermore, $a_{1,k,n} = a_{k,n}$ and $b_{1,k,n} = b_{k,n}$ for all n, k.

The following lemma allows us to evaluate the asymptotic behavior of the moments (5.15). The proof, which is rather difficult, is based on Lemma 5.2.

Lemma 5.5 *For any $p > 0$ and $0 < x < 1$ we have*

$$\lim_{n\to+\infty} n\sum_{k=0}^{n} a_{p,k,n}(x)B_{k,n}(x) = \frac{1-2x}{2}, \tag{5.18}$$

$$\lim_{n\to+\infty} n\sum_{k=0}^{n} b_{p,k,n}(x)B_{k,n}(x) = x(1-x). \tag{5.19}$$

Proof We start to prove (5.18). Let $0 < x < 1$, $n \geq 1$ and $k = 0, \ldots, n$ be fixed. There are different possibilities about the respective positions of x, $\frac{k}{n+1}$ and $\frac{k+1}{n+1}$, so we need to consider different cases.

Firstly, let us suppose that $x \leq \frac{k}{n+1}$. Then, a direct calculation from (5.16) leads to

$$a_{p,k,n}(x) = \left(\frac{n+1}{p+1}\right)^{\frac{1}{p}} \left[\left(\frac{k+1}{n+1}-x\right)^{p+1} - \left(\frac{k}{n+1}-x\right)^{p+1}\right]^{\frac{1}{p}}.$$

Since

$$\left(\frac{k+1}{n+1}-x\right)^{p+1} = \left(\frac{2k+1}{2(n+1)}-x\right)^{p+1}\left(1+\frac{1}{2(n+1)\left(\frac{2k+1}{2(n+1)}-x\right)}\right)^{p+1},$$

by applying Lemma 5.2 with

$$y_{k,n} = \frac{1}{2(n+1)\left(\frac{2k+1}{2(n+1)} - x\right)}, \tag{5.20}$$

$\alpha = p+1$ and $q = 2$, we get

$$\begin{aligned}\left(\frac{k+1}{n+1} - x\right)^{p+1} &= \left(\frac{2k+1}{2(n+1)} - x\right)^{p+1} + \frac{p+1}{2(n+1)}\left(\frac{2k+1}{2(n+1)} - x\right)^{p} \\ &+ \frac{(p+1)p}{8(n+1)^2}\left(\frac{2k+1}{2(n+1)} - x\right)^{p-1} + \left(\frac{2k+1}{2(n+1)} - x\right)^{p+1} R_2(y_{k,n}),\end{aligned}$$

where

$$|R_2(y_{k,n})| \le r_1(|y_{k,n}|^\beta + |y_{k,n}|^3), \tag{5.21}$$

$\beta \ge \max(p+1, 3)$ and r_1 depends only on p, β. Similarly,

$$\begin{aligned}\left(\frac{k}{n+1} - x\right)^{p+1} &= \left(\frac{2k+1}{2(n+1)} - x\right)^{p+1} - \frac{p+1}{2(n+1)}\left(\frac{2k+1}{2(n+1)} - x\right)^{p} \\ &+ \frac{(p+1)p}{8(n+1)^2}\left(\frac{2k+1}{2(n+1)} - x\right)^{p-1} + \left(\frac{2k+1}{2(n+1)} - x\right)^{p+1} R_2(\widetilde{y}_{k,n}),\end{aligned}$$

where $\widetilde{y}_{k,n} = -\frac{1}{2(n+1)\left(\frac{2k+1}{2(n+1)} - x\right)}$. Therefore,

$$\begin{aligned}a_{p,k,n}(x) &= \left[\left(\frac{2k+1}{2(n+1)} - x\right)^{p} + \frac{n+1}{p+1}\left(\frac{2k+1}{2(n+1)} - x\right)^{p+1}(R_2(y_{k,n}) - R_2(\widetilde{y}_{k,n}))\right]^{\frac{1}{p}} \\ &= \left(\frac{2k+1}{2(n+1)} - x\right)\left[1 + \frac{n+1}{p+1}\left(\frac{2k+1}{2(n+1)} - x\right)(R_2(y_{k,n}) - R_2(\widetilde{y}_{k,n}))\right]^{\frac{1}{p}} \\ &= \left(\frac{2k+1}{2(n+1)} - x\right) + \left(\frac{2k+1}{2(n+1)} - x\right)R_0(Y_{k,n}),\end{aligned} \tag{5.22}$$

where in the last row we applied again Lemma 5.2 with $\alpha = \frac{1}{p}$, $q = 0$ and

$$Y_{k,n} = \frac{n+1}{p+1}\left(\frac{2k+1}{2(n+1)} - x\right)(R_2(y_{k,n}) - R_2(\widetilde{y}_{k,n})). \tag{5.23}$$

We now estimate the last term in (5.22). Firstly, we note that for any $\gamma \ge \max(\frac{1}{p}, 1)$ and some r_2 depending only on p, γ,

$$|R_0(Y_{k,n})| \le r_2(|Y_{k,n}|^\gamma + |Y_{k,n}|).$$

Hence, taking also (5.23), (5.21) and (5.20) into account, by Jensen's inequality there exists c_1 independent of k, n, x such that

$$\begin{aligned}
&\left|\left(\frac{2k+1}{2(n+1)}-x\right)R_0(Y_{k,n})\right| \\
&\leq c_1\Bigg((n+1)^{-(\beta-1)\gamma}\left|\frac{2k+1}{2(n+1)}-x\right|^{1-(\beta-1)\gamma}+(n+1)^{-2\gamma}\left|\frac{2k+1}{2(n+1)}-x\right|^{1-2\gamma} \\
&+(n+1)^{1-\beta}\left|\frac{2k+1}{2(n+1)}-x\right|^{2-\beta}+(n+1)^{-2}\left|\frac{2k+1}{2(n+1)}-x\right|^{-1}\Bigg).
\end{aligned} \tag{5.24}$$

In the case $x \geq \frac{k+1}{n+1}$ similar arguments lead to

$$a_{p,k,n}(x) = -\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(x-t)^p\,\mathrm{d}t\right)^{\frac{1}{p}} = \left(\frac{2k+1}{2(n+1)}-x\right)+\left(\frac{2k+1}{2(n+1)}-x\right)R_0(\widetilde{Y}_{k,n}),$$

where $\left(\frac{2k+1}{2(n+1)}-x\right)R_0(\widetilde{Y}_{k,n})$ satisfies an estimate similar to (5.24). Therefore, for any $0<x<1$,

$$\begin{aligned}
&\lim_{n\to+\infty} n\left(\sum_{k=0}^{n} a_{p,k,n}(x)B_{k,n}(x)\right) = \lim_{n\to+\infty} n\left(\sum_{k=0}^{n}\left(\frac{2k+1}{2(n+1)}-x\right)B_{k,n}(x)\right) \\
&+\lim_{n\to+\infty} n\left(\sum_{k\neq\widetilde{k}}\left(\frac{2k+1}{2(n+1)}-x\right)R_0(Y_{k,n})B_{k,n}(x)\right) \\
&+\lim_{n\to+\infty} n\left(a_{p,\widetilde{k},n}(x)-\left(\frac{2\widetilde{k}+1}{2(n+1)}-x\right)\right)B_{\widetilde{k},n}(x),
\end{aligned} \tag{5.25}$$

where $\widetilde{k}$ is the unique integer (depending on n) satisfying $\frac{\widetilde{k}}{n+1} < x < \frac{\widetilde{k}+1}{n+1}$.
Now, we prove that the second and third limits after the equality sign of (5.25) are null.
In particular, for the second one we use the inequality (5.24), in which the right hand side is a sum of terms of the form $(n+1)^{-m}\left|\frac{2k+1}{2(n+1)}-x\right|^{1-m}B_{k,n}(x)$, for suitable $m>1$. This enables us to evaluate the general terms of that type. Let $\frac{1}{2}<\alpha<1$ and define

$$I_1 = \left\{k=0,\ldots,n : k\neq\widetilde{k}, \left|\frac{2k+1}{2(n+1)}-x\right| \leq n^{-\alpha}\right\},$$

$$I_2 = \left\{k=0,\ldots,n : k\neq\widetilde{k}, \left|\frac{2k+1}{2(n+1)}-x\right| > n^{-\alpha}\right\}.$$

Since

$$\left|\frac{2k+1}{2(n+1)}-x\right| \geq \frac{1}{2(n+1)} \quad \text{for } k\neq\widetilde{k}$$

and $1-m<0$, we have

$$
\begin{aligned}
&(n+1)^{-m}\sum_{k\neq\widetilde{k}}\left|\frac{2k+1}{2(n+1)}-x\right|^{1-m}B_{k,n}(x)\\
&=(n+1)^{-m}\sum_{k\in I_1}\left|\frac{2k+1}{2(n+1)}-x\right|^{1-m}B_{k,n}(x)+(n+1)^{-m}\sum_{k\in I_2}\left|\frac{2k+1}{2(n+1)}-x\right|^{1-m}B_{k,n}(x)\\
&\leq\frac{2^{m-1}}{n+1}\sum_{k\in I_1}B_{k,n}(x)+(n+1)^{m(\alpha-1)-\alpha}.
\end{aligned}
$$

Therefore, for some C independent of k,n,x,

$$
n\sum_{k\neq\widetilde{k}}\left|\left(\frac{2k+1}{2(n+1)}-x\right)R_0(Y_{k,n})\right|B_{k,n}(x)\leq C\frac{n}{n+1}\sum_{k\in I_1}B_{k,n}(x)+C(n+1)^{(m-1)(\alpha-1)}.
$$

Now, by a result that can be found in [26] (after Theorem 1.5.2, page 18),

$$
\lim_{n\to+\infty}\sum_{k\in I_1}B_{k,n}(x)=0.
$$

Hence, recalling that $m>1$ and $\alpha<1$,

$$
\lim_{n\to+\infty}n\sum_{k\neq\widetilde{k}}\left|\left(\frac{2k+1}{2(n+1)}-x\right)R_0(Y_{k,n})\right|B_{k,n}(x)=0.
$$

Finally, about the last limit in (5.25), we note that $\frac{\widetilde{k}}{n+1}-x\leq a_{p,k,n}(x)\leq\frac{\widetilde{k}+1}{n+1}-x$, i.e.

$$
\left|a_{p,\widetilde{k},n}(x)-\left(\frac{2\widetilde{k}+1}{2(n+1)}-x\right)\right|\leq\frac{1}{2(n+1)}.
$$

Moreover,

$$
\left|\frac{\widetilde{k}}{n}-x\right|<\frac{1}{n}. \tag{5.26}
$$

By [26, Theorem 1.5.2], we obtain $B_{\widetilde{k},n}(x)=\mathcal{O}(n^{-\frac{1}{2}})$ as $n\to+\infty$ uniformly w.r.t. $\widetilde{k}$ satisfying (5.26). These remarks imply that

$$
\lim_{n\to+\infty}n\left(a_{p,\widetilde{k},n}(x)-\left(\frac{2\widetilde{k}+1}{2(n+1)}-x\right)\right)B_{\widetilde{k},n}(x)=0.
$$

In conclusion, taking formula (5.11) into account, (5.25) reduces to (5.18).

Now we move to (5.19). For $x\leq\frac{k}{n+1}$, the quantity

$$
\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^{p+1}\,\mathrm{d}t\right)\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^{p}\,\mathrm{d}t\right)^{-1}=\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^{p}(t-x)\,\mathrm{d}t\right)\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^{p}\,\mathrm{d}t\right)^{-1}
$$

is between $\frac{k}{n+1} - x$ and $\frac{k+1}{n+1} - x$. The same holds for $\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^p \, \mathrm{d}t\right)^{\frac{1}{p}}$ and hence

$$\left(\frac{k}{n+1} - x\right)^2 \le b_{p,k,n}(x) \le \left(\frac{k+1}{n+1} - x\right)^2,$$

so that $b_{p,k,n}(x) = (\xi_{k,n} - x)^2$ for some $\xi_{k,n} \in \left]\frac{k}{n+1}, \frac{k+1}{n+1}\right[$. This also holds if $x \ge \frac{k+1}{n+1}$, since analogous steps can be done in that case. With the calculation in Lemma 5.4, one can see that

$$\left| b_{p,k,n}(x) - \left(\frac{k}{n} - x\right)^2 \right| \le \frac{1}{(n+1)^2} + \frac{1}{n+1}\left|\frac{k}{n} - x\right| < \frac{3}{(n+1)^2}.$$

Finally, if $\frac{k}{n+1} < x < \frac{k+1}{n+1}$,

$$\left| b_{p,k,n}(x) - \left(\frac{k}{n} - x\right)^2 \right| \le \left| b_{p,k,n}(x) \right| + \left(\frac{k}{n} - x\right)^2$$
$$\le \left(\frac{k}{n+1} - x\right)^2 + \left(\frac{k+1}{n+1} - x\right)^2 + \left(\frac{k}{n} - x\right)^2 \le \frac{3}{(n+1)^2}.$$

In conclusion,

$$\left| \sum_{k=0}^{n} b_{p,k,n}(x) B_{k,n}(x) - \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 B_{k,n}(x) \right| \le \frac{3}{(n+1)^2},$$

which proves (5.19). □

Finally, other moments that will appear in the proof of Theorem 5.1 have a decay faster than $\frac{1}{n}$ as $n \to +\infty$. Therefore, we collect here some useful results.

Lemma 5.6 *Let $\ell \in L^\infty(0,1)$ be such that $\lim_{s \to 0} \ell(s) = 0$. For any $j \ge 0$ and $p > 0$, we define*

$$\theta_{j,k,n}(x) = \max\left(\left|\frac{k}{n+1} - x\right|^j, \left|\frac{k+1}{n+1} - x\right|^j\right)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \ell(t-x)(t-x)^2 \, \mathrm{d}t,$$

$$\nu_{p,k,n}(x) = \max\left(\left|\frac{k}{n+1} - x\right|^2, \left|\frac{k+1}{n+1} - x\right|^2\right) \frac{\displaystyle\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \ell(t-x)|t-x|^p \, \mathrm{d}t}{\displaystyle\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |t-x|^p \, \mathrm{d}t}, \tag{5.27}$$

$$\sigma_{p,k,n}(x) = \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |\ell(t-x)||t-x|^{2p}\,\mathrm{d}t \right)^{\frac{1}{p}}.$$

Then,

$$\lim_{n\to+\infty} n \sum_{k=0}^{n} \theta_{j,k,n}(x) B_{k,n}(x) = 0, \qquad \lim_{n\to+\infty} n \sum_{k=0}^{n} \nu_{p,k,n}(x) B_{k,n}(x) = 0,$$

$$\lim_{n\to+\infty} n \sum_{k=0}^{n} \sigma_{p,k,n}(x) B_{k,n}(x) = 0,$$

uniformly w.r.t. $x \in [0, 1]$.

Proof We prove the result only for $\theta_{j,k,n}$ because of the similarity of arguments. Moreover, note that it is sufficient to prove the statement for $j = 0$.
For every $\epsilon > 0$, there exists $\delta > 0$ such that $|\ell(s)| < \epsilon$ for $|s| < \delta$. We define $J_1 = \{k = 0, \ldots, n : \max(|\frac{k}{n+1} - x|, |\frac{k+1}{n+1} - x|) < \delta\}$ and $J_2 = \{k = 0, \ldots, n : \max(|\frac{k}{n+1} - x|, |\frac{k+1}{n+1} - x|) \geq \delta\}$. By splitting the sum for $k \in J_1$ and $k \in J_2$, applying the estimate

$$|b_{k,n}(x)| \leq \max\left(\left|\frac{k}{n+1} - x\right|^2, \left|\frac{k+1}{n+1} - x\right|^2 \right),$$

and Lemma 4.1, we obtain

$$\begin{aligned}
\left| \sum_{k=0}^{n} \theta_{0,k,n}(x) B_{k,n}(x) \right| &\leq \epsilon \sum_{k\in J_1} b_{k,n} B_{k,n}(x) + \|\ell\|_\infty \sum_{k\in J_2} b_{k,n} B_{k,n}(x) \\
&\leq \epsilon \frac{x(1-x)}{n} + \|\ell\|_\infty \sum_{k\in J_2} \max\left(\left|\frac{k}{n+1} - x\right|^2, \left|\frac{k+1}{n+1} - x\right|^2 \right) B_{k,n}(x) \\
&\leq \epsilon \frac{x(1-x)}{n} + \frac{\|\ell\|_\infty}{\delta^2} \sum_{k\in J_2} \max\left(\left|\frac{k}{n+1} - x\right|^4, \left|\frac{k+1}{n+1} - x\right|^4 \right) B_{k,n}(x) \\
&\leq \epsilon \frac{x(1-x)}{n} + \frac{C}{n^2}
\end{aligned}$$

for some C independent of n. Hence, for $n \to +\infty$ and given the arbitrariness of ϵ, we get the conclusion. □

5.3 The proof of the asymptotic formula

After the previous preparatory results, we are now able to prove the asymptotic formula for the operators $K_{p,n}$.

Proof of Theorem 5.1 Let $f \in L^\infty(0,1)$ and $x \in [0,1]$ such that f is two times differentiable at x. As a preliminary fact, the following Taylor's formula,

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + h(t-x)(t-x)^2, \tag{5.28}$$

holds for all $t \in [0,1]$, where h is an essentially bounded function such that $h(s) \to 0$ as $s \to 0$. When $x = 0$ or 1, clearly we mean the derivatives on one side only.
In the first part we assume that f is non-negative and we divide the proof according to these cases:

1) $x \in [0,1]$ and $f(x) \neq 0$;
2) $x = 0$ or $x = 1$, $f(x) = 0$ and $f'(x) \neq 0$;
3) $x \in [0,1]$, $f(x) = f'(x) = 0$ and $f''(x) > 0$;
4) $x \in [0,1]$ and $f(x) = f'(x) = f''(x) = 0$.

Note that the above conditions cover any possibility for a non-negative function. Indeed, if $0 < x < 1$ and $f(x) = 0$, then $f'(x) = 0$; if $x \in [0,1]$ and $f(x) = f'(x) = 0$ then $f''(x) \geq 0$.

Let us start with case 1). From (5.28) it follows that

$$f(t)^p = f(x)^p \left(1 + \frac{f'(x)}{f(x)}(t-x) + \frac{1}{2}\frac{f''(x)}{f(x)}(t-x)^2 + \frac{h(t-x)}{f(x)}(t-x)^2\right)^p .$$

By applying Lemma 5.2 with $\alpha = p$, $q = 2$ and

$$y = y(t,x) = \frac{f'(x)}{f(x)}(t-x) + \frac{1}{2}\frac{f''(x)}{f(x)}(t-x)^2 + \frac{h(t-x)}{f(x)}(t-x)^2, \tag{5.29}$$

we obtain

$$\begin{aligned} f(t)^p = f(x)^p &\left(1 + p\frac{f'(x)}{f(x)}(t-x) + \frac{p}{2}\frac{f''(x)}{f(x)}(t-x)^2 + p\frac{h(t-x)}{f(x)}(t-x)^2\right) \\ &+ \frac{p(p-1)}{2} f(x)^p \left(\frac{f'(x)}{f(x)}(t-x) + \frac{1}{2}\frac{f''(x)}{f(x)}(t-x)^2 + \frac{h(t-x)}{f(x)}(t-x)^2\right)^2 \\ &+ f(x)^p R_2(y(t,x)). \end{aligned}$$

Therefore,

$$\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p \,\mathrm{d}t\right)^{\frac{1}{p}} = f(x)(1 + Y_{k,n}(x))^{\frac{1}{p}}, \tag{5.30}$$

putting

$$Y_{k,n}(x) = p\frac{f'(x)}{f(x)} a_{k,n}(x) + \left(\frac{p(p-1)}{2}\frac{f'(x)^2}{f(x)^2} + \frac{p}{2}\frac{f''(x)}{f(x)}\right) b_{k,n}(x) + c_{k,n}(x), \tag{5.31}$$

where $a_{k,n}$ and $b_{k,n}$ are defined by (5.8) and (5.9), respectively, and $c_{k,n}$ is a combination of integrals of the functions $(t-x)^3$, $(t-x)^4$, $h(t-x)(t-x)^2$, $h(t-x)(t-x)^3$, $h(t-x)(t-x)^4$, $h(t-x)^2(t-x)^4$ and $R_2(y(t,x))$. Using, in particular, the estimate for the reminder R_2 in Lemma 5.2, we note that

$$|c_{k,n}(x)| \leq (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \ell(t-x)(t-x)^2\,\mathrm{d}t \tag{5.32}$$

for some essentially bounded non-negative function ℓ such that $\ell(s) \to 0$. By applying again Lemma 5.2 with $\alpha = \frac{1}{p}$, $q = 2$, and $y = Y_{k,n}(x)$, we find that

$$\begin{aligned}(1+Y_{k,n}(x))^{\frac{1}{p}} &= 1 + \frac{f'(x)}{f(x)}a_{k,n}(x) + \left(\frac{p-1}{2}\frac{f'(x)^2}{f(x)^2} + \frac{1}{2}\frac{f''(x)}{f(x)}\right)b_{k,n}(x) + \frac{1}{p}c_{k,n}(x)\\ &+ \frac{1-p}{2}\left(\frac{f'(x)}{f(x)}a_{k,n}(x) + \left(\frac{p-1}{2}\frac{f'(x)^2}{f(x)^2} + \frac{1}{2}\frac{f''(x)}{f(x)}\right)b_{k,n}(x) + \frac{1}{p}c_{k,n}(x)\right)^2 + R_2(Y_{k,n}(x)).\end{aligned} \tag{5.33}$$

Hence, by (5.30) and (5.33), the expression of $(K_{p,n}f)(x) - f(x)$ is a linear combination of terms of the form

$$\sum_{k=0}^{n} a_{k,n}(x)^{i_1} b_{k,n}(x)^{i_2} c_{k,n}(x)^{i_3} B_{k,n}(x), \tag{5.34}$$

for integers $i_1, i_2, i_3 \geq 0$ such that $i_1 + i_2 + i_3 \leq 2$, and of $\sum_{k=0}^{n} R_2(Y_{k,n}(x))B_{k,n}(x)$.

Now, formula (5.32), the estimates

$$|a_{k,n}(x)| \leq \max\left(\left|\frac{k}{n+1} - x\right|, \left|\frac{k+1}{n+1} - x\right|\right),$$

$$|b_{k,n}(x)| \leq \max\left(\left|\frac{k}{n+1} - x\right|^2, \left|\frac{k+1}{n+1} - x\right|^2\right),$$

$$|c_{k,n}(x)| \leq \|\ell\|_\infty \max\left(\left|\frac{k}{n+1} - x\right|^2, \left|\frac{k+1}{n+1} - x\right|^2\right),$$

together with Lemmas 4.1 and 5.6, imply that all terms of the form (5.34) are $o\left(\frac{1}{n}\right)$ as $n \to +\infty$, except for

$$\sum_{k=0}^{n} a_{k,n}(x)B_{k,n}(x), \quad \sum_{k=0}^{n} a_{k,n}(x)^2 B_{k,n}(x), \quad \sum_{k=0}^{n} b_{k,n}(x)B_{k,n}(x).$$

Moreover, also $\sum_{k=0}^{n} R_2(Y_{k,n}(x))B_{k,n}(x) = o\left(\frac{1}{n}\right)$ as $n \to +\infty$, by (5.31), the estimate for R_2 of Lemma 5.2, Jensen's inequality, the estimates for $a_{k,n}$, $b_{k,n}$, $c_{k,n}$ above and Lemma 4.1. In conclusion, making use of (5.11)-(5.13), we have that

$$\begin{aligned}
&(Vf)(x)\\
&= \lim_{n\to+\infty} n \sum_{k=0}^{n}\left(f'(x)a_{k,n}(x) + \left(\frac{p-1}{2}\frac{f'(x)^2}{f(x)} + \frac{1}{2}f''(x)\right)b_{k,n}(x) + \frac{1-p}{2}\frac{f'(x)^2}{f(x)}a_{k,n}(x)^2\right)B_{k,n}(x)\\
&= f'(x)\frac{1-2x}{2} + \left(\frac{p-1}{2}\frac{f'(x)^2}{f(x)} + \frac{1}{2}f''(x)\right)x(1-x) + \frac{1-p}{2}\frac{f'(x)^2}{f(x)}x(1-x)\\
&= f'(x)\frac{1-2x}{2} + f''(x)\frac{x(1-x)}{2}.
\end{aligned}$$

In case 2), let $x = 0$. We first note that

$$K_{p,n}f(0) = \left((n+1)\int_0^{\frac{1}{n+1}} f(t)^p \,\mathrm{d}t\right)^{\frac{1}{p}}.$$

Moreover, by hypothesis, f is non-negative, so $f'(0) > 0$. From (5.28) we get

$$f(t)^p = f'(0)^p t^p \left(1 + \frac{f''(0)}{2f'(0)}t + \frac{h(t)}{f'(0)}t\right)^p, \qquad t \in [0,1].$$

By Lemma 5.2 with $\alpha = p$ and $q = 0$, we can write

$$\begin{aligned}
(n+1)\int_0^{\frac{1}{n+1}} f(t)^p \,\mathrm{d}t &= (n+1)f'(0)^p \int_0^{\frac{1}{n+1}} t^p\,(1 + R_0(y(t)))\,\mathrm{d}t\\
&= \frac{f'(0)^p}{(p+1)(n+1)^p} + (n+1)f'(0)^p \int_0^{\frac{1}{n+1}} t^p R_0\,(y(t))\,\mathrm{d}t,
\end{aligned}$$

where $y(t) = \frac{f''(0)}{2f'(0)}t + \frac{h(t)}{f'(0)}t$. By the estimate $|R_0(y(t))| \le r(|y(t)|^\beta + |y(t)|)$, for some r depending only on p and $\beta \ge \max(p,1)$, it is possible to see that there exists C independent of n such that

$$(n+1)f'(0)^p \int_0^{\frac{1}{n+1}} t^p R_0\,(y(t))\,\mathrm{d}t \le C\left(\frac{1}{(n+1)^{p+\beta}} + \frac{1}{(n+1)^{p+1}}\right).$$

All these considerations prove (5.3). The arguments when $x = 1$ are similar (note that in this case $f'(1) < 0$).

Now we move to case 3), where Taylor's formula (5.28) reduces to

$$f(t) = \frac{1}{2}f''(x)(t-x)^2 + h(t-x)(t-x)^2.$$

Therefore,

$$f(t)^p = \frac{f''(x)^p}{2^p}\left(1+\frac{2h(t-x)}{f''(x)}\right)^p |t-x|^{2p},$$

and, by Lemma 5.2 with $\alpha = p$,

$$f(t)^p = \frac{f''(x)^p}{2^p}\left(1+R_0\left(\frac{2h(t-x)}{f''(x)}\right)\right)|t-x|^{2p},$$

where R_0 is a function satisfying $|R_0(y)| \le r(|y|^\beta + |y|)$ for some $\beta \ge \max(p,1)$, r (depending only on p,β) and all $y \ge -1$. Hence

$$\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p\,\mathrm{d}t\right)^{\frac{1}{p}} = \frac{f''(x)}{2}d_{p,k,n}(x)(1+Y_{k,n}(x))^{\frac{1}{p}},$$

where $d_{p,k,n}$ is defined by (5.10) and

$$Y_{k,n}(x) = \left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} R_0\left(\frac{2h(t-x)}{f''(x)}\right)|t-x|^{2p}\,\mathrm{d}t\right)\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |t-x|^{2p}\,\mathrm{d}t\right)^{-1}.$$

Another use of Lemma 5.2, with $\alpha = \frac{1}{p}$ and $q = 0$, yields to

$$\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p\,\mathrm{d}t\right)^{\frac{1}{p}} = \frac{f''(x)}{2}d_{p,k,n}(x) + \frac{f''(x)}{2}d_{p,k,n}(x)R_0(Y_{k,n}(x)).$$

Therefore, $(Vf)(x)$ is the sum of two terms. The first one is

$$\lim_{n\to+\infty}\left(n\frac{f''(x)}{2}\sum_{k=0}^{n} d_{p,k,n}(x)B_{k,n}(x)\right) = f''(x)\frac{x(1-x)}{2},$$

by Lemma 5.4, while the second contribution is null by Lemma 5.6 (see (5.27)). In conclusion, (5.2) holds true, i.e., $(Vf)(x) = f''(x)\frac{x(1-x)}{2}$.

Finally, the case 4). Since $(K_{p,n}f)(x) = \sum_{k=0}^{n}\sigma_{p,k,n}B_{k,n}(x)$, where

$$\sigma_{p,k,n}(x) = \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} h(t-x)^p|t-x|^{2p}\,\mathrm{d}t\right)^{\frac{1}{p}},$$

Lemma 5.6 gives $(Vf)(x) = 0$, hence (5.2) holds true.

This concludes the proof when f is non-negative, so we now move to a function f with an arbitrary sign (possibly changing in the interval). Continuing the enumeration started earlier in the proof, we may fall into one of the following cases:

5) $x \in [0,1]$ and $f(x) \neq 0$;
6) $x = 0$ or $x = 1$, $f(x) = 0$ and $f'(x) \neq 0$;
7) $0 < x < 1$, $f(x) = 0$ and $f'(x) \neq 0$;
8) $x \in [0,1]$, $f(x) = f'(x) = 0$ and $f''(x) \neq 0$;
9) $x \in [0,1]$ and $f(x) = f'(x) = f''(x) = 0$.

About cases 5), 6) and 8), we first note that f_+ and f_- are two times differentiable at x (actually, a function between f_+ and f_- is null in a neighborhood of x and the other one coincides with f in the same neighborhood). Then, the asymptotic formula simply follows by the identity

$$(K_{p,n}f)(x) - f(x) = (K_{p,n}f_+)(x) - f_+(x) - (K_{p,n}f_-)(x) + f_-(x)$$

and by the case proved for non-negative functions.

As far as case 9) is concerned, we observe that from (5.28)

$$f_+(t) = h_+(t-x)(t-x)^2, \qquad f_-(t) = h_-(t-x)(t-x)^2, \qquad t \in [0,1],$$

and $|h_+(t-x)|, |h_-(t-x)| \leq |h(t-x)|$, so it is sufficient to apply Lemma 5.6,

$$|(K_{p,n}f)(x) - f(x)| = |(K_{p,n}f)(x)| \leq 2\sum_{k=0}^{n}\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} |h(t-x)|^p |t-x|^{2p}\,\mathrm{d}t\right)^{\frac{1}{p}} B_{k,n}(x).$$

Finally, the proof of case 7) needs a significant elaboration, because the asymptotic formula cannot be derived from the non-negative case (indeed f_+ and f_- are not differentiable at x). We start by pointing out that in the last steps of the proof we will make use of the rapid decay of the series $\sum_{k \in J_2} B_{k,n}(x)$ as $n \to +\infty$, where $J_2 = \{k = 0, \ldots, n : \max(|\frac{k}{n+1} - x|, |\frac{k+1}{n+1} - x|) \geq \delta\}$ and δ is fixed. This allows us to confine our arguments on a neighborhood $]x-\delta, x+\delta[$ of x. In particular, without loss of generality, we may suppose that $f'(x) > 0$ and then we can choose δ so that $f(t) \geq 0$ for $x < t < x+\delta$, while $f(t) \leq 0$ for $x - \delta < t < x$.
Now, if $k \in J_1 = \{k = 0, \ldots, n : \max(|\frac{k}{n+1} - x|, |\frac{k+1}{n+1} - x|) < \delta\}$, then three instances may occur:

$$\text{A)}\ \ x \leq \frac{k}{n+1}, \qquad \text{B)}\ \ \frac{k}{n+1} < x < \frac{k+1}{n+1}, \qquad \text{C)}\ \ \frac{k+1}{n+1} \geq x.$$

From now on, we set

$$F_{k,n} = \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_+(t)^p\,\mathrm{d}t\right)^{\frac{1}{p}} - \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f_-(t)^p\,\mathrm{d}t\right)^{\frac{1}{p}}.$$

We will prove that, for $k \in J_1$ and in any of cases A), B) and C),

$$F_{k,n} = f'(x)a_{p,k,n}(x) + \frac{f''(x)}{2}b_{p,k,n}(x) + c_{p,k,n}(x), \tag{5.35}$$

where $a_{p,k,n}$ and $b_{p,k,n}$ are defined in (5.16)-(5.17) and $c_{p,k,n}$ gives a negligible contribution. We make again use of Taylor's formula (5.28) of f which can be written as

$$f(t) = f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + h(t-x)(t-x)^2.$$

Let us start by case A). First of all, it follows that

$$F_{k,n} = \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)^p \,\mathrm{d}t \right)^{\frac{1}{p}}.$$

Moreover, by applying Lemma 5.2 with $\alpha = p$ and $q = 1$,

$$\begin{aligned} f(t)^p &= f'(x)^p (t-x)^p \left(1 + \frac{f''(x)}{2f'(x)}(t-x) + \frac{h(t-x)}{f'(x)}(t-x) \right)^p \\ &= f'(x)^p (t-x)^p \left(1 + p\frac{f''(x)}{2f'(x)}(t-x) + p\frac{h(t-x)}{f'(x)}(t-x) + R_1(y(t,x)) \right), \end{aligned}$$

where $y = y(t,x) = \dfrac{f''(x)}{2f'(x)}(t-x) + \dfrac{h(t-x)}{f'(x)}(t-x)$ and $|R_1(y)| \le r(|y|^\beta + |y|^2)$ for some $\beta \ge \max(p, 2)$. Thus, we obtain

$$\begin{aligned} F_{k,n} &= f'(x)\Bigg((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^p \,\mathrm{d}t + p\frac{f''(x)}{2f'(x)}(n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^{p+1} \,\mathrm{d}t \\ &+ p\frac{n+1}{f'(x)} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} h(t-x)(t-x)^{p+1} \,\mathrm{d}t + (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} R_1(y(t,x))(t-x)^p \,\mathrm{d}t \Bigg)^{\frac{1}{p}} \\ &= f'(x) a_{p,k,n}(x) \left(1 + p\frac{f''(x)}{2f'(x)} \frac{b_{p,k,n}(x)}{a_{p,k,n}(x)} + \frac{p}{f'(x)} \frac{c_{p,k,n}(x)}{a_{p,k,n}(x)} \right)^{\frac{1}{p}}, \end{aligned}$$

where $a_{p,k,n}$ and $b_{p,k,n}$ are given by (5.16)-(5.17) and

$$\begin{aligned} c_{p,k,n}(x) &= \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^p \,\mathrm{d}t \right)^{\frac{1}{p}-1} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} h(t-x)(t-x)^{p+1} \,\mathrm{d}t \\ &+ \frac{1}{p} \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (t-x)^p \,\mathrm{d}t \right)^{\frac{1}{p}-1} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} R_1(y(t,x))(t-x)^p \,\mathrm{d}t. \end{aligned}$$

By applying again Lemma 5.2 with $\alpha = \frac{1}{p}$ and $q = 1$, we get

$$F_{k,n} = f'(x) a_{p,k,n}(x) + \frac{f''(x)}{2} b_{p,k,n}(x) + c_{p,k,n}(x) + f'(x) a_{p,k,n}(x) R_1(Y_{k,n}),$$

where

$$Y_{k,n} = p\frac{f''(x)}{2f'(x)}\frac{b_{p,k,n}(x)}{a_{p,k,n}(x)} + \frac{p}{f'(x)}\frac{c_{p,k,n}(x)}{a_{p,k,n}(x)}$$

and $|R_1(Y_{k,n})| \le r_2(|Y_{k,n}|^\gamma + |Y_{k,n}|^2)$ for some $\gamma \ge \max(\frac{1}{p}, 2)$. Thus, we have proved (5.35). We further make a remark about $\widetilde{c}_{p,k,n}(x) := c_{p,k,n}(x) + f'(x)a_{p,k,n}(x)R_1(Y_{k,n})$. The term $c_{p,k,n}(x)$ can be bounded by a linear combination of elements of the form (5.27). For this step it is useful to write the terms in $c_{p,k,n}$ in a suitable way: for instance

$$\begin{aligned}
&\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^p\,\mathrm{d}t\right)^{\frac{1}{p}-1}(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} h(t-x)(t-x)^{p+1}\,\mathrm{d}t\\
&=\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} h(t-x)(t-x)^{p+1}\,\mathrm{d}t\right)\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^p\,\mathrm{d}t\right)^{-1}\\
&\le \max\left(\left|\frac{k}{n+1}-x\right|^2, \left|\frac{k+1}{n+1}-x\right|^2\right)\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} h(t-x)(t-x)^{p}\,\mathrm{d}t\right)\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(t-x)^p\,\mathrm{d}t\right)^{-1},
\end{aligned}$$

and so on. In addition, $|a_{p,k,n}(x)R_1(Y_{k,n})| \le C(\frac{k+1}{n+1} - x)^3$ for some C independent of k, n, x. Moving to case C), we get

$$F_{k,n} = -\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}(-f(t))^p\,\mathrm{d}t\right)^{\frac{1}{p}};$$

then from (5.28) writing

$$-f(t) = f'(x)(x-t) - \frac{1}{2}f''(x)(x-t)^2 - h(t-x)(x-t)^2,$$

we have by Lemma 5.2 that

$$\begin{aligned}
(-f(t))^p &= f'(x)^p(x-t)^p\left(1 - \frac{f''(x)}{2f'(x)}(x-t) - \frac{h(t-x)}{f'(x)}(x-t)\right)^p\\
&= f'(x)^p(x-t)^p\left(1 - p\frac{f''(x)}{2f'(x)}(x-t) - p\frac{h(t-x)}{f'(x)}(x-t) + R_1(y(t,x))\right),
\end{aligned}$$

where $y = y(t,x) = \dfrac{f''(x)}{2f'(x)}(t-x) + \dfrac{h(t-x)}{f'(x)}(t-x)$ and $|R_1(y)| \le r_1(|y|^\beta + |y|^2)$ for some $\beta \ge \max(p, 2)$. Proceeding with similar steps as in the previous case, we arrive to

$$F_{k,n} = f'(x)a_{p,k,n}(x) + \frac{f''(x)}{2}b_{p,k,n}(x) + c_{p,k,n}(x) + f'(x)a_{p,k,n}(x)R_1(Y_{k,n}),$$

where $a_{p,k,n}$ and $b_{p,k,n}$ are given by (5.16)-(5.17),

$$c_{p,k,n}(x) = \left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (x-t)^p \, \mathrm{d}t\right)^{\frac{1}{p}-1} (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} h(t-x)(x-t)^{p+1} \, \mathrm{d}t$$
$$+ \frac{1}{p}\left((n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} (x-t)^p \, \mathrm{d}t\right)^{\frac{1}{p}-1} (n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} R_1(y(t,x))(x-t)^p \, \mathrm{d}t,$$

and $\widetilde{c}_{p,k,n}(x) := c_{p,k,n}(x) + f'(x)a_{p,k,n}(x)R_1(Y_{k,n})$ can be bounded following a similar argument as before. Finally, reasoning as above and gathering the similar terms, (5.35) and a corresponding bound for $\widetilde{c}_{p,k,n}(x)$ hold also in case B).

Once we have proved that (5.35) is always true, we note that by Lemma 4.1

$$\lim_{n\to+\infty} n \left|\sum_{k\in J_2} F_{k,n} B_{k,n}(x)\right| \le \|f\|_\infty \lim_{n\to+\infty} n \sum_{k\in J_2} B_{k,n}(x)$$
$$\le \frac{\|f\|_\infty}{\delta^4} \lim_{n\to+\infty} n \sum_{k\in J_2} \max\left(\left|\frac{k}{n+1} - x\right|^4, \left|\frac{k+1}{n+1} - x\right|^4\right) B_{k,n}(x) = 0,$$

and

$$\lim_{n\to+\infty} n \left|\sum_{k\in J_2} \left(f'(x)a_{p,k,n}(x) + \frac{f''(x)}{2} b_{p,k,n}(x)\right) B_{k,n}(x)\right|$$
$$\le \left(\frac{|f'(x)|}{\delta} + \frac{|f''(x)|}{2}\right) \lim_{n\to+\infty} n \sum_{k\in J_2} \max\left(\left|\frac{k}{n+1} - x\right|^2, \left|\frac{k+1}{n+1} - x\right|^2\right) B_{k,n}(x)$$
$$\le \left(\frac{|f'(x)|}{\delta^3} + \frac{|f''(x)|}{2\delta^2}\right) \lim_{n\to+\infty} n \sum_{k\in J_2} \max\left(\left|\frac{k}{n+1} - x\right|^4, \left|\frac{k+1}{n+1} - x\right|^4\right) B_{k,n}(x) = 0.$$

Moreover, by the property of $\widetilde{c}_{p,k,n}(x)$ and Lemma 5.6,

$$\lim_{n\to+\infty} n \sum_{k\in J_1} \widetilde{c}_{p,k,n}(x) B_{k,n}(x) = 0. \tag{5.36}$$

Therefore, taking (5.35) and (5.36) into account, by means of Lemma 4.1 with $j = 3$ and Lemma 5.5, we have

$$(Vf)(x) = \lim_{n\to+\infty} n(K_{p,n} f)(x) = \lim_{n\to+\infty} n \left(\sum_{k=0}^{n} F_{k,n} B_{k,n}(x)\right)$$
$$= \lim_{n\to+\infty} n \left(\sum_{k\in J_1} F_{k,n} B_{k,n}(x) + \sum_{k\in J_2} F_{k,n} B_{k,n}(x)\right) = \lim_{n\to+\infty} n \left(\sum_{k\in J_1} F_{k,n} B_{k,n}(x)\right)$$

$$= \lim_{n\to+\infty} n\left(\sum_{k\in J_1}\left(f'(x)a_{p,k,n}(x)+\frac{f''(x)}{2}b_{p,k,n}(x)\right)B_{k,n}(x)\right)$$

$$= \lim_{n\to+\infty} n\left(\sum_{k=0}^{n}\left(f'(x)a_{p,k,n}(x)+\frac{f''(x)}{2}b_{p,k,n}(x)\right)B_{k,n}(x)\right)=f'(x)\frac{1-2x}{2}$$

$$+ f''(x)\frac{x(1-x)}{2}.$$

The proof of 7) is concluded and this completes the entire proof of the asymptotic formula. □

6 Comparisons between $K_{p,n}$ and K_n

In this final section we establish some comparison results about the approximation of functions on a given point by means of the operators $K_{p,n}$ and K_n. We will make use of the monotonicity property of the operators $K_{p,n}$ with respect to the parameter p stated in Remarks 3.2 and consequence of Holder's inequality: for any $n \geq 1$ and $f \in L^{\infty}(0,1)$ such that $f \geq 0$,

$$K_{p,n}f \leq K_{q,n}f, \qquad 0 < p < q. \tag{6.1}$$

Changing the sign and order, an analogous monotonicity result holds for non-positive functions. This proposition implies a benefit in the use of $K_{p,n}$ rather than K_n in the pointwise approximation of a non-negative function in many situations. Before stating the results, we specify the terminology. Let $x \in [0,1]$; we say that $K_{p,n}f$ *approximates* f *at* x *better than* $K_n f$ if the approximation error by $K_{p,n}f$ is less than the one by $K_n f$, i.e.

$$|(K_{p,n}f)(x) - f(x)| < |(K_n f)(x) - f(x)|.$$

Corollary 6.1 *Let* $f \in L^{\infty}(0,1)$, $f \geq 0$, *be such that it is not constant a.e. on* [0,1] *and it is continuous at* $x \in [0,1]$. *It results that*

(i) *if* $(K_n f)(x) < f(x)$, *then* $K_{p,n}f$ *approximates* f *at* x *better than* $K_n f$, *for all* $1 < p < \overline{p}$;
(ii) *if* $(K_n f)(x) > f(x)$, *then* $K_{p,n}f$ *approximates* f *at* x *better than* $K_n f$, *for all* $\widetilde{p} < p < 1$,

where $\overline{p}$, $\widetilde{p}$ *are such that* $1 < \overline{p} \leq +\infty$, $0 \leq \widetilde{p} < 1$ *and are depending on* f *and* x. *In particular,*

(iii) *if* $f(x)$ *is the maximum of* f, *then* $K_{p,n}f$ *approximates* f *at* x *better than* $K_n f$, *for all* $n \geq 1$ *and* $1 \leq p < \infty$;
(iv) *if* $f(x)$ *is the minimum of* f, *then* $K_{p,n}f$ *approximates* f *at* x *better than* $K_n f$, *for all* $n \geq 1$ *and* $0 < p < 1$.

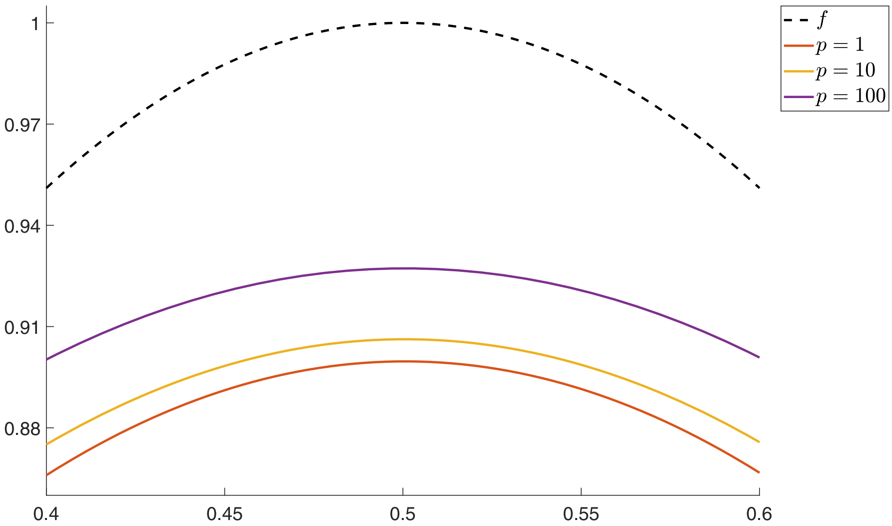

Fig. 1 Comparison between the graphs of the function f, given by $f(x) = \sin(\pi x)$, and its approximations $K_{p,n} f$ for different values of $p \geq 1$ (and $n = 10$) in a neighborhood of the maximum point $x = \frac{1}{2}$. The best approximation is a consequence of the monotonicity with respect to p

Proof Let us consider statement (i). Inequality (6.1) implies two possibilities about the function $E : [1, +\infty[\to \mathbb{R}$, $E(p) = |(K_{p,n} f)(x) - f(x)|$, i.e.

(a) there exists $p_1 > 1$ such that E is decreasing in $[1, p_1]$, $E(p_1) = 0$, and E is increasing in $[p_1, +\infty[$ or,
(b) E is decreasing in $[1, +\infty[$.

In the first case, let $\overline{p}$ be the real number greater than p_1 such that $E(\overline{p}) = |(K_n f)(x) - f(x)|$ if this number exists, otherwise let $\overline{p} = +\infty$. Then, $E(p) < E(1) = |(K_n f)(x) - f(x)|$ for all $1 < p < \overline{p}$. In the second case we define $\overline{p} = +\infty$ and obtain the same conclusion.
Statement (ii) can be proved in analogous way. The final statements are direct consequences of the previous ones and of Proposition 3.3. ◻

Figure 1 shows an application of this corollary for the function $f(x) = \sin(\pi x)$.

The following comparison result between the nonlinear and linear Kantorovich operators is instead a consequence of inequality (6.1) and the asymptotic formula (5.2).

Corollary 6.2 *Let $f \in L^\infty(0, 1)$ be a non-negative function such that f is two times differentiable at x where $0 < x < 1$ or $x = 0, 1$ and $f(x) \neq 0$. Moreover, let $p > 0$. If one of the following conditions is satisfied*

(i) $f'(x)\frac{1-2x}{2} + f''(x)\frac{x(1-x)}{2} > 0$ *and* $0 < p < 1$,
(ii) $f'(x)\frac{1-2x}{2} + f''(x)\frac{x(1-x)}{2} < 0$ *and* $p > 1$,

then $K_{p,n} f$ approximates f at x better than $K_n f$ for n large enough. In particular, if x is a relative extreme point and one of the following conditions is satisfied

(iii) f is strictly convex in a neighborhood of x and $0 < p < 1$,
(iv) f is strictly concave in a neighborhood of x and $p > 1$,

then $K_{p,n}f$ approximates f at x better than $K_n f$ for n large enough.

Remarks 6.3 Theorem 5.1 and inequality 6.1 imply also the following consideration. Let $f \in L^\infty(0,1)$ be a non-negative function such that f is two times differentiable at x, where $0 < x < 1$, or $x = 0, 1$ and $f(x) \neq 0$.

(i) If $f'(x)\frac{1-2x}{2} + f''(x)\frac{x(1-x)}{2} > 0$, then the lower p is, the better $(K_{p,n}f)(x)$ approximates $f(x)$ for n large enough.
(ii) If $f'(x)\frac{1-2x}{2} + f''(x)\frac{x(1-x)}{2} < 0$, then the higher p is, the better $(K_{p,n}f)(x)$ approximates $f(x)$ for n large enough.

Example 6.4 Let us consider the function $f : [0,1] \to \mathbb{R}$ given by $f(x) = x^\alpha$ with $\alpha > 0$ and its approximations $K_{p,n}f$ for $p > 0$ which are given by

$$(K_{p,n}f)(x) = \sum_{k=0}^{n} \frac{\left((k+1)^{\alpha p+1} - k^{\alpha p+1}\right)^{\frac{1}{p}}}{(\alpha p+1)^{\frac{1}{p}}(n+1)^\alpha} B_{k,n}(x)\,.$$

Since

$$f'(x)\frac{1-2x}{2} + f''(x)\frac{x(1-x)}{2} = \frac{\alpha x^{\alpha-1}}{2}(\alpha - (\alpha+1)x),$$

we can make the following consideration by Corollary 6.2:

(i) $K_{p,n}f$ with $0 < p < 1$, gives for n large enough, a better approximation than $K_n f$ at any point x such that $0 < x < \frac{\alpha}{\alpha+1}$;
(ii) $K_{p,n}f$ with $p > 1$, gives for n large enough, a better approximation than $K_n f$ at any point x such that $\frac{\alpha}{\alpha+1} < x < 1$.

This result can be seen in Figures 2a and 2b for the function $f(x) = x^2$. The graphs of f and some approximations $K_{p,n}f$ are plotted around the points $x = 0.655$ and $x = 0.705$. For $x = 0.655$ (which is less than $\frac{\alpha}{\alpha+1} = \frac{2}{3}$), the operator $K_{p,n}$ with $p = 0.01, 0.1, 0.5 < 1$ approximates $f(x)$ better than K_n (Figure 2a), while for $x = 0.705$ (which is greater than $\frac{\alpha}{\alpha+1} = \frac{2}{3}$), the operator $K_{p,n}$ with $p = 2, 5, 10 > 1$ approximates $f(x)$ better than K_n (Figure 2b). More generally, in Figure 2a the best approximation is given by the lowest value of p, while in Figure 2b the best approximation is given by the highest value of p, as observed by Remark 6.3.

Finally, concerning an interval endpoint and a function vanishing on it, with derivative not null on it, we find that the best values of p are those less than 1. This is stated in the next result which is a consequence of Theorem 5.1 and the fact that the function

$$p \mapsto \frac{1}{(p+1)^{\frac{1}{p}}}$$

is increasing. It gives additional interest to the case $0 < p < 1$, that curiously is the case of L^p-spaces which are not normed.

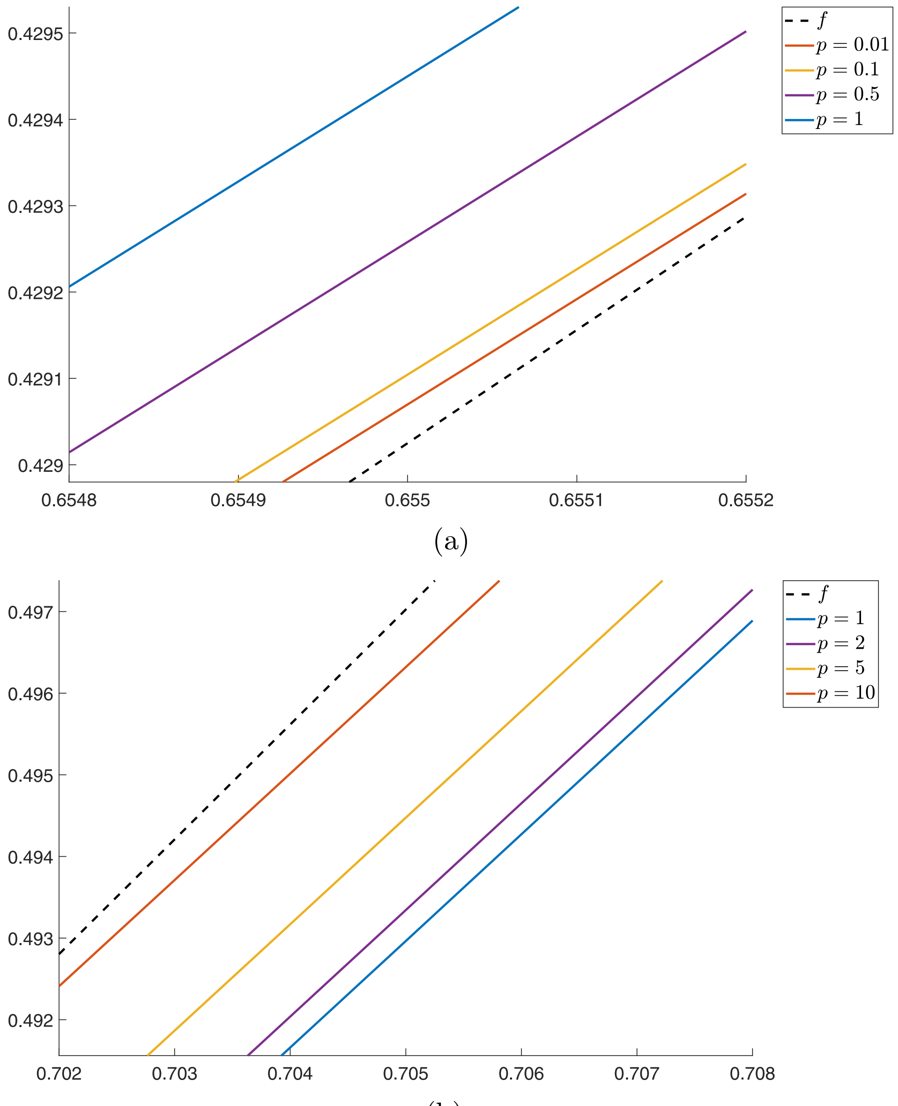

Fig. 2 Comparison between the graphs of the function f, given by $f(x) = x^2$, and its approximations $K_{p,n}f$ for different values of p (and $n = 20$) in neighborhoods of $x = 0.655$ and $x = 0.705$, which are on opposite sides with respect to $\frac{\alpha}{\alpha+1} = \frac{2}{3}$

Corollary 6.5 *Let $f \in L^\infty(0,1)$ be two times differentiable at $x = 0$ or 1 and such that $f(x) = 0$ and $f'(x) \neq 0$. Then, for any $0 < p < 1$, $K_{p,n}f$ approximates f at x better than $K_n f$ for n large enough.*

As an example of application, we consider the function f defined by $f(x) = -x + 2x^2$, for $x \in [0,1]$. It satisfies the assumptions of Corollary 6.5 and Figure 3

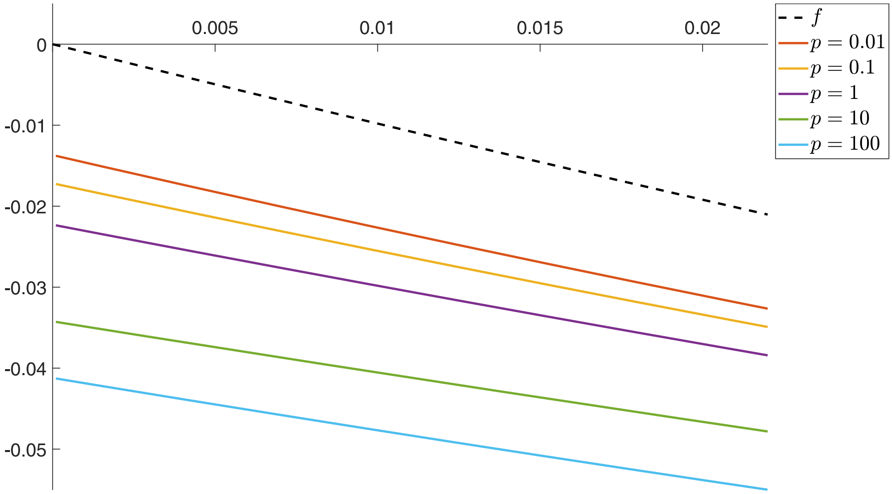

Fig. 3 Comparison between the graphs of the function f, given by $f(x) = -x+2x^2$, and its approximations $K_{p,n}f$ for different values of p (and $n = 20$) in a neighborhood of $x = 0$. The best approximation is a consequence of the asymptotic formula at $x = 0$

shows that the values of p less than 1 give approximations $(K_{p,n}f)(0)$ of $f(0) = 0$ better than $(K_n f)(0)$. Note that for this example Corollary 6.1 cannot be applied since, first of all, f is not non-negative (actually, neither non-positive) and moreover f at $x = 0$ has a local but not global maximum.

7 Conclusion

In this paper we studied operators of Kantorovich-type on [0, 1], denoted by $K_{p,n}$ involving new kind of averages (the p-averages) in their definitions. In other words, these operators provide approximations in terms of L^p-norms (or quasi-norms) of the functions on subintervals. We did not confine ourselves to the case $p \geq 1$, but we actually considered $p > 0$, noting the universality of many results (like Theorems 4.2 and 5.1) and the duality of others (like Corollary 6.2) between the cases $0 < p < 1$ and $p > 1$. Furthermore, we remark that the p-averages could be considered in generalizing other classes of approximation processes, acting on spaces of functions on unbounded intervals or on the real line.

As a crucial property, the operators under investigation in the present paper are nonlinear for $p \neq 1$ and consequently their study presents more challenges. Among the results in this paper, the main one is an asymptotic formula covering the complete setting and, to the best of our knowledge, it is the first one in its kind provided for nonlinear operators in Approximation Theory. Finally, we would like to mention that there are other topics to examine concerning the operators $K_{p,n}$ (for instance, their rate of convergence) and this will be the subject of a forthcoming paper.

Acknowledgements All authors are members of the "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni" (GNAMPA-INdAM), of the UMI-TAA group "Teoria dell'Approssimazione e Applicazioni" and of the "Research ITalian network on Approximation" (RITA) network.

Author Contributions M.C.M.: Conceptualization, Investigation, Methodology, Writing – original draft, review and editing. R.C.: Conceptualization, Investigation, Methodology, Software, Writing – original draft, review and editing V.L.: Conceptualization, Investigation, Methodology, Writing – original draft, review and editing.

Funding Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement. No funding was received for conducting this study.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

References

1. Acu, A.-M., Raşa, I., Şteopoaie, A.E.: Bernstein-Kantorovich operators, approximation and shape preserving properties. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. **118**(3), 107 (2024)
2. Adell, J.A., Bustamante, J., Quesada, J.M.: Estimates for the moments of Bernstein polynomials. J. Math. Anal. Appl. **432**, 114–128 (2015)
3. Altomare, F., Cappelletti Montano, M., Leonessa, V., Raşa, I.: A generalization of Kantorovich operators for convex compact subsets. Banach J. Math. Anal. **11**(3), 591–614 (2017)
4. Angeloni, L., Vici, F.: Approximation methods in BV for nonlinear generalized sampling. Result Math **80**, 99 (2025)
5. Artamonov, S., Runovski, K.V., Schmeisser, H.J.: Approximation by families of generalized sampling series, realizations of generalized K-functionals and generalized moduli of smoothness. J. Math. Anal. Appl. **489**(1), 124138 (2020)
6. Asdrubali, F., Baldinelli, G., Bianchi, F., Costarelli, D., Rotili, A., Seracini, M., Vinti, G.: Detection of thermal bridges from thermographic images by means of image processing approximation algorithms. Appl. Math. Comput. **317**, 160–171 (2018)
7. Bardaro, C., Mantellini, I.: A Voronovskaya-type theorem for a general class of discrete operators. Rocky Mountain J. Math. **39**(5), 1411–1442 (2009)
8. Bede, B., Nobuhara, H., Daňkovác, M., Di Nola, A.: Approximation by pseudo-linear operators. Fuzzy Sets Syst. **159**, 804–820 (2008)
9. Boccali, L., Costarelli, D., Vinti, G.: Convergence results in Orlicz spaces for sequences of max-product sampling Kantorovich operators. J. Comput. Appl. Math. **449**, 115957 (2024)
10. Butzer, P.L., Splettstößer, W., Stens, R.L.: The sampling theorem and linear prediction in signal analysis. Jahresber. Dtsch. Math.-Ver. **90**, 1–70 (1988)
11. Cappelletti Montano, M., Leonessa, V.: A sequence of Kantorovich-type operators on mobile intervals. Constr. Math. Anal. **2**(3), 130–143 (2019)

12. Conway, E.D.: Generalized solutions of linear differential equations with discontinuous coefficients and the uniqueness question for multidimensional quasilinear conservation laws. J. Math. Anal. Appl. **18**(2), 238–251 (1967)
13. Coroianu, L., Gal, S.G.: L^p-approximation by truncated max-product sampling operators of Kantorovich-type based on Fejér kernel. J. Integral Equ. Appl. **29**(2), 349–364 (2017)
14. Costarelli, D., Natale, M.: Advancements in nonlinear exponential sampling: convergence, quantitative analysis and Voronovskaya-type formula. Commun. Nonlinear Sci. Numer. Simulat. **152**, 109202 (2026)
15. Costarelli, D., Natale, M., Vinti, G.: Estimations for the convex modular of the aliasing error of nonlinear sampling Kantorovich operators. Nonlinear Anal. Model. Control. **30**(2), 270–290 (2025)
16. Cybenko, G.: Approximation by superpositions of a sigmoidal function. Math. Control Signals Systems **2**(4), 303–314 (1989)
17. de la Cal, J., Valle, A.M.: A generalization of Bernstein-Kantorovich operators. J. Math. Anal. Appl. **252**(2), 750–766 (2000)
18. Derrienic, M.M.: Sur l'approximation des fonctions integrables sur [0, 1] par des polynomes de Bernstein modifies. J. Approx. Theory **31**, 325–343 (1981)
19. Ditzian, Z., Hristov, V.H., Ivanov, K.G.: Moduli of smoothness and K-functionals in L_p, $0<p<1$. Constr. Approx. **11**, 67–83 (1995)
20. Durrmeyer, J.-L.: Une formule d'inversion de la transformée de Laplace: Applications à la théorie des moments. Thèse de 3e cycle, Faculté des Sciences de l'Université de Paris, 1967
21. Farin, G.: Curves and Surfaces for CAGD, a Practical Guide, 5th edn. Academic Press, San Diego USA (2002)
22. Gökçer, T.Y., Aslan, I.: Approximation by Kantorovich-type max-min operators and its applications. App. Math. Comp. **423**, 127011 (2022)
23. Kantorovich, L.V.: Sur certains développements suivant les polynômes de la forme de S. Bernstein, I, II. C. R. Acad. URSS, 563–568 and 595–600 (1930)
24. Karsli, H.: Voronovskaya-type theorems for Urysohn type nonlinear Bernstein operators. Math. Met. Appl. Sci. **42**(16), 5190–5198 (2018)
25. Kolomoitsev, Y., Tikhonov, S.: Properties of moduli of smoothness in $L_p(\mathbb{R}^d)$. J. Approx. Theory **257**, 105423 (2020)
26. Lorentz, G.G.: Bernstein Polynomials, 2nd edn. AMS Chelsea Publishing, New York (1986)
27. Lyu, Q., Lin, Z., She, Y., Zhang, C.: A comparison of typical ℓ_p minimization algorithms. Neurocomputing **119**, 413–424 (2013)
28. Musielak, J.: Nonlinear approximation in some modular function spaces. I. Math. Japonica **38**, 83–90 (1993)
29. Nagel, J.: Asymptotic properties of powers of Kantorovic operators. J. Approx. Theory **36**(3), 268–275 (1982)
30. Nagel, J.: Kantorovich operators of second order. Monatsh. Math. **95**, 33–44 (1983)
31. Păltănea, R.: Sur un opérateur polynomial defini sur l'ensemble des fonctions intégrables. "Babeş-Bolyai" Univ. Fac. Math. Res. Semin. **2**, 101–106 (1983)
32. Vinti, G., Zampogni, L.: A general method to study the convergence of nonlinear operators in Orlicz spaces. Adv. Nonlinear Stud. **22**, 594–618 (2022)
33. Wiegmann, A., Bube, K.P.: The immersed interface method for nonlinear differential equations with discontinuous coefficients and singular sources. SIAM J. Numer. Anal. **35**(1), 177–200 (1998)