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Harmonic Morphisms from Fefferman Spaces

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Abstract
We study a ramification of a phenomenon discovered by Baird and Eells (in: Looijenga et al (eds) Geometry Symposium Utrecht 1980. Lecture Notes in Mathematics, Springer, Berlin, 1981) i.e. that non-constant harmonic morphisms $\Phi : \mathfrak{M}^{\rm N} \to N^2$ from a N-dimensional (N $\geq$ 3) Riemannian manifold $\mathfrak{M}^{\rm N}$, into a Riemann surface N^2, can be characterized as those horizontally weakly conformal maps having minimal fibres. We recover Baird–Eells' result for S^1 invariant harmonic morphisms $\Phi : \mathfrak{M}^{2n+2} \to N^2$ from a class of Lorentzian manifolds arising as total spaces $\mathfrak{M} = C(M)$ of canonical circle bundles $S^1 \to \mathfrak{M} \to M$ over strictly pseudoconvex CR manifolds M^{2n+1}. The corresponding base maps $\phi : M^{2n+1} \to N^2$ are shown to satisfy $\lim_{\epsilon \to 0^+} \pi_{\mathscr{H}^\phi} \mu_\epsilon^{\mathscr{V}^\phi} = 0$, where $\mu_\epsilon^{\mathscr{V}^\phi}$ is the mean curvature vector of the vertical distribution $\mathscr{V}^\phi = \mathrm{Ker}(d\phi)$ on the Riemannian manifold (M, g_ϵ), and $\{g_\epsilon\}_{0<\epsilon<1}$ is a family of contractions of the Levi form of the pseudohermitian manifold (M, θ).

Keywords Harmonic morphism · Fefferman metric · Cauchy–Riemann manifold · Contraction of Levi form · Subelliptic harmonic morphism · Mean curvature

Mathematics Subject Classification 32V05 · 53C12 · 53C17 · 53C43 · 53C50 · 58E20

1 Harmonic Morphisms in Riemannian Geometry Versus Lorentzian Geometry

A harmonic morphism is a continuous mapping $\Phi : \mathfrak{M}^{\rm N} \to N^m$ of semi-Riemannian manifolds $(\mathfrak{M}, g)$ and (N, h) such that for every solution $v : V \to \mathbb{R}$ to $\Delta_h v =$

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0, defined on an open subset $V \subset N$, the pullback $u = v \circ \Phi$ is a distribution-solution to $\Delta_g u = 0$ on $U = \Phi^{-1}(V)$, where Δ_g and Δ_h are the Laplace–Beltrami operators of the given semi-Riemannian manifolds. Considerable attention has been given to the study of harmonic morphisms within the Riemannian category, and the main results up to 2003 were reported on in the wonderful monograph by Baird and Wood [5]. By a result of Fuglede (cf. [34]) every non-constant harmonic morphism of Riemannian manifolds is an open map. Remarkably, Fuglede's result relies on the Harnack inequality for positive harmonic functions on a Riemannian manifold (cf. e.g. Serrin [59]) thus establishing a solid bond between the geometry of harmonic morphisms and elliptic theory (cf. e.g. [39]). However, harmonic morphisms $\Phi : \mathfrak{M}^2 \to N$, from a Lorentzian surface $\mathfrak{M}$ into a semi-Riemannian manifold, that aren't open maps, do exist (cf. [5, pp. 446–448]). For morphisms from Fefferman spaces we may state (leaving definitions momentarily aside)

Theorem 1 *Let M be a strictly pseudoconvex CR manifold, equipped with the positively oriented contact form $\theta \in \mathscr{P}_+(M)$, and let N be a Riemannian manifold. Any nonconstant S^1 invariant harmonic morphism $\Phi : C(M) \to N$ from the total space of the canonical circle bundle $S^1 \to C(M) \to M$, endowed with the Lorentzian metric F_θ [the Fefferman metric of (M, θ)], is an open map. Moreover, if M is compact and N is connected then N is compact and Φ is surjective.*

As another contrasting feature of the semi-Riemannian case, harmonic morphisms of semi-Riemannian manifolds may be non smooth. Indeed the proof that a continuous harmonic morphism $\Phi : \mathfrak{M} \to N$ of Riemannian manifolds is actually C^∞ relies (cf. [5, p. 111]) on two ingredients i.e. (i) the existence of harmonic local coordinate systems on the target manifold N, and (ii) the hypoellipticity of the Laplace–Beltrami operator Δ_g of $(\mathfrak{M}, g)$, itself following from the ellipticity of Δ_g. The known proof of the existence of harmonic local coordinates is tied (cf. DeTurck and Kazdan [22]) to the ellipticity of Δ_h, although harmonic local coordinate systems on Lorentzian manifolds were used in spacetime physics as early as the work by Lanczos (cf. [47]) and Einstein himself (cf. [30]), yet without questioning their existence. Moreover, if say $(\mathfrak{M}, g)$ is a Lorentzian manifold, then its Laplace–Beltrami operator is the geometric wave operator $\Box_g$ which is not hypoelliptic. For morphisms from Fefferman spaces we may state

Theorem 2 *Any continuous S^1 invariant harmonic morphism $\Phi : C(M) \to N$ from the Lorentzian manifold $(C(M), F_\theta)$ into the Riemannian manifold N is smooth.*

Unique continuation (cf. [5, pp. 111–112]) doesn't hold for harmonic maps and morphisms of semi-Riemannian manifolds (cf. [5, p. 448]). It should also be mentioned that J.H. Sampson's unique continuation theorem for harmonic maps of Riemannian manifolds (cf. Theorem 1 in [58, p. 213]) relies on a unique continuation result for solutions to elliptic equations due to N. Aronszajn (cf. [2]) whose proof is believed to be wrong, cf. Appendix A in [25, pp. 433–434], (although the very result in [2] may hold true, at least for solutions to $\Delta_g u = 0$). Apart from a brief conjectural discussion in § 6, unique continuation of subelliptic harmonic maps and morphisms will be addressed in further work.

By a celebrated result of Baird and Eells (cf. [3]) a smooth non-constant horizontally weakly conformal map $\Phi : \mathfrak{M}^{\mathrm{N}} \to N^m$ of Riemannian manifolds is a harmonic morphism if and only if

$$(m-2)\,\mathscr{H}\big\{\nabla \log \lambda(\Phi)\big\} + (N-m)\,\Phi_*\,\mu^{\mathscr{V}} = 0, \tag{1}$$

where $\lambda(\Phi)$ and $\mu^{\mathscr{V}}$ are respectively the dilation of Φ and the mean curvature vector of its fibres. Cf. also [16] for the case $m = \mathrm{N}-1$. In particular, if the target manifold is a real surface ($m = 2$) then harmonic morphisms $\Phi : \mathfrak{M} \to N^2$ have minimal fibres. The case where $\mathfrak{M}$ is Lorentzian has not been studied, and a Lorentzian analog to the fundamental equation

$$\tau_g(\Phi) = -(m-2)\,\Phi_*\,\big\{\nabla \log \lambda(\Phi)\big\} - (N-m)\,\Phi_*\,\mu^{\mathscr{V}},$$

(cf. (4.5.2) in [5, p. 120]) and to the characterization (1) are not known, so far.

Given a Riemannian manifold N^m, the purpose of the present paper is to analyze harmonic morphisms $\Phi : \mathfrak{M}^{2n+2} \to N^m$ from the total space $\mathfrak{M} = C(M)$ of the canonical circle bundle $S^1 \to \mathfrak{M} \to M^{2n+1}$ over a strictly pseudoconvex CR manifold M, of CR dimension n. Here M is equipped with a fixed positively oriented contact form θ, so that $\mathfrak{M}$ is a Lorentzian manifold with the corresponding Fefferman metric $g = F_\theta$. The discussion is confined to S^1 invariant harmonic morphisms Φ of $(\mathfrak{M},\ F_\theta)$ into $(N^m,\ h)$, whose associated base maps $\phi : M \to N$ turn out to be subelliptic harmonic morphisms, in the sense of Dragomir and Lanconelli [25]. Our result in this direction is

Theorem 3 *Let M be a strictly pseudoconvex CR manifold, of CR dimension n, equipped with the positively oriented contact form $\theta \in \mathcal{P}_+(M)$, and let $(N,\ h)$ be a m-dimensional Riemannian manifold. Let $\Phi : C(M) \to N$ be a continuous S^1 invariant map, and let $\phi : M \to N$ be the corresponding base map. The following statements are equivalent*

(i) *Φ is a harmonic morphism of the Lorentzian manifold $(C(M),\ F_\theta)$ into $(N,\ h)$, of square dilation $\Lambda(\phi)\circ\pi$.*

(ii) *ϕ is a subelliptic harmonic morphism of the pseudohermitian manifold $(M,\ \theta)$ into $(N,\ h)$, of θ-dilation $\sqrt{\Lambda(\phi)}$.*

If this is the case then

(a) *Φ is nondegenerate at $p \Longleftrightarrow \pi(p) \in \Omega(\phi) := M\setminus Z\big[\Lambda(\phi)\big]$.*

(b) *$p \in \mathrm{Crit}(\Phi) \Longleftrightarrow \pi(p) \in \mathrm{Crit}(\phi)$.*

(c) *Φ is degenerate at $p \Longleftrightarrow$ either $m = 1$ and $\pi(p) \in \mathrm{II}_1(\phi)$, or $m \geq 2$ and $\pi(p) \in M\setminus S(\phi)$.*

(d) *Φ is a harmonic map of the Lorentzian manifold $(C(M),\ F_\theta)$ into the Riemannian manifold $(N,\ h)$, while ϕ is a subelliptic harmonic map of the pseudohermitian manifold (M, θ) into $(N,\ h)$.*

(e) *Φ is horizontally weakly conformal, while ϕ is Levi conformal.*

(f) *If $m = 2$ i.e. $(N,\ h)$ is a real surface, then every leaf of the pullback foliation $\pi^*\mathscr{F}$ of $S(\Phi)$ [the foliation of $S(\Phi)$ tangent to $\mathscr{V}^\Phi$] is a minimal submanifold of $\big(C(M),\ F_\theta\big)$.*

The equivalence (i) $\Longleftrightarrow$ (ii) in Theorem 3 was first observed by Barletta (cf. [6]) for the particular case of the Heisenberg group $M = \mathbb{H}_n$. The more general case at hand is treated in Section § 4 of the present paper.

Let $\mu^{\mathscr{V}^\phi}\big(g_\theta\,,\,\nabla\big) \in C^\infty\big(\mathscr{H}^\phi\big)$ be defined by formally replacing the Levi-Civita connection ∇^{g_θ} (of the Webster metric g_θ) by the Tanaka–Webster connection ∇ (of the pseudohermitian manifold $(M,\,\theta)$) in the ordinary mean curvature vector $\mu^{\mathscr{V}^\phi} \equiv \mu^{\mathscr{V}^\phi}\big(g_\theta\,,\,\nabla^{g_\theta}\big) \in C^\infty\big(\mathscr{H}^\phi\big)$ of the vertical distribution $\mathscr{V}^\phi = \operatorname{Ker}(d\phi)$ thought of as a distribution on the Riemannian manifold $(M,\ g_\theta)$. Let $\Phi = \phi \circ \pi$ be the vertical lift of ϕ to the total space $C(M)$ of the canonical circle bundle over M, equipped with the Fefferman metric F_θ. To some surprise, while the tension field $\tau_{F_\theta}(\Phi)$ projects on the pseudohermitian tension field $\tau_b(\phi)$, the square dilation $\ell(\Phi)$ is the vertical lift of the square dilation $\Lambda(\phi)$, and gradients with respect to F_θ project on horizontal gradients on $(M,\,\theta)$, the term $\Phi_*\ \mu^{\mathscr{V}^\Phi}$ (appearing in the fundamental equation (50)) doesn't project on $\phi_*\ \mu^{\mathscr{V}^\phi}\big(g_\theta\,,\,\nabla\big)$, as one might have hoped for, to start with. In a quest for the "correct" pseudohermitian analog to the mean curvature vector of $\mathscr{V}^\phi$, we endow M with the family

$$g_\epsilon = g_\theta + \Big(\frac{1}{\epsilon^2} - 1\Big)\,\theta \otimes \theta, \quad 0 < \epsilon < 1, \tag{2}$$

of contractions (in the sense of Strichartz [60]) of the Levi form G_θ, and analyze the behavior of $\mu_\epsilon^{\mathscr{V}^\phi}$ [the mean curvature vector of $\mathscr{V}^\phi$ as a distribution on the Riemannian manifold $(M,\,g_\epsilon)$] in the limit as $\epsilon \to 0^+$. The family of Riemannian metrics $\{g_\epsilon\}_{0<\epsilon<1}$ is devised such that $\big(M,\,d_\epsilon\big) \to \big(M,\,d_H\big)$ as $\epsilon \to 0^+$, in the Gromov–Hausdorff distance. Here d_ϵ and d_H are respectively the distance function of the Riemannian manifold $\big(M,\ g_\epsilon\big)$, and the Carnot–Carathéodory distance function associated to the sub-Riemannian structure $\big(H(M),\,G_\theta\big)$ (the maximally complex distribution of the CR manifold M, equipped with the Levi form, cf. [29, 60]). A comparison to the works by Barone-Adesi et al. [13], Cheng et al. [20], Malchiodi et al. [19], Danielli et al. [21], Garofalo et al. [37] and Pauls et al. [38], may reveal $\mu_{\rm hor}^{\mathscr{V}^\phi} := \lim_{\epsilon\to 0^+} \mu_\epsilon^{\mathscr{V}^\phi}$ as the appropriate candidate for the mean curvature vector [of a leaf of the foliation $\mathscr{F}$ tangent to $\mathscr{V}^\phi$, as a submanifold of the pseudohermitian manifold $(M,\,\theta)$]. For the time being, we establish (in the spirit of the work by Ni [53])

Theorem 4 *Let $\phi : M^{2n+1} \to N^2$ be a non-constant subelliptic harmonic morphism, of the pseudohermitian manifold $(M,\,\theta)$ into the real surface $(N,\,h)$. Let $\mu_\epsilon^{\mathscr{V}^\phi}$ be the mean curvature vector of $\mathscr{V}^\phi$, as a distribution on the Riemannian manifold $(M,\,g_\epsilon)$. Then $\pi_{\mathscr{H}^\phi}\ \mu_\epsilon^{\mathscr{V}^\phi} \to 0$ as $\epsilon \to 0^+$, uniformly on any relatively compact domain $U \subset M$.*

We revisit the notion of *horizontal mean curvature* of a real hypersurface in a Carnot group (cf. Capogna et al. [18], Capogna and Citti [17]) in the context of subelliptic harmonic morphisms $\phi : M^{2n+1} \to N^1$ from a pseudohermitian manifold $(M,\,\theta)$ into a $m = 1$ dimensional Riemannian manifold N^1. We compute the horizontal mean curvature of every leaf of the foliation $\mathscr{F}$ by real hypersurfaces of $S(\phi)$ [where, by Theorem 6, $S(\phi) = M\backslash\mathrm{Crit}(\phi)$, an open set] determined by the submersion $\phi : S(\phi) \to N^1$. Precisely, let $\{g_\epsilon\}_{0<\epsilon<1}$ be the family of contractions of G_θ given by (2), and let $\mathscr{H}_\epsilon^\phi$ be the g_ϵ-orthogonal complement of $\mathscr{V}^\phi$ in $(T(M),\, g_\epsilon)$. Let $\mathbf{n}^\epsilon \in C^\infty\big(\mathscr{H}_\epsilon^\phi\big)$ such that $g_\epsilon\big(\mathbf{n}^\epsilon\,,\ \mathbf{n}^\epsilon\big) = 1$. The horizontal normal $\mathbf{n}^0$ is

$$\mathbf{n}^0 = \frac{1}{g_\epsilon\big(\nu^\epsilon\,,\ \nu^\epsilon\big)^{1/2}}\,\nu^\epsilon\,, \qquad \nu^\epsilon \equiv \Pi_H\,\mathbf{n}^\epsilon = \mathbf{n}_\epsilon - \theta\big(\mathbf{n}_\epsilon\big)\,T,$$

and the horizontal mean curvature K_0 of the leaves of $\mathscr{F}$ is

$$K_0 = \mathrm{div}\big(\mathbf{n}^0\big) \in C^\infty(\Omega),$$

where $\Omega = M \setminus Z(\Lambda)$ (an open set) and the divergence is computed with respect to the volume form $\Psi = \theta \wedge (d\theta)^n$. The horizontal normal and mean curvature are well defined on Ω because $\Sigma(\mathscr{F}) \subset Z(\Lambda)$ [by Theorem 6 below, and our discussion in § 7] where $\Sigma(\mathscr{F})$ is the set of all characteristic points of the leaves of $\mathscr{F}$.

Theorem 5 *Let $\phi : M \to N^1$ be a subelliptic harmonic morphism, of square dilation Λ. Then*

(i) *For every local coordinate system $(V,\ y^1)$ on N such that $U = \phi^{-1}(V) \subset \Omega$*

$$\mathbf{n}^0 = \frac{1}{\sqrt{\Lambda_0}}\,\nabla^H\phi^1\,, \quad \Lambda_0 = \frac{\Lambda}{h_{11}\circ\phi}\,, \quad \phi^1 = y^1\circ\phi,$$

so that

$$K_0 = \mathrm{div}\Big(\frac{1}{\sqrt{\Lambda_0}}\,\nabla^H\phi^1\Big)$$
$$= -\frac{1}{\sqrt{\Lambda_0}}\Big\{\Delta_b\phi^1 + \big(\nabla^H\phi^1\big)\,\log\sqrt{\Lambda_0}\Big\}, \tag{3}$$

everywhere in U.

(ii) *The vector field*

$$\mu_{\mathrm{hor}}^{\mathscr{V}\phi} \equiv \mu^{\mathscr{V}\phi}\big(g_\theta\,,\ \nabla\big) + \frac{1}{2n}\,\pi_{\mathscr{H}\phi}\Big\{\frac{2}{\theta(\mathscr{T})}\,J\mathscr{T} - \nabla_{\mathscr{T}}\mathscr{T}\Big\} \in C^\infty\big(\mathscr{H}^\phi\big),$$

and the mean curvature K_0 are related by

$$2n\,g_\theta\big(\mu_{\mathrm{hor}}^{\mathscr{V}\phi}\,,\ \mathbf{n}^0\big) = \big\{\varphi\,T(\phi^1) - 1\big\}\,K_0\,,$$
$$\varphi^2\big\{\Lambda_0 + T(\phi^1)^2\big\} = 1 - \theta(\mathscr{T})^2\,, \quad \mathscr{T} = \|T^{\mathscr{V}}\|^{-1}\,T^{\mathscr{V}}. \tag{4}$$

Consequently

$$2n\,\mu_{\mathrm{hor}}^{\mathscr{V}\phi} = \alpha\,\nabla\phi^1\,, \quad \alpha := -\frac{\Delta_b\phi^1 + \sqrt{\Lambda_0}\,K_0}{\Lambda_0 + T(\phi^1)^2}. \tag{5}$$

In particular, for any local harmonic coordinate system $(V,\ y^1)$ *on* N [*i.e.* $\Delta_h y^1 = 0$ *in* V] *with* $U = \phi^{-1}(V) \subset \Omega$

$$2n\, \Pi_H\, \mu^{\mathscr{V}\phi}_{\rm hor} = -\frac{\Lambda_0}{\Lambda_0 + T(\phi^1)^2}\, K_0\, \mathbf{n}^0, \tag{6}$$

everywhere in U.

By a result in [5, p. 448], to every non-constant harmonic morphism $\Phi : C(M) \to N$ of class C^∞, there corresponds a symbol $\sigma_p(\Phi) : \big(T_p(C(M)),\ F_{\theta,\,p}\big) \to \big(T_{\Phi(p)}(M),\ h_{\Phi(p)}\big)$ which is a harmonic morphism (this may fail if $\Phi \in C^\ell$ for some $2 \le \ell < \infty$ yet $\Phi \notin C^{\ell+1}$). The CR structure on M induces a natural strictly pseudoconvex CR structure on the tangent space $T_x(M)$ at every point $x = \pi(p)$, yet the properties of the symbol $\sigma_x(\phi) : T_x(M) \to T_{\phi(x)}(N)$ of a subelliptic harmonic morphism are not known, so far.

The paper is organized as follows. Section 2 recalls the essentials of CR and pseudohermitian geometry (by following [29]) and of subelliptic harmonic maps $\phi : M \to N$, from a pseudohermitian manifold $(M,\ \theta)$ into a Riemannian manifold $(N,\ h)$ (cf. [9, 44]). The needed material on subelliptic harmonic morphisms is reviewed in Sect. 3 (cf. [7, 25]). Theorems 1, 2 and 3 are proved in Sect. 5. The Lorentzian and pseudohermitian ramifications of the result by Baird and Eells (cf. [3]) are treated in Sect. 3, where we also prove Theorem 4. In Sect. 7 we prove Theorem 5. Sect. 8 exhibits a few examples i.e. subellitic harmonic morphisms from the Heisenberg group and from Rossi spheres. The study of the properties of the symbol of a subelliptic harmonic morphism $\phi : M \to N$ is relegated to a further paper.

Given a 3-dimensional nondegenerate CR manifold M^3, let $\Phi : \mathfrak{M}^4 \to N^2$ be a harmonic morphism from a 4-manifold $\mathfrak{M}^4$ equipped with the Lorentizan metric g, into the Riemann surface N^2, such that the vertical spaces $\mathscr{V}^\Phi_p = \operatorname{Ker}\big(d_p\Phi\big)$ are nondegenerate for every $p \in \mathfrak{M}^4$, and let $\mathscr{H}^\Phi$ be the g-orthogonal complement of $\mathscr{V}^\Phi$. By a result of J. Ventura (cf. [62]) the Ricci curvature of $\big(\mathfrak{M}^4,\ g\big)$ may be computed in terms of i) the (square) dilation of Φ, ii) the first fundamental forms of $\mathscr{H}^\Phi$ and $\mathscr{V}^\Phi$, iii) the second fundamental forms of $\mathscr{H}^\Phi$ and $\mathscr{V}^\Phi$ and their adjoints, iv) the sectional curvature of the fibers of Φ, v) the mean curvature of $\mathscr{H}^\Phi$, vi) the mean curvature of the fibers of Φ, and vii) the integrability 1-form of $\mathscr{H}^\Phi$. Let $g^{\mathscr{V}}$ and $g^{\mathscr{H}}$ be the bundle metrics induced on $\mathscr{V}^\Phi$ and $\mathscr{H}^\Phi$, respectively. Given C^∞ functions $\sigma,\ \rho : \mathfrak{M}^4 \to (0,\ +\infty)$ the Lorentzian metric

$$\tilde{g} := \frac{1}{\sigma^2}\, g^{\mathscr{H}} + \frac{1}{\rho^2}\, g^{\mathscr{V}}, \tag{7}$$

is a *biconformal deformation* of g. The Einstein equation

$$\operatorname{Ric}_{\tilde{g}} = \Lambda\, \tilde{g}, \tag{8}$$

recasts as a PDE system in the unknown functions σ and ρ, and solving (8) for σ and ρ amounts to producing solutions (Einstein metrics) $\tilde{g}$ by biconformal deformations of an *a priori* given Lorentzian metric g. The approach was devised by Ventura (cf. [62]) and the method was applied to a number of space-times and morphisms $\Phi : \mathfrak{M}^4 \to N^2$ e.g. for the Schwarzschild metric g and a projection Φ from the Schwarzschild space-time onto $N^2 = S^2$. Cf. also Baird and Ventura [4], where the approach is however confined to the case of a Riemannian 4-manifold $\mathfrak{M}^4$. Given a positively oriented contact form θ on M^3, the Fefferman metric $g = F_\theta$ is never Einstein (cf. Lee [49]). The curvature calculations in [62] and the resulting attempt to solve (8) is then liable to produce Einstein metrics on the total space $\mathfrak{M}^4 = C(M^3)$ of the canonical circle bundle over M, by a biconformal deformation [associated to a given harmonic morphism Φ from $C(M^3)$ into a Riemann surface] of the Fefferman metric.

A similar problem was solved (outside harmonic morphisms theory) by Leitner (cf. [50]) who built pseudo-Einstein (cf. [29]) contact forms θ of vanishing pseudohermitian torsion, and observed that the corresponding Fefferman metric is conformally Einstein i.e. there is a C^∞ function $\sigma : C(M^3) \to (0, +\infty)$ such that $\tilde{g} = \left(1/\sigma^2\right) F_\theta$ is an Einstein metric [cf. (7) with $\sigma \equiv \rho$, as $F^\theta = F_\theta^{\mathscr{H}} + F_\theta^{\mathscr{V}}$ for any harmonic morphism Φ with a nondegenerate vertical distribution]. The results in [50, 62] were not paralleled so far. It is an open problem, suggested by the Reviewer, to build examples of Einstein metrics on $C(M^3)$ by a biconformal deformation of the Fefferman metric F_θ, associated to a contact form θ that is neither pseudo-Einstein nor transversally symmetric (and compensating said obstructions by an appropriate choice of harmonic morphism Φ).

2 Subelliptic Harmonic Maps

For notations, conventions and basic results in CR and pseudohermitian geometry, we follow the monograph by Dragomir and Tomassini [29]. Let M be a strictly pseudoconvex CR manifold, of CR dimension n, equipped with a positively oriented contact form θ, and let N be a m-dimensional Riemannian manifold, with the Riemannian metric h. Let $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ be the CR structure on M and let $H(M) = \mathrm{Re}\left\{T_{1,0}(M) \oplus T_{0,1}(M)\right\}$ be the corresponding maximally complex, or Levi, distribution. Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$ and overbars denote complex conjugates. Let $J : H(M) \to H(M)$ be the natural complex structure i.e.

$$J\left(Z + \overline{Z}\right) = \sqrt{-1}\left(Z - \overline{Z}\right), \quad Z \in T_{1,0}(M).$$

Let $H(M)^\perp \subset T^*(M)$ be the conormal bundle i.e. the real line bundle

$$H(M)_x^\perp = \left\{\omega \in T_x^*(M) \ : \ \mathrm{Ker}(\omega) \supseteq H(M)_x\right\}, \quad x \in M.$$

As is well known (cf. e.g. [29, pp. 8–9]) under the mild assumption that M is orientable, the conormal bundle is trivial i.e. $H(M)^\perp \simeq M \times \mathbb{R}$ (a vector bundle isomorphism). The set of all globally defined nowhere zero C^∞ sections in $H(M)^\perp$ is denoted by $\mathscr{P}(M)$. For every $\theta \in \mathscr{P}(M)$ let G_θ be the Levi form i.e.

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M).$$

Cf. [29, pp. 5–7]. Let $\mathscr{P}_+(M)$ denote the set of all $\theta \in \mathscr{P}(M)$ such that G_θ is positive definite. By its very definition, strict pseudoconvexity of the given CR structure $T_{1,0}(M)$ is equivalent to $\mathscr{P}_+(M) \neq \emptyset$. A contact form $\theta \in \mathscr{P}_+(M)$ is termed *positively oriented*. Let us consider the functional $E_b : C^\infty(M, N) \to \mathbb{R}$ given by

$$E_b(\phi) = \frac{1}{2} \int_\Omega \text{Trace}_{G_\theta}\Big(\Pi_H \, \phi^* \, h\Big) \, \theta \wedge (d\theta)^n.$$

Here $\Omega \subset\subset M$ is a relatively compact domain and $\Pi_H \, B$ denotes the restriction to $H(M) \otimes H(M)$ of the bilinear form B. A C^∞ map $\phi : M \to N$ is *subelliptic harmonic map* if it is a critical point of E_b i.e.

$$\frac{d}{dt}\Big\{E_b(\phi_t)\Big\}_{t=0} = 0,$$

for any smooth 1-parameter variation $\{\phi_t\}_{|t|<\epsilon} \subset C^\infty(M, N)$ of $\phi_0 = \phi$ with $\text{Supp}(V) \subset \Omega$, where $V = \partial\phi_t/\partial t \in C^\infty\big(\phi^{-1}TN\big)$ is the infinitesimal variation induced by $\{\phi_t\}_{|t|<\epsilon}$. Subelliptic harmonic maps [from a pseudohermitian manifold (M, θ) into a Riemannian manifold (N, h)] were first introduced by E. Barletta et al. [9], under the name *pseudoharmonic maps*. Cf. also [24]. Let $\{X_a : 1 \le a \le 2n\}$ be a local G_θ-orthonormal frame of $H(M)$, defined on the open set U which is also the domain of a local chart $\chi : U \to \mathbb{R}^{2n+1}$. Then $X \equiv \{\chi_* \, X_a : 1 \le a \le 2n\}$ is a Hörmander system of vector fields on $\chi(U)$ and for any pseudoharmonic map ϕ the map $\phi \circ \chi^{-1}$ is subellitic harmonic in the sense of Jost and Xu [44] i.e. as a map of $\chi(U)$ [an open set in $\mathbb{R}^{2n+1}$ equipped with the Hörmander system X] into the Riemannian manifold N (thus motivating the adopted terminology).

Let ∇ and ∇^h be respectively the Tanaka–Webster connection (cf. [29, Theorem 1.3, Definition 1.25, pp. 25–31]; see also Eq. (11) below) of (M, θ) and the Levi–Civita connection of (N, h). For every C^∞ map $\phi : M \to N$ let

$$B_b(\phi)(X, Y) = D^\phi_X \phi_* \, Y - \phi_* \nabla_X Y \,, \quad X, Y \in \mathfrak{X}(M),$$

be the *pseudohermitian* second fundamental form of ϕ. Here $\phi_* X$ is the C^∞ section in the pullback bundle $\phi^{-1}T(N) \to M$ defined by

$$\big(\phi_* \, X\big)(x) = (d_x\phi)X_x \,, \quad x \in M.$$

Also $D^\phi = \phi^{-1}\nabla^h$ is the pullback of ∇^h by ϕ [a connection in the vector bundle $\phi^{-1}T(N) \to M$ parallelizing the bundle metric $h^\phi = \phi^{-1}h$ (the pullback of h by ϕ)]. Let us set

$$\tau_b(\phi) = \text{trace}_{G_\theta}\big[\Pi_H \, B_b(\phi)\big] \in C^\infty\big(\phi^{-1}TN\big), \tag{9}$$

(the *pseudohermitian tension field* of ϕ) so that

$$\frac{d}{dt}\Big\{E_b(\phi_t)\Big\}_{t=0} = -\int_\Omega h^\phi\big(V, \, \tau_b(\phi)\big) \, \theta \wedge (d\theta)^n,$$

(the *first variation* formula for E_b) for any smooth 1-parameter variation $\{\phi_t\}_{|t<\epsilon}$ of ϕ supported in Ω. A C^∞ map ϕ is subelliptic harmonic if and only if

$$\tau_b(\phi) = 0. \tag{10}$$

Note that $\tau_b(\phi)$ is not the full trace of $\phi^* h$, but rather the trace (with respect to the Levi form G_θ) of $\Pi_H \phi^* h$ [the restriction of $\phi^* h$ to $H(M) \otimes H(M)$]. Omitting a direction in the calculation of the trace [as in (9)] has far reaching consequences, as explained by Dragomir and Perrone (cf. [28]): the principal part in the subelliptic harmonic map system (10) is the sublaplacian Δ_b of (M, θ), a degenerate elliptic operator whose ellipticity degenerates at the cotangent directions spanned by θ.

Pseudohermitian second fundamental forms were introduced by Petit (cf. [56]) who formally modified the definition of the second fundamental form (of a map of Riemannian manifolds) by replacing the Levi–Civita connection of the source manifold with the Tanaka–Webster connection. Nevertheless M does carry a natural Riemannian metric g_θ, springing from the given structure $\big(T_{1,0}(M),\, \theta\big)$, and $B_b(\phi)$ is related to the ordinary second fundamental form $B(\phi)$ of ϕ, as a map between the Riemannian manifolds $(M,\, g_\theta)$ and $(N,\, h)$. Precisely, let $T \in \mathfrak{X}(M)$ be the Reeb vector field of $(M,\, \theta)$ i.e. the globally defined, nowhere zero tangent vector field on M, transverse to $H(M)$, determined by $\theta(T) = 1$ and $T \rfloor d\theta = 0$. Profiting from the direct sum decomposition $T(M) = H(M) \oplus \mathbb{R}T$ one may extend the Levi form G_θ to a Riemannian metric g_θ on M [the *Webster metric* of $(M,\, \theta)$] by postulating that

$$g_\theta = G_\theta \;\text{ on }\; H(M) \otimes H(M), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X \in H(M)$. Then g_θ is a *contraction* of the sub-Riemannian structure $\big(H(M),\, G_\theta\big)$ (cf. Strichartz [60]) i.e. $d(x, y) \le \rho(x, y)$ for any $x, y \in M$, where d and ρ are respectively the Riemannian distance (associated to the Webster metric) and the Carnot–Carathéodory distance (associated to the sub-Riemannian structure). Let ∇^{g_θ} be the Levi–Civita connection of $(M,\, g_\theta)$ and let

$$B(\phi)(X, Y) = D^\phi_X \phi_* Y - \phi_* \nabla^{g_\theta}_X Y, \quad X, Y \in \mathfrak{X}(M),$$

be the second fundamental form of ϕ as a map of $(M,\, g_\theta)$ into $(N,\, h)$. The tension field of ϕ is

$$\tau(\phi) = \operatorname{trace}_{g_\theta} B(\phi) \in C^\infty\big(\phi^{-1} T N\big).$$

The Levi–Civita and Tanaka–Webster connections ∇^{g_θ} and ∇ are related by (cf. [29, p. 46])

$$\nabla^{g_\theta} = \nabla + \big(\Omega - A\big) \otimes T + \tau \otimes \theta + 2\,(\theta \odot J), \tag{11}$$

where τ is the pseudohermitian torsion of ∇ and

$$\Omega = -d\theta, \quad A(X, Y) = g_\theta(X,\, \tau Y).$$

A calculation relying on (11) shows that

$$\tau(\phi) = \tau_b(\phi) + D^{\phi}_T \, \phi_* \, T, \tag{12}$$

so that the notions of a harmonic map and a subelliptic harmonic map are logically inequivalent. Let div : $\mathfrak{X}(M) \to C^\infty(M)$ be the divergence operator with respect to the volume form $\Psi = \theta \wedge (d\theta)^n$ i.e.

$$\mathcal{L}_X \Psi = \operatorname{div}(X)\, \Psi,$$

where $\mathcal{L}_X$ is the Lie derivative. The *sublaplacian* is the formally self-adjoint, positive, second order operator Δ_b given by

$$\Delta_b u = -\operatorname{div}\big(\nabla^H u\big), \quad u \in C^2(M),$$
$$\nabla^H u = \Pi_H \nabla u, \quad \Pi_H = I - \theta \otimes T,$$
$$g_\theta(X, \nabla u) = X(u), \quad u \in C^1(M), \quad X \in \mathfrak{X}(M).$$

Let $\phi : M \to N$ be a C^∞ map. Let $\{X_a \; : \; 1 \leq a \leq 2n\} \subset C^\infty(U, \; H(M))$ be a G_θ-orthonormal [i.e. $G_\theta(X_a \, , \, X_b) = \delta_{ab}$] local frame and let $(V, \; y^\alpha)$ be a local coordinate system on N such that $\phi(U) \subset V$. The subelliptic harmonic map system (10) may be written locally as

$$-\Delta_b \phi^\alpha + \sum_{a=1}^{2n} \left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} X_a\big(\phi^\beta\big)\, X_a\big(\phi^\gamma\big) = 0, \tag{13}$$

where $\phi^\alpha = y^\alpha \circ \phi$ and $\left\{ \begin{array}{c} \alpha \\ \beta\gamma, \end{array} \right\}$ are the Christoffel symbols of the second kind of $h_{\alpha\beta}$. The sublaplacian is degenerate elliptic, yet subelliptic of order $1/2$ and hence hypoelliptic (cf. Hörmander [42]). The study of subelliptic harmonic maps, and then the study of subelliptic harmonic morphisms (a particular sort of subelliptic harmonic maps, as introduced by Dragomir and Lanconelli [25]) fits into the larger program of Jost and Xu (cf. [44]) devoted to the study of second order quasi-linear PDE systems of variational origin whose principal part is at least hypoelliptic.

The sublaplacian Δ_b may be thought of as the linear operator of Hilbert spaces

$$\Delta_b = \big(\nabla^H\big)^* \circ \nabla^H : \mathscr{D}\big(\Delta_b\big) \subset L^2(M) \to L^2(M),$$

with domain

$$\mathscr{D}\big(\Delta_b\big) = \big\{u \in \mathscr{D}\big(\nabla^H\big) \; : \; \nabla^H u \in \mathscr{D}\big[(\nabla^H)^*\big]\big\},$$

where ∇^H is the weak horizontal gradient and $\big(\nabla^H\big)^*$ is its adjoint. Then, although the subelliptic harmonic map system is but quasi-linear, weak solutions may be defined as maps $\phi \in W_H^{1,2}(M, N)$ such that for any $\varphi \in C_0^\infty(M)$

$$\int_U \Big\{ g_\theta\big(\nabla^H \phi^\alpha\,,\ \nabla^H \varphi\big) + \sum_{a=1}^{2n} \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} X_a\big(\phi^\beta\big)\, X_a\big(\phi^\gamma\big)\, \varphi \Big\}\, \theta \wedge (d\theta)^n = 0.$$

To make sense of the Sobolev type spaces $W_H^{1,2}(M, N)$ (the Folland–Stein spaces) the definition is either confined to target manifolds N which may be covered by a single coordinate neighborhood, or one first embeds (isometrically) N into a sufficiently high dimensional Euclidean space (by using Nash's embedding theorem [52]). The generalized Dirichlet problem for the PDE system (13) was solved by Jost and Xu (cf. [44]) who also proved interior continuity of solutions. Finally, existence of C^∞ subelliptic harmonic maps may be established by applying a result by Xu and Zuily (cf. [63]) who proved smoothness of continuous solutions to a class of PDE systems including the subelliptic harmonic map system.

3 Subelliptic Harmonic Morphisms

Definition 1 A continuous map ϕ of (M, θ) into (N, h) is a *subelliptic harmonic morphism* if for every open subset $V \subset N$, and every C^2 function $v : V \to \mathbb{R}$, if $\Delta_h v = 0$ in V then the pullback function $u = v \circ \phi$ is a distribution-solution to $\Delta_b u = 0$ in $U = \phi^{-1}(V)$. □

Cf. Dragomir and Lanconelli [25]. Here

$$\Delta_h v \equiv -\frac{1}{\sqrt{\mathfrak{h}}} \frac{\partial}{\partial y^\alpha} \left\{ \sqrt{\mathfrak{h}}\, h^{\alpha\beta}\, \frac{\partial v}{\partial y^\beta} \right\}, \quad \mathfrak{h} = \det\left[h_{\alpha\beta}\right],$$

is the Laplacian on (V, h).

Proposition 1 *Every subelliptic harmonic morphism ϕ of the pseudohermitian manifold $(M,\ \theta)$ into the Riemannian manifold $(N,\ h)$ is smooth.*

Proof For every point $x_0 \in M$ let $(V,\ y^\alpha)$ be a harmonic local coordinate system with $\phi(x_0) \in V$, and let us set $\phi^\alpha = y^\alpha \circ \phi$. Then $\Delta_b \phi^\alpha = 0$ in $U = \phi^{-1}(V)$ hence (as Δ_b is hypoelliptic) $\phi^\alpha \in C^\infty(U)$. □

Definition 2 A C^∞ map $\phi : M \to N$ is *Levi conformal* if there is a continuous map $\lambda = \lambda(\phi) : M \to [0, +\infty)$ (the θ-*dilation* of ϕ) such that λ^2 is C^∞ and

$$G_\theta\big(\nabla^H \phi^\alpha\,,\ \nabla^H \phi^\beta\big)_x = \lambda(x)^2\, \delta^{\alpha\beta}, \tag{14}$$

for any $x \in M$ and any local normal coordinate system $(V,\ y^\alpha)$ on N with center at $\phi(x) \in V$. □

We set as customary $\Lambda(\phi) = \lambda(\phi)^2$ (the *square* θ-*dilation*). For a fixed Levi conformal map $\phi : M \to N$ we abbreviate the notation to $\Lambda = \Lambda(\phi)$. Let $\phi : M \to N$ be a Levi conformal map, of (M, θ) into (N, h), and let $x \in M$ be an arbitrary point. Let $\left(V', y'^{\alpha}\right)$ be an arbitrary local coordinate system on N such that $\phi(x) \in V'$, and let us set $\phi'^{\alpha} = y'^{\alpha} \circ \phi$. Then

$$\nabla^H \phi'^{\alpha} = \left(\frac{\partial y'^{\alpha}}{\partial y^{\beta}}\right)^{\phi} \nabla^H \phi^{\beta}, \tag{15}$$

on $\phi^{-1}(V \cap V')$. Moreover [by (15) and (14)]

$$G_\theta\left(\nabla^H \phi'^{\alpha}, \nabla^H \phi'^{\beta}\right)_x = \Lambda(x)\, \delta^{\mu\nu} \frac{\partial y'^{\beta}}{\partial y^{\mu}}\big(\phi(x)\big) \frac{\partial y'^{\beta}}{\partial y^{\nu}}\big(\phi(x)\big) = \Lambda(x)\, h'^{\alpha\beta}\big(\phi(x)\big),$$

i.e. if ϕ is Levi conformal then for any $x \in M$ and *any* local coordinate system (V, y^{α}) about $\phi(x)$

$$G_\theta\left(\nabla^H \phi^{\alpha}, \nabla^H \phi^{\beta}\right) = \Lambda\, h^{\alpha\beta} \circ \phi, \tag{16}$$

everywhere in $\phi^{-1}(V)$.

By a result of Barletta [7] (revisited as in Appendix B of [25]) a C^∞ map $\phi : M \to N$ is a subelliptic harmonic morphism of (M, θ) into (N, h) if and only if ϕ is Levi conformal and a subelliptic harmonic map. Moreover (again by [7]) if $m > 2n$ then every subelliptic harmonic morphism is a constant, while if $m \le 2n$ then for every point $x \in M$ with $\lambda(x) \neq 0$ there is an open neighborhood U of x such that $\phi : U \to N$ is a C^∞ submersion. Barletta's result is a pseudohermitian analog to the Fuglede–Ishihara characterization (cf. Fuglede [34], Ishihara [43]) of harmonic morphisms between Riemannian manifolds.

Let $\phi : M \to N$ be a C^∞ map and let us set

$$\mathscr{V}_x^{\phi} = \operatorname{Ker}(d_x \phi), \quad \mathscr{H}_x^{\phi} = \left(\mathscr{V}_x^{\phi}\right)^{\perp}, \quad x \in M,$$

where the orthogonal complement is meant with respect to the inner product $g_{\theta, x}$.

Lemma 1 *Let M be a strictly pseudoconvex CR manifold, of CR dimension n, endowed with the positively oriented contact form θ, and let (N, h) be a m-dimensional Riemannian manifold. Let $\phi : M \to N$ be a C^∞ map. Then*

(i) *For every $x \in \phi^{-1}(V)$*

$$\left\{\left(\nabla \phi^{\alpha}\right)_x : 1 \le \alpha \le m\right\} \subset \mathscr{H}_x^{\phi}.$$

(ii) *Let $\phi : M \to N$ be a Levi conformal map, and let $Z(\Lambda) = \{x \in M : \Lambda(x) = 0\}$ be the zero set of its θ-dilation. Then*

$$\operatorname{Crit}(\phi) \subset Z(\Lambda). \tag{17}$$

Also

$$T_x \in \mathscr{H}_x^{\phi} \tag{18}$$

for any $x \in Z(\Lambda) \setminus \mathrm{Crit}(\phi)$.

(iii) *Let us assume that* $m \leq 2n$, *and let* $\phi : M \to N$ *be a subelliptic harmonic morphism. Then for every* $x \in M \setminus Z(\Lambda)$

$$\left\{ \left(\nabla \phi^{\alpha}\right)_x \ : \ 1 \leq \alpha \leq m \right\}, \tag{19}$$

is a linear basis in $\mathscr{H}_x^{\phi}$. *In particular*

$$\left\{ x \in M \ : \ T_x \in \mathscr{H}_x^{\phi} \right\} \subset Z(\Lambda).$$

Proof (i) Let $x \in M$ and $v \in \mathscr{V}_x^{\phi}$. Then

$$g_{\theta\,,\,x}\big(v,\ \left(\nabla \phi^{\alpha}\right)_x\big) \left(\frac{\partial}{\partial y^{\alpha}}\right)_{\phi(x)} = v(\phi^{\alpha}) \left(\frac{\partial}{\partial y^{\alpha}}\right)_{\phi(x)} = (d_x \phi) v = 0,$$

that is

$$\left(\nabla \phi^{\alpha}\right)_x \in \left(\mathscr{V}_x^{\phi}\right)^{\perp} = \mathscr{H}_x^{\phi}.$$

□

(ii) Let $x \in \mathrm{Crit}(\phi)$. Then

$$0 = \left(\nabla \phi^{\alpha}\right)_x = \left(\nabla^H \phi^{\alpha}\right)_x + T\left(\phi^{\alpha}\right)_x T_x \Longrightarrow,$$

[by the uniqueness of the direct sum decomposition $T_x(M) = H(M)_x \oplus \mathbb{R} T_x$]

$$\Longrightarrow \left(\nabla^H \phi^{\alpha}\right)_x = 0,$$

hence [by (14)] $x \in Z(\Lambda)$, accounting for (17).

Next, let $x \in Z(\Lambda) \setminus \mathrm{Crit}(\phi)$. Then [by (14)]

$$\left(\nabla^H \phi^{\alpha}\right)_x = 0, \quad 1 \leq \alpha \leq m,$$

so that

$$\begin{aligned} \exists\ \alpha \in \{1, \ldots, m\} \ &: \ 0 \neq (\nabla \phi^{\alpha})_x = T_x\left(\phi^{\alpha}\right) T_x \\ \exists\ \alpha \in \{1, \ldots, m\} \ &: \ T_x\left(\phi^{\alpha}\right) \neq 0. \end{aligned} \tag{20}$$

Finally, for every $v \in \mathscr{V}_x^{\phi}$ [by statement (i) in Lemma 1]

$$T_x\left(\phi^{\alpha}\right) g_{\theta\,,\,x}\big(v,\ T_x\big) = g_{\theta\,,\,x}\big(v,\ (\nabla \phi^{\alpha})_x\big) = 0,$$

yielding [by (20)] $T_x \in \left(\mathscr{V}_x^\phi\right)^\perp = \mathscr{H}_x^\phi$. □

(iii) Let $x_0 \in M \setminus Z(\Lambda)$. By a result of Barletta [7], there is an open neighborhood $U \subset M$ of x_0 such that $\phi : U \to N$ is a C^∞ submersion. Hence, $d_x\phi : \mathscr{H}_x^\phi \to T_{\phi(x)}(N)$ is a $\mathbb{R}$-linear isomorphism, for any $x \in U$. By statement (i) in Lemma 1 it suffices to show that the system $\left\{(\nabla\phi^\alpha)_{x_0} \; : \; 1 \le \alpha \le m\right\} \subset \mathscr{H}_x^\phi$ is free. Indeed if for some $\mu^\alpha \in \mathbb{R}$, $1 \le \alpha \le m$,

$$0 = \mu_\alpha (\nabla\phi^\alpha)_{x_0} = \mu_\alpha \left\{ (\nabla^H \phi^\alpha)_{x_0} + T_{x_0}(\phi^\alpha)\, T_{x_0} \right\},$$

then $\mu_\alpha \left(\nabla^H\phi^\alpha\right)_{x_0} = 0$ yielding $\mu^\alpha = 0$, because [by (14)] the vectors

$$\left\{ (\nabla^H \phi^\alpha)_{x_0} \; : \; 1 \le \alpha \le m \right\}, \tag{21}$$

are linearly independent.

Next, let $x \in M$ be a point such that $T_x \in \mathscr{H}_x^\phi$. Either x is a critical point of ϕ, so that [by (17)] $x \in Z(\Lambda)$, or $x \in M \backslash \mathrm{Crit}(\phi)$. For the remainder of the proof we argue by contradiction, i.e. let us assume that $\Lambda(x) \neq 0$. If this is the case, for any $1 \le \alpha \le m$

$$\mathscr{H}_x^\phi \ni (\nabla\phi^\alpha)_x - T_x(\phi^\alpha)\, T_x = (\nabla^H\phi^\alpha)_x,$$

hence (21) is a linear basis of $\mathscr{H}_x^\phi$, too, yielding $\mathscr{H}_x^\phi \subset H(M)_x$ and in particular $T_x \in H(M)_x$, a contradiction. □

Let $\phi : M \to N$ be a subelliptic harmonic morphism, of the pseudohermitian manifold (M, θ), into the Riemannian manifold (N, h). For each $x \in M$ we set

$$\mathscr{V}_{H,x}^\phi = H(M)_x \cap \mathscr{V}_x^\phi, \quad \mathscr{H}_{H,x}^\phi = H(M)_x \cap \mathscr{H}_x^\phi.$$

If $x \in \mathrm{Crit}(\phi)$ then

$$\mathscr{V}_{H,x}^\phi = H(M)_x, \quad \mathscr{H}_{H,x}^\phi = \{0\}.$$

If $x \in M \backslash \mathrm{Crit}(\phi)$ then the differential $d_x\phi : T_x(M) \to T_{\phi(x)}(N)$ may, or may not, be an epimorphism.

Definition 3 A regular point in the set

$$S(\phi) = \left\{ x \in M \setminus \mathrm{Crit}(\phi) \; : \; d_x\phi \quad \text{is on-to} \right\},$$

is called a *submersive point* of the morphism ϕ. □

At every submersive point $x \in S(\phi)$

$$\dim_{\mathbb{R}} \mathscr{H}_x^\phi = m, \quad \dim_{\mathbb{R}} \mathscr{V}_x^\phi = 2n - m + 1.$$

Lemma 2 *Let M and N be a strictly pseudoconvex CR manifold, of CR dimension n, and let N be an m-dimensional Riemannian manifold, such that $m \leq 2n$. Let $\phi : M \to N$ be a subelliptic harmonic morphism, of θ-dilation $\sqrt{\Lambda}$. Then* (i)

$$M \setminus Z(\Lambda) \subset S(\phi). \tag{22}$$

(ii) *For every submersive point $x \in S(\phi)$*

$$m - 1 \leq \dim_{\mathbb{R}} \mathscr{H}^{\phi}_{H,x} \leq m, \tag{23}$$

$$2n - m \leq \dim_{\mathbb{R}} \mathscr{V}^{\phi}_{H,x} \leq 2n - m + 1. \tag{24}$$

Proof (i) Let $x \in M$ with $x \notin Z(\Lambda)$. Then, on one hand [by (ii) in Lemma 1] x is a regular point of ϕ. On the other hand $m \leq 2n$ and $\Lambda(x) \neq 0$ so that, by a result of Barletta (cf. [7]) ϕ is a submersion on some neighborhood of x, and in particular x is a submersive point. □

(ii) For instance, for every $x \in S(\phi)$ the relations

$$\dim_{\mathbb{R}} \left[H(M)_x + \mathscr{H}^{\phi}_x \right] = 2n + m - \dim_{\mathbb{R}} \mathscr{H}^{\phi}_{H,x} ,$$

$$2n \leq \dim_{\mathbb{R}} \left[H(M)_x + \mathscr{H}^{\phi}_x \right] \leq 2n + 1,$$

yield (19). □

Note that [by taking complements in (22)] $M \setminus S(\phi) \subset Z(\Lambda)$. Next, as a consequence of (23) and (24), the set of submersive points of ϕ admits the natural partition

$$\begin{aligned} S(\phi) &= \mathrm{I}_m(\phi) \cup \mathrm{II}_m(\phi) \cup \mathrm{III}_m(\phi), \\ \mathrm{I}_m(\phi) &= \left\{ x \in S(\phi) \; : \; \dim_{\mathbb{R}} \mathscr{H}^{\phi}_{H,x} = m, \;\; \dim_{\mathbb{R}} \mathscr{V}^{\phi}_{H,x} = 2n - m \right\}, \\ \mathrm{II}_m(\phi) &= \left\{ x \in S(\phi) \; : \; \dim_{\mathbb{R}} \mathscr{H}^{\phi}_{H,x} = m - 1, \;\; \dim_{\mathbb{R}} \mathscr{V}^{\phi}_{H,x} = 2n - m + 1 \right\}, \\ \mathrm{III}_m(\phi) &= \left\{ x \in S(\phi) \; : \; \dim_{\mathbb{R}} \mathscr{H}^{\phi}_{H,x} = m - 1, \;\; \dim_{\mathbb{R}} \mathscr{V}^{\phi}_{H,x} = 2n - m \right\}. \end{aligned} \tag{25}$$

Indeed, case (IV) where $\dim_{\mathbb{R}} \mathscr{H}^{\phi}_{H,x} = m$ and $\dim_{\mathbb{R}} \mathscr{V}^{\phi}_{H,x} = 2n - m + 1$ is ruled out by $\mathscr{H}^{\phi}_{H,x} \oplus \mathscr{V}^{\phi}_{H,x} \subset H(M)_x$.

The main difficulties one encounters are related to the presence of *two* pairs of complementary distributions on M [rather than just $\big(H(M), \mathbb{R}T\big)$ as in CR geometry, or just $\big(\mathscr{V}^{\phi}, \mathscr{H}^{\phi}\big)$ as in the theory of harmonic morphisms between Riemannian manifolds]. These distributions intersect, and the dimension of the intersections may vary from a point to another, requiring a classification of types of points, relative to a fixed subelliptic harmonic morphism ϕ, as captured by the partition (25). Our conclusive finding is

Theorem 6 *Let M be a strictly pseudoconvex CR manifold, equipped with the contact form $\theta \in \mathscr{P}_+(M)$, and let (N, h) be a m-dimensional Riemannian manifold. Let $\phi : M \to N$ be a subelliptic harmonic morphism, from (M, θ) into (N, h).*

(i) *If* $m=1$ *then*

$$Z(\Lambda)=\mathrm{II}_1(\phi)\cup \mathrm{Crit}(\phi), \quad M\setminus S(\phi)=\mathrm{Crit}(\phi). \tag{26}$$

(ii) *If* $m\geq 2$ *then*

$$\mathrm{II}_m(\phi)=\emptyset, \quad Z(\Lambda)=M\setminus S(\phi). \tag{27}$$

Consequently, for every subelliptic harmonic morphism $\phi: M\to N^m$ *all points in* M *are submersive, except for the critical points of* ϕ *when* $m=1$*, or for the zeros of the square dilation* $\Lambda=\Lambda(\phi)$ *when* $m\geq 2$.

The proof of Theorem 6 requires a number of lemmas.

Lemma 3 *Under the assumptions of Lemma 2, let* $x\in S(\phi)$ *be a submersive point. Then*

(i) $x\in \mathrm{I}_m(\phi)\Longleftrightarrow T_x\in \mathscr{V}_x^{\phi}$.

(ii) $x\in \mathrm{II}_m(\phi)\Longleftrightarrow T_x\in \mathscr{H}_x^{\phi}$.

Proof (i) If $x\in \mathrm{I}_m(\phi)$ then $\dim_{\mathbb{R}}\mathscr{H}_x^{\phi}=m$ hence $\mathscr{H}_x^{\phi}=\mathscr{H}_{H,x}^{\phi}\subset H(M)_x$ yielding $\mathscr{V}_x^{\phi}=\left(\mathscr{H}_x^{\phi}\right)^{\perp}\supset H(M)_x^{\perp}=\mathbb{R}T_x$. □

Viceversa, if $T_x\in\mathscr{V}_x^{\phi}$ then T_x is orthogonal to $\mathscr{H}_x^{\phi}$ i.e. $\mathscr{H}_x^{\phi}\subset H(M)_x$ yielding $\mathscr{H}_x^{\phi}=\mathscr{H}_{H,x}^{\phi}$, and consequently $\dim_{\mathbb{R}}\mathscr{H}_{H,x}^{\phi}=m$. The sets $\mathrm{I}_m(\phi)$, $\mathrm{II}_m(\phi)$ and $\mathrm{III}_m(\phi)$ are mutually disjoint, so it must be that $x\in\mathrm{I}_m(\phi)$. □

(ii) If $x\in\mathrm{II}_m(\phi)$ then $\dim_{\mathbb{R}}\mathscr{V}_{H,x}^{\phi}=2n-m+1$, hence $\mathscr{V}_x^{\phi}=\mathscr{V}_{H,x}^{\phi}\subset H(M)_x$ implying that T_x is orthogonal to $\mathscr{V}_x^{\phi}$ i.e. $T_x\in\mathscr{H}_x^{\phi}$. □

Viceversa, if $T_x\in\mathscr{H}_x^{\phi}$ then T_x is orthogonal to $\mathscr{V}_x^{\phi}$ i.e. $\mathscr{V}_x^{\phi}\subset H(M)_x$. It follows that the subspaces $\mathscr{V}_{H,x}^{\phi}$ and $\mathscr{V}_x^{\phi}$ coincide, yet the space $\mathscr{V}_x^{\phi}$ is $(2n-m+1)$-dimensional, so that $x\in\mathrm{II}_m(\phi)$, again because (25) is a partition.

Lemma 4 *Under the assumptions of Lemma 2*

$$\mathrm{II}_m(\phi)=S(\phi)\cap Z(\Lambda). \tag{28}$$

Consequently

$$\mathrm{I}_m(\phi)\cap Z(\Lambda)=\emptyset, \quad \mathrm{II}_m(\phi)\subset Z(\Lambda)\setminus\mathrm{Crit}(\phi), \quad \mathrm{III}_m(\phi)\cap Z(\Lambda)=\emptyset.$$

Proof For every $x\in\mathrm{II}_m(\phi)$ [by statement (ii) in Lemma 2] $T_x\in\mathcal{H}_x^{\phi}$ and then [by statement (iii) in Lemma 1] $x\in Z(\Lambda)$. □

To prove the opposite inclusion, let $x\in S(\phi)\cap Z(\Lambda)$. Then $\Lambda(x)=0$ so that [by (14)]

$$\left(\nabla^H\phi^{\alpha}\right)_x=0, \quad 1\leq\alpha\leq m.$$

Consequently [by statement (i) in Lemma 1]

$$T_x(\phi^\alpha)\, T_x = \left(\nabla\phi^\alpha\right)_x \in \mathcal{H}_x^\phi, \quad 1 \le \alpha \le m,$$

and there is $\alpha \in \{1, \ldots, m\}$ such that $T_x(\phi^\alpha) \neq 0$, because x is a regular point. Therefore $T_x \in \mathcal{H}_x^\phi$. □

Let us set $\Omega = \Omega(\phi) := M \setminus Z(\Lambda)$ (an open subset of M). Then

$$\partial\Omega \subset Z(\Lambda). \tag{29}$$

Lemma 5 *Under the assumptions of Lemma* 2

$$S(\phi) = \Omega(\phi) \cup \mathrm{II}_m(\phi). \tag{30}$$

Proof Let $x \in S(\phi)$. We distinguish two cases, as A) $x \in Z(\Lambda)$, or B) $z \notin Z(\Lambda)$. In case (A) [by Lemma 4]

$$x \in S(\phi) \cap Z(\Lambda) = \mathrm{II}_m(\phi).$$

In case (B)

$$x \in M \setminus Z(\Lambda) = \Omega(\phi).$$

To check the opposite inclusion, let $x \in \Omega(\phi) \cup \mathrm{II}_m(\phi)$. Then [by (28) and (22)] either

$$x \in \mathrm{II}_m(\phi) = S(\phi) \cap Z(\Lambda) \subset S(\phi),$$

or

$$x \in \Omega(\phi) = M \setminus Z(\Lambda) \subset S(\phi).$$

□

Lemma 6 *Under the assumptions of Lemma* 2

(1) $\mathrm{I}_m(\phi) \subset \Omega$,

(2) $\mathrm{III}_m(\phi) = \Omega \backslash \mathrm{I}_m(\phi)$.

Proof (1) Given any $x \in \mathrm{I}_m(\phi)$, let us show that $x \notin Z(\Lambda)$. We argue by contradiction. If $x \in Z(\Lambda)$ then [by statement (ii) in Lemma 1, as x is a regular point] $T_x \in \mathscr{H}_x^\phi$. Yet [by statement (i) in Lemma 3] $T_x \in \mathscr{V}_x^\phi$, implying that $T_x = 0$, a contradiction.

(2) By (30) and the first statement in the current lemma

$$S(\phi) = \Omega \cup \mathrm{II}_m(\phi) = \mathrm{I}_m(\phi) \cup \mathrm{II}_m(\phi) \cup \left[\Omega \setminus \mathrm{I}_m(\phi)\right],$$

implying [by (25)] $\Omega \backslash \mathrm{I}_m(\phi) = \mathrm{III}_m(\phi)$. □

Lemma 7 *Let $\phi : M \to N$ be a subelliptic harmonic morphism, from (M, θ) into (N, h). Then for every $x \in Z(\Lambda)\backslash \mathrm{Crit}(\phi)$*

$$\mathscr{V}_x^{\phi} = H(M)_x, \tag{31}$$

and in particular $\dim_{\mathbb{R}} (d_x\phi)T_x(M) = 1$.

Proof Let $x \in Z(\Lambda) \setminus \mathrm{Crit}(\phi)$. By $\Lambda(x) = 0$ and (14) [as ϕ is Levi conformal]

$$(\nabla^H \phi^\alpha)_x = 0, \quad 1 \le \alpha \le m.$$

Let $\{E_a : 1 \le a \le 2n\}$ be a local G_θ-orthonormal frame of $H(M)$, defined on an open neighborhood $U \subset M$ of x. Then

$$0 = (\nabla^H \phi^\alpha)_x = \sum_{a=1}^{2n} E_{a,x}(\phi^\alpha)\, E_{a,x} \Longrightarrow E_{a,x}(\phi^\alpha) = 0,$$

hence

$$0 = E_{a,x}(\phi^\alpha) \left(\frac{\partial}{\partial y^\alpha}\right)_{\phi(x)} = (d_x\phi)E_{a,x} \Longrightarrow E_{a,x} \in \mathscr{V}_x^{\phi},$$

that is

$$H(M)_x \subset \mathscr{V}_x^{\phi}. \tag{32}$$

As $x \notin \mathrm{Crit}(\phi)$ it must be

$$T_x \notin \mathscr{V}_x^{\phi}. \tag{33}$$

Indeed, if $T_x \in \mathscr{V}_x^{\phi}$ then [by (32)] $T_x(M) \subset \mathscr{V}_x^{\phi}$ i.e. $d_x\phi = 0$, a contradiction. Next, let $v \in \mathscr{V}_x^{\phi}$ so that

$$v = \lambda^a\, E_{a,x} + \theta_x(v)\, T_x,$$

for some $\lambda^a \in \mathbb{R}$, $1 \le a \le 2n$. By applying $d_x\phi$ to both members [and using (32) and (33)]

$$0 = (d_x\phi)v = \theta_x(v)\,(d_x\phi)T_x \Longrightarrow \theta_x(v) = 0 \Longrightarrow v \in H(M)_x,$$

that is $\mathscr{V}_x^{\phi} \subset H(M)_x$, yielding equality in (32). □ In particular [by (31)]

$$\dim_{\mathbb{R}} T_x(M) = \dim_{\mathbb{R}} \mathscr{V}_x^{\phi} + \dim_{\mathbb{R}}(d_x\phi)T_x(M) \Longrightarrow \dim_{\mathbb{R}}(d_x\phi)T_x(M) = 1.$$

□

At this point we may complete the proof of Theorem 6.

(i) By Lemmas 1 and 4 the inclusion

$$Z(\Lambda) \supset \mathrm{II}_m(\phi) \cup \mathrm{Crit}(\phi),$$

holds for arbitrary $m \geq 1$. To check the opposite inclusion, let $x \in Z(\Lambda)$. Then either $x \in \mathrm{Crit}(\phi)$ and we are done, or $x \notin \mathrm{Crit}(\phi)$ and then we may apply Lemma 7 to conclude that the space $(d_x\phi)\, T_x(M)$ is 1-dimensional. Hence (as $m = 1$) $d_x\phi$ is an epimorphism, implying that $x \in S(\phi)$, and then

$$x \in S(\phi) \cap Z(\Lambda) = \mathrm{II}_1(\phi),$$

(according to Lemma 4). □

To prove the second equality in (26) note first that

$$M \setminus S(\phi) \supset \mathrm{Crit}(\phi),$$

[by its very definition, $S(\phi)$ lies in the complementary of $\mathrm{Crit}(\phi)$]. To check the opposite inclusion, let $x \in M \backslash S(\phi)$. We argue by contradiction i.e. we assume that $x \notin \mathrm{Crit}(\phi)$. On the other hand [by $x \notin S(\phi)$] the differential $d_x\phi$ is not on-to, implying that $\Lambda(x) = 0$ [otherwise ϕ is a submersion on some neighborhood of x, a contradiction]. At this point we may apply Lemma 7 to conclude that $(d_x\phi)T_x(M)$ is 1-dimensional, so that $d_x\phi$ is surjective i.e. $x \in S(\phi)$, a contradiction.

(ii) The proof of $\mathrm{II}_m(\phi) = \emptyset$ is by contradiction. If $\mathrm{II}_m(\phi) \neq \emptyset$, let $x \in \mathrm{II}_m(\phi)$ i.e. (by Lemma 4) $x \in S(\phi) \cap Z(\Lambda)$. Therefore $d_x\phi \neq 0$ and $\Lambda(x) = 0$ so we may apply Lemma 7 to conclude that $T_{\phi(x)}(N) = (d_x\phi)T_x(M)$ is 1-dimensional i.e. $m = 1$, a contradiction. □

To prove the second statement in (27), let $x \in Z(\Lambda)$. Then either $x \in \mathrm{Crit}(\phi)$, implying that $d_x\phi$ is not on-to i.e. $x \in M \backslash S(\phi)$, or $x \notin \mathrm{Crit}(\phi)$ and one may apply Lemma 7 to conclude that

$$\dim_{\mathbb{R}}(d_x\phi)T_x(\phi) = 1 < m,$$

hence $d_x\phi$ is not on-to i.e. $x \notin S(\phi)$. The inclusion $Z(\Lambda) \subset M \setminus S(\phi)$ is proved. As to the opposite inclusion, let $x \in M \setminus S(\phi)$ and let us assume that $x \notin Z(\Lambda)$. Then ϕ is a submersion on some neighborhood of x, and in particular $x \in S(\phi)$, a contradiction. This yields $M \backslash S(\phi) \subset Z(\Lambda)$. □

4 Harmonic Morphisms from Fefferman Spaces

Let M be a strictly pseudoconvex CR manifold, of CR dimension n, and let θ be a positively oriented contact form on M. A complex p-form η on M is of *type* $(p, 0)$, or a $(p, 0)$-*form*, if $T_{0,1}(M) \rfloor \eta = 0$. Let $\Lambda^{p,0}(M) \subset \Lambda^p T^*(M) \otimes \mathbb{C}$ be the relevant bundle. Unlike the case of complex geometry, top degree $(p, 0)$-forms are $(n+1, 0)$-forms [rather than $(n, 0)$-forms, due to the presence of the additional real cotangent direction θ]. Then $\mathbb{R}_+ = \mathrm{GL}^+(1, \mathbb{R})$ acts freely on $K^0(M) = \Lambda^{n+1,0}(M) \backslash \{\text{zero section}\}$ and

$C(M) = K^0(M)/\mathbb{R}_+$ is a principal S^1 bundle over M (the *canonical circle bundle*). The $(2n+2)$-dimensional manifold $C(M)$ carries the Lorentzian metric F_θ (the *Fefferman metric*) naturally associated to θ

$$F_\theta = \pi^* \tilde{G}_\theta + 2(\pi^*\theta) \odot \sigma \, . \tag{34}$$

Cf. Lee [49] (or [29, pp. 128–129]). Here $\tilde{G}_\theta$ is the (degenerate) bilinear form on $T(M)$ got by requiring that $\tilde{G}_\theta = G_\theta$ on $H(M) \otimes H(M)$ and $\tilde{G}_\theta(T, V) = 0$ for any $V \in \mathfrak{X}(M)$. Also σ is the (globally defined) real 1-form on $C(M)$ given by

$$\sigma = \frac{1}{n+2}\left[d\gamma + \pi^* \left(i\, \omega_\alpha{}^\alpha - \frac{i}{2}\, g^{\alpha\overline{\beta}}\, dg_{\alpha\overline{\beta}} - \frac{\rho}{4(n+1)}\, \theta \right)\right], \tag{35}$$

where $\pi : C(M) \to M$ is the projection and $\gamma : \pi^{-1}(U) \to \mathbb{R}$ is a local fiber coordinate on $C(M)$. Also, for any local frame $\{T_\alpha \, : \, 1 \leq \alpha \leq n\} \subset C^\infty\big(U, \, T_{1,0}(M)\big)$

$$\nabla T_\beta = \omega_\beta{}^\alpha \otimes T_\alpha \, , \quad g_{\alpha\overline{\beta}} = G_\theta\big(T_\alpha \, , \, T_{\overline{\beta}}\big),$$

$$\big[g^{\alpha\overline{\beta}}\big] = \big[g_{\alpha\overline{\beta}}\big]^{-1}, \quad R_{\alpha\overline{\beta}} = R_\alpha{}^\gamma{}_{\gamma\overline{\beta}}\, , \quad \rho = g^{\alpha\overline{\beta}}\, R_{\alpha\overline{\beta}},$$

and ρ is the *pseudohermitian scalar* curvature (cf. [29, p. 50]). By a result of Graham (cf. [40]) σ is a connection 1-form on the canonical circle bundle [the *Graham connection* on $C(M)$]. For every tangent vector field $X \in \mathfrak{X}(M)$ let $X^\uparrow \in \mathfrak{X}(C(M))$ be the horizontal lift of X with respect to the Graham connection i.e.

$$X^\uparrow_p \in \operatorname{Ker}(\sigma_p), \quad (d_p\pi) X^\uparrow_p = X_{\pi(p)},$$

for any $p \in C(M)$. Let $S \in \mathfrak{X}(C(M))$ be the tangent to the S^1 action. Locally $S = [(n+2)/2]\,\big(\partial/\partial\gamma\big)$. Then $T^\uparrow - S$ is a globally defined time-like vector field on $C(M)$, hence the Lorentzian manifold $(C(M), F_\theta)$ is time oriented.

Let $\Box$ be the Laplace–Beltrami operator of the Lorentzian manifold $(C(M), F_\theta)$ (the geometric wave operator). By a result of Lee (cf. [49]) the pushforward of $\Box$ is precisely the sublaplacian Δ_b of (M, θ) i.e.

$$\pi_* \Box = \Delta_b. \tag{36}$$

By a result of Barletta et al. (cf. [9]) a C^∞ map $\phi : (M, \theta) \to (N, h)$ is subelliptic harmonic if and only if its vertical lift $\Phi = \phi \circ \pi : (C(M), F_\theta) \to (N, h)$ is a harmonic map.

Our main purpose in the present section is to relate (S^1 invariant) harmonic morphisms from $(C(M), \; F_\theta)$ to subelliptic harmonic morphisms from (M, θ), in the spirit of the geometric interpretation of subelliptic harmonic maps provided in [8], and prove Theorem 3.

The equivalence (i) $\Longleftrightarrow$ (ii) in Theorem 3 may be accounted for, as follows. Let $v : V \to \mathbb{R}$ be a C^2 solution to $\Delta_h v = 0$ with $V \subset N$ open, and let $\mathscr{U} = \Phi^{-1}(V)$ and $U = \phi^{-1}(V)$. For any $x \in U$ and $p \in \pi^{-1}(x) \subset \mathscr{U}$ [by (36)]

$$\Box(v \circ \Phi)\,(p) = \Box(v \circ \phi \circ \pi)(p) = (\pi_* \Box)(v \circ \phi)\,(x) = \Delta_b(v \circ \phi)\,(x),$$

[hence $\Box(v \circ \Phi) = 0$ in $\mathscr{U} \Longleftrightarrow \Delta_b(v \circ \phi) = 0$ in U]. □

Proof of Theorem 1 Follows from statement (i) $\Longrightarrow$ (ii) in Theorem 3, a result by S. Dragomir & E. Lanconelli (cf. Corollary 4 in [25, p. 421]), and the fact that $\pi : C(M) \to M$ is an open map. □

Proof of Theorem 2 Follows from Theorem 3 and Proposition 1. □

The study of harmonic morphisms in the semi-Riemannian category was started by Fuglede [35] (cf. also Parmar [54]) and the state-of-the-art up to 2003 is described in the monograph [5], where from we recall a few basic notions, confined to our needs i.e. to the case of harmonic morphisms from the Lorentzian manifold $(C(M),\, F_\theta)$ into the Riemannian manifold $(N,\, h)$.

Definition 4 A C^∞ map $\Phi : C(M) \to N$ is *harmonic* if it is a critical point of the energy functional

$$\mathbb{E}_D(\Phi) = \frac{1}{2}\int_D \text{Trace}_{F_\theta}\big(\Phi^* h\big)\, d\,\text{vol}(F_\theta),$$

for any relatively compact domain $D \subset\subset C(M)$. That is

$$\frac{d}{dt}\Big\{\mathbb{E}_D\big(\Phi_t\big)\Big\}_{t=0} = 0,$$

for every smooth 1-parameter variation $\{\Phi_t\}_{|t|<\epsilon} \subset C^\infty\big(C(M),\, N\big)$ of $\Phi_0 = \Phi$ supported in Ω i.e. $\text{Supp}(\mathbb{V}) \subset D$. □

Here $\mathbb{V} \in C^\infty\big(\Phi^{-1} TN\big)$ is the infinitesimal variation induced by $\{\Phi_t\}_{|t|<\epsilon}$ i.e.

$$\mathbb{V}_p = (d_{(p,0)}\Phi)\Big(\frac{\partial}{\partial t}\Big)_{(p,0)}, \quad p \in C(M).$$

Also

$$d\,\text{vol}(F_\theta) = \sqrt{-G}\, d\gamma \wedge du^1 \wedge \cdots \wedge du^{2n+1},$$

is the canonical volume form on $C(M)$, associated to the Lorentzian metric F_θ, where we have set

$$G = \det\left[g_{rs}\right], \quad g_{rs} = F_\theta\left(\partial_r\,,\ \partial_s\right),$$
$$\partial_s \equiv \frac{\partial}{\partial u^s}\,, \quad 0 \leq s \leq 2n+1, \quad u^0 = \gamma\,,$$
$$u^A = x^A \circ \pi\,, \quad 1 \leq A \leq 2n+1,$$

and $(U,\ x^A)$ is an arbitrary local coordinate system on M. The Euler–Lagrange equations of the variational principle $\delta\,\mathbb{E}_D(\Phi) = 0$ are

$$\tau_{F_\theta}(\Phi) = 0, \tag{37}$$

where $\tau_{F_\theta}(\Phi) \in C^\infty(\Phi^{-1}TN)$ is the tension field of Φ i.e.

$$\tau_{F_\theta}(\Phi) = \operatorname{trace}_{F_\theta}\left[\beta_{F_\theta}(\Phi)\right],$$
$$\beta_{F_\theta}(\Phi)(A, B) = D^\Phi_A \Phi_* B - \Phi_* \nabla^{F_\theta}_A B,$$
$$D^\Phi = \Phi^{-1}\nabla^h\,, \quad A,\ B \in \mathfrak{X}\left(C(M)\right).$$

Let $\Phi : C(M) \to N$ be a C^∞ map. For each point $p \in C(M)$ we set

$$\mathscr{V}^\Phi_p = \operatorname{Ker}\left(d_p\Phi\right), \quad \mathscr{H}^\Phi_p = \left(\mathscr{V}^\Phi_p\right)^\perp\,,$$

(the perp space is meant with respect to F_θ).

Definition 5 $\Phi : C(M) \to N$ is *nondegenerate* at $p \in C(M)$ if $\mathscr{V}^\Phi_p$ is a nondegenerate subspace of the inner product space $\left(T_p(C(M)),\ F_{\theta,\,p}\right)$. Otherwise Φ is *degenerate* at $p \in C(M)$. □

We also recall (cf. [5, p. 444], or Fuglede [35]).

Definition 6 Let $\Phi : C(M) \to N$ be a C^1 map, and let $p \in C(M)$ be a point. Φ is *horizontally weakly conformal* at p provided that

(i) If $p \in C(M) \setminus \operatorname{Crit}(\Phi)$ and $\mathscr{V}^\Phi_p$ is nondegenerate, then the differential $d_p\Phi : \mathcal{H}^\Phi_p \to T_{\Phi(p)}(N)$ is on-to, and there is a unique nonzero number $L(p) \in \mathbb{R}\backslash\{0\}$ such that

$$h_{\Phi(p)}\left((d_p\Phi)X,\ (d_p\Phi)Y\right) = L(p)\, F_{\theta,\,p}(X, Y), \tag{38}$$

for any $X, Y \in \mathcal{H}^\Phi_p$.

(ii) If $p \in C(M)$ and $\mathscr{V}^\Phi_p$ is degenerate, then

$$\mathscr{H}^\Phi_p \subset \mathscr{V}^\Phi_p, \tag{39}$$

[i.e. $F_{\theta,\,p}(X, Y) = 0$ for any $X, Y \in \mathscr{H}^\Phi_p$]. The number $L(p)$ is the (*square*) *dilation* at p. □

It is customary to set $L(p) = 0$ when $p \in \text{Crit}(\Phi)$ or $\mathscr{V}_p^\Phi$ is degenerate. The resulting function $L : C(M) \to \mathbb{R}$ [the (*square*)[1] *dilation* of Φ] is continuous. Also $\Phi \in C^\infty \Longrightarrow L \in C^\infty$. Occasionally we refine the notation to $L = L(\Phi)$. We shall need the following characterization of horizontal weak conformality (cf. [5, pp. 444–445]).

Lemma 8 *Let $\Phi : C(M) \to N$ be a C^1 map, and let $p \in C(M)$. The following statements are equivalent*

(i) *Φ is horizontally weakly conformal at $p \in C(M)$, with dilation $L(p)$.*

(ii) *There is $L(p) \in \mathbb{R}$ such that*

$$(\Phi^* h)_p = L(p)\, F_{\theta,\, p} \quad \text{on} \quad \mathscr{H}_p^\Phi \times \mathscr{H}_p^\Phi,$$

and $L(p) \neq 0 \Longrightarrow d_p \Phi$ is on-to.

(iii) *There is $L(p) \in \mathbb{R}$ such that, for every local coordinate system $(V,\, y^i)$ on N about $\Phi(p)$*

$$F_\theta(\nabla \Phi^i, \nabla \Phi^j)_p = L(p)\, h^{ij}\big(\Phi(p)\big), \quad 1 \leq i,\, j \leq m,$$

where $\Phi^i = y^i \circ \Phi$.

Here $\nabla u = \nabla^{F_\theta} u$ is the gradient of $u \in C^1(C(M))$ with respect to the Fefferman metric i.e. $F_\theta(\nabla u,\, X) = X(u)$ for any $X \in \mathfrak{X}(C(M))$.

We shall need the following result (the semi-Riemannian version of the *Fuglede–Ishihara theorem*, cf. Theorem 14.6.2 in [5, p. 447], or Fuglede [35])

Theorem 7 *A C^2 map $\Phi : C(M) \to N$ is a harmonic morphism of the Lorentzian manifold $(C(M),\, F_\theta)$ into the Riemannian manifold $(N,\, h)$ if and only if Φ is both a harmonic map, and a horizontally weakly conformal map.*

We now attack the remaining part of the proof of Theorem 3. We start by observing that

Lemma 9 *The dilation $L(\Phi)$ is S^1-invariant.*

Proof Indeed the distributions $\mathscr{V}^\Phi$ and $\mathscr{H}^\Phi$ are invariant by right translations with respect to the natural action of S^1 on $C(M)$ i.e. for every $p \in C(M)$ and $a \in S^1$

$$(d_p R_a)\, \mathscr{V}_p^\Phi = \mathscr{V}_{p \cdot a}^\Phi \quad \text{and} \quad (d_p R_a)\, \mathscr{H}_p^\Phi = \mathscr{H}_{p \cdot a}^\Phi.$$

Here $R_a : C(M) \to C(M)$ denotes the right translation with $a \in S^1$. Next [as Φ is horizontally weakly conformal]

$$(\Phi^* h)_p = L(\Phi)_p\, F_{\theta,p} \quad \text{on} \quad \mathscr{H}_p^\Phi \times \mathscr{H}_p^\Phi. \tag{40}$$

[1] Which is the same as the terminology adopted in the Riemannian case [keeping in mind that eventually $L\big(C(M)\big) \cap (-\infty,\, 0) \neq \emptyset$].

Let $a \in S^1$ and $u,\ v \in \mathscr{H}_p^{\Phi}$, and let us set

$$u' = (d_p R_a)u, \quad v' = (d_p R_a)v.$$

Then [by (40)]

$$\begin{aligned} L(\Phi)_{p\cdot a}\, F_{\theta,\, p\cdot a}(u',\ v') &= h_{\Phi(p)}\big(d_p(\Phi \circ R_a)u,\ d_p(\Phi \circ R_a)v\big) \\ &= h_{\phi(p)}\big((d_p\Phi)u,\ (d_p\Phi)v\big) = L(\Phi)_p\, F_{\theta,p}(u, v), \end{aligned}$$

and [by $S^1 \subset \mathrm{Isom}(C(M), F_\theta)$]

$$F_{\theta,\, p\cdot a}(u',\ v') = \big(R_a^* F_\theta\big)_p\big(u,\ v\big) = F_{\theta,\, p}(u, v),$$

yielding

$$\big[L(\Phi)_{p\cdot a} - L(\Phi)_p\big]\, F_{\theta,\, p}(u,\ v) = 0,$$

so that $L(\Phi)_{p\cdot a} = L(\Phi)_p$ when $\mathscr{H}_p^{\Phi}$ is nondegenerate, and $L(\Phi)_p = 0$ when $\mathscr{H}_p^{\Phi}$ is degenerate. Once again, as the right translation R_a is an isometry, the degeneracy of $\mathscr{H}_p^{\Phi}$ implies that of $\mathscr{H}_{p\cdot a}^{\Phi}$, and hence $L(\Phi)_{p\cdot a} = 0$.

□

Next, we relate the horizontal weak conformality condition on $\Phi = \phi \circ \pi$ to the Levi conformality condition on ϕ. Let us set

$$\Phi^j = y^j \circ \Phi, \quad \phi^j = y^j \circ \phi, \quad 1 \leq j \leq m.$$

Let $\{E_a\ :\ 1 \leq a \leq 2n\}$ be a local G_θ-orthonormal frame of $H(M)$

$$G_\theta\big(E_a\ ,\ E_b\big) = \delta_{ab}\ , \quad 1 \leq a,\ b \leq 2n,$$

defined on the open set $U \subset M$. Then

$$\begin{aligned} &\{\mathcal{E}_\alpha\ :\ 0 \leq \alpha \leq 2n+1\}, \\ &\mathcal{E}_0 = T^\uparrow - S, \quad \mathcal{E}_a = E_a^\uparrow\ , \quad 1 \leq a \leq 2n, \quad \mathcal{E}_{2n+1} = T^\uparrow + S, \end{aligned}$$

is a local F_θ-orthonormal frame of $T(C(M))$ i.e.

$$F_\theta\big(\mathcal{E}_\alpha\ ,\ \mathcal{E}_\beta\big) = \epsilon_\alpha\, \delta_{\alpha\beta}\ , \quad \epsilon_0 = -1, \quad \epsilon_j = 1, \quad 1 \leq j \leq 2n+1,$$

on $\pi^{-1}(U)$. Then

$$\nabla \Phi^j = \lambda^\alpha\, \mathcal{E}_\alpha,$$

for some $\lambda^\alpha \in C^\infty\big(\pi^{-1}(U)\big)$. Contracting with F_θ one gets

$$\lambda^0 = -T^\uparrow(\Phi^j), \quad \lambda^a = E_a^\uparrow(\Phi^j), \quad \lambda^{2n+1} = T^\uparrow(\Phi^j),$$

namely

$$\nabla \Phi^j = \sum_{a=1}^{2n} E_a^\uparrow(\Phi^j) E_a^\uparrow + 2T^\uparrow(\Phi^j) S. \tag{41}$$

Consequently

$$\pi_* \nabla \Phi^j = \left(\nabla^H \phi^j\right)^\pi . \tag{42}$$

We recall the following (cf. Proposition 14.5.4 in [5, p. 445])

Proposition 1 *A C^∞ map $\Phi : C(M) \to N$ is horizontally weakly conformal at $p \in C(M)$ with (square) dilation $L(p)$ if and only if one of the following statements holds*

(i) $L(p) \neq 0$, *$d_p\Phi$ is on-to, and $(\Phi^* h)_p = L(p)\, F_{\theta,\, p}$ on $\mathscr{H}_p^\Phi \times \mathscr{H}_p^\Phi$.*

(ii) $p \in \mathrm{Crit}(\Phi)$ [*so that $L(p) = 0$ and $d_p\Phi = 0$*].

(iii) *$\mathscr{V}_p^\Phi$ is degenerate and $\mathscr{H}_p^\Phi \subset \mathscr{V}_p^\Phi$* [*so that $L(p) = 0$ yet $d_p\Phi \neq 0$*].

Statement (a) in Theorem 3 is proved in two steps i.e. we show that

Lemma 10 *Φ is nondegenerate at $p \Longleftrightarrow L(\Phi)_p > 0 \Longleftrightarrow \pi(p) \in \Omega(\phi)$.*

We proceed by distinguishing between the cases contemplated by Proposition 1. To start with, let us assume that $L(p) \neq 0$. Then $d_p\Phi \neq 0$, the restriction of $d_p\Phi$ to $\mathscr{H}_p^\Phi$ is surjective, and (38) holds. Moreover Φ is nondegenerate at p and [by (iii) in Lemma 8] $\{(\nabla\Phi^j)_p \;\; 1 \leq j \leq m\}$ is a linear basis in $\mathscr{H}_p^\Phi$. Once again by (iii) in Lemma 8

$$L(\Phi)_p\; h^{jj}(\Phi(p)) = F_{\theta,\, p}(\nabla\Phi^j, \nabla\Phi^j) =,$$

[by (41) together with the fact that S is lightlike and F_θ-orthogonal to each $E_a^\uparrow$]

$$= \sum_{a=1}^{2n} \left[E_a^\uparrow(\Phi^j)(p)\right]^2 \geq 0.$$

Hence [as h is Riemannian] $0 \neq L(\Phi)_p \geq 0$ i.e. $L(\Phi)_p > 0$. Thus $L(\Phi)_p > 0$ is a necessary condition for the nondegeneracy of Φ at p. Clearly, it also suffices [if $L(\Phi)_p > 0$ then $d_p\Phi$ is onto and $\mathscr{H}_p^\Phi$ is space-like by (ii)of Lemma 8]. So Φ is nondegenerate at $p \Longleftrightarrow L(\Phi)_p > 0$. Next [by Lemma 9] there is a C^∞ function $\ell(\Phi) : M \to (0, +\infty)$ such that $L(\Phi) = \ell(\Phi) \circ \pi$. The horizontal weak conformality condition on Φ is then

$$F_\theta\big(\nabla\Phi^i\,,\; \nabla\Phi^j\big)_p = \ell(\Phi)_{\pi(p)}\, h^{ij}(\Phi(p)), \tag{43}$$

or [by (34)]

$$\begin{aligned}&\left(\pi^*\tilde{G}_\theta\right)(\nabla\Phi^i, \nabla\Phi^j)_p + (\pi^*\theta)(\nabla\Phi^i)_p\, \sigma(\nabla\Phi^j)_p + (\pi^*\theta)(\nabla\Phi^j)_p\, \sigma(\nabla\Phi^i)_p \\ &\quad = \ell(\Phi)_{\pi(p)}\, h^{ij}(\phi(\pi(p))),\end{aligned}$$

hence [by (42), and then by $\left(\pi^*\theta\right)(\nabla\Phi^j) = 0$]

$$G_\theta(\nabla^H\phi^i\,,\ \nabla^H\phi^j)_x = \ell(\Phi)_x\, h^{ij}(\phi(x)), \quad x = \pi(p). \tag{44}$$

Next [by the Levi-conformality condition on ϕ]

$$G_\theta(\nabla^H\phi^i\,,\ \nabla^H\phi^j)_x = \Lambda(x)\, h^{ij}(\phi(x)),$$

hence [as h is positive definite and $\Lambda(\phi)_x = \ell(\Phi)_x$]

$$L(\Phi)_p = \Lambda(\phi)_{\pi(p)},$$

so that $L(\Phi)(p) > 0 \Longleftrightarrow x \in M\backslash[Z(\Lambda(\phi))]$. □

Let us now examine the case $L(p) = 0$, when either $p \in \text{Crit}(\Phi)$ or $d_p\Phi \neq 0$, $\mathscr{V}_p^\Phi$ is degenerate, and $\mathscr{H}_p^\Phi \subset \mathscr{V}_p^\Phi$.

(b) If $p \in \text{Crit}(\Phi)$ then $x = \pi(p) \in \text{Crit}(\phi)$ and conversely. Indeed let $d_p\Phi = 0$. Then $\left(\nabla\Phi^j\right)_p = 0$ hence [by (41)]

$$0 = (d_p\pi)(\nabla\Phi^j)_p = (\nabla^H\phi^j)_x,$$

so that $(\nabla\phi^j)_x = T(\phi^j)_x\, T_x$. Yet

$$0 = (d_p\Phi^j)T_p^\uparrow = (d_x\phi^j)T_x = T(\phi^j)_x,$$

so that $(\nabla\phi^j)_x = 0$ i.e. $d_x\phi = 0$. □

(c) If $d_p\Phi \neq 0$, $\mathscr{V}_p^\Phi$ is degenerate, and $\mathscr{H}_p^\Phi \subset \mathscr{V}_p^\Phi$, then (equivalently) $x \in Z(\Lambda) \setminus \text{Crit}(\phi)$. Indeed, if $x \notin Z(\Lambda)$ then [by our discussion of the case $L(p) \neq 0$] Φ is nondegenerate at p, while if $x \in \text{Crit}(\phi)$ then $p \in \text{Crit}(\Phi)$ [by statement (b) in Theorem 3]. The proof of statement (c) in Theorem 3 is now completed by applying Theorem 6 to $x = \pi(p) \in Z(\Lambda)\backslash\text{Crit}(\phi)$.

5 Harmonic Morphisms Within Foliation Theory

Let $\phi : M \to N$ be a subelliptic harmonic morphism of (M, θ) into (N, h), of θ-dilation $\lambda(\phi)$, and let $\Phi = \phi \circ \pi$ be its vertical lift [a harmonic morphism of square dilation $L(\Phi) = \lambda^2(\phi) \circ \pi$]. Let $S(\phi)$ be the set of all submersive points of the morphism ϕ (cf. Definition 3 above). The connected components of the fibres of

$\phi : S(\phi) \to N$ are the leaves of a foliation $\mathscr{F}$ of $S(\phi)$. Let us set

$$S(\Phi) := \pi^{-1}\big[S(\phi)\big] \subset C(M).$$

Then $\Phi : S(\Phi) \to N$ is a submersion and the corresponding foliation of $S(\Phi)$ is the pullback of $\mathscr{F}$ by π i.e. the foliation $\pi^*\mathscr{F}$ of $C(M)$ whose tangent bundle is

$$T(\pi^*\mathscr{F}) = T(\mathscr{F})^{\uparrow} \oplus \mathrm{Ker}(d\pi).$$

The horizontal lift is meant with respect to the Graham connection σ. Cf. Molino [51, p. 54], and Dragomir and Nishikawa [26]. Cf. also [10].

Let $Q = \nu(\mathscr{F}) = T(M)/\mathscr{V}^{\phi}$ be the transverse bundle, and let $\Pi : T(M) \to Q$ be the projection. Let $\sigma_Q : Q \to \mathscr{H}^{\phi}$ be the vector bundle isomorphism

$$\sigma_Q(s) = \mathcal{H}Y = Y^{\mathcal{H}}, \quad s \in Q, \quad Y \in T(M), \quad \Pi(Y) = s,$$

and let g_Q be the Riemannian bundle metric

$$g_Q\big(s,\ r\big) = g_\theta\big(\sigma_Q(s),\ \sigma_Q(r)\big), \quad s,\ r \in Q.$$

Let us consider the Q-valued symmetric 2-form α on $\mathscr{V}^{\phi} \otimes \mathscr{V}^{\phi}$, the bundle endomorphism $W(Z) : \mathscr{V}^{\phi} \to \mathscr{V}^{\phi}$, and the basic 1-form $\kappa \in \Omega^1_B(\mathscr{F})$, given by

$$\begin{aligned} g_{\mathscr{F}}\big(X,\ X'\big) &= g_\theta\big(X,\ X'\big), \quad \alpha(X,\ X') = \Pi\, \nabla^{g_\theta}_X X', \\ g_{\mathscr{F}}\big(W(Z)X,\ X'\big) &= g_Q\big(\alpha(X,\ X'),\ Z\big), \\ \kappa(Z) &= \mathrm{Trace}\, W(Z), \end{aligned}$$

for any $X,\ X' \in \mathscr{V}^{\phi}$ and $Z \in \mathscr{H}^{\phi}$. We follow the notations and conventions in Tondeur [61]. Let $\chi_{\mathscr{F}} \in \Omega^p(M)$ [with $p = 2n - m + 1$] be the tangential volume form i.e.

$$\begin{aligned} &\chi_{\mathscr{F}}\big(Y_1,\ \ldots,\ Y_p\big) = \det\Big[g_\theta\big(Y_i,\ E_j\big)\Big]_{1 \le i,\ j \le p}, \\ &\big\{E_i\ :\ 1 \le i \le p\big\} \equiv \big\{V_j,\ \mathscr{T}\ :\ 1 \le j \le 2n - m\big\}, \\ &Y_1,\ \ldots,\ Y_p \in T(M). \end{aligned}$$

Note that $\mathscr{H}^{\phi} \rfloor \chi_{\mathscr{F}} = 0$. Rummler's formula is (cf. Eq. (6.17) in [61, p. 66])

$$\big(\mathscr{L}_Z\ \chi_{\mathscr{F}}\big)\big|_{\mathscr{V}^{\phi}} = -\kappa(Z)\ \chi_{\mathscr{F}}\,\big|_{\mathscr{V}^{\phi}}, \tag{45}$$

where $\mathscr{L}_Z$ is the Lie derivative. Next, let

$$\mathrm{div}_{\mathscr{F}}(X) = \sum_{i=1}^{p} g_\theta\big(\nabla^{g_\theta}_{E_i} X,\ E_i\big), \quad X \in \mathscr{V}^{\phi},$$

[globally defined, as the trace of $X' \in \mathscr{V}^\phi \mapsto \pi^{\mathscr{V}^\phi} \nabla^{g_\theta}_{X'} X$] be the divergence operator along the leaves. Similar to (45)

$$\begin{aligned}&\big(\mathscr{L}_X\, \chi_{\mathscr{F}}\big)\big(X_1\,,\,\dots\,,\,X_p\big) = \operatorname{div}_{\mathscr{F}}(X)\;\chi_{\mathscr{F}}\big(X_1\,,\,\dots\,,\,X_p\big),\\ &X_1\,,\,\dots\,,\,X_p \in \mathscr{V}^\phi.\end{aligned}$$

Indeed

$$\begin{aligned}&\big(\mathscr{L}_X\, \chi_{\mathscr{F}}\big)\big(E_1\,,\,\dots\,,\,E_p\big) = X\big(\chi_{\mathscr{F}}\big(E_1\,,\,\dots\,,\,E_p\big)\big)\\ &\quad + - \sum_{j=1}^{p} \chi_{\mathscr{F}}\big(E_1\,,\,\dots\,,\,E_{j-1}\,,\,\big[X,\,E_j\big],\,E_{j+1}\,,\,\dots\,,\,E_p\big) =,\end{aligned}$$

[by $\chi_{\mathscr{F}}\big(E_1\,,\,\dots\,,\,E_p\big) = 1$ and $\pi_{\mathscr{V}^\phi}\,\big[X,\,E_j\big] = \sum_{i=1}^{p} g_\theta\big(\big[X,\,E_j\big],\,E_i\big)\,E_i$]

$$= -\sum_{i=1}^{p} g_\theta\big(\big[X,\,E_i\big],\,E_i\big) = -\sum_{i=1}^{p} g_\theta\big(\nabla^{g_\theta}_X E_i - \nabla^{g_\theta}_{E_i} X,\,E_i\big) =,$$

[by $\nabla^{g_\theta} g_\theta = 0$ and $\|E_i\| = 1$]

$$= \sum_{i=1}^{p} g_\theta\big(\nabla^{g_\theta}_{E_i} X,\,E_i\big) = \operatorname{div}_{\mathscr{F}}(X).$$

□

5.1 Mean Curvature of Fibres

Let $(\mathfrak{M}^{\mathrm{N}},\,g)$ be a N-dimensional semi-Riemannian manifold, equipped with the semi-Riemannian metric g, and let D be a linear connection on $\mathfrak{M}$. Let $\mathscr{D}$ be a C^∞ distribution on $\mathfrak{M}$, of rank $1 \le r \le \mathrm{N} - 1$, and such that $\mathscr{D}_x$ is a nondegenerate subspace of $(T_x(\mathfrak{M}),\,g_x)$, for any $x \in \mathfrak{M}$. Let $\mathscr{D}^\perp$ be the orthogonal complement of $\mathscr{D}$, and let $\pi^\perp : T(\mathfrak{M}) \to \mathscr{D}^\perp$ be the projection associated to the direct sum decomposition $T(M) = \mathscr{D} \oplus \mathscr{D}^\perp$. Let us consider the bilinear form $B_{\mathscr{D}} = B_{\mathscr{D}}(g,\,D)$ given by

$$B_{\mathscr{D}}(X,\,Y) = \pi^\perp\,\nabla_X Y, \quad X,\,Y \in \mathscr{D}.$$

Next, let $\mu^{\mathscr{D}} = \mu^{\mathscr{D}}(g,\,D)$ be given by

$$\mu^{\mathscr{D}} = \frac{1}{r}\,\mathrm{Trace}_g\,B_{\mathscr{D}} \in C^\infty\big(\mathscr{D}^\perp\big). \tag{46}$$

When $D = \nabla^g$ [the Levi-Civita connection of $(\mathfrak{M}, g)$] $\mu^{\mathscr{D}} = \mu^{\mathscr{D}}(g, \nabla^g)$ is the *mean curvature* vector of $\mathcal{D}$ (cf. e.g. Definition 1.26 in [29, p. 37]). Given a subelliptic harmonic morphism $\phi : M \to N$ under the assumptions of Theorem 6, we consider both the mean curvature vector of $\mathscr{V}^\phi$ in (M, g_θ)

$$\mu^{\mathscr{V}^\phi} \equiv \mu^{\mathscr{V}^\phi}\big(g_\theta\,,\ \nabla^{g_\theta}\big) \in C^\infty\big(\mathscr{H}^\phi\big), \tag{47}$$

and its pseudohermitian analog [got by replacing the Levi–Civita connection of (M, g_θ) by the Tanaka–Webster connection ∇ of (M, θ)]

$$\mu^{\mathscr{V}^\phi}\big(g_\theta\,,\ \nabla\big) \in C^\infty\big(\mathscr{H}^\phi\big).$$

From now on, let us assume that $m \geq 2$ so that [by Theorem 6]

$$S(\phi) = \Omega(\phi) = M \setminus Z\big(\Lambda(\phi)\big).$$

By arguing as in the proof of Theorem 3 [case $L(p) \neq 0$] for every $p \in S(\Phi) = \pi^{-1}\big[S(\phi)\big]$ the horizontal space $\mathscr{H}^\Phi_p$ is space-like i.e. $F_{\theta,\,p}$ is positive definite on $\mathscr{H}^\Phi_p$. Consequently, for every $p \in S(\Phi)$ the vertical space $\mathscr{V}^\Phi_p$ has index $\operatorname{ind} \mathscr{V}^\Phi_p = 1$, i.e. $F_{\theta,\,p}$ has signature $(1,\ 2n - m + 1)$ on $\mathscr{V}^\Phi_p$. Therefore

Lemma 11 *$C(M)$ is foliated by $(2n-m+2)$-dimensional Lorentzian manifolds, whose normal bundles are spacelike.*

Let β_p be the inverse of $d_p\Phi : \mathscr{H}^\Phi_p \simeq T_{\Phi(p)}(N)$ [$\beta : \Phi^{-1}T(N) \to \mathscr{H}^\Phi$ is the *horizontal lift*, a vector bundle isomorphism].

Lemma 12 *Let $\phi : M \to N$ be a subelliptic harmonic morphism of (M, θ) into (N, h), of θ-dilation $\lambda(\phi)$, and let $\Phi = \phi \circ \pi : C(M) \to N$ be its vertical lift $\big($a harmonic morphism of square dilation $\ell(\Phi)^2 = \big[\lambda(\phi)\circ\pi\big]^2\big)$. The second fundamental form $B_{F_\theta}(\Phi)$ of Φ satisfies*

$$\begin{aligned} &B_{F_\theta}(\Phi)(H, H)\\ &\quad = 2\,H\,(\log\,\ell(\Phi))\ \ \Phi_* H - F_\theta(H, H)\ \Phi_*\nabla\,(\log\,\ell(\Phi))\,, \end{aligned} \tag{48}$$

$$\begin{aligned} &B_{F_\theta}(\Phi)(V, V) = -\Phi_* \nabla^{F_\theta}_V V\,,\\ &\quad H \in C^\infty\big(S(\Phi),\ \mathscr{H}^\Phi\big), \quad V \in C^\infty\big(S(\Phi),\ \mathscr{V}^\Phi\big), \end{aligned} \tag{49}$$

everywhere in $S(\Phi)$.

The proof of Lemma 12 is a *verbatim* repetition of the arguments in [5, pp. 119–120]. Let us check for instance (48). To this end, let $\{Z_\alpha\ :\ 1 \leq \alpha \leq m\}$ be a local h-orthonormal frame of $T(N)$, defined on the open set $V \subset N$. Then [by (ii) in Lemma 8]

$$\big\{\ell(\Phi)\,\beta\,Z_\alpha\ :\ \alpha \leq j \leq m\big\},$$

is a local F_θ-orthonormal frame for $\mathscr{H}^\Phi$, defined on $\mathscr{U} = \Phi^{-1}(V)$. Let $Y \in \mathfrak{X}(V)$ and let $H = \beta Y \in C^\infty(\mathscr{U}, \mathscr{H}^\Phi)$. Then

$$2\,\pi_{\mathscr{H}^\Phi}\left\{\nabla^{F_\theta}_H H\right\} = 2\,\ell(\Phi)^2 \sum_{\alpha=1}^m F_\theta\big(\nabla^{F_\theta}_H H,\, \beta Z_\alpha\big)\, \beta Z_\alpha =,$$

(by the explicit expression of ∇^{F_θ} as a Levi–Civita connection, cf. e.g. Proposition 2.3 in [45, 46, Vol. I, p. 160])

$$= \ell(\Phi)^2 \sum_{\alpha=1}^m \Big\{2\,H\big(F_\theta(H,\, \beta Z_\alpha)\big) - (\beta Z_\alpha)\big(F_\theta(H, H)\big) + \\ -2\,F_\theta(H,\, [H,\, \beta Z_\alpha])\Big\}\, \beta Z_\alpha =,$$

[by (ii) in Lemma 8 i.e.

$$F_\theta(H,\, \beta Z_\alpha) = \ell(\Phi)^{-2}\, h(H,\, Z_\alpha) \circ \Phi,$$

and by

$$\Phi_*\big[H,\, \hat{Z}_\alpha\big] = \big[Y,\, Z_\alpha\big] \circ \Phi,$$

cf. e.g. Proposition B.1 in[2] [27, pp. 303–304]]

$$= \ell(\Phi)^2 \sum_{\alpha=1}^m \Big\{2\,H\big(\ell(\Phi)^{-2}\big)\, h(Y,\, Z_\alpha) \circ \Phi \\ +2\,\ell(\Phi)^{-2}\, H\big(h(Y,\, Z_\alpha) \circ \Phi\big) - (\beta Z_\alpha)\big(\ell(\Phi)^{-2}\big)\, h(Y,\, Y) \circ \Phi\ + \\ -\ell(\Phi)^{-2}\,(\beta Z_\alpha)\big(h(Y,\, Y) \circ \Phi\big) - 2\,\ell(\Phi)^{-2} h(Y,\, [Y,\, Z_\alpha]) \circ \Phi\Big\}\, \beta Z_\alpha.$$

Next [again by Proposition 2.3 in [45, 46, Vol. I, p. 160], applied to the Levi–Civita connection ∇^h]

$$\pi_{\mathscr{H}^\Phi}\left\{\nabla^{F_\theta}_H H\right\} = \left(\nabla^h_Y Y\right) \circ \Phi + \sum_{\alpha=1}^m \Big\{ -2\,H(\log\, \ell(\Phi))\, h(Y,\, Z_\alpha) \\ +(\beta Z_\alpha)(\log\, \ell(\Phi))\, h(Y,\, Y)^\Phi\Big\}\, \beta Z_\alpha,$$

and

$$\sum_{\alpha=1}^m (\beta Z_\alpha)(\log\, \ell(\Phi))\, h(Y,\, Y)^\Phi\, \hat{Z}_\alpha = F_\theta(H,\, H)\, \pi_{\mathscr{H}^\Phi}\, \nabla\, \log\, \ell(\Phi),$$

[2] Appendix B in [27, pp. 303–306], is concerned with the geometry of *Riemannian* submersions, yet Proposition B.1 in there transposes *ad literam* to semi-Riemannian submersions (submersions of semi-Riemannian manifolds, with nondegenerate fibres) and in particular to $\Phi : \big(C(M),\, F_\theta\big) \to (N,\, h)$.

so that

$$\beta_{F_\theta}(\Phi)(H, H) = D^{\Phi}_H \Phi_* H - \Phi_* \pi_{\mathscr{H}^\Phi} \left\{ \nabla^{F_\theta}_H H \right\}$$
$$= 2\, H\big(\log\, \ell(\Phi)\big)\Phi_* H - F_\theta(H,\ H)\, \Phi_* \nabla\, \log\, \ell(\Phi).$$

□

Lemma 13 *Under the assumptions of Lemma 12, the tension field of* Φ *is given by*

$$\tau_{F_\theta}(\Phi) = -(m-2)\, \Phi_* \nabla\, \log\, \ell(\Phi) - (2n - m + 2)\, \Phi_*\, \mu^{\mathscr{V}^\Phi}, \tag{50}$$

everywhere in $S(\Phi)$ [*the set of submersive points of* Φ].

Proof Let

$$\left\{ H_\alpha\ :\ 1 \le \alpha \le m \right\} \subset \mathscr{H}^\Phi,$$
$$\left\{ V_k\ :\ 0 \le k \le 2n - m + 1 \right\} \subset \mathscr{V}^\Phi, \quad F_\theta(V_0\,,\ V_0) = -1,$$

be local F_θ-orthonormal frames. Then

$$\tau_{F_\theta}(\Phi) = \text{Trace}_{F_\theta} \left\{ \beta_{F_\theta}(\Phi) \right\}$$
$$= \sum_{\alpha=1}^{m} \beta_{F_\theta}(\Phi)(H_\alpha\,,\ H_\alpha) + \sum_{k=1}^{2n-m+1} \beta_{F_\theta}(\Phi)(V_k\,,\ V_k) - \beta_{F_\theta}(\Phi)(V_0\,,\ V_0),$$

and [by Lemma 12]

$$\beta_{F_\theta}(\Phi)\big(H_\alpha\,,\ H_\alpha\big) = 2\, H_\alpha\big(\log\, \ell(\Phi)\big)\, \Phi_*\, h_\alpha - \Phi_* \nabla\, \log\, \ell(\Phi),$$

so that

$$\sum_{\alpha=1}^{m} \beta_{F_\theta}(\Phi)(H_\alpha\,,\ H_\alpha) = 2\, \Phi_* \nabla\, \log\, \ell(\Phi) - m\, \Phi_* \nabla\, \log\, \ell(\Phi).$$

Also

$$\beta_{F_\theta}(\Phi)(V_k\,,\ V_k) = -\Phi_* \nabla^{F_\theta}_{V_k} V_k\,,$$
$$\sum_{k=1}^{2n-m+1} \beta_{F_\theta}(\Phi)(V_k\,,\ V_k) - \beta_{F_\theta}(\Phi)(V_0\,,\ V_0) = -(2n - m + 2)\, \Phi_*\, \mu^{\mathscr{V}^\Phi}.$$

□

Next, we project (50) on the base manifold M, so that to get a subelliptic version of the fundamental equation for a harmonic morphism (cf. e.g. Eq. (4.5.2) in [5, p. 129]), applying to the base map ϕ. We start by recalling the following result[3] (relating the Levi–Civita connection ∇^{F_θ} of $(C(M),\ F_\theta)$ to the Tanaka–Webster connection ∇ of $(M,\ \theta)$, cf. [9, p. 26] or [1])

Lemma 14 *For any $X, Y \in H(M)$*

$$\nabla^{F_\theta}_{X^\uparrow} Y^\uparrow = (\nabla_X Y)^\uparrow + \Omega(X, Y)\, T^\uparrow - \Big[A(X, Y) + (d\sigma)(X^\uparrow, Y^\uparrow)\Big] S\,, \tag{51}$$

$$\nabla^{F_\theta}_{X^\uparrow} T^\uparrow = (\tau X + \varphi X)^\uparrow\,, \tag{52}$$

$$\nabla^{F_\theta}_{T^\uparrow} X^\uparrow = (\nabla_T X + \varphi X)^\uparrow + 2(d\sigma)(X^\uparrow, T^\uparrow) S, \tag{53}$$

$$\nabla^{F_\theta}_{X^\uparrow} S = \nabla^{F_\theta}_{S} X^\uparrow = \frac{1}{2}\,(JX)^\uparrow, \tag{54}$$

$$\nabla^{F_\theta}_{T^\uparrow} T^\uparrow = V^\uparrow\,, \tag{55}$$

$$\nabla^{C(M)}_{S} S = \nabla^{C(M)}_{S} T^\uparrow = \nabla^{C(M)}_{T^\uparrow} S = 0, \tag{56}$$

where $\varphi : H(M) \to H(M)$ and $V \in H(M)$ are given by

$$G_\theta(\varphi X,\ Y) = (d\sigma)(X^\uparrow\,,\ Y^\uparrow), \quad G_\theta(V,\ Y) = 2\,(d\sigma)(T^\uparrow, Y^\uparrow).$$

The tension field $\tau_{F_\theta}(\Phi)$ may be shown to project on $\tau_b(\phi)$ i.e.

$$\tau_{F_\theta}(\Phi) = \tau_b(\phi) \circ \pi. \tag{57}$$

To prove (57) let $\{E_a\ :\ 1 \le a \le 2n\}$ be a local G_θ-orthonormal frame of $H(M)$, defined on the open set $U \subset M$. Then

$$\big\{E^\uparrow_a\,,\ T^\uparrow \pm S\ :\ 1 \le a \le 2n\big\},$$

is a local F_θ-orthonormal frame for $T\big(C(M)\big)$, defined on $\mathscr{U} = \pi^{-1}(U)$, so that

$$\tau_{F_\theta}(\Phi) = \sum_{a=1}^{2n} \Big\{D^\Phi_{E^\uparrow_a} \Phi_* E^\uparrow_a - \Phi_* \nabla^{F_\theta}_{E^\uparrow_a} E^\uparrow_a\Big\} + D^\Phi_{T^\uparrow+S} \Phi_*(T^\uparrow + S)$$
$$- \Phi_* \nabla^{F_\theta}_{T^\uparrow+S}(T^\uparrow + S) - D^\Phi_{T^\uparrow-S}\Phi_*(T^\uparrow - S) + \Phi_* \nabla^{C(M)}_{T^\uparrow-S}(T^\uparrow - S).$$

Also, for every $X \in H(M)$ [by Lemma 14 and $\Phi_* S = 0$]

$$D^\Phi_{X^\uparrow} \Phi_* X^\uparrow = \Big\{D^\phi_X \phi_* X\Big\} \circ \pi\,,$$
$$\Phi_* \nabla^{F_\theta}_{X^\uparrow} X^\uparrow = (\phi_* \nabla_X X) \circ \pi\,,$$
$$D^\Phi_{T^\uparrow+S} \Phi_* (T^\uparrow + S) - D^\Phi_{T^\uparrow-S} \Phi_* (T^\uparrow - S) = 2\, D^\Phi_S\, \Phi_* T^\uparrow = 0,$$
$$-\nabla^{F_\theta}_{T^\uparrow+S}(T^\uparrow + S) + \nabla^{F_\theta}_{T^\uparrow-S}(T^\uparrow - S) = -2\Big\{\nabla^{F_\theta}_{T^\uparrow} S + \nabla^{F_\theta}_{S} T^\uparrow\Big\} = 0.$$

[3] A missing $1/2$ factor in Lemma 2 of [9, pp. 083504-26], is added here, cf. our identity (54) in Lemma 14.

Finally

$$\tau_{F_\theta}(\Phi) = \sum_{a=1}^{2n} \left\{ D^{\phi}_{E_a} \phi_* \, E_a - \phi_* \, \nabla_{E_a} \, E_a \right\} \circ \pi = \tau_b(\phi) \circ \pi.$$

□

The gradient $\nabla \log \ell(\Phi)$ may be shown to project on the horizontal gradient $\nabla^H \log \lambda(\phi)$. Indeed [by arguing as in the proof of (41)]

$$\nabla \log \ell(\Phi) = \sum_{a=1}^{2n} E_a^{\uparrow}\big(\log \ell(\Phi)\big) \, E_a^{\uparrow} + 2\, T^{\uparrow}\big(\log \ell(\Phi)\big)\, S.$$

Also [by $\ell(\Phi) = \lambda(\phi) \circ \pi$] $X^{\uparrow}(\log \ell(\Phi)) = X(\log \lambda(\phi)) \circ \pi$ hence

$$\Phi_* \, \nabla \log \ell(\Phi) = \left\{ \phi_* \, \nabla^H \log \lambda(\phi) \right\} \circ \pi.$$

□

Next, we seek to project the mean curvature vector

$$\mu^{\mathscr{V}^\Phi} = \mu^{\mathscr{V}^\Phi}\big(F_\theta \, , \, \nabla^{F_\theta}\big) \in C^\infty\big(\mathscr{H}^\Phi\big).$$

We need to produce a local F_θ-orthonormal frame of $\mathscr{V}^\Phi$, adapted to the decomposition

$$T\big(C(M)\big) = H(M)^{\uparrow} \oplus \mathbb{R}T^{\uparrow} \oplus \mathbb{R}S,$$

[and allowing for the use of Lemma 14]. We start from building a local g_θ-orthonormal frame of $\mathscr{V}^\phi$, adapted to the decomposition $T(M) = H(M) \oplus \mathbb{R}$. Once again, difficulties arise from the fact that the pairs of complementary distributions $(H(M), \, \mathbb{R}T)$ and $\big(\mathscr{V}^\phi \, , \, \mathscr{H}^\phi\big)$ intersect, and then the use of Theorem 6 is crucial in ascertaining that the intersections have constant ranks on certain open sets. Indeed

$$\dim_{\mathbb{R}} \big(\mathscr{V}^\phi_H\big)_x = 2n - m,$$

at every point $x \in S(\phi) = \Omega(\phi) = \mathrm{I}_m(\phi) \cup \mathrm{III}_m(\phi)$, provided that $m \geq 2$. Let then

$$\big\{ V_j \, : \, 1 \leq j \leq 2n - m \big\} \subset C^\infty(U, \, \mathscr{V}^\phi_H),$$

be a G_θ-orthonormal frame, defined on the open set $U \subset \Omega(\phi)$. Let us set

$$T^{\mathscr{V}} := \pi_{\mathscr{V}^\phi} \, T \in \mathscr{V}^\phi \, , \quad T^{\mathscr{H}} := T - T^{\mathscr{V}} \in \mathscr{H}^\phi.$$

Our discussion in Sect. 3 shows that $T_x^{\mathscr{V}} \neq 0$ for every $x \in S(\phi)$, hence one may set

$$\mathscr{T} := \frac{1}{\|T^{\mathscr{V}}\|} \, T^{\mathscr{V}} \in C^\infty\big(\Omega(\phi), \, \mathscr{V}^\phi\big).$$

Lemma 15 $\big\{V_j \, , \; \mathscr{T} \, : \, 1 \leq j \leq 2n - m\big\}$ *is a g_θ-orthonormal frame for $\mathscr{V}^\phi$, defined on $U \subset \Omega(\phi)$.*

Proof Note that $\mathscr{T}_x \neq 0$ for every $x \in \Omega(\phi)$ [otherwise $T_{x_0} \in \mathscr{H}^\phi_{x_0}$ for some $x_0 \in \Omega(\phi)$ i.e. $\mathrm{II}_m(\phi) \neq \emptyset$, a contradiction] and $\|\mathscr{T}\| = 1$. Also

$$g_\theta(V_j \, , \, \mathscr{T}) = \frac{1}{\|T^{\mathscr{V}}\|} \, g_\theta\big(V_j \, , \, T^{\mathscr{V}} + T^{\mathscr{H}}\big) = \frac{1}{\|T^{\mathscr{V}}\|} \, g_\theta(V_j \, , \, T) = 0.$$

Also $\theta(\mathscr{T}) \neq 0$ everywhere in $\Omega(\phi)$. Indeed, if $\theta(\mathscr{T})_{x_0} = 0$ for some $x_0 \in \Omega(\phi)$ then

$$\mathscr{T}_{x_0} \in H(M)_{x_0} \cap \mathscr{V}^\phi_{x_0} = \big(\mathscr{V}^\phi_H\big)_{x_0} \Longrightarrow \dim_{\mathbb{R}} \big(\mathscr{V}^\phi_H\big)_{x_0} = 2n - m + 1,$$

a contradiction. □

Using the local frame provided by Lemma 15 one may relate the mean curvature vector $\mu^{\mathscr{V}^\phi} \equiv \mu^{\mathscr{V}^\phi}\big(g_\theta \, , \, \nabla^{g_\theta}\big)$ to its pseudohermitian analog $\mu^{\mathscr{V}^\phi}\big(g_\theta \, , \, \nabla\big)$ i.e.

$$\begin{aligned} (2n - m + 1) \, \mu^{\mathscr{V}^\phi} &= (2n - m + 1) \, \mu^{\mathscr{V}^\phi}\big(g_\theta \, , \, \nabla\big) \\ &+ \pi_{\mathscr{H}^\phi}\Big\{\theta(\mathscr{T}) \, \big(\tau + 2 \, J\big) \, \mathscr{T} - \Big(\mathrm{Trace}_{g_\theta} \, \Pi_{\mathscr{V}^\phi} \, A\Big) \, T\Big\}, \end{aligned} \tag{58}$$

where $\Pi_{\mathscr{V}^\phi} \, A$ denotes the restriction of A to $\mathscr{V}^\phi \otimes \mathscr{V}^\phi$. Indeed [by (11)]

$$\begin{aligned} \nabla^{g_\theta}_V V &= \nabla_V V - A(V, \, V) \, T + \theta(V) \, \tau \, V + 2 \, \theta(V) \, J \, V, \\ \nabla^{g_\theta}_{\mathscr{T}} \mathscr{T} &= \nabla_{\mathscr{T}} \mathscr{T} - A(\mathscr{T}, \, \mathscr{T}) \, T + u\big(\tau + 2 \, J\big) \, \mathscr{T}, \end{aligned}$$

for any $V \in \mathscr{V}^\phi_H$. Hence

$$\begin{aligned} &= \pi_{\mathscr{H}^\phi}\Big\{ \sum_{j=1}^{2n-m} \nabla^{g_\theta}_{V_j} V_j + \nabla^{g_\theta}_{\mathscr{T}} \mathscr{T}\Big\} \\ &= \pi_{\mathscr{H}^\phi}\Big\{ \sum_{j=1}^{2n-m} \nabla_{V_j} V_j + \nabla_{\mathscr{T}} \mathscr{T} - \big(\mathrm{Trace}_{g_\theta} \, \Pi_{\mathscr{V}^\phi} \, A\big) \, T + u \, (\tau + 2 \, J\big) \, \mathscr{T}\Big\}. \end{aligned}$$

□

For further use, note that [by (11)]

$$\sum_{j=1}^{2n-m} g_\theta\big(\nabla_{V_j}\mathscr{T},\ V_j\big) = \operatorname{div}_{\mathscr{F}}(\mathscr{T}) + \theta(\mathscr{T})\big[A(\mathscr{T},\ \mathscr{T}) - \operatorname{Trace}_{g_\theta}\Pi_{\mathscr{V}\phi} A\big]. \tag{59}$$

Using the Graham connection σ to lift $\{V_j,\ \mathscr{T}\}$, one produces the local frame $\big\{V_j^\uparrow,\ \mathscr{T}^\uparrow,\ S\ :\ 1 \le j \le 2n-m\big\}$ for $\mathscr{V}^\Phi$, defined on $\mathscr{U} = \pi^{-1}(U) \subset S(\Phi)$.

Lemma 16 *Let us set*

$$u := \theta(\mathscr{T}), \qquad v := \frac{2-u^2}{u}, \qquad u,\ v \in C^\infty\big(\Omega(\phi)\big).$$

Then

$$\Big\{V_j^\uparrow,\ \ \mathscr{T}^\uparrow + u\,S,\ \ \mathscr{T}^\uparrow - v\,S\ :\ 1 \le j \le 2n-m\Big\},$$

is a F_θ-orthonormal frame for $\mathscr{V}^\Phi$ with $\mathscr{T}^\uparrow - v\,S$ timelike.

The proof is straightforward. Note that

$$|u| = \big|g_\theta\big(\mathscr{T},\ T\big)\big| \le \|\mathscr{T}\|\ \|T\| = 1,$$

and in particular $|v| \ge 1$.

Lemma 17

$$\begin{aligned}(2n-m+2)\ \Phi_*\ \mu^{\mathscr{V}^\Phi} &= (2n-m+1)\,\phi_*\,\mu^{\mathscr{V}^\phi}\big(g_\theta\,,\ \nabla\big)\\ &\quad + \phi_*\left\{\frac{2}{\theta(\mathscr{T})}\,J\,\mathscr{T} - \nabla_{\mathscr{T}}\mathscr{T}\right\}\circ\pi.\end{aligned} \tag{60}$$

Proof We start by computing the needed components of $B_{\mathscr{V}^\Phi}$. By Lemma 14 and $S \in \mathscr{V}^\Phi$

$$\begin{aligned} B_{\mathscr{V}^\Phi}\big(X^\uparrow,\ X^\uparrow\big) &= \pi_{\mathscr{H}^\Phi}\,\nabla^{F_\theta}_{X^\uparrow} X^\uparrow\\ &= \pi_{\mathscr{H}^\Phi}\Big\{\big(\nabla_X X\big)^\uparrow - A(X,\ X)\,S\Big\} = \pi_{\mathscr{H}^\Phi}\,\big(\nabla_X X\big)^\uparrow,\\ B_{\mathscr{V}^\Phi}(S,\ S) &= \pi_{\mathscr{H}^\Phi}\,\nabla^{F_\theta}_S S = 0.\end{aligned}$$

The calculation of $B_{\mathscr{V}^\Phi}\big(\mathscr{T}^\uparrow,\ S\big)$ is a bit trickier. One first decomposes $\mathscr{T} = \Pi_H\,\mathscr{T} + u\,T$ and then [again by Lemma 14 and $S(u\circ\pi) = 0$]

$$\nabla^{F_\theta}_S\,\mathscr{T}^\uparrow = \frac{1}{2}\big(J\,\Pi_H\,\mathscr{T}\big)^\uparrow,$$

i.e. (as $J\,T = 0$)

$$B_{\mathscr{V}^\Phi}\big(\mathscr{T}^\uparrow\,,\ S\big) = \frac{1}{2}\,\pi_{\mathscr{H}^\Phi}\,\big(J\,\mathscr{T}\big)^\uparrow.$$

Next (by Lemma 16 and $u + v = 2/u$)

$$\begin{aligned}
&(2n - m + 2)\,\mu^{\mathscr{V}^\Phi} \\
&\quad = \operatorname{Trace}_{F_\theta}\,B_{\mathscr{V}^\Phi} = \sum_{j=1}^{2n-m} B_{\mathscr{V}^\Phi}\big(V_j^\uparrow\,,\ V_j^\uparrow\big) + B_{\mathscr{V}^\Phi}\big(\mathscr{T}^\uparrow + u\,S,\ \mathscr{T}^\uparrow + u\,S\big) \\
&\qquad - B_{\mathscr{V}^\Phi}\big(\mathscr{T}^\uparrow - v\,S,\ \mathscr{T}^\uparrow - v\,S\big) \\
&\quad = \sum_{j=1}^{2n-m} B_{\mathscr{V}^\Phi}\big(V_j^\uparrow\,,\ V_j^\uparrow\big) + \big(u^2 - v^2\big)\,B_{\mathscr{V}^\Phi}(S,\ S) + 2(u+v)\,B_{\mathscr{V}^\Phi}\big(\mathscr{T}^\uparrow\,,\ S\big) \\
&\quad = \pi_{\mathscr{H}^\Phi}\Big[\sum_{j=1}^{2n-m} \nabla_{V_j} V_j + \frac{2}{u}\,J\,\mathscr{T}\Big]^\uparrow.
\end{aligned}$$

Substitution from

$$\pi_{\mathscr{H}^\phi}\sum_{j=1}^{2n-m} \nabla_{V_j} V_j = (2n - m + 1)\,\mu^{\mathscr{V}^\phi}\big(g_\theta\,,\ \nabla\big) - \pi_{\mathscr{H}^\phi}\,\nabla_{\mathscr{T}}\mathscr{T},$$

yields (60). □

Summing up [by Lemmas 13 to 17] the fundamental equation (50) projects on

$$\begin{aligned}
\tau_b(\phi) = &-\frac{m-2}{2}\,\phi_*\,\nabla^H \log\,\Lambda(\phi) - (2n - m + 1)\,\phi_*\,\mu^{\mathscr{V}^\phi}\big(g_\theta\,,\ \nabla\big) \\
&+ - \phi_*\left\{\frac{2}{\theta(\mathscr{T})}\,J\,\mathscr{T} - \nabla_{\mathscr{T}}\mathscr{T}\right\}.
\end{aligned} \tag{61}$$

Besides from the foliation $\mathscr{F}$ tangent to $\mathscr{V}^\phi\big|_{S(\phi)}$ [the portion of the vertical bundle $\mathscr{V}^\phi$ over the (open) set $\Omega(\phi) = S(\phi) = \mathrm{I}_m(\phi) \cup \mathrm{III}_m(\phi)$ of all submersive points], the manifold M comes equipped with the Reeb foliation i.e. the codimension $2n$ foliation $\mathscr{R}$ of M tangent to T. The case where $\mathscr{R}$ is a subfoliation of $\mathscr{F}$ is closest to the Riemannian case i.e. (61) becomes

$$\tau(\phi) = -\frac{m-2}{2}\,\phi_*\,\nabla \log \Lambda(\phi) - (2n - m + 1)\,\phi_*\,\mu^{\mathscr{V}^\phi}, \tag{62}$$

which is the fundamental Eq. (4.5.2) in [5, p. 120], for $\phi : M \to N$ as a map of the Riemannian manifolds $(M,\ g_\theta)$ and $(N,\ h)$. Indeed, at each point $x \in \mathrm{I}_m(\phi)$ [equivalently $(d_x\phi)\,T_x = 0$] one has $\tau_b(\phi)_x = \tau(\phi)_x$ [by (12)] and $\mu_x^{\mathscr{V}^\phi} = \mu^{\mathscr{V}^\phi}\big(g_\theta\,,\ \nabla\big)_x$

[by (60)] hence (61) becomes

$$\tau(\phi) = -\frac{m-2}{2}\,\phi_*\,\nabla\log\Lambda(\phi) - (2n-m+1)\,\phi_*\,\mu^{\mathscr{V}^\phi} + \phi_*\,\nabla_{\mathscr{T}}\mathscr{T},$$

along $\mathrm{I}_m(\phi)$. If $\mathscr{R}\subset\mathscr{F}$, i.e. $\phi_*\,T=0$ everywhere in $\Omega(\phi)$, then $T^{\mathscr{V}}=T$ and $u=1$ on the whole open set $\Omega(\phi)$, hence $\nabla T=0$ yields $\nabla_{\mathscr{T}}\mathscr{T}=0$ on $\Omega(\phi)$. □

Corollary 1 *Let $\phi : M \to N$ be a subelliptic harmonic morphism of the pseudohermitian manifold $(M,\,\theta)$ into the real surface $(N,\,h)$.*

(i) *If the Reeb foliation is a subfoliation of $\mathscr{F}$ [the foliation of $\Omega(\phi)$ tangent to $\mathscr{V}^\phi$] then every leaf of $\mathscr{F}$ is a minimal submanifold of the Riemannian manifold $(M,\,g_\theta)$.*

(ii) *If $(d_x\phi)T_x\neq 0$ for some $x\in M\backslash\mathrm{Crit}(\phi)$, then*

$$(2n-1)\,\mu^{\mathscr{V}^\phi}\big(g_\theta\,,\,\nabla\big) = \pi_{\mathscr{H}^\phi}\left\{\nabla_{\mathscr{T}}\mathscr{T} - \frac{2}{\theta(\mathscr{T})}\,J\,\mathscr{T}\right\}.$$

Proof (i) If $m=2$ then (62) yields $\mu^{\mathscr{V}^\phi}=0$. □

(ii) Follows from (61).

5.2 ϵ-Contractions

Let $p = 2n-m+1$. To some surprise, the term $p\ \ \phi_*\,\mu^{\mathscr{V}^\phi}\big(g_\theta\,,\,\nabla\big) + \phi_*\big\{2\,\theta(\mathscr{T})^{-1}\,J\,\mathscr{T} - \nabla_{\mathscr{T}}\mathscr{T}\big\}$ replaces the term $p\ \ \phi_*\,\mu^{\mathscr{V}^\phi}$ [occurring in the fundamental equation (62), in the Riemannian setting]. Besides the term $\phi_*\,\mu^{\mathscr{V}^\phi}(g_\theta\,,\,\nabla)$ that one might have hoped for to start with, the fundamental equation for a subelliptic harmonic morphism contains the additional term $\phi_*\big\{2\,\theta(\mathscr{T})^{-1}\,J\,\mathscr{T}-\nabla_{\mathscr{T}}\mathscr{T}\big\}$ whose geometric meaning is so far unknown. So, given an immersion $f : L\to M$ of a p-dimensional manifold L into the pseudohermitian manifold $(M,\,\theta)$, what is the “correct” pseudohermitian analog to the mean curvature vector (of an isometric immersion)?

Pseudohermitian geometry (on a strictly pseudoconvex CR manifold) embeds into sub-Riemannian geometry. One may construct families of contractions $\{g_\epsilon\}_{0<\epsilon<1}$ of the Levi form G_θ [so that the norm of the Reeb vector T is $O(\epsilon^{-1})$] and examine Riemannian geometric objects in the limit as $\epsilon\to 0^+$, in an attempt to discover new pseudohermitian invariants. Cf. e.g. the approaches by Barletta et al. [12] and Capogna and Citti [17].

Let $0<\epsilon<1$ and let g_ϵ be the Riemannian metric

$$g_\epsilon(X,Y) = G_\theta(X,Y),\quad g_\epsilon(X,T)=0,\quad g_\theta(T,T)=\epsilon^{-2},$$

for any $X,\ Y\in H(M)$. Equivalently

$$g_\epsilon = g_\theta + \left(\frac{1}{\epsilon^2}-1\right)\,\theta\otimes\theta, \tag{63}$$

(the ϵ-*contraction* of G_θ, cf. Strichartz [60], Barletta et al. [12]). To illustrate our strategy, let us assume that, for every $0 < \epsilon < 1$, the map $\phi : (M,\ g_\epsilon) \to (N,\ h)$ is horizontally weakly conformal, with square dilation Λ_ϵ i.e. for any $x_0 \in M \setminus \mathrm{Crit}(\phi)$ and any local coordinate system $\big(V,\ y^\alpha\big)$ on N with $\phi(x_0) \in V$

$$m\,\Lambda_\epsilon = \big(h_{\alpha\beta} \circ \phi\big)\, g_\epsilon\big(\nabla^\epsilon\,\phi^\alpha\,,\ \nabla^\epsilon\,\phi^\beta\big). \tag{64}$$

Here ∇^ϵ is the gradient with respect to g_ϵ. Choose $V \subset N$ such that $U = \phi^{-1}(V) \subset M$ is a relatively compact domain. A straightforward calculation (relying on (63), cf. also [12]) leads to

$$\nabla^\epsilon\,\phi^\alpha = \nabla^H\,\phi^\alpha + \epsilon^2\,\theta(\nabla\,\phi^\alpha)\,T,$$

yielding

$$g_\epsilon\big(\nabla^\epsilon\,\phi^\alpha\,,\ \nabla^\epsilon\,\phi^\beta\big) = G_\theta\big(\nabla^H\,\phi^\alpha\,,\ \nabla^H\,\phi^\beta\big) + \epsilon^2\,\theta\big(\nabla\phi^\alpha\big)\,\theta\big(\nabla\phi^\beta\big),$$

and in particular

$$\sup_{x\in U}\,\big|m\,\Lambda_\epsilon(x) - G_\theta\big(\nabla^H\,\phi^\alpha\,,\ \nabla^H\,\phi^\beta\big)_x\,h_{\alpha\beta}\big(\phi(x)\big)\big| \le C(\phi)\,\epsilon^2\,,$$
$$C(\phi) = \sup_{x\in U}\,\theta\big(\nabla\phi^\alpha\big)_x\,\theta\big(\nabla\phi^\beta\big)_x\,h_{\alpha\beta}\big(\phi(x)\big),$$

hence

$$\Lambda_\epsilon \to \frac{1}{m}\,G_\theta\big(\nabla^H\,\phi^\alpha\,,\ \nabla^H\,\phi^\beta\big)\,h_{\alpha\beta}\circ\phi\,,\quad \epsilon \to 0^+,$$

uniformly on U, and the Levi conformality condition (14) is got, in the limit as $\epsilon \to 0^+$, from the horizontal weak conformality condition on $\phi : (M,\ g_\epsilon) \to (N,\ h)$.

Let $\phi : M \to N$ be a subelliptic harmonic morphism of the pseudohermitian manifold $(M,\ \theta)$ into the Riemannian manifold $(N,\ h)$, of θ-dilation $\lambda(\phi)$. We shall compute the mean curvature vector $\mu_\epsilon^{\mathscr{V}^\phi}$ of the vertical distribution $\mathscr{V}^\phi$ on the Riemannian manifold $(M,\ g_\epsilon)$, and examine the behavior of $\mu_\epsilon^{\mathscr{V}^\phi}$ as $\epsilon \to 0^+$, in an attempt to discover the "correct" pseudohermitian analog to the ordinary mean curvature vector. To this end, let $\mathscr{H}_\epsilon^\phi$ be the g_ϵ-orthogonal complement of $\mathscr{V}^\phi$. Let us set

$$B_\epsilon(X, Y) = \pi_{\mathscr{H}_\epsilon^\phi}\,\nabla^\epsilon_X Y,\quad X, Y \in \mathscr{V}^\phi\,,$$
$$\mu_\epsilon^{\mathscr{V}^\phi} = \frac{1}{2n - m + 1}\,\mathrm{Trace}_{g_\epsilon}\,B_\epsilon,$$

i.e. $\mu_\epsilon^{\mathscr{V}^\phi}$ is the mean curvature vector of $\mathscr{V}^\phi$. Here ∇^ϵ is the Levi-Civita connection of (M, g_ϵ). Let us set

$$\mathscr{T}_\epsilon = \frac{1}{g_\epsilon(\mathscr{T},\ \mathscr{T})^{1/2}}\ \mathscr{T} = \frac{\epsilon}{\sqrt{\epsilon^2 + (1-\epsilon^2)\, u}}\ \mathscr{T}.$$

Then $\{V_j\ ,\ \mathscr{T}_\epsilon\ :\ 1 \le j \le 2n-m\}$ is a local g_ϵ-orthonormal frame of $\mathscr{V}^\phi$, adapted to the decomposition $T(M) = H(M) \oplus \mathbb{R}T$ [which is both g_θ and g_ϵ orthogonal]. Let

$$B_\epsilon(X,\ Y) = \pi_{\mathscr{H}_\epsilon^\phi}\ \nabla_X^\epsilon Y, \quad X,\ Y \in \mathscr{V}^\phi,$$

[the second fundamental form of $L \hookrightarrow (M,\ g_\epsilon)$, for every leaf $L \in \big[M \backslash \mathrm{Crit}(\phi)\big]\big/\mathscr{F}$].

Lemma 18 *For every* $X \in \mathfrak{X}(M)$

$$\pi_{\mathscr{H}_\epsilon^\phi}\ X = \pi_{\mathscr{H}^\phi}\ X + \frac{(1-\epsilon^2)u}{\epsilon^2 + (1-\epsilon^2)\, u^2}\ \Big\{u\, g_\theta(\mathscr{T},\ X) - \theta(X)\Big\}\ \mathscr{T}. \tag{65}$$

Proof Let $\pi_{\mathscr{V}^\phi}^\epsilon\ :\ T(M) \to \mathscr{V}^\phi$ be the projection associated with the direct sum decomposition $T(M) = \mathscr{V}^\phi \oplus \mathscr{H}_\epsilon^\phi$. For every $X \in T(M)$

$$\begin{aligned}\pi_{\mathscr{H}_\epsilon^\phi}\ X\ &= X - \pi_{\mathscr{V}^\phi}^\epsilon\ X \\ &= X - \sum_{j=1}^{2n-m} g_\epsilon\big(V_j\ ,\ X\big)\ V_j - g_\epsilon\big(\mathscr{T}_\epsilon\ ,\ X\big)\ \mathscr{T}_\epsilon =,\end{aligned}$$

[by (63)]

$$\begin{aligned}&= X - \sum_{j=1}^{2n-m} g_\theta\big(V_j\ ,\ X\big)\ V_j \\ &\quad + - \frac{\epsilon^2}{\epsilon^2 + (1-\epsilon^2)\, u^2}\ \Big\{g_\theta(\mathscr{T},\ X) + \Big(\frac{1}{\epsilon^2} - 1\Big)\, u\, \theta(X)\Big\}\ \mathscr{T} \\ &= X - \sum_{j=1}^{2n-m} g_\theta\big(V_j\ ,\ X\big)\ V_j - g_\theta(\mathscr{T},\ X)\ \mathscr{T} \\ &\quad + \frac{(1-\epsilon^2)u}{\epsilon^2 + (1-\epsilon^2)\, u^2}\ \Big\{u\, g_\theta(\mathscr{T},\ X) - \theta(X)\Big\}\ \mathscr{T}.\end{aligned}$$

□

We shall need (cf. Lemma 2 in [12, pp. 11–12])

Lemma 19 *The Levi–Civita connection* ∇^ϵ *of* $(M,\ g_\epsilon)$ *and the Tanaka–Webster connection* ∇ *of* $(M,\ \theta)$ *are related by*

$$\nabla^\epsilon_X Y = \nabla_X Y + \Big\{\Omega(X, Y) - \epsilon^2 A(X, Y)\Big\}\, T\,,$$
$$\nabla^\epsilon_X T = \tau X + \frac{1}{\epsilon^2}\, JX, \quad \nabla^\epsilon_T X = \nabla_T X + \frac{1}{\epsilon^2}\, JX, \quad \nabla^\epsilon_T T = 0,$$

for any $X, Y \in H(M)$.

Lemma 20

$$\begin{aligned}(2n-m+1)\,\mu^{\mathscr{V}\phi}_\epsilon &= (2n-m+1)\,\mu^{\mathscr{V}\phi}\big(g_\theta\,,\ \nabla\big)\\ &+ - \frac{(1-\epsilon^2)\,u^2}{\epsilon^2+(1-\epsilon^2)\,u^2}\,\Big\{\pi_{\mathscr{H}\phi}\,\nabla_{\mathscr{T}}\,\mathscr{T} + \mathrm{div}_{\mathscr{F}}\,(\mathscr{T})\,\mathscr{T}\Big\}\\ &+ - \frac{(1-\epsilon^2)\,u^3}{\epsilon^2+(1-\epsilon^2)\,u^2}\,\Big\{A(\mathscr{T},\ \mathscr{T}) - \mathrm{Trace}_{g_\theta}\,\Pi_{\mathscr{V}\phi}\,A\Big\}\,\mathscr{T}\\ &+ \frac{2\,u}{\epsilon^2+(1-\epsilon^2)\,u^2}\,\pi_{\mathscr{H}\phi}\,J\,\mathscr{T}\\ &+ \epsilon^2\,\big(\mathrm{Trace}_{g_\theta}\,\Pi_{\mathscr{V}\phi}\,A\big)\,\Big\{-T + \frac{u}{\epsilon^2+(1-\epsilon^2)\,u^2}\,\mathscr{T}\Big\}\\ &+ \frac{(1-\epsilon^2)\,\epsilon^2\,u^2}{\epsilon^2+(1-\epsilon^2)\,u^2}\,A(\mathscr{T},\ \mathscr{T})\,\mathscr{T}\\ &+ \frac{\epsilon^2\,u}{\epsilon^2+(1-\epsilon^2)u^2}\,\pi_{\mathscr{H}\phi}\,\tau\,\mathscr{T} - \frac{\epsilon^2\,(1-\epsilon^2)\,u}{\big[\epsilon^2+(1-\epsilon^2)u^2\big]^2}\,\mathscr{T}(u)\,\mathscr{T}.\end{aligned} \tag{66}$$

Proof

$$\begin{aligned}(2n-m+1)\,\mu^{\mathscr{V}\phi}_\epsilon &= \sum_{j=1}^{2n-m} B_\epsilon\big(V_j\,,\ V_j\big) + B_\epsilon\big(\mathscr{T}_\epsilon\,,\ \mathscr{T}_\epsilon\big),\\ B_\epsilon\big(V_j\,,\ V_j\big) &= \pi_{\mathscr{H}^\phi_\epsilon}\,\nabla^\epsilon_{V_j} V_j =,\end{aligned} \tag{67}$$

[by Lemma 19, as $V_j \in H(M)$]

$$= \pi_{\mathscr{H}^\phi_\epsilon}\,\Big\{\nabla_{V_j} V_j - \epsilon^2\,A\big(V_j\,,\ V_j\big)\,T\Big\} =,$$

[by (65) in Lemma 18 with $X = \nabla_{V_j} V_j - \epsilon^2\,A\big(V_j\,,\ V_j\big)\,T$]

$$\begin{aligned}&= \pi^{\mathscr{H}\phi}\,\nabla_{V_j} V_j - \epsilon^2\,A\big(V_j\,,\ V_j\big)\,T^{\mathscr{H}} + \frac{(1-\epsilon^2)\,u}{\epsilon^2+(1-\epsilon^2)\,u^2}\\ &\times\Big\{u\,g_\theta\big(\mathscr{T},\ \nabla_{V_j} V_j\big) + \epsilon^2\,\big(1-u^2\big)\,A\big(V_j\,,\ V_j\big)\Big\}\,\mathscr{T},\end{aligned}$$

or

$$B_\epsilon\big(V_j\,,\;V_j\big) = \pi^{\mathscr{H}^\phi}\,\nabla_{V_j} V_j + \frac{(1-\epsilon^2)\,u^2}{\epsilon^2 + (1-\epsilon^2)\,u^2}\;g_\theta\big(\mathscr{T},\;\nabla_{V_j} V_j\big)\,\mathscr{T}$$
$$+\epsilon^2\,A\big(V_j\,,\;V_j\big)\,\Big\{-T + \frac{u}{\epsilon^2 + (1-\epsilon^2)\,u^2}\;\mathscr{T}\Big\}. \tag{68}$$

Next

$$B_\epsilon\big(\mathscr{T}_\epsilon\,,\;\mathscr{T}_\epsilon\big) = \frac{\epsilon^2}{\epsilon^2 + (1-\epsilon^2)u^2}\;B_\epsilon\big(\mathscr{T},\;\mathscr{T}\big),$$
$$B_\epsilon\big(\mathscr{T},\;\mathscr{T}\big) = \pi_{\mathscr{H}^\phi_\epsilon}\;\nabla^\epsilon_{\mathscr{T}}\mathscr{T},$$

and [by Lemma 19, and $\mathscr{T} = \Pi_H\,\mathscr{T} + u\,T$, and $T\,\rfloor\,A = 0$, $\tau\,T = 0$, $J\,T = 0$]

$$\nabla^\epsilon_{\mathscr{T}}\mathscr{T} = \nabla_{\Pi_H\,\mathscr{T}}\,\Pi_H\,\mathscr{T} + u\,\nabla_T\,\Pi_H\,\mathscr{T} + u\left(\tau + \frac{2}{\epsilon^2}\,J\right)\mathscr{T}$$
$$+\big\{\mathscr{T}(u) - \epsilon^2\,A(\mathscr{T},\;\mathscr{T})\big\}\;T,$$
$$\nabla_T\,\Pi_H\,\mathscr{T} = \nabla_T\mathscr{T} - T(u)\,T,$$
$$\nabla_{\Pi_H\,\mathscr{T}}\,\Pi_H\,\mathscr{T} = \nabla_{\mathscr{T}}\mathscr{T} - u\,\nabla_T\,\mathscr{T} + \big\{u\,T(u) - \mathscr{T}(u)\big\}\,T,$$

or

$$\nabla^\epsilon_{\mathscr{T}}\,\mathscr{T} = \nabla_{\mathscr{T}}\,\mathscr{T} + u\left(\tau + \frac{2}{\epsilon^2}\,J\right)\mathscr{T} - \epsilon^2\,A(\mathscr{T},\;\mathscr{T})\;T. \tag{69}$$

Note that [by (69)]

$$\pi_{\mathscr{H}^\phi}\,\nabla^\epsilon_{\mathscr{T}}\mathscr{T} = \pi_{\mathscr{H}^\phi}\,\nabla_{\mathscr{T}}\,\mathscr{T} + u\,\pi_{\mathscr{H}^\phi}\left(\tau + \frac{2}{\epsilon^2}\,J\right)\mathscr{T} - \epsilon^2\,A(\mathscr{T},\;\mathscr{T})\;T^{\mathscr{H}},$$
$$g_\theta\big(\mathscr{T},\;\nabla^\epsilon_{\mathscr{T}}\,\mathscr{T}\big) = \big(1-\epsilon^2\big)\,u\,A(\mathscr{T},\;\mathscr{T}),$$
$$\theta\big(\nabla^\epsilon_{\mathscr{T}}\,\mathscr{T}\big) = \theta\big(\nabla_{\mathscr{T}}\,\mathscr{T}\big) - \epsilon^2\,A(\mathscr{T},\;\mathscr{T}),$$
$$\theta\big(\nabla_{\mathscr{T}}\,\mathscr{T}\big) = \mathscr{T}(u).$$

Let us apply $\pi^{\mathscr{H}^\phi_\epsilon}$ to both sides of (69) so that [by (65) in Lemma 18]

$$B_\epsilon(\mathscr{T},\;\mathscr{T}) \;= \Pi_{\mathscr{H}^\phi}\,\nabla_{\mathscr{T}}\,\mathscr{T} + u\,\pi_{\mathscr{H}^\phi}\circ\left(\tau + \frac{2}{\epsilon^2}\,J\right)\mathscr{T}$$
$$+A(\mathscr{T},\;\mathscr{T})\,\big(u\,\mathscr{T} - \epsilon^2\,T\big) - \frac{(1-\epsilon^2)\,u}{\epsilon^2 + (1-\epsilon^2)\,u^2}\;\mathscr{T}(u)\,\mathscr{T}. \tag{70}$$

Then [by (68) and (70)]

$$
\begin{aligned}
(2n-m+1)\,\mu_\epsilon^{\mathscr{V}\phi} \;=\; & \sum_{j=1}^{2n-m} \Big\{ \pi_{\mathscr{H}\phi}\, \nabla_{V_j} V_j + \frac{(1-\epsilon^2)\,u^2}{\epsilon^2 + (1-\epsilon^2)\,u^2}\, g_\theta\big(\mathscr{T},\, \nabla_{V_j} V_j\big)\, \mathscr{T} \\
& + \epsilon^2\, A(V_j\,,\, V_j) \Big[-T + \frac{u}{\epsilon^2 + (1-\epsilon^2) u^2}\, \mathscr{T} \Big] \Big\} \\
& + \frac{\epsilon^2}{\epsilon^2 + (1-\epsilon^2)\,u^2} \Big\{ \pi_{\mathscr{H}\phi}\, \nabla_{\mathscr{T}} \mathscr{T} + u\, \pi_{\mathscr{H}\phi} \circ \Big(\tau + \frac{2}{\epsilon^2}\, J \Big)\, \mathscr{T} \\
& + + A(\mathscr{T},\, \mathscr{T})\, \big(u\, \mathscr{T} - \epsilon^2\, T \big) - \frac{(1-\epsilon^2)\,u}{\epsilon^2 + (1-\epsilon^2)\,u^2}\, \mathscr{T}(u)\, \mathscr{T} \Big\}.
\end{aligned}
\tag{71}
$$

Using

$$
(2n-m+1)\;\mu^{\mathscr{V}\phi}\big(g_\theta\,,\,\nabla\big) = \pi_{\mathscr{H}\phi} \Big\{ \sum_{j=1}^{2n-m} \nabla_{V_j} V_j + \nabla_{\mathscr{T}} \mathscr{T} \Big\},
$$

$$
\mathrm{Trace}_{g_\theta}\; \Pi_{\mathscr{V}\phi}\, A = \sum_{j=1}^{2n-m} A\big(V_j\,,\,V_j\big) + A(\mathscr{T},\,\mathscr{T}),
$$

$$
\begin{aligned}
& \sum_{j=1}^{2n-m} g_\theta\big(\mathscr{T},\, \nabla_{V_j} V_j\big)\, \mathscr{T} \\
& \quad = -\Big\{ \mathrm{div}_{\mathscr{F}}\,(\mathscr{T}) + u \Big[A(\mathscr{T},\,\mathscr{T}) - \mathrm{Trace}_{g_\theta}\; \Pi_{\mathscr{V}\phi}\, A \Big] \Big\}\; \mathscr{T},
\end{aligned}
$$

equation (71) simplifies to (66). □

Proof of Theorem 4 Next [by (66) with $p = 2n - m + 1$]

$$
\begin{aligned}
& \Big\| p \Big[\pi_{\mathscr{H}\phi}\; \mu_\epsilon^{\mathscr{V}\phi} - \mu^{\mathscr{V}\phi}\big(g_\theta\,,\,\nabla\big) \Big] + \pi_{\mathscr{H}\phi} \Big[\nabla_{\mathscr{T}}\, \mathscr{T} - \frac{2}{u}\, J\, \mathscr{T} \Big] \Big\| \\
& \leq \epsilon^2 \Big| \mathrm{Trace}_{g_\theta}\; \Pi_{\mathscr{V}\phi}\; A \Big|\; \big\| \pi_{\mathscr{V}\phi}\, T \big\| + \frac{\epsilon^2}{\epsilon^2 + (1-\epsilon^2) u^2} \Big\{ \Big\| \pi_{\mathscr{H}\phi}\; \nabla_{\mathscr{T}} \mathscr{T} \Big\| \\
& \quad + |u| \Big[\Big\| \pi_{\mathscr{H}\phi}\; \tau\; \mathscr{T} \Big\| + \frac{2(1-u^2)}{u^2}\; \Big\| \pi_{\mathscr{H}\phi}\;\; J\; \mathscr{T} \Big\| \Big] \Big\}.
\end{aligned}
$$

Let $U \subset M$ be a relatively compact domain and $a = \inf_U u^2$, so that $0 < a \leq 1$. Indeed if $a = 0$ then for every $\nu \in \mathbb{N}$

$$
\frac{1}{\nu} > 0 = \inf_U u^2,
$$

hence there is $x_\nu \in U$ such that $u(x_\nu)^2 < 1/\nu$. There is a subsequence, denoted by the same symbol x_ν, such that $x_\nu \to x_0$ as $\nu \to \infty$, for some $x_0 \in \overline{U}$. Thus

$$
0 = u(x_0) = \big\| T^{\mathscr{V}} \big\|_{x_0} \Longrightarrow T_{x_0} \in \mathscr{H}^\phi_{x_0},
$$

a contradiction. Let $\varphi(t) = 2(1-t)/t$, $0 < t \le 1$, so that $\sup_U \varphi(u) = \varphi(a)$. Note that

$$\left\|\pi_{\mathscr{H}\phi}\, T\right\| = \left\|T - u\,\mathscr{T}\right\| \le 1 + |u|\,\|\mathscr{T}\| \le 2,$$
$$\left\|\pi_{\mathscr{H}\phi}\, \nabla_{\mathscr{T}}\mathscr{T}\right\| \le \left\|\pi_{\mathscr{H}\phi}\right\|\,\|\nabla_{\mathscr{T}}\mathscr{T}\|,$$
$$\left\|\pi_{\mathscr{H}\phi}\, \tau\,\mathscr{T}\right\| \le \left\|\pi_{\mathscr{H}\phi}\right\|\,\|\tau\|, \quad \left\|\pi_{\mathscr{H}\phi}\, J\,\mathscr{T}\right\| \le \left\|\pi_{\mathscr{H}\phi}\right\|,$$

where $\left\|\pi_{\mathscr{H}\phi}\right\|$ is the operator norm. Finally

$$\left\|p\left[\pi_{\mathscr{H}\phi}\,\mu_\epsilon^{\mathscr{V}\phi} - \mu^{\mathscr{V}\phi}\left(g_\theta\,,\,\nabla\right)\right] + \pi_{\mathscr{H}\phi}\left[\nabla_{\mathscr{T}}\mathscr{T} - \frac{2}{u}\,J\,\mathscr{T}\right]\right\|$$
$$\le 2\,\epsilon^2\left|\mathrm{Trace}_{g_\theta}\,\Pi_{\mathscr{V}\phi}\,A\right| + \frac{\epsilon^2}{a+\epsilon^2(1-a)}\left\|\pi_{\mathscr{H}\phi}\right\|\left\{\|\nabla_{\mathscr{T}}\mathscr{T}\| + \|\tau\| + \varphi(a)\right\}.$$

Consequently

$$(2n-m+1)\,\pi_{\mathscr{H}\phi}\,\mu_\epsilon^{\mathscr{V}\phi} \to (2n-m+1)\,\mu^{\mathscr{V}\phi}\left(g_\theta\,,\,\nabla\right)$$
$$+\pi_{\mathscr{H}\phi}\left\{\frac{2}{u}\,J\,\mathscr{T} - \nabla_{\mathscr{T}}\mathscr{T}\right\}, \quad \epsilon \to 0^+,$$

uniformly on $U \subset M$. □

Let us set by definition

$$\mu_{\mathrm{hor}}^{\mathscr{V}\phi} := \pi_{\mathscr{H}\phi}\,H\left(\mathscr{V}^\phi\right),\ (2n-m+1)\,H\left(\mathscr{V}^\phi\right)$$
$$:= (2n-m+1)\,\mu^{\mathscr{V}\phi}\left(g_\theta\,,\,\nabla\right) + \frac{2}{u}\,\pi_{\mathscr{H}\phi}\,J\,\mathscr{T} - \pi_{\mathscr{H}\phi}\,\nabla_{\mathscr{T}}\mathscr{T}$$
$$+ - \left\{\mathrm{div}_{\mathscr{F}}(\mathscr{T}) + u\left[A(\mathscr{T},\,\mathscr{T}) - \mathrm{Trace}_{g_\theta}\,\Pi_{\mathscr{V}\phi}\,A\right]\right\}\mathscr{T}. \quad (72)$$

When T is tangent to the leaves of $\mathscr{F}$

$$\mu_{\mathrm{hor}}^{\mathscr{V}\phi} = H\left(\mathscr{V}^\phi\right) = \mu^{\mathscr{V}\phi}\left(g_\theta\,,\,\nabla\right).$$

By Lemma 20

$$\lim_{\epsilon\to 0^+}\mu_\epsilon^{\mathscr{V}\phi} = \mu_{\mathrm{hor}}^{\mathscr{V}\phi}.$$

On the other hand, by definition (72) Eq. (61) becomes

$$\tau_b(\phi) = -\frac{m-2}{2}\,\phi_*\,\log\Lambda(\phi) - (2n-m+1)\,\phi_*\,\mu_{\mathrm{hor}}^{\mathscr{V}\phi},$$

so that $\tau_b(\phi) = 0$ and $m = 2$ yield $\mu_{\mathrm{hor}}^{\mathscr{V}\phi} = 0$. Theorem 4 is proved.

6 Unique Continuation

Let $\Omega \subset \mathbb{R}^N$ be a domain, $\alpha > 0$, $x_0 \in \Omega$, and $1 \le p \le \infty$. A measurable function $u : \Omega \to \mathbb{C}$ has a zero of order α at x_0 in the p-mean if

$$\int_{B_r(x_0)} \big|u(x)\big|^p \, dx = O\big(r^{p\alpha+N}\big).$$

An ordinary zero of order α [i.e. except on a set of measure zero $\big|u(x)\big| = O\big(\big|x - x_0\big|^\alpha\big)$ as $x \to x_0$] corresponds to a zero in the p-mean for $p = \infty$. Let L be a second order linear elliptic operator. By a result of N. Aronszajn (cf. [2]) if u is a solution to

$$\big|Lu(x)\big|^2 \le M \left\{ \sum_{i=1}^N \left|\frac{\partial u}{\partial x^i}\right|^2 + |u(x)|^2 \right\},$$

and u has a zero of infinite order in the 1-mean at some $x_0 \in \Omega$, then[4] $u \equiv 0$ in Ω. Aronszajn's proof to his result was criticized in [25, pp. 433–434], because of Aronszajn's claim that a pair of conformally related Riemannian metrics (associated to the symbol of L) have the same geodesics. We conjecture that the arguments in [2] may be reconsidered within conformal Riemannian geometry i.e. by understanding the conformal properties of geodesic spheres, based on the use of *conformal geodesics* (cf. e.g. [33]). The result itself in [2] may nevertheless be true, and if that is the case it yields Sampson's unique continuation theorem for harmonic maps of Riemannian manifolds (cf. Theorem 1 in [58, p. 213]). Let Let $\{X_a \, : \, 1 \le a \le 2n\}$ be a G_θ-orthonormal frame of $H(M)$, defined on the open set $U \subset M$, and let $\Omega \subset U$ be a domain. Let $u = \big(u^1, \ldots, u^m\big) : \Omega \to \mathbb{R}^m$ be a solution to

$$\big|\Delta_b u^\alpha\big| \le C \left\{ \sum_{a,\beta} \big|X_a\big(u^\beta\big)\big| + \sum_\beta \big|u^\beta\big| \right\}. \tag{73}$$

We conjecture that, if u has a zero of infinite order at some point of Ω then $u \equiv 0$ in Ω. Should the conjecture be true, one has

Corollary 2 *Let $\phi, \psi : M \to N$ be two subelliptic harmonic maps, from the connected pseudohermitian manifold (M, θ) into the Riemannian manifold (N, h). If ϕ and ψ agree on some open set, then they are identical.*

Proof Let $\chi = (y^1, \ldots, y^m)$ be a local coordinate system on N, whose domain is a ball $V = \chi^{-1}\big[B_r(\xi_0)\big]$, and let $\{X_a \, : \, 1 \le a \le 2n\}$ be a local G_θ-orthonormal frame of $H(M)$, defined on the open set $U \subset M$ such that $\phi(U) \cup \psi(U) \subset V$. Let us set $u^\alpha := \phi^\alpha - \psi^\alpha$ so that [as both ϕ and ψ are harmonic maps]

$$\Delta_b u^\alpha = \sum_{a=1}^{2n} \left[\left(\left\{ {\alpha \atop \beta\gamma} \right\} \circ \phi \right) X_a(\phi^\beta) \, X_a(\phi^\gamma) \right.$$

[4] Subject to a number of structural assumptions on L, cf. [2, p. 236].

$$+ - \left(\left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} \circ \psi \right) X_a(\psi^\beta)\, X_a(\psi^\gamma) \Big]$$

$$= \sum_{a=1}^{2n} \Big[\left(\left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} \circ \phi \right) X_a(u^\beta) \left[X_a(\phi^\gamma) + X_a(\psi^\gamma) \right]$$

$$+ - \left[\left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} \circ \psi - \left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} \circ \phi \right] X_a(\psi^\beta)\, X_a(\psi^\gamma) \Big].$$

Let $x \in U$ and let $\xi = \phi(x)$ and $\eta = \psi(x)$. By the mean value theorem, there is $0 < \tau < 1$ such that

$$\left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} \big(\chi^{-1}(\eta)\big) - \left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\} \big(\chi^{-1}(\xi)\big)$$

$$= \frac{\partial \left\{ \begin{array}{c} \alpha \\ \beta\gamma \end{array} \right\}}{\partial y^\mu} \left[\chi^{-1}\big((1-\tau)\,\xi + \tau\,\eta\big) \right] \big(\eta^\mu - \xi^\mu\big).$$

By eventually shrinking U the derivatives $X_a(\phi^\beta)$ and $X_a(\psi^\beta)$ are bounded, so that u^α satisfy (73). □

We conjecture that the uniqueness continuation result by Garofalo and Lanconelli (cf. Theorem 1.2 in [36, p. 319]) on the solutions to $\Delta_b u + V(x)\, u = 0$ carries over to equations of the form $\Delta_b u + f(u) = 0$ [with the nonlinear term $f(u)$ as considered by Birindelli and Prajapat [15] (cf. also Birindelli and Lanconelli [14]) for different purposes] with applications to the unique continuation of subelliptic harmonic maps.

7 Horizontal Mean Curvature

Let $\phi : M \to N$ be a subelliptic harmonic morphism, of the pseudohermitian manifold (M, θ) into the Riemannian manifold (N, h). Let $\mathscr{F}$ be the foliation of $S(\phi)$ by maximal integral manifolds of $\mathscr{V}^\phi$. A point $x \in S(\phi)$ is a *characteristic point* of $\mathscr{F}$ if

$$H(M)_x \subset \mathscr{V}^\phi_x. \tag{74}$$

Let $\Sigma(\mathscr{F})$ be the set of all characteristic points of $\mathscr{F}$. If $x \in \Sigma(\mathscr{F})$ and $L \in S(\phi)/\mathscr{F}$ is the leaf of $\mathscr{F}$ passing through x, then x is a characteristic point of L, e.g. in the sense of L. Capogna & G. Citti (cf. [17, p. 7]). The inclusion (74) yields

$$2n \leq \dim_{\mathbb{R}} \mathscr{V}^\phi_x \leq 2n + 1, \tag{75}$$

hence one has equality in (74) unless $x \in \mathrm{Crit}(\phi)$. Yet $S(\phi)$, and hence $\Sigma(\mathscr{F})$, contains no critical points. Also [by (75)]

$$\Sigma(\mathscr{F}) \neq \emptyset \Longrightarrow m = 1.$$

This limitation doesn't occur in [17] (where the ambient space M is a Carnot group, the rank of whose first stratus, or horizontal plane, is in general smaller than the dimension of $\mathscr{F}$). For the remainder of the present section we confine ourselves to subelliptic harmonic morphisms $\phi : M^{2n+1} \to N^1$ i.e. $m = 1$ (so that every leaf of $\mathscr{F}$ is a real hypersurface in $S(\phi)$). By Theorem 6 (with $m = 1$)

$$Z(\Lambda) = \mathrm{II}_1(\phi) \cup \mathrm{Crit}(\phi), \quad S(\phi) = M \setminus \mathrm{Crit}(\phi),$$
$$\Omega = S(\phi) \setminus \mathrm{II}_1(\phi), \quad \mathrm{I}_1(\phi) \subset \Omega\,, \quad \mathrm{III}_1(\phi) = \Omega \setminus \mathrm{I}_1(\phi),$$

where [by Lemmas 5, 6]

$$\mathrm{I}_1(\phi) = \left\{x \in S(\phi) \ : \ \mathscr{H}^\phi_x \subset H(M)_x\,, \ \ \dim_{\mathbb{R}} \mathscr{V}^\phi_{H,\,x} = 2n-1\right\},$$
$$\mathrm{II}_1(\phi) = \left\{x \in S(\phi) \ : \ \mathscr{V}^\phi_x = H(M)_x\right\},$$
$$\mathrm{III}_1(\phi) = \left\{x \in S(\phi) \ : \ \mathscr{H}^\phi_{H,\,x} = (0), \ \ \dim_{\mathbb{R}} \mathscr{V}^\phi_{H,\,x} = 2n-1\right\},$$
$$\mathrm{I}_1(\phi) \cup \mathrm{III}_1(\phi) = \Omega, \quad \mathrm{II}_1(\phi) = \Sigma(\mathscr{F}).$$

Let $\{g_\epsilon\}_{0<\epsilon<1}$ be the family of contractions of the Levi form G_θ given by (2), and let $\mathbf{n}^\epsilon \in C^\infty\big(S(\phi),\ \mathscr{H}^\phi_\epsilon\big)$ such that $g_\epsilon\big(\mathbf{n}^\epsilon\,,\ \mathbf{n}^\epsilon\big) = 1$. Next, let

$$\nu^\epsilon := \Pi_H\, \mathbf{n}^\epsilon = \mathbf{n}^\epsilon - \theta\big(\mathbf{n}^\epsilon\big)\, T, \tag{76}$$

be the projection of $\mathbf{n}^\epsilon$ on $H(M)$.

Lemma 21 *For every $x \in S(\phi)$, the following statements are equivalent*
(i) $x \in \Sigma(\mathscr{F})$.
(ii) $\nu^\epsilon_x = 0$ *for any* $0 < \epsilon < 1$.

Proof (i) $\Longrightarrow$ (ii). Let $x \in \Sigma(\mathscr{F})$, so that

$$H(M)_x = \mathscr{V}^\phi_x \perp_{g_{\epsilon,x}} \mathscr{H}^\phi_{\epsilon\,,\,x} \ni \mathbf{n}^\epsilon,$$

i.e. for every $X \in H(M)_x$

$$0 = g_{\epsilon\,,\,x}\big(X,\ \mathbf{n}^\epsilon_x\big) = g_{\theta\,,\,x}\big(X,\ \mathbf{n}^\epsilon_x\big) + \left(\frac{1}{\epsilon^2} - 1\right)\, \theta_x(X)\, \theta\big(\mathbf{n}^\epsilon\big)_x,$$

yielding [by $\theta_x(X) = 0$]

$$\mathbf{n}^\epsilon_x \perp_{g_{\theta,x}} H(M)_x,$$

or $\mathbf{n}^\epsilon_x = \lambda\, T_x$ for some $\lambda \in \mathbb{R}$. Then [by (76)] $\nu^\epsilon_x = 0$. □

(ii) $\Longrightarrow$ (i). Let $\nu^\epsilon_x = 0$. Then [by (76)] $\mathbf{n}^\epsilon_x = \lambda\, T_x$ with $\lambda := \theta\big(\mathbf{n}^\epsilon\big)_x$. As $\mathrm{Sing}(\mathbf{n}^\epsilon) = \emptyset$ and $\mathrm{Sing}(T) = \emptyset$, it must be $\lambda \neq 0$. Hence $\mathbf{n}^\epsilon_x \perp_{g_{\theta,x}} H(M)_x$ implying [by (2)] that $\mathbf{n}^\epsilon_x \perp_{g_{\epsilon,x}} H(M)_x$. On the other hand $\mathbf{n}^\epsilon_x \perp_{g_{\epsilon,x}} \mathscr{V}^\phi_x$ so that (by the uniqueness of the $g_{\epsilon\,,\,x}$-orthogonal complement of $\mathbf{n}^\epsilon_x$) it must be $H(M)_x = \mathscr{V}^\phi_x$ i.e. $x \in \Sigma(\mathscr{F})$. □

Let us set

$$\mathbf{n}^0(x) := \frac{1}{\sqrt{f_\epsilon(x)}}\, \nu_x^\epsilon\,, \quad x \in \Omega \setminus \Sigma(\mathscr{F}),$$
$$f_\epsilon := g_\epsilon\big(\nu^\epsilon\,,\ \nu^\epsilon\big) \in C^\infty\big(\Omega,\ \mathbb{R}_+\big), \tag{77}$$

with $\mathbb{R}_+ = [0,\ +\infty)$. According to the terminology by L. Capogna et al. (cf. [17, p. 7]) $\mathbf{n}^0$ is the *horizontal normal* (on the leaves of $\mathscr{F}$).

Lemma 22 *For each $x \in \Omega$ the function*

$$\epsilon \in (0,\ 1) \longmapsto f_\epsilon(x)^{-1/2}\ \nu_x^\epsilon \in H(M)_x,$$

is constant i.e. $\mathbf{n}^0(x)$ *in* (77) *doesn't depend on* $0 < \epsilon < 1$.

Proof Note that [by (76)]

$$f_\epsilon = 1 - \frac{1}{\epsilon^2}\, \theta\big(\mathbf{n}^\epsilon\big)^2. \tag{78}$$

Also, for any $\epsilon,\ \epsilon' \in (0,\ 1)$

$$g_{\epsilon'} = g_\epsilon + \left(\frac{1}{\epsilon'^2} - \frac{1}{\epsilon^2}\right)\theta \otimes \theta. \tag{79}$$

Also [by $\dim_{\mathbb{R}} \mathscr{H}_x^\phi = 1$ for any $x \in \Omega$] $\mathbf{n}^{\epsilon'} = \lambda\, \mathbf{n}^\epsilon$ for some C^∞ function $\lambda : \Omega \to \mathbb{R}\backslash\{0\}$. Thus [by (79)]

$$1 = g_{\epsilon'}\big(\mathbf{n}^{\epsilon'}\,,\ \mathbf{n}^{\epsilon'}\big) = \lambda^2\left\{1 + \left(\frac{1}{\epsilon'^2} - \frac{1}{\epsilon^2}\right)\theta\big(\mathbf{n}^\epsilon\big)^2\right\},$$

yielding $f_{\epsilon'}(x)^{-1/2}\ \nu_x^{\epsilon'} = f_\epsilon(x)^{-1/2}\ \nu_x^\epsilon$ for every $x \in \Omega$. □

For every C^1 vector field X on M, its divergence with respect to the volume form $\Psi = \theta \wedge (d\theta)^n$ is given by

$$\mathscr{L}_X\, \Psi = \operatorname{div}(X)\ \Psi,$$

where $\mathscr{L}_X$ is the Lie derivative at the direction X.

Definition 7 The *horizontal mean curvature* of $\mathcal{F}$ is

$$K_0 = \operatorname{div}\big(\mathbf{n}^0\big) \in C^\infty(\Omega). \tag{80}$$

□

Let

$$d\,\mathrm{vol}(g_\theta) = \sqrt{\mathfrak{g}}\, dx^1 \wedge \cdots \wedge dx^{2n+1}\,,$$

$$\mathfrak{g} = \det\big[g_{jk}\big], \quad g_{jk} = g_\theta\big(\partial_j\,,\,\partial_k\big), \quad \partial_j \equiv \frac{\partial}{\partial x^j},$$

be the canonical volume form of the oriented Riemannian manifold $(M,\ g_\theta)$. Then (cf. e.g. [9])

$$d\,\mathrm{vol}(g_\theta) = C_n\,\Psi,$$

for some constant $C_n > 0$ depending only on the CR dimension n. Hence the divergence operator in (80) is the ordinary Riemannian divergence on $(M,\ g_\theta)$. The volume form Ψ is parallel with respect to the Tanaka–Webster connection ∇ of $(M,\ \theta)$, hence $\mathrm{div}(X)$ can be computed as the trace of the covariant derivative ∇X. Therefore, if $x \in \Omega$ and $\{X_a\ :\ 1 \le a \le 2n\}$ is a local G_θ-orthonormal frame of $H(M)$, defined on a neighborhood $U \subset \Omega$ of x, then [as $\{X_a\,,\ T\ :\ 1 \le a \le 2n\}$ is a local orthonormal frame of $T(M)$ on U]

$$K_0(x) = \sum_{a=1}^{2n} g_\theta\big(\nabla_{X_a}\mathbf{n}^0,\ X_a\big)_x + g_\theta\big(\nabla_T\mathbf{n}^0\,,\ T\big)_x,$$

hence [by $\nabla_T\mathbf{n}^0 \in H(M)$, as $H(M)$ is parallel with respect to ∇, and by $\nabla g_\theta = 0$]

$$K_0(x) = \sum_{a=1}^{2n}\Big\{X_a\big(g_\theta(\mathbf{n}^0\,,\ X_a)\big)_x - g_\theta\big(\mathbf{n}^0\,,\ \nabla_{X_a}X_a\big)_x\Big\}. \tag{81}$$

To draw a parallel between the considerations in the present paper and those in the work by Capogna et al. (cf. [17]) let $M = \mathbb{H}_n$ be the Heisenberg group i.e. the noncommutative Lie group $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$ with the group law

$$(z,\ t)\cdot(w,\ s) = \big(z + w,\ t + s + 2\,\mathrm{Im}(z\cdot\overline{w})\big),$$
$$t,\ s \in \mathbb{R}, \quad z,\ w \in \mathbb{C}^n\,, \quad z\cdot\overline{w} = \delta^{\alpha\beta}\,z_\alpha\,\overline{w}_\beta,$$

equipped with the strictly pseudoconvex, left invariant, CR structure $T_{1,0}(\mathbb{H}_n)$ spanned by

$$L_\alpha \equiv \frac{\partial}{\partial z_\alpha} + i\,\overline{z}_\alpha\,\frac{\partial}{\partial t}\,, \quad 1 \le \alpha \le n,$$

[so that $\overline{L}_\alpha$ are the *Lewy operators*] and with the contact form

$$\theta_0 = dt + i\sum_{\alpha=1}^{n}\big(z_\alpha\,d\overline{z}_\alpha - \overline{z}_\alpha\,dz_\alpha\big) \in \mathscr{P}_+(\mathbb{H}_n).$$

The work [17] deals with an arbitrary Carnot group G, yet in general the horizontal plane H of G may lack a complex structure. Also, if the horizontal plane admits a complex structure $J : H \to H$, and the corresponding almost CR structure $\text{Eigen}\big(J^{\mathbb{C}}, +i\big) \subset H \otimes \mathbb{C}$ is formally integrable, then in general the CR codimension of the resulting CR structure is > 1. So for comparison reasons, between the theory developed here and the geometric foundations on which [17] relies, we confine ourselves to the Heisenberg group $G = \mathbb{H}_n$, which is both a Carnot group and a strictly pseudoconvex CR manifold (isomorphic to the boundary of the Siegel domain in $\mathbb{C}^{n+1}$). If this is the case

$$X_\alpha \equiv \frac{1}{\sqrt{2}}\Big(L_\alpha + \overline{L}_\alpha\Big) = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial x_\alpha} - 2\, y_\alpha \frac{\partial}{\partial t}\Big),$$

$$X_{n+\alpha} \equiv \frac{i}{\sqrt{2}}\Big(L_\alpha - \overline{L}_\alpha\Big) = \frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial y_\alpha} + 2\, x_\alpha \frac{\partial}{\partial t}\Big),$$

is a (globally defined) G_{θ_0}-orthonormal frame of the Levi distribution $H(\mathbb{H}_n)$. The Reeb vector field and the Tanaka–Webster connection of $\big(\mathbb{H}_n\,,\ \theta_0\big)$ are $T \equiv \partial/\partial t$ and

$$\nabla_{L_A} L_B = 0, \quad A,\ B \in \big\{0,\ 1,\ \cdots,\ n,\ \overline{1},\ \cdots,\ \overline{n}\big\},$$
$$L_0 \equiv T, \quad L_{\overline{\alpha}} \equiv \overline{L}_\alpha\,, \quad 1 \leq \alpha \leq n.$$

Consequently $\nabla_{X_a} X_a = 0$, so that for every subelliptic harmonic morphism $\phi : \mathbb{H}_n \to N^1$ our formula (81) becomes

$$K_0 = \sum_{a=1}^{2n} X_a\big(g_{\theta_0}(\mathbf{n}^0\,,\ X_a)\big)$$

which is formula (2.2) in [17, p. 7]. Going back to the general case, let us observe that $T_x^{\mathscr{V}} \neq 0$ for every $x \in \Omega$. Otherwise $T_x \in \mathscr{H}_x^\phi$ for some $x \in \Omega$, hence [by Lemma 3] $x \in \text{II}_1(\phi) = \Sigma(\mathscr{F})$, a contradiction. Therefore the vector field

$$\mathscr{T} = \big\|T^{\mathscr{V}}\big\|^{-1}\, T^{\mathscr{V}} \in C^\infty\big(\Omega \setminus \Sigma(\mathscr{F}),\ \mathscr{V}^\phi\big),$$

(considered by us in Sect. 5.1, though confined to the case $m \geq 2$) is well defined for $m = 1$, as well. In particular, Lemma 15 applies to the case $m = 1$, producing a local G_θ-orthonormal frame

$$\big\{V_j\,,\ \mathscr{T}\ :\ 1\le j\le 2n-1\big\}\subset C^\infty\big(U,\ \mathscr{V}^\phi\big),$$
$$V_j\in C^\infty\big(U,\ \mathscr{V}_H^\phi\big),\quad U\subset\Omega.$$

Let us complete $\{V_j\ :\ 1\le j\le 2n-1\}$ to a local G_θ-orthonormal frame

$$\{V_a\ :\ 1\le a\le 2n\}\subset C^\infty(U,\ H(M)).$$

If $(V,\ y^1)$ is a local coordinate system on N^1 and $U=\phi^{-1}(V)$ then

$$V_{2n}=\frac{1}{\sqrt{\Lambda_0}}\,\nabla^H\phi^1\,,\quad \phi^1\equiv y^1\circ\phi,\quad \Lambda_0\equiv\frac{\Lambda}{h_{11}\circ\phi}. \tag{82}$$

Indeed $\nabla^H\phi^1=\lambda^a\,V_a$ and

$$0=\phi_*\,V_j=V_j\big(\phi^1\big)\Big(\frac{\partial}{\partial y^1}\Big)^\phi=g_\theta\big(V_j\,,\ \nabla^H\phi^1\big)\Big(\frac{\partial}{\partial y^1}\Big)^\phi$$
$$=\lambda^j\,\Big(\frac{\partial}{\partial y^1}\Big)^\phi\Longrightarrow\lambda^j=0,$$

i.e. $\nabla^H\phi^1=\lambda^{2n}\,V_{2n}$. By the Levi conformality condition

$$\Lambda(x)\,h^{11}\big(\phi(x)\big)=G_\theta\big(\nabla^H\phi^1\,,\ \nabla^H\phi^1\big)_x,$$

for every $x\in U$, hence $\lambda^{2n}=\sqrt{\Lambda_0}$. □

Lemma 23 *For every subelliptic harmonic morphism $\phi: M^{2n+1}\to N^1$*

$$\mathbf{n}^\epsilon=\frac{1}{\sqrt{1+\epsilon^2\,f^2}}\,\Big\{V_{2n}-\epsilon^2\,f\,T\Big\},\quad 0<\epsilon<1,$$
$$f\equiv\frac{g_\theta\big(V_{2n}\,,\ \mathscr{T}\big)}{\theta(\mathscr{T})}\in C^\infty(U), \tag{83}$$

everywhere in $U=\phi^{-1}(V)$, where $V_{2n}\equiv\Lambda_0^{-1/2}\,\nabla^H\phi^1$.

Proof As $\mathbf{n}^\epsilon\in\mathscr{H}_\epsilon^\phi\subset T(M)=H(M)\oplus\mathbb{R}T$

$$\mathbf{n}^\epsilon=\sum_{a=1}^{2n}f_a\,V_a+f_0\,T,$$

with $f_j=0$ [because of $g_\epsilon\big(\mathbf{n}^\epsilon\,,\ V_j\big)=0$] i.e. $\mathbf{n}^\epsilon=\lambda\,V_{2n}+f_0\,T$ with $\lambda:=f_{2n}$. On the other hand

$$g_\epsilon(T,\ \mathscr{T})=\frac{1}{u}\,g_\epsilon\big(T,\ T^\mathscr{V}\big)=\frac{1}{u}\Big\{g_\theta\big(T,\ T^\mathscr{V}\big)+\Big(\frac{1}{\epsilon^2}-1\Big)\,\theta\big(T^\mathscr{V}\big)\Big\}=\frac{u}{\epsilon^2},$$

[because of $\theta\big(T^{\mathscr{V}}\big) = g_\theta\big(T, \ T^{\mathscr{V}}\big) = \|T^{\mathscr{V}}\|^2 = u^2$]. Here $u = \theta(\mathscr{T})$. Then

$$0 = g_\epsilon\big(\mathbf{n}^\epsilon \,,\ \mathscr{T}\big) = \lambda \, g_\epsilon\big(V_{2n} \,,\ \mathscr{T}\big) + f_0 \, \frac{u}{\epsilon^2}$$

$$\mathbf{n}^\epsilon = \lambda \left\{ V_{2n} - \epsilon^2 \, f \ T \right\}. \tag{84}$$

Finally [by (84)]

$$1 = g_\epsilon\big(\mathbf{n}^\epsilon \,,\ \mathbf{n}^\epsilon\big) = g_\theta\big(\mathbf{n}^\epsilon \,,\ \mathbf{n}^\epsilon\big) + \Big(\frac{1}{\epsilon^2} - 1\Big)\, \theta\big(\mathbf{n}^\epsilon\big)^2$$
$$= \lambda^2 \left\{ 1 + \frac{\epsilon^2}{u^2}\, g_\theta\big(V_{2n} \,,\ \mathscr{T}\big)^2 \right\} = \lambda^2 \left\{ 1 + \epsilon^2 \, f^2 \right\}.$$

□

As a corollary of (83)

Proposition 2 (i) *The function $f_\epsilon \in C^\infty(\Omega)$ in (77) is given by*

$$f_\epsilon = \frac{1}{1 + \epsilon^2 \, f^2}.$$

(ii) *The vector field $\nu_\epsilon \in C^\infty\big(\Omega, \ H(M)\big)$ is given by*

$$\nu_\epsilon = \frac{1}{\sqrt{1 + \epsilon^2 \, f^2}}\, V_{2n} = \frac{1}{\sqrt{\big(1 + \epsilon^2 \, f^2\big)\, \Lambda_0}}\, \nabla^H \phi^1,$$

everywhere on $U = \phi^{-1}(V)$.

(iii) *The horizontal normal $\mathbf{n}^0$ is given by*

$$\mathbf{n}^0 = V_{2n} = \frac{1}{\sqrt{\Lambda_0}}\, \nabla^H \phi^1,$$

on U.

(iv) *The horizontal mean curvature of $\mathcal{F}$ is given by*

$$K_0 = \operatorname{div}\Big(\frac{1}{\sqrt{\Lambda_0}}\, \nabla^H \phi^1\Big) = -\frac{1}{\sqrt{\Lambda_0}} \left\{ \Delta_b \phi^1 + \big(\nabla^H \phi^1\big) \, \log \sqrt{\Lambda_0} \right\},$$

on U.

Proof (i) By (83)

$$\theta\big(\mathbf{n}^\epsilon\big) = -\frac{\epsilon^2 \, f}{\sqrt{1 + \epsilon^2 \, f^2}}\,,$$

$$g_\epsilon\big(\mathbf{n}^\epsilon\,,\,T\big) = \frac{1}{\epsilon^2}\,\theta\big(\mathbf{n}^\epsilon\big), \quad g_\epsilon(T\,,\,T) = \frac{1}{\epsilon^2},$$

and then

$$\begin{aligned} f_\epsilon &= g_\epsilon\big(\nu_\epsilon\,,\,\nu_\epsilon\big) = 1 - 2\,\theta\big(\mathbf{n}^\epsilon\big)\,g_\epsilon\big(\mathbf{n}^\epsilon\,,\,T\big) + \theta\big(\mathbf{n}^\epsilon\big)^2\,g_\epsilon(T\,,\,T) \\ &= \frac{1}{1+\epsilon^2 f^2}. \end{aligned}$$

The remainder of the section is devoted to the proof of Theorem 5. Statement (i) was proved in Proposition 2.

(ii) The horizontal mean curvature is given by

$$K_0 = \operatorname{div}\big(\mathbf{n}^0\big) = \sum_{a=1}^{2n} g_\theta\big(\nabla_{V_a}\mathbf{n}^0\,,\,V_a\big) =,$$

[by $g_\theta\big(\nabla_{V_{2n}}\mathbf{n}^0\,,\,V_{2n}\big) = 0$ and $\nabla g_\theta = 0$]

$$= -\sum_{j=1}^{2n-1} g_\theta\big(\mathbf{n}^0\,,\,\nabla_{V_j}V_j\big).$$

On the other hand

$$2n\,\mu^{\mathscr{V}\phi}\big(g_\theta\,,\,\nabla\big) = \pi_{\mathscr{H}\phi}\Big\{\sum_{j=1}^{2n-1}\nabla_{V_j}V_j + \nabla_{\mathscr{T}}\mathscr{T}\Big\},$$

hence [by taking the inner product with $\mathbf{n}^0$]

$$g_\theta\big(\mathbf{n}^0\,,\,2n\,\mu^{\mathscr{V}\phi}(g_\theta\,,\,\nabla) - \pi_{\mathscr{H}\phi}\,\nabla_{\mathscr{T}}\mathscr{T}\big) = -K_0 - \sum_{j=1}^{2n-1} g_\theta\big(\mathbf{n}^0\,,\,\pi_{\mathscr{V}\phi}\,\nabla_{V_j}V_j\big). \tag{85}$$

Next [by (iii) in Proposition 2]

$$\mathbf{n}^0 = \frac{1}{\sqrt{\Lambda_0}}\,\big\{\nabla\phi^1 - T(\phi^1)\,T\big\}, \tag{86}$$

and [by Lemma 1] $\nabla\phi^1 \in \mathscr{H}^\phi$. Consequently [by (86)]

$$\pi_{\mathscr{V}\phi}\,\mathbf{n}^0 = -\frac{1}{\sqrt{\Lambda_0}}\,T(\phi^1)\,T^{\mathscr{V}} = -\frac{\theta(\mathscr{T})}{\sqrt{\Lambda_0}}\,T(\phi^1)\,\mathscr{T}. \tag{87}$$

As $\dim_{\mathbb{R}} \mathscr{H}_x^\phi = 1$, there is a unique function $\varphi \in C^\infty(\Omega)$ such that

$$T^{\mathscr{H}} = \varphi \, \nabla \phi^1 = \varphi \left\{ \nabla^H \phi^1 + T(\phi^1) \, T \right\}. \tag{88}$$

To compute φ one starts from $T^{\mathscr{H}} = T - T^{\mathscr{V}}$, yielding

$$\| T^{\mathscr{H}} \|^2 = 1 - u^2.$$

On the other hand [by (88)]

$$\| T^{\mathscr{H}} \|^2 = \varphi^2 \left\{ \left\| \nabla^H \phi^1 \right\|^2 + T(\phi^1)^2 \right\},$$

so that [by the Levi conformality property]

$$\varphi^2 \left\{ \Lambda \left(h^{11} \circ \phi \right) + T(\phi^1)^2 \right\} = 1 - \theta(\mathscr{T})^2. \tag{89}$$

We may now compute the last term in (85) i.e.

$$\sum_{j=1}^{2n-1} g_\theta\big(\mathbf{n}^0 \, , \, \pi_{\mathscr{V}\phi} \, \nabla_{V_j} V_j\big) = \sum_{j=1}^{2n-1} g_\theta\big(\pi_{\mathscr{V}\phi} \, \mathbf{n}^0 \, , \, \nabla_{V_j} V_j\big) =,$$

[by (88)]

$$= -\frac{1}{\sqrt{\Lambda_0}} \, T(\phi^1) \sum_{j=1}^{2n-1} g_\theta\big(T^{\mathscr{V}} \, , \, \nabla_{V_j} V_j\big) =,$$

[by $T^{\mathscr{V}} = T - T^{\mathscr{H}}$ and $T \perp H(M) \ni \nabla_{V_j} V_j$]

$$= \frac{1}{\sqrt{\Lambda_0}} \, T(\phi^1) \sum_{j=1}^{2n-1} g_\theta\big(T^{\mathscr{H}} \, , \, \nabla_{V_j} V_j\big) =,$$

[by (88)]

$$= \frac{\varphi}{\sqrt{\Lambda_0}} \, T(\phi^1) \sum_{j=1}^{2n-1} g_\theta\big(\nabla \phi^1 \, , \, \nabla_{V_j} V_j\big)$$

$$= \frac{\varphi}{\sqrt{\Lambda_0}} \, T(\phi^1) \sum_{j=1}^{2n-1} g_\theta\big(\nabla^H \phi^1 \, , \, \nabla_{V_j} V_j\big) =,$$

[by $\nabla g_\theta = 0$]

$$= \frac{\varphi}{\sqrt{\Lambda_0}} T(\phi^1) \sum_{j=1}^{2n-1} \Big\{ V_j\big(g_\theta\big(\nabla^H \phi^1 , V_j\big)\big) - g_\theta\big(\nabla_{V_j} \nabla^H \phi^1 , V_j\big)\Big\} =,$$

[as $\nabla^H \phi^1 \in \mathscr{H}^\phi \perp \mathscr{V}^\phi \ni V_j$]

$$= -\frac{\varphi}{\sqrt{\Lambda_0}} T(\phi^1) \sum_{j=1}^{2n-1} g_\theta\big(\nabla_{V_j} \nabla^H \phi^1 , V_j\big) =,$$

[by (iii) in Proposition 2]

$$= -\frac{\varphi}{\sqrt{\Lambda_0}} T(\phi^1) \Big\{ \mathrm{div}\,\big(\nabla^H \phi^1\big) - g_\theta\big(\nabla_{V_{2n}}\big(\sqrt{\Lambda_0}\, V_{2n}\big) , V_{2n}\big)\Big\} =,$$

[as $g_\theta\big(\nabla_{V_{2n}} V_{2n} , V_{2n}\big) = 0$ and $g_\theta\big(V_{2n} , V_{2n}\big) = 1$]

$$= \frac{\varphi}{\sqrt{\Lambda_0}} T(\phi^1) \Big\{ \Delta_b \phi^1 + V_{2n}\big(\sqrt{\Lambda_0}\big)\Big\}$$
$$= \frac{\varphi}{\sqrt{\Lambda_0}} T(\phi^1) \Big\{ \Delta_b \phi^1 + \big(\nabla^H \phi^1\big) \log \sqrt{\Lambda_0}\Big\},$$

hence [by (iv) in Proposition 2]

$$\sum_{j=1}^{2n-1} g_\theta\big(\mathbf{n}^0 , \pi_{\mathscr{V}^\phi} \nabla_{V_j} V_j\big) = -\varphi\, T(\phi^1)\, K_0. \tag{90}$$

Equation (85) becomes [by (90)]

$$g_\theta\big(\mathbf{n}^0 , 2n\,\mu^{\mathscr{V}^\phi}(g_\theta , \nabla) - \pi_{\mathscr{H}^\phi} \nabla_{\mathscr{T}} \mathscr{T}\big) = \big\{\varphi\, T(\phi^1) - 1\big\}\, K_0. \tag{91}$$

This yields (4) in Theorem 5 because of

$$g_\theta\big(\mathbf{n}^0 , \pi_{\mathscr{H}^\phi}\, J\mathscr{T}\big) = 0. \tag{92}$$

Indeed, the identity $\mathscr{T} = u^{-1}\big\{T - \varphi\, \nabla\phi^1\big\}$ implies

$$J\mathscr{T} = -\frac{\varphi}{u} \sqrt{\Lambda_0}\, J\mathbf{n}^0. \tag{93}$$

Also [by (87)]

$$\pi_{\mathscr{H}^\phi}\, \mathbf{n}^0 = \mathbf{n}^0 + \frac{u}{\sqrt{\Lambda_0}}\, T(\phi^1)\, \mathscr{T}. \tag{94}$$

Finally

$$g_\theta\big(\mathbf{n}^0\,,\ \pi_{\mathscr{H}\phi}\ J\mathscr{T}\big) = g_\theta\big(\pi_{\mathscr{H}\phi}\,\mathbf{n}^0\,,\ J\mathscr{T}\big) =,$$

[by (94), as $g_\theta(\mathscr{T},\ J\mathscr{T}) = 0$]

$$= g_\theta\big(\mathbf{n}^0\,,\ J\mathscr{T}\big) = \quad \text{[by (93)]}$$
$$= -\frac{\varphi}{u}\sqrt{\Lambda_0}\,g_\theta\big(\mathbf{n}^0\,,\ J\mathbf{n}^0\big) = 0,$$

and (92) is proved.

Let us recall (61). This was stated for $m \geq 2$ yet it is easily seen to hold for any $m \geq 1$, everywhere in Ω. Then [by (61) with $m = 1$]

$$\tau_b(\phi) = \phi_*\,\nabla^H \log\sqrt{\Lambda} - 2n\,\phi_*\,\mu_{\text{hor}}^{\mathscr{V}\phi},$$

so that [by $\tau_b(\phi) = 0$]

$$2n\,\mu_{\text{hor}}^{\mathscr{V}\phi} = \pi_{\mathscr{H}\phi}\,\nabla^H\,\log\sqrt{\Lambda}. \tag{95}$$

Note that $\Lambda_0 = \Lambda\,\big(h^{11}\big)^\phi$ yields

$$\nabla^H \log\sqrt{\Lambda} = \nabla^H \log\sqrt{\Lambda_0} + \left\{\begin{matrix}1\\11\end{matrix}\right\}^\phi \nabla^H\phi^1. \tag{96}$$

Then [by (95) and (96)]

$$2n\,g_\theta\big(\mathbf{n}^0\,,\ \mu_{\text{hor}}^{\mathscr{V}\phi}\big) = g_\theta\big(\pi_{\mathscr{H}\phi}\,\mathbf{n}^0\,,\ \nabla^H\log\sqrt{\Lambda_0}\big) + \left\{\begin{matrix}1\\11\end{matrix}\right\}^\phi g_\theta\big(\pi_{\mathscr{H}\phi}\,\mathbf{n}^0\,,\ \nabla^H\phi^1\big). \tag{97}$$

The right hand side in (97) may be computed as follows

$$g_\theta\big(\pi_{\mathscr{H}\phi}\,\mathbf{n}^0\,,\ \nabla^H\log\sqrt{\Lambda_0}\big) = \frac{1}{\sqrt{\Lambda_0}}\big[1 - \varphi\,T(\phi^1)\big]\big(\nabla^H\phi^1\big)\,\log\sqrt{\Lambda_0}\,,$$
$$g_\theta\big(\pi_{\mathscr{H}\phi}\,\mathbf{n}^0\,,\ \nabla^H\phi^1\big) = \sqrt{\Lambda_0}\big[1 - \varphi\,T(\phi^1)\big],$$

hence (97) becomes

$$2n\,g_\theta\big(\mathbf{n}^0\,,\ \mu_{\text{hor}}^{\mathscr{V}\phi}\big) = \big\{1 - \varphi\,T(\phi^1)\big\}\Big[\frac{1}{\sqrt{\Lambda_0}}\big(\nabla^H\phi^1\big)\ \log\sqrt{\Lambda_0} + \sqrt{\Lambda_0}\left\{\begin{matrix}1\\11\end{matrix}\right\}^\phi\Big],$$

or [by (3)]

$$2n\,g_\theta\big(\mathbf{n}^0\,,\ \mu_{\text{hor}}^{\mathscr{V}\phi}\big) = \big\{\varphi\,T(\phi^1) - 1\big\}\Big[K_0 + \frac{1}{\sqrt{\Lambda_0}}\,\Delta_b\phi^1 - \sqrt{\Lambda_0}\left\{\begin{matrix}1\\11\end{matrix}\right\}^\phi\Big],$$

where $-\Delta_b \phi^1 + \Lambda_0 \left\{ \begin{smallmatrix} 1 \\ 11 \end{smallmatrix} \right\}^\phi = 0$ (as ϕ is also a subelliptic harmonic map). Therefore, unlike the case $m = 2$, the fundamental equation (61) for a subelliptic harmonic morphism is equivalent to (4) in Theorem 5 and implies no further restrictions on K_0.

8 Examples

8.1 Morphisms from the Heisenberg Group

Let us set $f(z,t) = |z|^2 - i\,t$, so that f is a CR function on $\mathbb{H}_n$ i.e. $\overline{L}_\alpha f = 0$ for any $1 \le \alpha \le n$.

Theorem 8 *Let* $\phi : \mathbb{H}_n \setminus \{0\} \to \mathbb{R}$ *be the* C^∞ *map given by*

$$\phi = 1 \big/ \big(f\,\overline{f}\big)^{n/2}. \tag{98}$$

Then

(i) ϕ *is a subelliptic harmonic morphism of the pseudohermitian manifold* $\big(\mathbb{H}_n \setminus \{0\},\ \theta_0\big)$ *into the Riemannian manifold* $\big(\mathbb{R},\ dy^1 \otimes dy^1\big)$.

(ii) $\mathrm{Crit}(\phi) = \emptyset$ *and* $S(\phi) = \mathbb{H}_n \setminus \{0\}$.

(iii) $\mathrm{I}_1(\phi) = \mathbb{C}^* \times \{0\}$ *where* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(iv) ϕ *is a subelliptic harmonic map of* $(\mathbb{H}_n \setminus \{0\},\ \theta_0)$ *into* $(\mathbb{R},\ dy^1 \otimes dy^1)$*, and a Levi conformal map of square dilation*

$$\Lambda(x) = \frac{2n^2\,|z|^2}{|x|^{2Q}}\,, \quad x = (z,t) \in \mathbb{H}_n\,, \quad x \neq 0. \tag{99}$$

Consequently

$$\mathrm{II}_1(\phi) = \{0\} \times \mathbb{R}^*\,, \quad \mathrm{III}_1(\phi) = \mathbb{C}^* \times \mathbb{R}^*\,, \quad \mathbb{R}^* = \mathbb{R} \setminus \{0\}. \tag{100}$$

(v) *The horizontal mean curvature of the leaves of* $\mathscr{F}$ *is*

$$K_0 = \frac{1}{2\sqrt{2}\,|z|}\,\big(f\,\overline{f}\big)^{-1/2}\big[f + \overline{f} - 2\,Q\,|z|^2\big] = -\frac{(Q-1)\,|z|}{\sqrt{2}\,|x|^2}. \tag{101}$$

Here $Q = 2n + 2$ (the *homogeneous dimension* of $\mathbb{H}_n$) and $|x| = \big(|z|^4 + t^2\big)^{1/4}$ [the *Heisenberg norm* of $x = (z,t) \in \mathbb{H}_n$].

Proof (i) By (98) $\phi(x) = |x|^{-Q+2}$ for any $x \in \mathbb{H}_n$. Then $\phi(x)$ is the fundamental solution to $\mathscr{L}_0 = \Delta_b$ discovered by Folland (cf. [31]) i.e. there is a constant $c_0 \neq 0$ such that $\Delta_b \phi = c_0\,\delta$, where δ is the Dirac distribution (concentrated in zero). In particular $\phi : \mathbb{H}_n \setminus \{0\} \to \mathbb{R}$ is a subelliptic harmonic morphism of $\big(\mathbb{H}_n \setminus \{0\},\ \theta_0\big)$ into $\big(\mathbb{R},\ dy^1 \otimes dy^1\big)$.

(ii) The Euclidean gradient of ϕ is

$$D\phi(x) = -\frac{(Q-2)\,|z|^2}{|x|^{Q+2}}\left(z,\ \frac{t}{2|z|^2}\right), \quad x = (z,\ t), \tag{102}$$

so that $\mathrm{Crit}(\phi) = \emptyset$, and then $S(\phi) = \mathbb{H}_n \setminus \{0\}$.
(iii) Note that

$$T(\phi)_x = -\frac{n\,t}{|x|^{Q+2}}, \quad x = (z,\ t).$$

Hence [by statement (i) in Lemma 3]

$$x \in \mathrm{I}_1(\phi) \iff (d_x\phi)T_x = 0 \iff t = 0.$$

□

(iv) The horizontal gradient of ϕ is

$$\nabla^H \phi = -\frac{n}{|x|^{Q+2}} \sum_{\alpha=1}^{n} \left\{ z^\alpha\, f\, L_\alpha + \overline{z}_\alpha\, \overline{f}\, \overline{L}_\alpha \right\}. \tag{103}$$

Then [by (16)] the square dilation is

$$\Lambda(z,t) = G_\theta(\nabla^H \phi, \nabla^H \phi)_{(z,t)} = \frac{2n^2\,|z|^2}{\left(|z|^4 + t^2\right)^{n+1}},$$

which is (99). Next [by (99) together with statements (ii) in Lemma 3 and (iii) in Lemma 1]

$$x \in \mathrm{II}_1(\phi) \iff \Lambda(x) = 0 \iff z = 0.$$

□

(v) The use of the CR function $f(z,t) = |z|^2 - it$ greatly simplifies calculations. One starts by rephrasing the square dilation as

$$\Lambda_0 = \Lambda = \frac{2\,n^2\,|z|^2}{(f\,\overline{f})^{Q/2}},$$

so that

$$L_\alpha\left(\log\sqrt{\Lambda_0}\right) = \frac{\overline{z}_\alpha}{2\,f\,|z|^2}\left(f - Q\,|z|^2\right),$$

yielding [by (103) and statement (i) in Theorem 5] (101). □

The sublaplacian Δ_b on $(\mathbb{H}_n\,,\ \theta_0)$ belongs to the family $\{\mathscr{L}_\gamma\}_{\gamma\in\mathbb{C}}$ of Folland-Stein operators (cf. e.g. [29, p. 177])

$$\mathscr{L}_\gamma \equiv -\frac{1}{2}\sum_{\alpha=1}^{n}\big(L_\alpha\,\overline{L}_\alpha + \overline{L}_\alpha\,L_\alpha\big) + i\,\gamma\,\frac{\partial}{\partial t}\,,\quad \gamma\in\mathbb{C}.$$

A C^∞ map $\phi:\mathbb{H}_n\backslash\{0\}\to N$ into a Riemannian manifold $(N,\ h)$ is a *$\mathscr{L}_\gamma$-morphism* if for any harmonic function $v: V\subset N\to\mathbb{R}$ [i.e. $\Delta_h v = 0$ in V] $\mathscr{L}_\gamma(v\circ\phi) = 0$ in $U=\phi^{-1}(V)$.

A complex number $\gamma\in\mathbb{C}$ is *admissible* if and only if $c_\gamma\neq 0$ where

$$c_\gamma = \frac{2^{2-2n}\,\pi^{n+1}}{\Gamma\Big(\dfrac{n+\gamma}{2}\Big)\,\Gamma\Big(\dfrac{n-\gamma}{2}\Big)}.$$

By a result of Folland and Stein (cf. [32], or [29, p. 179]) if γ is admissible (equivalently $\gamma\in\big\{\pm n,\ \pm(n+2),\ \pm(n+4),\ \ldots\}$)

$$\phi_\gamma = f^{-(n+\gamma)/2}\,\overline{f}^{-(n-\gamma)/2},$$

is a fundamental solution to $\mathscr{L}_\gamma$ i.e. $\mathscr{L}_\gamma\,\phi_\gamma = c_\gamma\,\delta$. In particular

$$\phi_{\pm p}:\mathbb{H}_n\setminus\{0\}\to\mathbb{R},\quad p\in\big\{n,\ n+2,\ n+4,\ \ldots\big\},$$

are $\mathscr{L}_{\pm p}$-morphisms, of $\mathbb{H}_n\backslash\{0\}$ into $\big(\mathbb{R},\ dy^1\otimes dy^1\big)$. Also $\mathscr{L}_\gamma$ is hypoelliptic if and only if γ is admissible. The study of $\mathscr{L}_\gamma$-morphisms into a general Riemannian manifold $(N,\ h)$ is an open problem.

8.2 Morphisms from Rossi Spheres

Let $S^2 = \big\{\big(y^1\,,\ y^2\,,\ y^3\big)\in\mathbb{R}^3\ :\ \sum_{j=1}^3 (y^j)^2 = 1\big\}$ and $S^3 = \big\{(z,\ w)\in\mathbb{C}^2\ :\ |z|^2 + |w|^2 = 1\big\}$, and let $\pi: S^3\to S^2$ be the Hopf fibration i.e. $\pi(z,\ w) = \big(y^1\,,\ y^2\,,\ y^3\big)$

$$\begin{cases} y^1 = |z|^2 - |w|^2\,,\\ y^2 = z\,\overline{w} + \overline{z}\,w,\\ y^3 = -i\ (z\overline{w} - \overline{z}\,w)\,. \end{cases} \tag{104}$$

Let $h_{S^N} = \mathbf{j}^*\,g_0$ be the first fundamental form of $\mathbf{j}: S^N\hookrightarrow\mathbb{R}^{N+1}$, where g_0 is the Euclidean metric on $\mathbb{R}^{N+1}$. Let S^3 be equipped with the standard CR structure $T_{1,0}(S^3)$ [induced by the complex structure of $\mathbb{C}^2$], and with the canonical contact form

$$\theta = \frac{i}{2}\big\{-\overline{z}\,dz + z\,d\overline{z} - \overline{w}\,dw + w\,d\overline{w}\big\}\in\mathscr{P}_+(S^3). \tag{105}$$

$T_{1,0}(S^3)$ is the span of $L = \overline{w}\,(\partial/\partial z) - \overline{z}\,(\partial/\partial w)$. Let us set

$$L_t = L + t\,\overline{L}, \quad |t| < 1,$$

and let H_t be CR structure on S^3 spanned by L_t [$\{(S^3\,,\ H_t)\}_{|t|<1}$ are the *Rossi spheres*]. By a result in [57], the CR manifold $(S^3\,,\ H_t)$ is globally embeddable in $\mathbb{C}^2$ if and only if $t = 0$.

Theorem 9 (i) *The Hopf map* $\pi\,:\,S^3 \to S^2$ *is a subelliptic harmonic morphism of* $\big(S^3\,,\ T_{1,0}(S^3),\ \theta\big)$ *into* $(S^2\,,\ h_{S^2})$.

(ii) π *is a subelliptic harmonic morphism of* $\big(S^3\,,\ H_t\,,\ \theta\big)$ *into* $\big(S^2\,,\ h_{S^2}\big)$ *if and only if* $t = 0$.

Note that θ is indeed a positively oriented contact form on each Rossi sphere $(S^3,\ H_t)$. The corresponding Levi form $G_\theta(t)$ is

$$G_\theta(t)\big(L_t\,,\ \overline{L}_t\big) = \frac{1-t^2}{2}\,, \quad |t| < 1.$$

The Reeb vector field of $(S^3\,,\ \theta)$ is

$$T = i\left\{z\,\frac{\partial}{\partial z} - \overline{z}\,\frac{\partial}{\partial \overline{z}} + w\,\frac{\partial}{\partial w} - \overline{w}\,\frac{\partial}{\partial \overline{w}}\right\}. \tag{106}$$

Also, the CR manifolds $\{\big(S^3\,,\ H_t\big)\}_{|t|<1}$ have the same Levi distribution as $\big(S^3\,,\ H_0\big)$ i.e. $H(S^3) = \mathrm{Re}\{H_0 \oplus \overline{H}_0\}$

Proof of Theorem 9 (i) A calculation shows that $\pi_* T = 0$. Let g_θ be the Webster metric of $(S^3\,,\ \mathscr{H}_0\,,\ \theta)$. Then $g_\theta = g_{S^3}$ (cf. e.g. [29]). The Hopf map π is an ordinary harmonic morphism of $\big(S^3\,,\ g_{S^3}\big)$ into $\big(S^2\,,\ g_{S^2}\big)$. Therefore [by (12)] π is a subelliptic harmonic morphism of $\big(S^3\,,\ H_0\,,\ \theta\big)$ into $\big(S^2\,,\ g_{S^2}\big)$.

(ii) By statement (i) in Theorem 9, the Hopf map π is a C^∞ submersion such that $\mathscr{V}^\pi = \mathbb{R}T$. In particular the Levi and horizontal distributions coincide i.e. $H(S^3) = \mathscr{H}^\pi$.

Let us assume that π is Levi conformal, as a map of $(S^3\,,\ H_t\,,\ \theta)$ into $(S^2\,,\ h_{S^2})$. The vector fields

$$X_1(t) = L_t + \overline{L}_t = (1+t)\,\big(L + \overline{L}\big),$$
$$X_2(t) = i\,\big(L_t - \overline{L}_t\big) = i\,(1-t)\,\big(L - \overline{L}\big),$$

span $H(S^3)$ and

$$G_\theta(t)\,\big(X_a^t\,,\ X_a^t\big) = 1 - t^2\,, \quad G_\theta(t)\big(X_1^t\,,\ X_2^t\big) = 0, \quad a \in \{1,\ 2\}.$$

Then [by (16)] for any $(z, w) \in S^3 \backslash \mathrm{Crit}(\pi)$

$$
\begin{aligned}
1-t^2 &= G_\theta(t)\big(X_1(t),\ X_1(t)\big)_{(z,w)} = \Lambda(z,w)\, h^{\pi}_{S^2}\big(\pi_* X_1(t),\ \pi_* X_1(t)\big)_{(z,w)} \\
&= 4(1+t)^2\, \Lambda(z,w),
\end{aligned}
$$

i.e.

$$
4\,\Lambda(z,w) = \frac{1-t}{1+t}.
$$

On the other hand

$$
\begin{aligned}
1-t^2 &= G_\theta(t)\big(X_2(t),\ X_2(t)\big)_{(z,w)} \\
&= \Lambda(z,w)\, h^{\pi}_{S^2}\big(\pi_* X_2(t),\ \pi_* X_2(t)\big)_{(z,w)} = 4(1-t)^2\, \Lambda(z,w),
\end{aligned}
$$

yields

$$
4\,\Lambda(z,w) = \frac{1+t}{1-t},
$$

hence $t = 0$. □

The Authors are grateful to the Referee for drawing their attention on the work by J. Ventura (cf. [62]) and the Ricci curvature calculations there *vis-a-vis* to a horizontally weakly conformal map $\Phi : \mathfrak{M} \to N^2$ from a 4-dimensional Lorentzian manifold $\mathfrak{M}$ to a surface N^2, together with the investigation (cf. *op. cit.*) of the behavior of Ricci curvature under biconformal deformations (7). It should be noticed that according to our Theorem 9 examples in that context are actually scarce (the only CR structure on S^3 in Rossi's family, with respect to which the Hopf map $\pi : S^3 \to S^2$ is a subelliptic harmonic morphism, is the standard CR structure on the sphere). Several new space-time models are built in [62] starting from classical examples of space-times and harmonic morphisms (cf. [5]) and it is a natural question, asked by the Reviewer, whether and how biconformal changes of the metric affect our examples in Sect. 8. We leave that as an open problem.

We close with the observation that, in a simple context such as $M^3 = \mathbb{H}_1$ (the lowest dimensional Heisenberg group) and $\mathfrak{M}^4 = C\big(\mathbb{H}_1\big)$ equipped with the Fefferman metric $g_0 = F_{\theta_0}$ [associated to the canonical contact form θ_0 in Sect. 7] looking for vacuum solutions to the gravitational field equations on $C(\mathbb{H}_1)$ by conformal or biconformal deformations of g_0, lacks a physical meaning. Indeed g_0 isn't flat and its curvature corresponds, by the General Relativity and Gravitation Theory, to the content of matter and energy of the region $\Omega \subset C(\mathbb{H}_1)$ where gravitational effects are perceived. Said matter-energy content of Ω is described by an energy-momentum tensor $T_{\lambda\mu}$ that is by definition the traceless Ricci tensor associated to g_0. The linearized Einstein equations (in the presence of the matter distribution assimilated with the non flat character of g_0 i.e. involving $T_{\lambda\mu}$) were solved by Barletta et al. [11].

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