A modification of Bernstein-Durrmeyer operators with Jacobi weights on the unit interval

Mirella Cappelletti Montano and Vita Leonessa

Abstract The present paper is devoted to the study of a sequence of positive linear operators, acting on the space of all continuous functions on $[0, 1]$ as well as on some weighted spaces of integrable functions on $[0, 1]$. These operators are, as a matter of fact, a generalization of the Bernstein-Durrmeyer operators with Jacobi weights. In particular, we present qualitative and approximation properties of these operators, also providing estimates of the rate of convergence. Moreover, by means of their asymptotic formula, we compare our operators with the Bernstein-Durrmeyer ones and a suitable modification of theirs, showing that, in suitable intervals, they provide a lower approximating error estimate.

2020 Mathematics Subject Classification. 41A36, 41A25.

Keywords and phrases. Bernstein-Durrmeyer-type operators. Jacobi weights. Positive approximation processes. Rate of convergence. Generalized convexity.

1 Introduction

In [18] a modification of the classical Bernstein operators B_n on $[0, 1]$ that fixes the constants and the function x^2 (instead of x) was introduced; the author in particular showed that this modification provides an error of approximation that is as least as good as the one of the Bernstein operators on certain subintervals of $[0, 1]$.

Mirella Cappelletti Montano
Department of Mathematics, University of Bari Aldo Moro, Bari, Italy, e-mail: mirella.cappellettimontano@uniba.it

Vita Leonessa
Department of Mathematics, Computer Science and Economics, University of Basilicata, Potenza, Italy, e-mail: name@email.address

Since then, many other mathematicians have undertaken the task to construct other modifications of well-known approximation processes in the same spirit of [18], in order to get better approximation results. For a survey on such type of operators we refer the interested readers to [4].

In particular, in [13], the authors introduced a modification of the Bernstein operators that fixes the constants and a given strictly increasing function, as follows: for any $n \geq 1$ and $f \in C([0,1])$,

$$B_n^\tau(f) = B_n(f \circ \tau^{-1}) \circ \tau,$$

where τ is a suitable strictly increasing C^∞-function on $[0,1]$. The authors studied qualitative and quantitative properties of such operators and compared their approximation error estimate with the one of the B_n's.

Subsequently (see [3]) this idea was applied in order to define a modification M_n^τ of the Bernstein-Durrmeyer operators on [0,1] introduced in [15], and independently in [20], which are a useful tool to study the approximation properties also of integrable functions.

During the years, Bernstein-Durrmeyer operators have been object of investigations by many authors (see, e.g. [14, 12]); in particular, in [21] the author studied a generalization $M_{n,a,b}$ of Bernstein-Durrmeyer operators acting on weighted spaces of integrable functions, where the considered one is the classical Jacobi weight $w_{a,b}$ on [0, 1]. Those operators have been intensely studied during the years in the one-dimensional and in multidimensional setting (see, e.g., [1, 23, 26]), also in connection with certain partial differential problems (see [5, 7]).

In this paper, we present a modification of the Bernstein-Durrmeyer operators with Jacobi weights $M_{n,a,b}^\tau$ in the same spirit of [13, 3].

We establish some qualitative properties of the operators $M_{n,a,b}^\tau$, such as their behaviour with respect to Lipschitz-continuous functions; moreover, we prove that they preserve some forms of convexity. We also prove that the sequence $(M_{n,a,b}^\tau)_{n\geq 1}$ is an approximation process in $C([0,1])$, as well in suitable spaces of integrable functions, and we evaluate the rate of convergence by means of appropriate moduli of smoothness.

Finally, we use an asymptotic formula for the operators $M_{n,a,b}^\tau$ in order to compare them with the $M_{n,a,b}$'s and the M_n^τ's, showing under which conditions the operators introduced in the present paper provide a lower approximating error estimate at least on certain subintervals of $[0,1]$.

2 Preliminaries

From now on fix $a, b \in]-1, +\infty[$ and consider the normalized Jacobi weight

$$w_{a,b}(x) := \frac{x^a(1-x)^b}{\int_0^1 y^a(1-y)^b\,dy} \quad (0 < x < 1). \tag{1}$$

Moreover, let us denote by $\mu_{a,b} \in M_1^+([0,1])$ the absolutely continuous measure with respect to the Borel-Lebesgue measure λ_1 on $[0,1]$ with density $w_{a,b}$. Obviously, if $a = b = 0$ then $\mu_{0,0} = \lambda_1$.

By means of such a measure, it is possible to define the so called Bernstein-Durrmeyer operators with Jacobi weights (see [21, 22]). More precisely, for every $f \in L^1([0,1], \mu_{a,b})$, $n \geq 1$ and $x \in [0,1]$, set

$$M_{n,a,b}(f)(x) := \sum_{h=0}^{n} \omega_{n,h}(f) \binom{n}{h} x^h (1-x)^{n-h}, \tag{2}$$

where

$$\begin{aligned} \omega_{n,h}(f) &:= \frac{1}{\int_0^1 t^h (1-t)^{n-h} \, d\mu_{a,b}} \int_0^1 t^h (1-t)^{n-h} f(t) \, d\mu_{a,b} \\ &= \frac{\Gamma(n+a+b+2)}{\Gamma(h+a+1)\Gamma(n-h+b+1)} \int_0^1 t^{h+a} (1-t)^{n-h+b} f(t) \, dt, \end{aligned} \tag{3}$$

Γ being the classical Euler Gamma function.

We also recall that in [7] it has been noted that

$$M_{n,a,b}(f) = B_n(D_{n,a,b}(f)) \quad (f \in L^1([0,1], \mu_{a,b})), \tag{4}$$

where B_n stand for the classical Bernstein operators on $[0,1]$ and $D_{n,a,b} : L^1([0,1], \mu_{a,b}) \longrightarrow L^1([0,1], \mu_{a,b})$ are the positive linear operators defined in [10, formula (4.6)] by

$$D_{n,a,b}(f)(x) = \frac{\Gamma(n+a+b+2)}{\Gamma(nx+a+1)\Gamma(n-nx+b+1)} \int_0^1 t^{nx+a} (1-t)^{n-nx+b} f(t) \, dt. \tag{5}$$

Given (4) and denoted by $e_m(x) = x^m$, $m \in \mathbb{N}$, it is possible to evaluate $M_n(e_m)$, $m \in \mathbb{N}$, since

$$D_{n,a,b}(e_m) = \frac{\Gamma(n+a+b+2)}{\Gamma(m+n+a+b+2)} (a+1+ne_1) \cdots (a+m+ne_1). \tag{6}$$

In particular (see [22, Section 5.2]),

$$M_{n,a,b}(e_0) = e_0 \tag{7}$$

$$M_{n,a,b}(e_1) = \frac{a+1+ne_1}{n+a+b+2}, \tag{8}$$

and

$$M_{n,a,b}(e_2) = \frac{(a+1)(a+2) + 2n(a+2)e_1 + n(n-1)e_2}{(n+a+b+2)(n+a+b+3)}. \tag{9}$$

Moreover, if, for a given $x \in [0,1]$ we denote by $\psi_x^i(t) = (t-x)^i$, $i \geq 1$, we have that

$$M_{n,a,b}(\psi_x)(x) = \frac{a+1-(a+b+2)x}{n+a+b+2} \tag{10}$$

and

$$\begin{aligned} M_{n,a,b}(\psi_x^2)(x) &= \frac{2nx(1-x)+x^2(a+b+2)(a+b+3)}{(n+a+b+2)(n+a+b+3)} \\ &- \frac{2x(a+1)(a+b+3)}{(n+a+b+2)(n+a+b+3)} + \frac{(a+1)(a+2)}{(n+a+b+2)(n+a+b+3)}. \end{aligned} \tag{11}$$

3 Modified Bernstein-Durrmeyer operators with Jacobi weights

In what follows, τ will be an infinitely differentiable function on $[0,1]$ such that $\tau(0)=0$, $\tau(1)=1$, and $\tau'(x)>0$ for $x\in[0,1]$.

Consider now the image measure $\mu_{a,b}^{\tau}$ of $\mu_{a,b}$ by means of τ and the corresponding Lebesgue space $L^p([0,1],\mu_{a,b}^{\tau})$, with $1\le p<+\infty$. Namely, a function f belongs to $L^p([0,1],\mu_{a,b}^{\tau})$ if

$$\int_0^1 |f|^p \, d\mu_{a,b}^{\tau} = \int_0^1 |f\circ\tau^{-1}|^p \, d\mu_{a,b} < +\infty\,.$$

Such a space is equipped with the norm $\|\cdot\|_{L^p([0,1],\mu_{a,b}^{\tau})}$. If $\tau=e_1$, $\mu_{a,b}^{e_1}=\mu_{a,b}$ and, if this is the case, we will omit the superscript e_1. Note that $f\in L^p([0,1],\mu_{a,b})$ if and only if $f\in L^p([0,1],\mu_{a,b}^{\tau})$. Moreover, also $\mu_{a,b}^{\tau}\in M_1^+([0,1])$. Finally, whenever $a=b=0$ and $\tau=e_1$, the corresponding space is indeed $L^p([0,1])$ endowed with the usual norm $\|\cdot\|_p$.

For every $n\ge1$, the positive linear operator

$$M_{n,a,b}^{\tau}: L^1([0,1],\mu_{a,b}^{\tau}) \longrightarrow L^1([0,1],\mu_{a,b}^{\tau})$$

that we are going to consider is defined by setting, for every $f\in L^1([0,1],\mu_{a,b}^{\tau})$, $0\le x\le1$,

$$M_{n,a,b}^{\tau}(f)(x) := \sum_{h=0}^{n}\binom{n}{h}\tau^h(x)(1-\tau(x))^{n-h}\omega_{n,h}(f\circ\tau^{-1}) \tag{12}$$

(see (3)). More precisely

$$M_{n,a,b}^{\tau}(f) = M_{n,a,b}(f\circ\tau^{-1})\circ\tau. \tag{13}$$

If we choose $\tau=e_1$ we get the original Bernstein-Durrmeyer operators with Jacobi weights $M_{n,a,b}$ [21].

Further, in the particular case of $a=b=0$, we get the modified Bernstein-Durrmeyer operators introduced and studied in [3]. Observe that, if in addition

$\tau = e_1$, then those operators turn into the classical Bernstein-Durrmeyer operators [15, 20].

If $f = \tau^m$ ($m \in \mathbb{N}$), then $f \circ \tau^{-1} = e_m$. Hence we have

$$M^{\tau}_{n,a,b}(\tau^m) = (M_{n,a,b}(e_m)) \circ \tau. \tag{14}$$

In particular, from (7)-(9), we get

$$M^{\tau}_{n,a,b}(e_0) = e_0 \tag{15}$$

$$M^{\tau}_{n,a,b}(\tau) = \frac{a+1+n\tau}{n+a+b+2}\,, \tag{16}$$

and

$$M^{\tau}_{n,a,b}(\tau^2) = \frac{(a+1)(a+2)+2n(a+2)\tau+n(n-1)\tau^2}{(n+a+b+2)(n+a+b+3)}\,. \tag{17}$$

For the operators $M^{\tau}_{n,a,b}$, a formula similar to (4) can be obtained considering the following modification B^{τ}_n of the Bernstein operators introduced in [13]:

$$B^{\tau}_n(f)(x) = \sum_{h=0}^{n} \binom{n}{h} \tau^h(x)(1-\tau(x))^{n-h}(f \circ \tau^{-1})\left(\frac{h}{n}\right)$$

($n \geq 1$, $f \in C([0,1])$, $0 \leq x \leq 1$). In fact, on account of (5),

$$M^{\tau}_{n,a,b}(f) = B^{\tau}_n(D_{n,a,b}(f \circ \tau^{-1}) \circ \tau) \quad (f \in L^1([0,1], \mu^{\tau}_{a,b}))\,. \tag{18}$$

Theorem 1 *For every $f \in C([0,1])$ we have*

$$\lim_{n\to\infty} M^{\tau}_{n,a,b}(f) = f \quad \textit{uniformly on } [0,1]\,.$$

Proof It is sufficient to note that $\{e_0, \tau, \tau^2\}$ is an extended complete Tchebychev system on $[0,1]$ and that, thanks to (15)-(17), $\lim_{n\to\infty} M^{\tau}_{n,a,b}(e_0) = e_0$, $\lim_{n\to\infty} M^{\tau}_{n,a,b}(\tau) = \tau$, and $\lim_{n\to\infty} M^{\tau}_{n,a,b}(\tau^2) = \tau^2$, uniformly on $[0,1]$. □

In order to get some estimates of the rate of convergence in Theorem 1, we use a general result (see [22, 8]) which involves the usual first modulus of continuity $\omega(f,\delta)$ and the second modulus of smoothness $\omega_2(f,\delta)$. To this end, we need some further notations.

For $x \in [0,1]$, let $\psi^x_{\tau,i}$ be the function defined by

$$\psi^x_{\tau,i}(t) = (\tau(t) - \tau(x))^i \quad (i = 0, 1, 2, \ldots)\,.$$

If $\tau = e_1$ we shall simply write $\psi^i_x(t) = (t-x)^i$.

For any $n \geq 1$ and $x \in [0,1]$ (see (11)), we have

$$M_{n,a,b}^{\tau}(\psi_{\tau,2}^{x})(x) = \frac{2n\tau(x)(1-\tau(x)) + \tau(x)^2(a+b+2)(a+b+3)}{(n+a+b+2)(n+a+b+3)} - \frac{2\tau(x)(a+1)(a+b+3)}{(n+a+b+2)(n+a+b+3)} + \frac{(a+1)(a+2)}{(n+a+b+2)(n+a+b+3)}. \tag{19}$$

Moreover, by using a result due to Freud (see [16]), we get that there exists a constant $K > 0$ such that

$$K\psi_x^2(t) \le \tau'(x)\psi_{\tau,2}^{x}(t) \quad \text{for every } x, t \in [0,1]\,. \tag{20}$$

Obviously, $K = 1$ if $\tau = e_1$.

We can now state the following result.

Proposition 1 *Consider* $n \ge 1$, $f \in C([0,1])$ *and* $0 \le x \le 1$. *Then*

$$|M_{n,a,b}^{\tau}(f)(x) - f(x)| \le \omega(f, \sigma_n^{\tau}(x)) + \frac{3}{2}\omega_2(f, \sigma_n^{\tau}(x))\,, \tag{21}$$

where

$$\sigma_n^{\tau}(x) = \frac{\sqrt{\tau'(x)}}{\sqrt{K}} \times \sqrt{\frac{2n\tau(x)(1-\tau(x)) + \tau(x)^2(a+b+2)(a+b+3) - 2\tau(x)(a+1)(a+b+3) + (a+1)(a+2)}{(n+a+b+2)(n+a+b+3)}}.$$

Moreover,

$$\|M_{n,a,b}^{\tau}(f) - f\|_{\infty} \le \omega(f, \delta_n^{\tau}) + \frac{3}{2}\omega_2(f, \delta_n^{\tau})\,, \tag{22}$$

where

$$\delta_n^{\tau} = \frac{\sqrt{\|\tau'\|_{\infty}}}{\sqrt{K}}\sqrt{\frac{n/2 + \max\{a^2+3a+2, b^2+3b+2\}}{(n+a+b+2)(n+a+b+3)}}.$$

Proof Let $n \ge 1$, $f \in C([0,1])$, $0 \le x \le 1$ and $\delta > 0$. Using [22, Theorem 2.2.1] (see also [8, Theorem 1.6.2]), we have that

$$\begin{aligned} |M_{n,a,b}^{\tau}(f)(x) - f(x)| &\le |f(x)||M_{n,a,b}^{\tau}(e_0)(x) - 1| + \frac{1}{\delta}|M_{n,a,b}^{\tau}(\psi_x)(x)|\omega(f,\delta) \\ &+ \left\{M_{n,a,b}^{\tau}(e_0)(x) + \frac{1}{2\delta^2}M_{n,a,b}^{\tau}(\psi_x^2)(x)\right\}\omega_2(f,\delta) \\ &= \frac{1}{\delta}|M_{n,a,b}^{\tau}(\psi_x)(x)|\omega(f,\delta) + \left\{1 + \frac{1}{2\delta^2}M_{n,a,b}^{\tau}(\psi_x^2)(x)\right\}\omega_2(f,\delta)\,. \end{aligned}$$

By Cauchy-Schwarz inequality we get

$$|M^{\tau}_{n,a,b}(\psi_x)| \leq \sqrt{M^{\tau}_{n,a,b}(\psi_x^2)},$$

therefore

$$\begin{aligned} &|M^{\tau}_{n,a,b}(f)(x) - f(x)| \leq \\ &\leq \frac{1}{\delta}\sqrt{M^{\tau}_{n,a,b}(\psi_x^2)(x)}\,\omega(f,\delta) + \left\{1 + \frac{1}{2\delta^2} M^{\tau}_{n,a,b}(\psi_x^2)(x)\right\} \omega_2(f,\delta)\,. \end{aligned}$$

From (20) and the positivity of the $M^{\tau}_{n,a,b}$'s, we get

$$M^{\tau}_{n,a,b}(\psi_x^2) \leq \frac{\tau'(x)}{K} M^{\tau}_{n,a,b}(\psi^x_{\tau,2}).$$

Taking (19) into account and setting $\delta = \sigma^{\tau}_n(x)$, we get (21).

To get (22), note that, for every $x \in [0,1]$, $2n\tau(x)(1-\tau(x)) \leq n/2$ and that the function

$$g(x) = \tau(x)^2(a+b+2)(a+b+3) - 2\tau(x)(a+1)(a+b+3) + (a+1)(a+2)$$

has a unique critical point at $x = \tau^{-1}\left(\dfrac{a+1}{a+b+2}\right)$, which is a local minimum point, so that its global maximum has to be $\max\{g(0), g(1)\}$. $\square$

We pass now to discuss approximation properties of $(M^{\tau}_{n,a,b}(f))_{n\geq 1}$ also in the space of $L^p([0,1],\mu^{\tau}_{a,b})$, $p \geq 1$. We note that these results seem to be new also in the context $a = b = 0$.

First we recall that a measure μ on $[0,1]$ is said to be *invariant* for an operator B of domain $D(B)$ if

$$\int_0^1 B(f)\,d\mu = \int_0^1 f\,d\mu \quad \text{for every } f \in D(B)$$

(see [19, Section 5.1, p. 178]).

Lemma 1 *The measure $\mu^{\tau}_{a,b}$ is an invariant measure for the operators $M^{\tau}_{n,a,b}$ on $L^1([0,1],\mu^{\tau}_{a,b})$, and in particular for their restrictions to $C([0,1])$. Moreover, each $M^{\tau}_{n,a,b}$ is a contraction from $L^p([0,1],\mu^{\tau}_{a,b})$ into itself.*

Proof Fix $f \in L^1([0,1],\mu^{\tau}_{a,b})$; then

$$\begin{aligned} \int_0^1 M^{\tau}_{n,a,b}(f)\,d\mu^{\tau}_{a,b} &= \sum_{h=0}^{n}\binom{n}{k}\omega_{n,h}(f\circ\tau^{-1})\int_0^1 \tau^h(1-\tau)^{n-h}\,d\mu^{\tau}_{a,b} \\ &= \frac{1}{\int_0^1 y^a(1-y)^b\,dy}\int_0^1\left[\sum_{h=0}^{n}\binom{n}{k}y^h(1-y)^{n-h}\right] y^a(1-y)^b(f\circ\tau^{-1})(y)\,dy \\ &= \frac{1}{\int_0^1 y^a(1-y)^b\,dy}\int_0^1 y^a(1-y)^b(f\circ\tau^{-1})(y)\,dy = \int_0^1 f\,d\mu^{\tau}_{a,b}\,, \end{aligned}$$

hence the first part of the claim is proven.

In order to prove the second part, first note that from Jensen's inequality it follows that, if $f \in L^p([0,1], \mu^\tau_{a,b})$, $|M^\tau_{n,a,b}(f)|^p \le M^\tau_{n,a,b}(|f|^p)$. Then

$$\int_0^1 |M^\tau_{n,a,b}(f)|^p \, d\mu^\tau_{a,b} \le \int_0^1 M^\tau_{n,a,b}(|f|^p) \, d\mu^\tau_{a,b} = \int_0^1 |f|^p \, d\mu^\tau_{a,b} \,,$$

that is $\|M^\tau_{n,a,b}\|_{L^p([0,1],\mu^\tau_{a,b})} \le 1$. □

Theorem 2 *For every* $f \in L^p([0,1], \mu^\tau_{a,b})$,

$$\lim_{n\to\infty} M^\tau_{n,a,b}(f) = f \quad \text{in } L^p([0,1], \mu^\tau_{a,b})\,.$$

Proof As a consequence of the previous lemma, the sequence $(M^\tau_{n,a,b})_{n\ge 1}$ is equibounded from $L^p([0,1], \mu^\tau_{a,b})$ into $L^p([0,1], \mu^\tau_{a,b})$. On account of Theorem 1 and recalling that $C([0,1])$ is dense in $L^p([0,1], \mu^\tau_{a,b})$ (see [11, Lemma 26.2 and Theorem 29.14]) the proof is given. □

We proceed by obtaining an estimate of the convergence in Theorem 2 in the particular case $a = b = 0$ for which, as quoted before, our operators turn into those ones considered in [3]. Let us denote by M^τ_n the operators $M^\tau_{n,0,0}$ and by μ^τ the measure $\mu^\tau_{0,0}$. We shall use a result due to Swetits and Wood [25, Theorem 1] which involves the second-order integral modulus of smoothness defined, for $f \in L^p([0,1])$, $1 \le p < +\infty$, as

$$\omega_{2,p}(f,\delta) := \sup_{0<t\le\delta} \|f(\cdot + t) - 2f(\cdot) + f(\cdot - t)\|_p \quad (\delta > 0).$$

In particular, recalling that $f \circ \tau^{-1} \in L^p([0,1])$, [25, Theorem 1] states that there exists a constant $C_p > 0$ such that

$$\begin{aligned}
&\|M^\tau_n(f) - f\|_{L^p([0,1],\mu^\tau)} = \|M_n(f\circ\tau^{-1}) - f\circ\tau^{-1}\|_p \\
&\le C_p\{\rho^2_{n,p}\|f\circ\tau^{-1}\|_p + \omega_{2,p}(f\circ\tau^{-1}, \rho_{n,p})\},
\end{aligned}$$

where the sequence $\rho_{n,p} \to 0$ as $n \to \infty$ and it is defined as follows:

$$\rho_{n,p} := \max\left\{\|M_n(\psi_x)\|_p^{1/2}, \|M_n(\psi_x^2)\|_p^{p/(2p+1)}\right\}.$$

From (10) we get $\|M_n(\psi_x)\|_p^{1/2} \le \dfrac{1}{\sqrt{n+2}(p+1)^{1/(2p)}} =: \beta_{n,p} \to 0$. Moreover,

$$\|M_n(\psi_x^2)\|_p^{p/(2p+1)} = \frac{\|2ne_1(1-e_1) - 6e_1(1-e_1) + 2\|_p^{p/(2p+1)}}{(n+2)^{p/(2p+1)}(n+3)^{p/(2p+1)}} =: \gamma_{n,p}\,.$$

Note that

$$0 \le \gamma_{n,p} \le \left(\frac{n+4}{2(n+2)(n+3)}\right)^{p/(2p+1)} \to 0\,.$$

By setting

$$\alpha_{n,p} := \max\{\beta_{n,p}, \gamma_{n,p}\}$$

we obtain that

$$\|M_n^\tau(f) - f\|_{L^p([0,1],\mu^\tau)} \le C_p(\alpha_{n,p}^2 \|f \circ \tau^{-1}\|_p + \omega_{2,p}(f \circ \tau^{-1}, \alpha_{n,p})). \quad (23)$$

We now present some shape preserving properties of the operators $M_{n,a,b}^\tau$.

For every $k \in \mathbb{N}$, consider the linear subspace $\mathbb{P}_{\tau,k}$ generated by the set $\{\tau^i : i = 0, \dots, k\}$. This space is invariant under our operators, i.e.

$$M_{n,a,b}^\tau(\mathbb{P}_{\tau,k}) \subset \mathbb{P}_{\tau,k} \qquad (k \in \mathbb{N}, n \ge 1).$$

Indeed, as shown in (6), $D_{n,a,b}$ maps polynomials on $[0, 1]$ into polynomials on $[0, 1]$ of the same degree; by this, (18), (14) and the fact that $B_n^\tau(\mathbb{P}_{\tau,k}) \subset \mathbb{P}_{\tau,k}$ (see [13, Section 2]), the statement easily follows.

We also prove that the $M_{n,a,b}^\tau$'s preserve some forms of convexity and we investigate their behaviour with respect to Lipschitz-continuous functions.

First of all, we point out that the operators $M_{n,a,b}^\tau$ do not preserve the usual convexity. For instance, if $\tau(x) = 4/\pi \arctan(x)$ $(0 \le x \le 1)$, then $M_{n,0,0}^\tau(e_1)$ is not convex for low values of n.

Anyway, the operators $M_{n,a,b}^\tau$ preserve other forms of convexity.

We recall (see [27]) that a function $f \in C([0, 1])$ is said to be *convex with respect to* τ if, whenever $0 \le x_0 < x_1 < x_2 \le 1$,

$$\begin{vmatrix} 1 & 1 & 1 \\ \tau(x_0) & \tau(x_1) & \tau(x_2) \\ f(x_0) & f(x_1) & f(x_2) \end{vmatrix} \ge 0.$$

In particular, f is convex with respect to τ if and only if $f \circ \tau^{-1}$ is convex.

We can state the following result.

Proposition 2 *Let $f \in C([0, 1])$ be convex with respect to τ. Then $M_{n,a,b}^\tau(f)$ is convex with respect to τ for any $n \ge 1$.*

Proof Since, for every $n \ge 1$, the operators $M_{n,a,b}$ map continuous convex functions into (continuous) convex functions (see, for example, [7, Proposition 2]), if $f \in C([0, 1])$ is continuous with respect to τ, then $M_{n,a,b}(f \circ \tau^{-1})$ is convex and, hence, $M_{n,a,b}^\tau(f)$ is convex with respect to τ by means of (13). □

Another form of convexity can be considered.

Let us fix $k \ge 1$ and $a_0 < a_1 < \dots < a_k \in \mathbb{R}$; moreover, for $x \in \mathbb{R}$ set $u(x) := (x - a_0) \cdots (x - a_k)$. If $f : [a_0, a_k] \to \mathbb{R}$, the *divided difference* of f with respect to $a_0, \dots, a_k$ is defined by

$$[a_0, \dots, a_k; f] := \sum_{h=0}^{k} \frac{f(a_h)}{u'(a_h)}.$$

A function $f : I \longrightarrow \mathbb{R}$ is said to be k-*convex* (see, e.g., [8, Appendix 2]) on the interval I if for all $a_0 < a_1 < \ldots < a_k$ in I one has $[a_0, \ldots, a_k; f] \geq 0$.

On the other hand, if $a \in I$ and $h > 0$ are such that $a, a+h, a+2h, \ldots, a+kh \in I$, then

$$[a, a+h, a+2h, \ldots, a+kh; f] = \frac{1}{k!h^k}\Delta_h^k f(a),$$

where $\Delta_h^k f(a)$ is the classical k-th difference of f with step h at point a.

Hence, if f is k-convex, then $\Delta_h^k f(a) \geq 0$ for all $a \in I$. Moreover, if f is continuous, it is always possible to choose the $a_0, \ldots, a_k$ in the definition of k-convex functions in such a way they are equally spaced (see [24]), so a continuous function is k-convex if and only if $\Delta_h^k f(a) \geq 0$ for all $a \in I$.

We remark that a function $f \in C^k([0,1])$ is k-convex if $f^{(k)} \geq 0$.

Obviously, 1-convex functions are just the increasing ones, while 2-convex functions are the usual convex ones.

It is possible to further extend the definition of k-convex functions following [17]. If $f \in C(I)$, set, for all $a \in I$,

$$\Delta_{h,\varphi}^k f(a) := \Delta_h^k (f \circ \varphi^{-1})(\varphi(a)),$$

φ being a C^∞-function on I such that $\varphi'(x) \neq 0$ for all $x \in I$ and that $\lim_{x\to 0}\varphi(x) = 0$, provided that 0 is a cluster point for I. f is said φ-*convex of order* k (see [17]) if $\Delta_{h,\varphi}^k f(a) \geq 0$.

If $f \in C^k([0,1])$, then f is φ-convex of order k if

$$D_\varphi^{(k)}(f)(x) := (f \circ \varphi^{-1})^{(k)}(\varphi(x)) \geq 0 \quad (x \in I).$$

It is easy to show that, given our assumptions on τ, a function $f \in C^k([0,1])$ is τ-convex of order k if and only if

$$(f \circ \tau^{-1})^{(k)} \geq 0;$$

in other words, f is τ-convex of order k if $f \circ \tau^{-1}$ is k-convex.

Many classical approximation processes preserve k-convex functions, like for example Bernstein operators (see [8, Prop. A.2.5]) or the classical Bernstein-Durrmeyer operators (see [2]). Also Berstein-Durrmeyer operators with Jacobi weights preserve k-convexity.

First off, given $k \in \mathbb{N}$ and $h = 0, \ldots, n$, we set

$$\Delta_1^k \omega_{n,h}(f) = \sum_{l=0}^{k} (-1)^{k-l}\binom{k}{l}\omega_{n,h+l}(f).$$

By induction, it is easy to prove that, for every $n \geq 1$, $k \in \mathbb{N}$, $f \in C([0,1])$, and $x \in [0,1]$,

$$M^{(k)}_{n,a,b}(f)(x) = n(n-1)\cdots(n-k+1)\sum_{h=0}^{n-k}\binom{n-k}{h}\Delta_1^k\omega_{n,h}(f)x^h(1-x)^{n-h-k}. \quad (24)$$

Following the same reasoning as in [2] (see also [7, Proposition 2.7]), we can prove the following result.

Proposition 3 *If $f \in C([0,1])$ is k-convex, then $M_{n,a,b}(f)$ is k-convex.*
As a consequence, the operators $M^{\tau}_{n,a,b}$ preserve τ-convexity of order k.

Proof Let us fix a k-convex function $f \in C([0,1])$. It is enough to assume $f \in C^k([0,1])$; in fact every continuous k-convex function is the uniform limit a sequence $(f_m)_{m\geq 1}$ of k-convex and C^k functions (take, for example $f_m = B_m(f)$ for all $m \geq 1$).

To show that $M_{n,a,b}(f)$ is k-convex, taking (24) into account, we have to prove that

$$\Delta_1^k\omega_{n,h}(f) \geq 0.$$

Indeed,

$$\Delta_1^k\omega_{n,h}(f) = (-1)^k\frac{\Gamma(n+a+b+2)}{\Gamma(h+a+1+k)\Gamma(n-h+b+1)}\int_0^1 F^{(k)}(x)f(x)\,dx,$$

where $F(x) = x^{h+a+k}(1-x)^{n-h+b}$. From this, integrating by parts,

$$(-1)^k\int_0^1 F^{(k)}(x)f(x)\,dx, = \int_0^1 F(x)f^{(k)}(x)\,dx \geq 0$$

and this completes the proof.

Consider now a τ-convex function f of order k; we have to show that $M^{\tau}_{n,a,b}(f)\circ\tau^{-1}$ is k-convex but this is a straightforward consequence of the previous considerations, (13) and the fact that $f\circ\tau^{-1}$ is k-convex. □

We pass now to investigate the behavior of the operators $M^{\tau}_{n,a,b}$ on Lipschitz-continuous functions. We first recall that we denote by $\mathrm{Lip}([0,1])$ the space consisting of those $f \in C([0,1])$ such that

$$|f|_{\mathrm{Lip}} := \sup_{\substack{x,y\in[0,1]\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|} < +\infty.$$

Moreover, for $M > 0$, $f \in \mathrm{Lip}_M 1$ if

$$|f(x)-f(y)| \leq M|x-y| \quad \text{for every } 0 \leq x, y \leq 1.$$

$\mathrm{Lip}_M 1$ is said to be the space of all Lipschitz continuous functions with Lipschitz constant M.

Finally, for $0 \leq \alpha \leq 1$, we shall write $f \in \mathrm{Lip}_M\alpha$ if

$$|f(x)-f(y)| \leq M|x-y|^{\alpha} \quad \text{for every } 0 \leq x, y \leq 1.$$

Observe that, both τ and τ^{-1} are Lipschitz continuous functions. More precisely, $\tau \in \mathrm{Lip}_L 1$ with $L := \|\tau'\|_\infty$ and $\tau^{-1} \in \mathrm{Lip}_N 1$ with $N := (\min_{[0,1]} \tau')^{-1}$.

Proposition 4 $M^\tau_{n,a,b}(f) \in \mathrm{Lip}([0,1])$ *for every* $n \geq 1$ *and* $f \in \mathrm{Lip}([0,1])$*; moreover*

$$|M^\tau_{n,a,b}(f)|_{\mathrm{Lip}} \leq \left(1 + \frac{\omega}{n}\right) LN|f|_{\mathrm{Lip}}. \tag{25}$$

where

$$\omega := -\frac{a+b+2}{a+b+3} < 0. \tag{26}$$

As a consequence

$$M^\tau_{n,a,b}(\mathrm{Lip}_M 1) \subset \mathrm{Lip}_{MLN} 1 \quad \textit{for every } n \geq 1\,. \tag{27}$$

Further, for every $n \geq 1$*,* $f \in C([0,1])$*,* $\delta > 0$*,* $M > 0$ *and* $0 < \alpha \leq 1$*,*

$$\omega(M^\tau_{n,a,b}(f), \delta) \leq (1 + LN)\omega(f, \delta) \quad \textit{and} \quad M^\tau_{n,a,b}(\mathrm{Lip}_M \alpha) \subset \mathrm{Lip}_{(LN)^\alpha M} \alpha\,. \tag{28}$$

Proof By recalling [7, Theorem 3.2], $M_{n,a,b}(\mathrm{Lip}([0,1])) \subset \mathrm{Lip}([0,1])$ and

$$|M_{n,a,b}(f)|_{\mathrm{Lip}} \leq \left(1 + \frac{\omega}{n}\right) |f|_{\mathrm{Lip}} \leq |f|_{\mathrm{Lip}},$$

hence we get $M^\tau_{n,a,b}(\mathrm{Lip}([0,1])) \subset \mathrm{Lip}([0,1])$ and (25) easily follows from

$$|M^\tau_{n,a,b}(f)|_{\mathrm{Lip}} \leq \left(1 + \frac{\omega}{n}\right) |f|_{\mathrm{Lip}} |\tau|_{\mathrm{Lip}} |\tau^{-1}|_{\mathrm{Lip}}\,.$$

As a consequence (27) is fulfilled.

Finally, taking [6, Cor. 6.1.20] into account and since $\|M^\tau_{n,a,b}\| = 1$ and property (27) holds, for every $n \geq 1$, $f \in C([0,1])$, $\delta > 0$, $M > 0$ and $0 < \alpha \leq 1$, (28) is proven. □

4 Asymptotic formula and its consequences

In this section we want to find a tool to compare the operators $M_{n,a,b}$ and M^τ_n with the operators $M^\tau_{n,a,b}$, showing under which conditions the latter perform better in order to approximate certain functions. A way to do that consists in comparing the corresponding asymptotic formulae.

We recall that (see [7]), for every $u \in C^2([0,1])$,

$$\lim_{n\to\infty} n(M_{n,a,b}(u) - u) = A_{a,b}(u) \tag{29}$$

uniformly in $[0,1]$, where, for every $u \in C^2([0,1])$,

$$A_{a,b}(u)(x) = x(1-x)u''(x) + (a+1-(a+b+2)x)u'(x).$$

On the other hand, also in view of [3], it is easy to obtain the following result.

Proposition 5 *For every $u \in C^2([0,1])$,*

$$\lim_{n\to\infty} n(M^{\tau}_{n,a,b}(u)(x) - u(x)) = (A_{a,b}(u\circ\tau^{-1})\circ\tau)(x) = \frac{\tau(x)(1-\tau(x))}{\tau'(x)^2}u''(x) + \frac{1}{\tau'(x)}\left((a+1)-(a+b+2)\tau(x) - \frac{\tau(x)(1-\tau(x))\tau''(x)}{\tau'(x)^2}\right)u'(x) \tag{30}$$

uniformly w.r.t. $x \in [0,1]$.

By comparing (29) and (30), we can infer the next theorem.

Theorem 3 *If $f \in C^2([0,1])$ and there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $x \in]0,1[$,*

$$f(x) \leq M^{\tau}_{n,a,b}(f)(x) \leq M_{n,a,b}(f)(x)\,,$$

then, for $x \in]0,1[$,

$$\begin{aligned} f''(x) &\geq \frac{\tau''(x)}{\tau'(x)}f'(x) - \frac{\tau'(x)}{\tau(x)(1-\tau(x))}\left((a+1)-(a+b+2)\tau(x)\right)f'(x) \\ &\geq \left(1 - \frac{x(1-x)\tau'(x)^2}{\tau(x)(1-\tau(x))}\right)f''(x) - \frac{(a+1)-(a+b+2)x}{\tau(x)(1-\tau(x))}\tau'(x)^2 f'(x) \end{aligned} \tag{31}$$

Conversely, if there exists $x_0 \in]0,1[$, in which (31) *holds with strict inequalities, then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,*

$$f(x_0) < M^{\tau}_{n,a,b}(f)(x_0) < M_{n,a,b}(f)(x_0)\,.$$

Example 1 Consider

$$\tau = \frac{e_2 + \alpha e_1}{1+\alpha} \quad (\alpha > 0)\,.$$

Moreover, for the sake of simplicity, let us suppose that $a = 1/2$, $b = -1/2$, and $f = e_2$.

We prove that, for a fixed $\alpha > 0$, there exist a subinterval J_α of $]0,5/6[$ and $n_0 \in \mathbb{N}$ such that, for each $x \in J_\alpha$ and $n \geq n_0$,

$$x^2 < M^{\tau}_{n,1/2,-1/2}(e_2)(x) < M_{n,1/2,-1/2}(e_2)(x)\,.$$

Taking Theorem 3 into account, we have to show that, in a suitable interval of $]0,5/6[$,

$$1 > \frac{\tau''(x)}{\tau'(x)}x - \frac{\tau'(x)\,(3-4\tau(x))}{2\tau(x)(1-\tau(x))}x > 1 - \frac{x(5-6x)\tau'(x)^2}{2\tau(x)(1-\tau(x))}$$

or, equivalently,

$$\begin{cases} F^\tau(x) = 1 - \dfrac{\tau''(x)}{\tau'(x)}x + \dfrac{\tau'(x)}{\tau(x)} \cdot \dfrac{3-4\tau(x)}{2(1-\tau(x))}x > 0\,, \\ G^\tau(x) = -F^\tau(x) + \dfrac{x(5-6x)\tau'(x)^2}{2\tau(x)(1-\tau(x))} > 0\,. \end{cases} \tag{32}$$

Direct calculations show that, for all $x \in]0, 5/6[$,

$$F^\tau(x) = \frac{\alpha}{2x+\alpha} + \frac{2x+\alpha}{x+\alpha} \cdot \frac{3(\alpha+1) - 4(x^2+\alpha x)}{2(1+\alpha-(x^2+\alpha x))} > 0\,.$$

Observe that $\lim_{x\to 0^+} F^\tau(x) = 5/2$. Moreover

$$\lim_{x\to 0^+} G^\tau(x) = -\frac{5}{2(1+\alpha)}\,, \quad \lim_{x\to 5/6^-} G^\tau(x) = -\frac{105\alpha+200}{(5+3\alpha)(5+6\alpha)(11+6\alpha)}\,.$$

Finally $G^\tau(2/3) > 0$.

In what follows we continue to denote by F^τ and G^τ respectively the extensions by continuity in 0 and in 5/6 of the functions in (32).

First we observe that from $G^\tau(0) < 0 < G^\tau(2/3)$ if follows that there exists $y_\alpha \in]0, 2/3[$ such that $G^\tau(y_\alpha) = 0$. Analogously, from $G^\tau(5/6) < 0 < G^\tau(2/3)$ it follows that there exists $z_\alpha \in]2/3, 5/6[$ such that $G^\tau(z_\alpha) = 0$.

It can be proven that G^τ is positive in $]y_\alpha, z_\alpha[$.

Such a neighborhood is contained in $]0, 5/6[$ and it is the desired subinterval in which (32) holds.

As an example we provide the plot of G^τ for $\alpha = 1$.

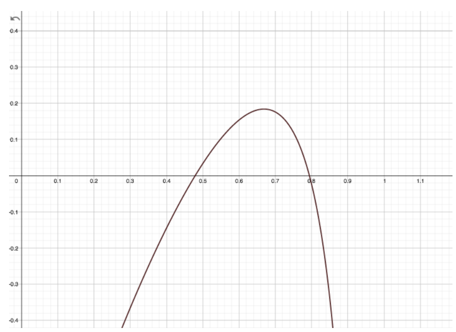

Fig. 1 The plot of G^τ for $\alpha = 1$.

Under suitable assumptions, the operators $M^\tau_{n,a,b}$ also perform better than the operators M^τ_n considered in [3], as showed in the following result. Note that the asymptotic formula for the operators M^τ_n is (30) for $a = b = 0$.

Theorem 4 *If $f \in C^2([0,1])$ and there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $x \in]0,1[$,*

$$f(x) \leq M^\tau_{n,a,b}(f)(x) \leq M^\tau_n(f)(x)\,,$$

then, for $x \in]0,1[$,

$$\begin{aligned} f''(x) &\geq \frac{\tau''(x)}{\tau'(x)} f'(x) - \frac{\tau'(x)}{\tau(x)(1-\tau(x))}((a+1)-(a+b+2)\tau(x))f'(x) \\ &\geq \frac{\tau''(x)}{\tau'(x)} f'(x) - \frac{\tau'(x)}{\tau(x)(1-\tau(x))}(1-2\tau(x))f'(x)\,. \end{aligned} \tag{33}$$

Conversely, if there exists $x_0 \in]0,1[$*, in which* (33) *holds with strict inequalities, then there exists* $n_0 \in \mathbb{N}$ *such that, for all* $n \geq n_0$,

$$f(x_0) < M^\tau_{n,a,b}(f)(x_0) < M^\tau_n(f)(x_0)\,.$$

Example 2 Fix $\tau = \frac{e_2+\alpha e_1}{1+\alpha}$ ($\alpha > 0$), $-1 < a = b < 0$ and let $f = e_2$. Then, (33) with strict inequalities becomes

$$1 > \frac{\tau''(x)}{\tau'(x)}x - \frac{\tau'(x)(a+1)(1-2\tau(x))}{\tau(x)(1-\tau(x))}x > \frac{\tau''(x)}{\tau'(x)}x - \frac{\tau'(x)(1-2\tau(x))}{\tau(x)(1-\tau(x))}x$$

and it is easy to see that it holds whenever $1 - 2\tau(x) > 0$, that is for every $0 < x < (\sqrt{\alpha^2+2\alpha+2}-\alpha)/2 < 1$.

A further consequence of the asymptotic formula (30) consists in finding a representation in terms of the operators $M^\tau_{n,a,b}$ of suitable semigroups acting on spaces of continuous as well as integrable functions. For similar results see, e.g. [8], where the reader can also find more details about semigroup theory.

Corollary 1 *There exists a Markov semigroup* $(T(t))_{t\geq 0}$ *such that for every* $f \in C([0,1])$, $t \geq 0$ *and for every sequence* $(k_n)_{n\geq 1}$ *of positive integers such that* $\lim_{n\to\infty} k_n/n = t$,

$$\lim_{n\to\infty} (M^\tau_{n,a,b})^{k_n}(f) = T(t)(f\circ\tau^{-1})\circ\tau \quad \textit{uniformly on } [0,1]. \tag{34}$$

Moreover, for every $f \in C([0,1])$,

$$\lim_{m\to\infty} (M^\tau_{n,a,b})^{m}(f) = \int_0^1 f\,d\mu^\tau_{a,b} = \lim_{t\to\infty} T(t)(f\circ\tau^{-1})\circ\tau \tag{35}$$

uniformly on $[0,1]$.

Further, for every $p \geq 1$, $(T(t))_{t\geq 0}$ *has a unique extention* $(T_p(t))_{t\geq 0}$ *which is a positive contraction semigroup on* $L^p([0,1],\mu_{a,b})$ *and, if* $t \geq 0$ *and* $(k_n)_{n\geq 1}$ *is a sequence of positive integers satisfying* $\lim_{n\to\infty} k_n/n = t$*, then for every* $f \in L^p([0,1],\mu^\tau_{a,b})$,

$$\lim_{n\to\infty} (M^\tau_{n,a,b})^{k_n}(f) = T_p(t)(f\circ\tau^{-1})\circ\tau \quad \textit{in } L^p([0,1],\mu^\tau_{a,b}). \tag{36}$$

Finally, if $f \in L^p([0,1],\mu^\tau_{a,b})$ *and* $n \geq 1$,

$$\lim_{m\to\infty}(M^{\tau}_{n,a,b})^m(f) = \int_0^1 f d\mu^{\tau}_{a,b} = \lim_{t\to\infty} T_p(t)(f\circ\tau^{-1})\circ\tau \tag{37}$$

in $L^p([0,1],\mu^{\tau}_{a,b})$.

Proof First note that, for every $k,n\geq 1$ and $f\in C([0,1])$ we put

$$(M^{\tau}_{n,a,b})^k(f) := M^k_{n,a,b}(f\circ\tau^{-1})\circ\tau.$$

Formula (34) follows directly from [9, Theorem 3.3]. Formula (35) is a consequence of [3, formula (8)] and [7, Theorem 4.2].

On the other hand, (36) derives from [7, Theorem 4.4], since $f\in L^p([0,1],\mu^{\tau}_{a,b})$ if and only if $f\in L^p([0,1],\mu_{a,b})$. Finally, formula (37) can be obtained from (35), since $C([0,1])$ is dense in $L^p([0,1],\mu^{\tau}_{a,b})$. □

Remark 1 We notice that the generator of the semigroup $(T(t))_{t\geq 0}$ in Corollary 1 is the closure of the differential operator $A_{a,b}(u\circ\tau^{-1})\circ\tau$ on $C^2([0,1])$ (see (30)).

Acknowledgements The paper has been performed within the activities of GNAMPA-INdAM (Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni dell' Istituto Nazionale di Alta Matematica), of the network RITA (Research ITalian network on Approximation), and of the UMI Group "Teoria dell'Approssimazione e Applicazioni".

References

1. Abel, U., Heilmann, M.: The complete asymptotic expansion for Bernstein–Durrmeyer operators with Jacobi weights. Mediterr. J. Math. **1**(4), 487–499 (2004)
2. Abel, U., Leviatan, D., Raşa, I.: On the q-monotonicity preservation of Durrmeyer-type operators. Mediterr. J. Math. **18**(4), Paper No. 173, 15 pp (2021)
3. Acar, T., Aral, A., Raşa, I.: Modified Bernstein-Durrmeyer operators. General Mathematics **22**(1), 27–41 (2014)
4. Acar, T., Cappelletti Montano, M., Garrancho, P., Leonessa, V.: On sequences of J. P. King-type operators. J. Funct. Spaces, vol. **2019**, Article ID 2329060, 12 pp (2019)
5. Albanese, A., Campiti, M., Mangino, E.: Approximation formulas for C_0-semigroups and their resolvent operators. J. Appl. Funct. Anal. **1**(3), 343–358 (2006)
6. Altomare, F., Campiti, M.: Korovkin-Type Approximation Theory and its Applications. De Gruyter Studies in Mathematics **17**, Walter de Gruyter, Berlin-New York (1994)
7. Altomare, F., Cappelletti Montano, M., Leonessa, V.: On the positive semigroups generated by Fleming-Viot type differential operators. Commun. Pure Appl. Anal. **18**(1), 323–340 (2019)
8. Altomare, F., Cappelletti Montano, M., Leonessa, V., Raşa, I.: Markov Operators, Positive Semigroups and Approximation Processes. De Gruyter Studies in Mathematics **61**, Walter de Gruyter GmbH, Berlin/Boston (2014)
9. Altomare, F., Raşa, I.: Lipschitz Contractions, unique ergodicity and asymptotics of Markov semigroups. Boll. U. M. I, Serie 9, **5**(1), 1–17 (2012)
10. Attalienti, A., Campiti, M.: Degenerate evolution problems and Beta-type operators. Studia Math. **140**, 117–139 (2000)
11. Bauer, H.: Measure and integration theory. De Gruyter Studies in Mathematics **26**, Walter de Gruyter & Co., Berlin (2001)

12. Berens, H., Xu, Y.: On Bernstein-Durrmeyer polynomials with Jacobi weights. In: Approximation Theory and Functional Analysis (ed. C. K. Chui), Academic Press, Boston, (1991), 25–46.
13. Cárdenas-Morales, D., Garrancho, P., Raşa, I.: Bernstein-type operators which preserve polynomials. Comput. Math. Appl. **62**(1), 158–163 (2011)
14. Derriennic, M. M.: Sur l'approximation des fonctions integrables sur $[0, 1]$ par des polynomes de Bernstein modifiés. J. Approx. Theory **31**, 325–343 (1981)
15. Durrmeyer, J. L.: Une formule d'inversion de la transformée de Laplace: application a la theorie des moments. These de 3e cycle, Faculté des Sciences de l'Université de Paris (1967)
16. Freud, G.: On approximation by positive linear methods I, II. Stud. Scin. Math. Hungar, **2**, 63–66 (1967), **3**, 365–370 (1968)
17. López-Moreno, A.-J., Muñoz-Delgado, F.-J.: Asymptotic expansion of multivariate conservative linear operators. J. Comp. and Applied Math. **150**, 219–251 (2003)
18. King, J. P.: Positive linear operators which preserve x^2. Acta Math. Hungar **99**(3), 203–208 (2003)
19. Krengel, U.: Ergodic Theorems. De Gruyter Studies in Mathematics **6**, Walter de Guyter, Berlin/New York (1985)
20. Lupaş, A.: Die Folge der Betaoperatoren. Dissertation, Universität Stuttgart (1972)
21. Păltănea, R.: Sur un opérateur polynomial défini sur l'ensemble des fonctions intégrables. (French) [A polynomial operator defined on the set of integrable functions] Itinerant seminar on functional equations, approximation and convexity (Cluj-Napoca, 1983), 101–106, Preprint, 83–2, Univ. Babeş-Bolyai, Cluj-Napoca (1983)
22. Păltănea, R.: Approximation Theory Using Positive Linear Operators. Birkhäuser-Verlag, Boston (2004)
23. Păltănea, R.: Durrmeyer type operators on a simplex. Constr. Math. Anal. **4**(2), 215–228 (2021)
24. Popoviciu, T.: Sur le reste dans certaines formules linéaires d'approximation de l'analyse. Mathematica (Cluj) **1** (24), 95–142 (1959)
25. Swetits, J. J. , Wood, B.: Quantitative estimates for L_p approximation with positive linear operators. J. Approx. Theory **38**, 81–89 (1983)
26. Waldron, S.: A generalized beta integral and the limit of Bernstein-Durrmeyer operator with Jacobi weights. J. Approx. Theory, **122**, 141–150 (2003)
27. Ziegler, Z.: Linear approximation and generalized convexity. J. Approx. Theory **1**, 420–433 (1968)