EXPLICIT TIME INTEGRATION ALGORITHM FOR THE FINITE ELEMENT SOLUTION OF TRANSMISSION LINES

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Abstract: The constitutive error based approach is applied for the analysis of lossy transmission lines in the time domain. Explicit and implicit time integration techniques are introduced and compared, with reference to a canonical examples. The extension of these techniques to the solution of 3D field problems is outlined.

Introduction

The interest for the simulation of lossy transmission lines has recently increased, due to the growing importance of the studies on the performance of fast chip to chip interconnections [1].

In this paper we discuss the extension of the constitutive error based approach [2] to the solution of lossy transmission lines in the time domain. This extension, although quite straightforward, at least in principle, is of interest for several reasons. It represents an interesting alternative for the numerical simulation of fast interconnections in the frame of the analysis of nonlinear circuits, due to its peculiarities; specifically, this technique is based on the numerical minimisation of an error functional, giving, as an additional outcome, global and local error estimates, also useful for time and space discretization adaptive refinements. Moreover, since the mathematical structure is substantially the same as the one-dimensional Maxwell equations, it sheds also some light on the main aspects of the method when applied in a more general context.

In this paper, after a description of the numerical formulation, explicit and implicit time integration techniques are introduced and compared, with reference to a canonical examples. The extension of these techniques to the solution of 3D field problems is finally outlined.

Numerical formulation

We refer to the classic circuit model of the transmission lines, whose elementary cell is shown in fig. 1. Having introduced nodal voltages v(x, t) and mesh currents i(x, t), the Kirchhoff laws are identically satisfied for any choice of these variables. As a consequence, the error is localized in the constitutive equations at every elementary circuit branch. In particular we have:

$$v(x+\Delta x, t)-v(x, t)= r\,\Delta x\, i(x, t) + l\,\Delta x\frac{\partial i}{\partial t} \qquad (1)$$

$$i(x+\Delta x, t)-i(x, t)= g\,\Delta x\, v(x+\Delta x, t) + c\,\Delta x\frac{\partial v}{\partial t} \qquad (2)$$

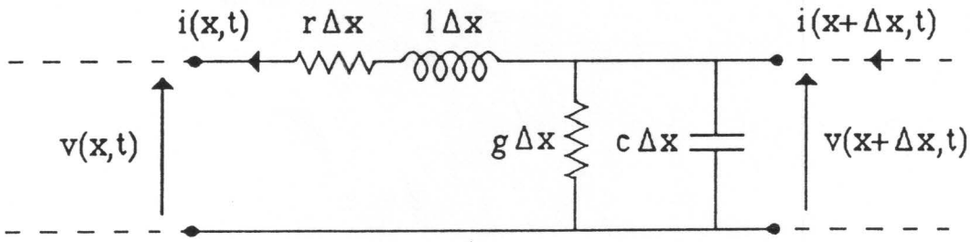

Fig. 1. Elementary cell of a transmission line

and in the limit $\Delta x \to 0$:

$$\frac{\partial v}{\partial x} = r\, i + l \frac{\partial i}{\partial t} \tag{3}$$

$$\frac{\partial i}{\partial x} = g\, v + c \frac{\partial v}{\partial t} \tag{4}$$

Of course, the constitutive error at the two terminations of the line should also be considered.

We notice that this approach has been already introduced for solving the full set of Maxwell equations [2]; in that case, having defined the vector potentials

$$\mathbf{A} = -\int_0^t \mathbf{E}\, dt \qquad \mathbf{F} = \int_0^t \mathbf{H}\, dt \tag{5}$$

then the fields

$$\mathbf{E}=-\partial\mathbf{A}/\partial t,\ \mathbf{H}=\partial\mathbf{F}/\partial t,\ \mathbf{B}=\mathbf{B}_o+\nabla\times\mathbf{A},\ \mathbf{D}=\mathbf{D}_o+\nabla\times\mathbf{F} \tag{6}$$

automatically satisfy Maxwell equations and the initial conditions for arbitrary values of **A** and **F**.

In analogy, although not strictly necessary for the analysis of the transmission lines, we introduce the nodal flux $\phi(x, t)$ and the mesh charge $q(x, t)$ such as $v=\partial\phi/\partial t$, $i=\partial q/\partial t$.

Then we proceed to definition of a local error functional $\lambda(x, t)\geq 0$ with strict inequality for flux and charge estimates which do not satisfy the constitutive equations:

$$\frac{\partial \phi}{\partial x} = r\, q + l \frac{\partial q}{\partial t} \tag{7}$$

$$\frac{\partial q}{\partial x} = g\, \phi + c \frac{\partial \phi}{\partial t} \tag{8}$$

A possible definition is:

$$\lambda = \lambda_{LR} + \lambda_{CG} \qquad (9)$$

with

$$\lambda_{LR} = \alpha_{LR}\, l \left[\frac{\partial \phi}{l \partial x} - \frac{\partial q}{\partial t} - \frac{r}{l} q \right]^2 \qquad (10)$$

$$\lambda_{CG} = \alpha_{CG}\, c \left[\frac{\partial q}{c \partial x} - \frac{\partial \phi}{\partial t} - \frac{g}{c} \phi \right]^2 \qquad (11)$$

In this way, the problem can be reduced to searching for the two functions $\phi(x, t)$ and $q(x, t)$ which minimize a global error functional Λ defined as

$$\Lambda = \int_0^T \int_0^D \lambda(x,t)\, dx\, dt \qquad (12)$$

where D is the length of the line and (0, T) is the time interval of interest.

The functions $\phi(x, t)$ and $q(x, t)$ are approximated by using standard finite element shape functions $N_i(x)$ as

$$\phi(x, t) = \Sigma\, \Phi_i(t)\, N_i(x) \qquad (13)$$

$$q(x, t) = \Sigma\, Q_i(t)\, N_i(x) \qquad (14)$$

Partitioning the interval of interest into a number of time steps (t_k, t_{k+1}), in which we assume linear variation, the solution at each time step (i.e. the set of coefficients at t_{k+1}) is obtained via minimization of an error Λ_k defined as

$$\Lambda_k = \int_{t_k}^{t_{k+1}} \int_0^D \lambda(x,t)\, dx\, dt \qquad (15)$$

In this way, the following set of equations for $\Phi_{k+1} = \{\Phi_i(t_{k+1})\}$ and $Q_{k+1} = \{Q_i(t_{k+1})\}$ is obtained under minimization of (15)

$$\begin{bmatrix} \dfrac{\partial \Lambda_k}{\partial \Phi_{k+1}} \\ \dfrac{\partial \Lambda_k}{\partial Q_{k+1}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (16)$$

that in the linear case yields

$$\begin{bmatrix} \dfrac{\partial^2 \Lambda_k}{\partial \Phi_{k+1} \partial \Phi_{k+1}} & \dfrac{\partial^2 \Lambda_k}{\partial \Phi_{k+1} \partial Q_{k+1}} \\ \dfrac{\partial^2 \Lambda_k}{\partial Q_{k+1} \partial \Phi_{k+1}} & \dfrac{\partial^2 \Lambda_k}{\partial Q_{k+1} \partial Q_{k+1}} \end{bmatrix} \begin{bmatrix} \Phi_{k+1} \\ Q_{k+1} \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial \Lambda_k}{\partial \Phi} \\ \dfrac{\partial \Lambda_k}{\partial Q} \end{bmatrix}_0 \qquad (17)$$

The solution matrix is then symmetric with respect to the Φ and the $\mathbf{Q}$ solutions. In addition, using the θ-method, i.e. approximating the time integral $\int_{t_k}^{t_{k+1}} \lambda(x,t)dt$ as $\lambda(x,t_\theta)\Delta t$ with $\Delta t=t_{k+1}-t_k$ and $t_\theta=\theta t_{k+1}+(1-\theta)t_k$, it is possible to have the splitting of the system of equations. In fact, for a suitable choice of the non-dimensional coefficients $[\alpha_{LR}/\alpha_{CG}=(1/\Delta t+\theta/\tau_{CG})/(1/\Delta t+\theta/\tau_{LR})$, $\tau_{CG}=c/g$, $\tau_{LR}=l/r]$, the unknown vectors Φ_{k+1} and $\mathbf{Q}_{k+1}$ can be obtained as the solutions of two independent systems of equations, since in this case

$$\left[\frac{\partial^2 \Lambda_k}{\partial \Phi_{k+1}\partial Q_{k+1}}\right]=\left[\frac{\partial^2 \Lambda_k}{\partial Q_{k+1}\partial \Phi_{k+1}}\right]^T=[\mathbf{0}] \tag{18}$$

This splitting is however different from solving two separate problems for Φ and $\mathbf{Q}$ in (0,T) because in our case each vector at t_{k+1} depends on both vectors at t_k.

As far as the boundary conditions are considered, it is convenient to enforce them explicitly if voltage or current are prescribed, i.e. if there is an independent generator. However, care must be taken when doing this. For instance, with the choice of the elementary cell made in Fig. 1, the connection of the right end to an independent voltage generator may cause numerical troubles because of the parallel with the capacitor of the elementary cell; dual problems may be caused by the insertion of an independent current generator at the left end. If instead the line end is connected to an impedance, then it is convenient to insert the error in the boundary condition into the error functional, otherwise the symmetry of the system can be lost.

Explicit time integration

Using the θ-method and selecting the coefficients α_{LR} and α_{LR} as indicated in the previous section, we obtain the splitting of the system of equations. In particular the i-th row of the system of equations for Φ_{k+1}, $A\Phi_{k+1}=\mathbf{b}$, contains only three nonzero coefficients:

$$A_{i,i-1} = k_1\theta^2<N'_{i-1}N'_i> + (1+k_2\theta)<N_{i-1}N_i> \tag{19}$$

$$A_{ii} = k_1\theta^2<N'_iN'_i> + (1+k_2\theta)<N_iN_i> \tag{20}$$

$$A_{i,i+1} = k_1\theta^2<N'_iN'_{i+1}> + (1+k_2\theta)<N_iN_{i+1}> \tag{21}$$

where $U'=dU/dx$, $<UV>=\int_0^D U(x)V(x)dx$, k_1 and k_2 are constants that depend on the parameters of the line.

We notice that even assuming $\theta=0$ the solution of a linear system is generally required. This solution is straightforward due to the tridiagonal form of the matrix. However, taking into account the analogy with the electromagnetic field analysis, it is interesting to see in which case we may have an explicit expression for Φ_{k+1}. For $\theta=0$ the off-diagonal terms are $A_{i,i-1}=<N_{i-1}N_i>$ and $A_{i,i+1}=<N_iN_{i+1}>$. These terms are non zero if the space-integrals are computed analytically or using Gauss rule. Instead they disappear when using the trapezoidal rule within each finite element in space. In fact the product of two different shape functions is zero at all nodes.

This treatment, similar to the "mass lumping" [3], might seem to be promising especially for field applications. However, at least in this form, it does not work for an ideal transmission line (r=0, g=0). In fact, performing Von Neumann stability analysis we found the algorithm unconditionally unstable, i.e. unstable for any time and space discretizations.

Results

We analyzed a transmission line with the following parameters: D = 1 m, $r = 10^{-5}\ \Omega m^{-1}$, $g = 10^{-5}\ \Omega^{-1}m^{-1}$, $l = 10^{-7}\ Hm^{-1}$, $c = 1.11\ 10^{-10}\ Fm^{-1}$. The right end is open whereas the left end is connected to a voltage generator in series with a resistance of 1 Ω. The applied voltage is $e(t)=E_o\sin(2\pi t/T_o)$ in $(0,T_o/2)$ with $e(t)=0$ for $t>T_o/2$ with $T_o=3ns$. For this line the losses are negligible, $R_o=(l/c)^{-1/2}=30\Omega$ and $v_p=(lc)^{-1/2}=3\ 10^{-8}\ ms^{-1}$.

Fig. 2 shows the results obtained with $\theta=1/2$, $Cou=v_p\Delta t/\Delta x=0.1$ and different values of Δx, the step size (i.e. the lenght of the elementary cell).

Fig. 3 shows the correlation between analytic and constitutive errors. The analytic error is defined as $\lambda_a=[\alpha_{LR}\Delta v^2/l+\alpha_{CG}\Delta i^2/c]\ T_oR_o/E_o{}^2$.

Numerical calculations have also been carried out with $\Delta x=D/100$ and $\theta=0$ to explore the stability properties. With Gauss integration in space, the numerical method results to be unstable for $\Delta t>3.3\ 10^{-11}$ (Cou=0.01), whereas the explicit scheme with lumping (trapezoidal rule) diverges for any Δt. The numerical stability obtained in case of Gauss integration for lower values of the time step can probably be ascribed to the nonzero values of r and g ($r = 10^{-4}\ \Omega m^{-1}$, $g = 10^{-4}\ \Omega^{-1}m^{-1}$).

Conclusions

The constitutive error based approach has been applied to the analysis of the transmission lines in the time domain, taking into account the strict analogy with the electromagnetic field problems.

The explicit time integration with lumping is unstable. Therefore it cannot be applied as it is. The θ-method is not very expensive for the transmission lines since the inversion of a tridiagonal matrix is straightforward. On the other hand leapfrog scheme or higher order methods should be introduced in view of an efficient solution of the 3D Maxwell equations in the time domain.

References

[1] J.S.Roychowdhury, A.R.Newton, D.O.Pederson,"Algorithms for the Transient Simulation of Lossy Interconnect", IEEE Trans.on Computer-Aided Design of Integrated Circuits and Systems, vol.13, pp.96-104, January 1994.

[2] R.Albanese, R.Fresa, R.Martone, G.Rubinacci,"An error based approach to the solution of full Maxwell Equations", IX Compumag Miami 1993- IEEE Trans.on Magnetics, MAG-30, No 5, 1994

[3] O.C. Zienkiewicz, "The Finite Element Method", McGraw-Hill Book Company, 1977

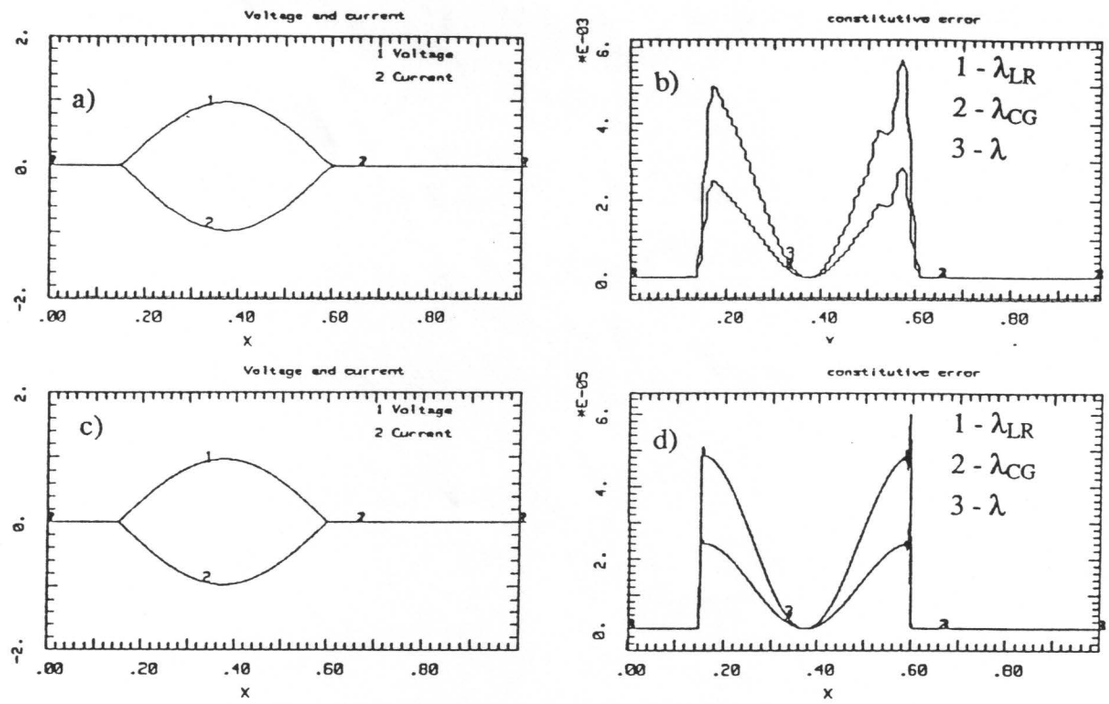

Fig. 2 Results obtained with $\theta=1/2$, $Cou=v_p\Delta t/\Delta x=0.1$ and different values of Δx at t=1.999 ns: (a) Voltage and current as a function of x, with $\Delta x=0.01$ m; (b) λ, λ_{LR}, λ_{CG} as a function of x, with $\Delta x=0.01$ m; (c) Voltage and current as a function of x, with $\Delta x=0.001$ m; (d) λ, λ_{LR}, λ_{CG} as a function of x, with $\Delta x=0.001$ m.

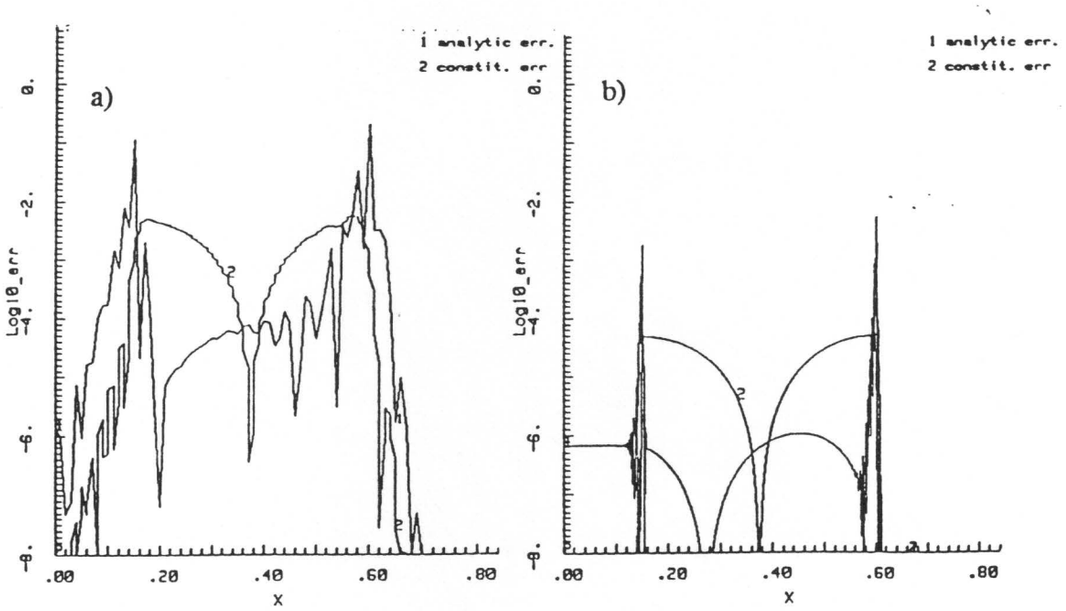

Fig. 3 Correlation between analytic and constitutive errors. The analytic error is defined as $\lambda_a=[\alpha_{LR}\Delta v^2/l+\alpha_{CG}\Delta i^2/c]\ To Ro/Eo^2$: (a) $\Delta x=0.01$ m; (b) $\Delta x=0.001$ m.