mathematics

MDPI

Article

Temperature Distribution in Graded Si Ge Thin Films: A Preliminary Study

Isabella Carlomagno [1], Vito Antonio Cimmelli [2,*] and David Jou [3]

[1] Department of Industrial Engineering, University of Salerno, Via Giovanni Paolo II, 132, 84084 Fisciano, Italy; icarlomagno@unisa.it
[2] Department of Fundamental and Applied Sciences, University of Basilicata, Viale dell'Ateneo Lucano, 10, 85100 Potenza, Italy
[3] Secció de Ciències i Tecnologia, Institut d'Estudis Catalans, Carme 47, 08001 Barcelona, Catalonia, Spain; david.jou.mirabent@gmail.com
* Correspondence: vito.cimmelli@unibas.it

Abstract

In this article we develop a model of a square-shaped heat conductor, with thermal conductivity taking different values along two orthogonal directions in the domain. In order to achieve a complete study of heat conduction in such a domain, preliminarily, we approximate the square-shaped domain with two non-graded Si Ge thin films with different constitutive equations for thermal conductivity. For both systems, by using Mathematica, we find the solution of a suitable boundary value problem arising in the problem under study and compare the corresponding temperature profiles. Such profiles can be concave or convex depending on the thermal properties of the conductor. The heat flux values and privileged direction of heat conduction in different physical situations are calculated as well. The obtained results can be regarded as a first step toward the solution of the complete problem.

Keywords: heat rectification; composition-graded alloys; silicon-germanium thin films; heat rectification coefficient

MSC: 74A15; 74A20; 80A19; 35Q79

1. Introduction

Nowadays, thermal transport in two-dimensional structures, such as thin films, has received new impetus due to the relevant advances in physics of low-dimensional systems [1–3], which enabled better understanding of how thermal conductivity depends on the geometry and stoichiometry of the materials. Within such a framework, intensive research efforts have been directed toward the design of thermal rectifiers, where heat can flow in an easier way in one verse of a given direction than in the opposite one [4–10].

The aim of the present investigation is to explore the temperature distribution in two different two-dimensional Si Ge alloys with different expressions for thermal conductivity. This is a preliminary study to an extensive investigation of heat conduction in graded two-dimensional systems, which strongly depends on the geometry of the system. Our analysis takes advantage of recent findings in phononics [5–8] that provide the theoretical basis for the construction of phononic diodes by determining a privileged direction or verse for heat conduction. In fact, in Si Ge alloys, it is possible that the heat flux in one verse is different from that in the opposite one under the same absolute value of the temperature

Check for updates

Academic Editor: Yang Liu

Received: 13 July 2026
Revised: 19 September 2026
Accepted: 25 September 2026
Published: 29 September 2026

Copyright: © 2026 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license.

difference imposed on the boundaries. This phenomenon, called heat flux rectification, arises from the dependency of thermal conductivity λ on some inhomogeneous parameters besides the temperature T. Usual parameters are the geometry of the system (the width or thickness of a layer, for instance) or the chemical composition as in $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$ alloys, with the stoichiometric variable c ranging in the interval $[0,1]$. Thus, for $c=0$ we have pure Ge, for $c=1$ we have pure Si, and an alloy of Si and Ge at the intermediate points [11–13]. Since the aforementioned material parameters cannot be changed once the system has been manufactured, it is important to design the system correctly in order to get the desired behavior.

For one-dimensional systems, in [9,10], we explored the consequences of the graduation in the composition on the value of the rectification coefficient $\mathrm{R}(\mathrm{T}_H,\mathrm{T}_C)$, with T_H and T_C the hottest and coldest temperatures, respectively (in K), applied to the ends of the system and R defined as the ratio of the value q_r of the reverse heat flux with respect to a given reference verse over the value q_d of the direct heat flux with respect to the same verse, i.e., $\mathrm{R}(\mathrm{T}_H,\mathrm{T}_C)\equiv q_r/q_d$ [7], for given values of T_H and T_C.

Here we analyze two-dimensional systems from a different perspective by exploring the temperature distribution in two square-shaped $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$ and $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$ thin films of side $\mathrm{L}=3\,\mathrm{mm}$ with different constitutive equations for the thermal conductivity $\lambda(c)$ (in $\mathrm{Wm}^{-1}\,\mathrm{K}^{-1}$). From now on we call direct the heat conduction from the Ge-richer region to the Si-richer region and reverse the heat conduction from the Si-richer region to the Ge-richer region. Accordingly, we call direct heat flux q_d (in Wm^{-2}) the heat flow from the Ge-richer region to the Si-richer region and reverse heat flux q_r the heat flow from the Si-richer region to the Ge-richer region. In Si Ge alloys the thermal conductivity depends on the composition, i.e., the percentages of silicon and germanium present in each point. The alloy can be functionally graded, with the composition varying with the position, or homogeneous, i.e., with a constant composition. In the last case, besides the geometry, the percentages of silicon and germanium present in each point play a central role in the determination of its thermal properties. As we will show in the following, the heat conduction in homogeneous alloys changes drastically when the composition is changed. Thus, it is important to know the dependency on the composition of the temperature distribution corresponding to fixed boundary conditions. Such knowledge can greatly aid in the design and fabrication of homogeneous alloys, whose composition could be chosen in order to guarantee the desired behavior. For Si Ge alloys, the constitutive equation for the thermal conductivity is not simple. In Ref. [14], the analysis of the data in [11–13] suggested to look for a best-fit curve of the form

$$\lambda(c)=f(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})e^{\mathrm{A}\,c^2+\mathrm{B}\,c}+g(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})e^{\mathrm{D}\,c^2+\mathrm{E}\,c}, \tag{1}$$

where $\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E}$ are material parameters depending on the temperature and size of the system and $f(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})$ and $g(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})$ are suitable parameter-dependent coefficients. Here and in the following we suppose that the thermal conductivity of the domains under investigation can be approximated by the thermal conductivity in the bulk.

It is worth observing that, in $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$ alloys, the heat flow faces a different thermal conductivity with respect to $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$ ones. For the first system type, the conditions $\lambda(0)=\lambda_{\mathrm{Ge}}$ and $\lambda(1)=\lambda_{\mathrm{Si}}$ give the following expressions of the functions $f(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})$ and $g(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})$:

$$f(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})=\frac{\lambda_{\mathrm{Si}}-\lambda_{\mathrm{Ge}}\,e^{\mathrm{D}+\mathrm{E}}}{e^{\mathrm{A}+\mathrm{B}}-e^{\mathrm{D}+\mathrm{E}}},\qquad g(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})=\frac{-\lambda_{\mathrm{Si}}+\lambda_{\mathrm{Ge}}\,e^{\mathrm{A}+\mathrm{B}}}{e^{\mathrm{A}+\mathrm{B}}-e^{\mathrm{D}+\mathrm{E}}}.$$

Therefore, we obtain

$$\lambda(c) = \frac{\lambda_{\mathrm{Si}} - \lambda_{\mathrm{Ge}}\, e^{\mathrm{D}+\mathrm{E}}}{e^{\mathrm{A}+\mathrm{B}} - e^{\mathrm{D}+\mathrm{E}}} e^{\mathrm{A}\,c^2+\mathrm{B}\,c} + \frac{-\lambda_{\mathrm{Si}} + \lambda_{\mathrm{Ge}}\, e^{\mathrm{A}+\mathrm{B}}}{e^{\mathrm{A}+\mathrm{B}} - e^{\mathrm{D}+\mathrm{E}}} e^{\mathrm{D}\,c^2+\mathrm{E}\,c}. \tag{2}$$

The values of A, B, D, E at T = 300 K for bulk $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$ are shown in Table 1. The values of the heat-conduction parameter for pure Si and pure Ge for bulk are shown in Table 2.

Table 1. Values of the parameters in the analytical expression (2) of thermal conductivity for bulk $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.

Temperature	A	B	D	E
T = 300 K	4.87	−3.76	109.45	−108.95

Table 2. Thermal conductivity in ($\mathrm{Wm}^{-1}\,\mathrm{K}^{-1}$) at T = 300 K for bulk Si and Ge.

Temperature	λ_{Si}	λ_{Ge}
T = 300 K	149.95	77.95

For the second system type, instead, the conditions $\lambda(0) = \lambda_{\mathrm{Si}}$ and $\lambda(1) = \lambda_{\mathrm{Ge}}$ yield

$$f(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E}) = \frac{\lambda_{\mathrm{Ge}} - \lambda_{\mathrm{Si}}\, e^{\mathrm{D}+\mathrm{E}}}{e^{\mathrm{A}+\mathrm{B}} - e^{\mathrm{D}+\mathrm{E}}}, \qquad g(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E}) = \frac{-\lambda_{\mathrm{Ge}} + \lambda_{\mathrm{Si}}\, e^{\mathrm{A}+\mathrm{B}}}{e^{\mathrm{A}+\mathrm{B}} - e^{\mathrm{D}+\mathrm{E}}}.$$

Then, the constitutive equation of thermal conductivity is similar to Equation (2) but with different expressions for $f(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})$ and $g(\mathrm{A},\mathrm{B},\mathrm{D},\mathrm{E})$, i.e.,

$$\lambda(c) = \frac{\lambda_{\mathrm{Ge}} - \lambda_{\mathrm{Si}}\, e^{\mathrm{D}+\mathrm{E}}}{e^{\mathrm{A}+\mathrm{B}} - e^{\mathrm{D}+\mathrm{E}}} e^{\mathrm{A}\,c^2+\mathrm{B}\,c} + \frac{-\lambda_{\mathrm{Ge}} + \lambda_{\mathrm{Si}}\, e^{\mathrm{A}+\mathrm{B}}}{e^{\mathrm{A}+\mathrm{B}} - e^{\mathrm{D}+\mathrm{E}}} e^{\mathrm{D}\,c^2+\mathrm{E}\,c}, \tag{3}$$

wherein the values of A, B, D, E at T = 300 K for bulk $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$ are shown in Table 3.

Table 3. Values of the parameters in the analytical expression (3) of thermal conductivity for bulk $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.

	A	*B*	*D*	*E*
T = 300 K	4.87	−5.98	109.46	−109.96

In stationary conditions, we solve the same boundary value problem for the temperature in two homogeneous thin films whose thermal conductivity is given by Equations (2) and (3), respectively, and compare the corresponding temperature distributions. The problem is solved for different values of c.

The paper has the following layout.

In Section 2, we develop the theoretical framework of asymmetric heat conduction in non-homogeneous and homogeneous square-shaped thin films. For the second system type, we establish the boundary value problem approximating the heat conduction in graded materials.

In Section 3, for different values of c, we solve a boundary value problem for the temperature in two different systems, i.e., $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$ and $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$ alloys, under the hypothesis that, in the first one, Equation (2) holds while, in the second one, Equation (3) holds.

In Section 4 and the Appendix A, we join the results for the two systems specified above and study the possible rectification of heat corresponding to the two problems solved in Section 3.

In Section 5, we discuss the obtained results and propose some future generalizations of the theory.

2. Theoretical Framework

2.1. The General Model

In this section we develop a general model of a two-dimensional heat conductor, represented in Figure 1 as square-shaped thin film D of side L, with $\mathrm{L} = 3\,\mathrm{mm}$.

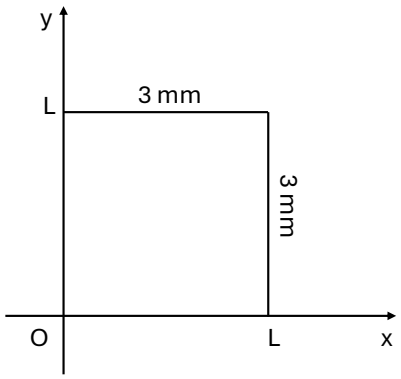


Figure 1. A sketch of the system under consideration, a square-shaped thin film D of side L with $\mathrm{L} = 3\,\mathrm{mm}$. The thickness of D is supposed to be such that its thermal conductivity can be approximated by the bulk thermal conductivity.

In the following we assume that c depends on x only and not y. In order to get asymmetric heat conduction, we assume that the thermal conductivity depends on the composition c according to Equation (2) for $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$ (direct heat conduction) and Equation (3) in $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$ (reverse heat conduction). In this way we have two asymmetries: the first one is a consequence of the validity of two different constitutive equations for the thermal conductivity in the direct and reverse heat flow; the second one is due to the assumption that c depends only on x and not y. We assume stationary conditions that, in the absence of external heat sources, yield $\nabla \cdot \mathbf{q} = 0$, with $\mathbf{q}$ as the heat flux vector. Let $\mathbf{q} = q_x(x,y)\mathbf{i} + q_y(x,y)\mathbf{j}$, with (O,x,y) a system of Cartesian axes on D, $(\mathbf{i},\mathbf{j})$ an orthonormal basis in the Euclidean two-dimensional vector space $\mathbf{E}_2$, and $q_x(x,y)$ and $q_y(x,y)$ the components of vector $\mathbf{q}$ in such a basis. Thus, we can rewrite the stationary condition in the form

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0. \tag{4}$$

We postulate for the components of the heat flux the following Fourier constitutive equations:

$$q_x = -\lambda(c(x))\frac{\partial \mathrm{T}}{\partial x}, \quad q_y = -\lambda(c(x))\frac{\partial \mathrm{T}}{\partial y}. \tag{5}$$

The constitutive equations above mean that D is represented as composed by an infinite number of one-dimensional slices of length L, going from $y = 0$ to $y = \mathrm{L}$, each of which occupies the position $x \in [0,\mathrm{L}]$ and has thermal conductivity $\lambda(c(x))$, which, for fixed x, is constant along the direction y. In practice, although λ does not vary with y, each slice has a constant thermal conductivity that depends on the position x of the slice. Furthermore, we assume that c is linear in x; i.e., $c = x\,\mathrm{L}^{-1}$.

Under the previous assumption, we get

$$\frac{\partial q_x}{\partial x} = -\frac{1}{L}\frac{\partial \lambda}{\partial c}\frac{\partial \mathrm{T}}{\partial x} - \lambda(c(x))\frac{\partial^2 \mathrm{T}}{\partial x^2}, \tag{6}$$

$$\frac{\partial q_y}{\partial y} = -\lambda(c(x))\frac{\partial^2 \mathrm{T}}{\partial y^2}. \tag{7}$$

Finally, once Equation (4) has been taken into account, the combination of Equations (6) and (7) yields

$$\frac{1}{L}\frac{\partial \lambda}{\partial c}\frac{\partial \mathrm{T}}{\partial x} + \lambda(c(x))\frac{\partial^2 \mathrm{T}}{\partial x^2} + \lambda(c(x))\frac{\partial^2 \mathrm{T}}{\partial y^2} = 0. \tag{8}$$

2.2. The Reduced Problem

Due to the complex constitutive equation of the thermal conductivity, Equation (8) can be solved only numerically. Thus, we make some additional hypotheses that allow us to reduce the problem to a couple of ordinary differential equations. For such equations we provide a semi-analytical solution by using Mathematica. To this end, we suppose that the temperature can be represented as the product of two positive functions $X(x)$ and $Y(y)$, namely $\mathrm{T}(x,y) = X(x)Y(y)$, with $X(x) \in C^2([0,\mathrm{L}])$ and $Y(y) \in C^2([0,\mathrm{L}])$. Under the hypotheses above, we can rewrite Equation (8) as

$$\frac{1}{L}\frac{\partial \lambda}{\partial c}\frac{dX(x)}{dx}Y(y) + \lambda(c)\frac{d^2X(x)}{dx^2}Y(y) + \lambda(c)X(x)\frac{d^2Y(y)}{dy^2} = 0. \tag{9}$$

By Equation (9) and the supposed positivity of $X(x)$ and $Y(y)$ it follows that

$$\frac{1}{L}\frac{1}{\lambda}\frac{\partial \lambda}{\partial c}\frac{1}{X(x)}\frac{dX(x)}{dx} + \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = 0. \tag{10}$$

Now we observe that, on the left-hand side of Equation (10), the first two terms depend on the variable x only, while the third term depends on the variable y only. Thus, Equation (10) is satisfied if and only if

$$\frac{d^2Y(y)}{dy^2} = CY(y), \tag{11}$$

$$\frac{1}{L}\frac{1}{\lambda(c(x))}\frac{\partial \lambda}{\partial c}\frac{dX(x)}{dx} + \frac{d^2X(x)}{dx^2} = -CX(x), \tag{12}$$

wherein C is a constant.

The conditions above mean that the two differential operators

$$\mathrm{D}_1 : Y(y) \in C^2[0,\mathrm{L}] \longrightarrow \frac{d^2Y}{dy^2} \in C^0[0,\mathrm{L}],$$

and

$$\mathrm{D}_2 : X(x) \in C^2[0,\mathrm{L}] \longrightarrow \frac{1}{L}\frac{1}{\lambda(c(x))}\frac{\partial \lambda}{\partial c}\frac{dX}{dx} + \frac{d^2X}{dx^2} \in C^0[0,\mathrm{L}],$$

have opposite eigenvalues, and this is a necessary and sufficient condition for the solvability of Equation (8) by the method of separation of the variables. It is worth noticing that, for technical reasons, we must suppose that the physical dimension of both $X(x)$ and $Y(y)$ is $\sqrt{\mathrm{K}}$ in such a way that T is measured in K. Hence, although the single functions $X(x)$ and $Y(y)$ do not correspond to any physically measurable quantity, their product yields a temperature. Such a consideration suggests the following boundary conditions for Equations (11) and (12):

$$Y(0) = \sqrt{300}\ \ \mathrm{K}^{1/2}, \quad Y(\mathrm{L}) = \sqrt{280}\ \ \mathrm{K}^{1/2}. \tag{13}$$

$$X(0) = \sqrt{300}\ \ \mathrm{K}^{1/2}, \quad X(\mathrm{L}) = \sqrt{280}\ \ \mathrm{K}^{1/2}. \tag{14}$$

The previous equations lead to the edge conditions

$$\mathrm{T}(0,0) = 300 \ \ \mathrm{K}, \quad \mathrm{T}(\mathrm{L},\mathrm{L}) = 280 \ \ \mathrm{K}, \tag{15}$$

which is suggested by the results obtained in Refs. [9,10].

The main advantage of this methodology is that it includes the boundary conditions for the whole domain. In fact, Equation (11), with the boundary condition (13), enables obtaining the function $Y(y)$ $\forall y \in [0, \mathrm{L}]$, while Equation (12), with the boundary condition (14), enables obtaining the function $X(x)$ $\forall x \in [0, \mathrm{L}]$. Thus, their product $\mathrm{T}(x,y) = X(x)Y(y)$ yields the temperature at any point $(x,y) \in [0,\mathrm{L}] \times [0,\mathrm{L}]$, namely the closure of D, without requiring any other boundary condition.

3. Solution of the Boundary Value Problem and Temperature Distribution

In this section we solve the boundary value problem (11)–(14) under the hypotheses that $\lambda(c)$ is given by Equation (2) in the direct case (from Ge to Si) and Equation (3) in the reverse case (from Si to Ge). As a further simplification, we evaluate both Equations (2) and (3) for constant c so that the different values of λ are the sole effect of the graduation.

In this way, we solve the boundary value problem for two different heat conductors, with different thermal conductivity, and compare the corresponding solutions. For the first one, λ is given by Equation (2); for the second one λ is given by Equation (3).

We choose $C = 1.1$ and $C = 0.4$ and calculate the semi-analytical solution of the problems (11) and (12), with the boundary conditions (13) and (14), by the software MATHEMATICA Version 14.3.0.0 2), which can be used for symbolic and numeric computation, visualization, and application development.

In Figures A1–A36 the temperature distribution in D is shown in the direct case (from Ge to Si) and the reverse case (from Si to Ge). A strong dependency is shown regarding the composition of both the shape and the temperature value. Moreover, for the same value of c, the temperature distribution is different in the direct and reverse cases. A dependency of the solution on the value of C can be observed as well. Thus, the choice of both c and C can be tuned in order to obtain the desired behavior of the heat conductor.

4. Rectification of the Heat Flux: Preliminary Results

As far as the heat rectification is concerned, we notice that, in Refs. [9,10], for graded alloys with $c = x\,\mathrm{L}^{-1}$, we have determined the rectification coefficient by:

- Applying a direct heat flux q_d on the Ge side at temperature T_H (hottest T) and obtaining the corresponding temperature T_C (coldest T) at the Si side (direct situation);
- Applying different values of the reverse heat flux on the Si side at the highest temperature T_H until finding the value q_r (reverse heat flux) for which the temperature at the Ge side is the lowest temperature T_C previously obtained for q_d (reverse situation);
- Calculating the rectification coefficient as $\mathrm{R}(\mathrm{T}_H, \mathrm{T}_C) \equiv q_r/q_d$.

Such a procedure cannot be applied here because we used the constitutive Equation (5) in order to eliminate **q** and obtain a partial differential equation for the temperature. Hence, both in the direct and reverse cases, the heat flux must be calculated by using the Fourier law and the temperature distribution obtained in Section 3.

The values of the thermal conductivity corresponding to different values of the composition that enter the Fourier law are shown in Tables 4–6.

Table 4. Values of the thermal conductivity (in $Wm^{-1}K^{-1}$) corresponding to the mentioned compositions at T = 300 K (from [11–13]).

	λ_{Si}	$\lambda_{Si_{0.95}}\lambda_{Ge_{0.05}}$	$\lambda_{Si_{0.90}}\lambda_{Ge_{0.10}}$	$\lambda_{Si_{0.85}}\lambda_{Ge_{0.15}}$	$\lambda_{Si_{0.75}}\lambda_{Ge_{0.25}}$	$\lambda_{Si_{0.65}}\lambda_{Ge_{0.35}}$	$\lambda_{Si_{0.6}}\lambda_{Ge_{0.4}}$
$T = 300\,K$	149.95	37.15	25.13	19.65	14.15	11.30	10.17

Table 5. Values of the thermal conductivity (in $Wm^{-1}K^{-1}$) corresponding to the mentioned compositions at T = 300 K (from [11–13]).

	$\lambda_{Si_{0.59}}\lambda_{Ge_{0.41}}$	$\lambda_{Si_{0.5}}\lambda_{Ge_{0.5}}$	$\lambda_{Si_{0.41}}\lambda_{Ge_{0.59}}$	$\lambda_{Si_{0.4}}\lambda_{Ge_{0.6}}$	$\lambda_{Si_{0.35}}\lambda_{Ge_{0.65}}$	$\lambda_{Si_{0.25}}\lambda_{Ge_{0.75}}$
$T = 300\,K$	10.33	8.97	8.22	8.16	7.94	7.93

Table 6. Values of the thermal conductivity (in $Wm^{-1}K^{-1}$) corresponding to the mentioned compositions at T = 300 K (from [11–13]).

	$\lambda_{Si_{0.15}}\lambda_{Ge_{0.85}}$	$\lambda_{Si_{0.10}}\lambda_{Ge_{0.90}}$	$\lambda_{Si_{0.05}}\lambda_{Ge_{0.95}}$	λ_{Ge}
$T = 300\,K$	8.94	10.46	14.52	77.95

In this situation, for two different values of the constant C entering Equations (11) and (12), namely $C = 1.1$ and $C = 0.4$, we have calculated different values of the temperature (in K) at any point of the conductor. Moreover, we have approximated the temperature gradient (in $K\,m^{-1}$) with the difference between the maximum and minimum values of the temperature along the x direction divided by the length of the conductor (L = 3 mm). In such a way, we obtained the corresponding values of the direct heat flux q_d and the reverse heat flux q_r (in $W\,m^{-2}$) and the rectification coefficient $R = q_r/q_d$. The parameter R can be interpreted as the rectification coefficient of a virtual heat conductor whose thermal conductivity is given by Equation (2) in the direct case and Equation (3) in the reverse case. Imagine the heat is flowing from Ge to Si along the $Si_c\,Ge_{1-c}$ alloy (direct heat flux). Then, for $C = 1.1$ and $c = 0.05$, the temperature distribution is the one shown in Figure A1. If, instead, the heat is flowing from Si to Ge along a $Si_{1-c}\,Ge_c$ alloy, the temperature distribution is the one shown in Figure A2. Due to the different thermal conductivities, we get different values of the temperature distribution and heat flux in the direct and reverse situations. Thus, a virtual conductor, with the thermal conductivity given by Equation (2) when the flow is from left to right and Equation (3) when the flow is from right to left, exactly as occurs in graded systems, is capable of rectifying the heat flux. In this way, we can obtain a preliminary understanding of the heat conduction in graded thin films without solving the problem with an inhomogeneous composition, which will be the subject of our future research.

By applying the procedure illustrated above we obtained the results shown in Tables 7 and 8. The details of the computation are shown in the Appendix A.

Table 7. Values of the rectification coefficient R corresponding to the mentioned compositions at T = 300 K for $C = 1.1$.

c	0.05	0.10	0.15	0.25	0.4	0.5	0.59	0.65	0.75
R	2.56	2.40	2.20	1.78	1.27	1	0.81	0.70	0.62

Table 8. Values of the rectification coefficient R corresponding to the mentioned compositions at $T = 300\,K$ for $C = 0.4$.

c	0.05	0.10	0.15	0.25	0.4	0.5	0.59	0.65	0.75
R	2.56	2.40	2.20	1.78	1.27	1	0.81	0.70	0.62

5. Discussion and Future Perspectives

The results in Sections 3 and 4 lead to the following conclusions:

- The value of the composition plays a fundamental role in determining the temperature distribution in D and the privileged direction of heat conduction. The temperature graphs for $C = 1.1$ and $C = 0.4$ are similar but not identical, both from the quantitative and qualitative points of view, and the value of C also influences the solution.
- For $C = 1.1$ and $c \in [0.05, 0.25]$ the surface representing the solution in D is concave, at $c = 0.25$ it becomes almost flat in the reverse case and, finally, it becomes convex for $c \in [0.25, 0.75]$ except for $c = 0.59$ in the reverse case. For $C = 0.4$ and $c \in [0.05, 0.4]$ in the direct case and $c = 0.5$ in the reverse case, the surface is concave, at $c = 0.4$ in the direct case and $c = 0.59$ in the reverse case it becomes almost flat and, finally, it becomes convex for $c \in [0.59, 0.75]$.
- Both for $C = 1.1$ and $C = 0.4$, R ranges from 2.56 for $c = 0.05$ to 0.62 for $c = 0.75$. Thus, we may conclude that, for $c \to 0$, the heat conduction from Si to Ge is prevalent with respect to that from Ge to Si $(R > 1)$. On the other hand, for $c \to 1$, the heat conduction from Ge to Si is prevalent with respect to that from Si to Ge $(R < 1)$.
- For $c = 0.5$ the symmetric heat conduction $(R = 1)$ takes over. It is worth observing that such a situation does not coincide with the flatness of the temperature graph.

  In our future research we aim to solve the two-dimensional problem for a single heat conductor under the hypothesis $c = x\,L^{-1}$. In Section 2 we have illustrated the advantages of the method of separation of the variables. Of course, if c depends on both x and y, the separation of the variables can be applied only for particular forms of $c(x, y)$; otherwise a different method of solution must be applied. Thus, although solutions by different methods, such as the finite difference scheme, are currently out of our scope, in future research, we do not exclude such a possibility. To the best of our knowledge, no other results on this problem can be found in the literature, so we cannot compare the present results with pre-existing ones (for a review on the state of the art of thermal diodes realized with functionally graded materials, see Ref. [5], Table 1 and Section 2.1.2).

  Finally, we aim to investigate the same phenomenon on a nanometric scale, for instance, for $L = 100\,nm$, for which the experimental values of $\lambda(c)$ at $T = 300\,K$ are available too.

Author Contributions: Conceptualization, I.C., V.A.C. and D.J.; methodology, I.C., V.A.C. and D.J.; formal analysis, I.C., V.A.C. and D.J.; writing—original draft, I.C., V.A.C. and D.J.; writing—review and editing, I.C., V.A.C. and D.J. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the University of Basilicata, the University of Salerno and the Italian Gruppo Nazionale per la Fisica Matematica (GNFM-INδAM).

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: Work performed under the auspices of the Italian National Group of Mathematical Physics (GNFM-INδAM). I.C. acknowledges the financial support from the University of Salerno

and the Italian Gruppo Nazionale per la Fisica Matematica (GNFM-INδAM). V.A.C. acknowledges the financial support from the University of Basilicata, the University of Salerno and the Italian Gruppo Nazionale per la Fisica Matematica (GNFM-INδAM).

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Appendix A.1. Rectification Coefficient for $C = 1.1$

Below we calculate the rectification coefficient R for different compositions and $C = 1.1$.

$c = 0.05$, $C = 1.1$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.
maximum value along x: 299.99
minimum value along x: 279.99
direct heat flux:

$$q_d = 14.52 \times \frac{299.99 - 279.99}{0.003} = 14.52 \times 6666.67 = 96{,}800.05$$

$c = 0.05$, $C = 1.1$; reverse case $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.
maximum value along x: 299.99
minimum value along x: 279.99
reverse heat flux:

$$q_r = 37.15 \times \frac{299.99 - 279.99}{0.003} = 37.15 \times 6666.67 = 247{,}666.79$$

Rectification coefficient:

$$\mathrm{R} = \frac{247{,}666.79}{96{,}800.05} = 2.56$$

$c = 0.10$, $C = 1.1$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.
maximum value along x: 299.99
minimum value along x: 279.99
direct heat flux

$$q_d = 10.46 \times \frac{299.99 - 279.99}{0.003} = 10.46 \times 6666.67 = 69{,}733.37$$

$c = 0.10$, $C = 1.1$; reverse case $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.
maximum value along x: 299.99
minimum value along x: 279.99
reverse heat flux

$$q_r = 25.13 \times \frac{299.99 - 279.99}{0.003} = 25.13 \times 6666.67 = 167{,}533.42$$

Rectification coefficient:

$$\mathrm{R} = \frac{167{,}533.42}{69{,}733.37} = 2.40$$

$c = 0.15$, $C = 1.1$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.
maximum value along x: 299.99
minimum value along x: 279.99
direct heat flux

$$q_d = 8.94 \times \frac{299.99 - 279.99}{0.003} = 8.94 \times 6666.67 = 59{,}600.03$$

$c = 0.15$, $C = 1.1$; reverse case $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.
maximum value along x: 299.99

minimum value along x: 279.99
reverse heat flux

$$q_r = 19.65 \times \frac{299.99 - 279.99}{0.003} = 19.65 \times 6666.67 = 131{,}000.07$$

Rectification coefficient

$$\mathrm{R} = \frac{131{,}000.07}{59{,}600.03} = 2.20$$

$c = 0.25$, $C = 1.1$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.
maximum value along x: 299.99
minimum value along x: 279.99
direct heat flux

$$q_d = 7.93 \times \frac{299.99 - 279.99}{0.003} = 7.93 \times 6666.67 = 52{,}866.69$$

$c = 0.25$, $C = 1.1$; reverse case $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.
maximum value along x: 299.99
minimum value along x: 279.99
reverse heat flux

$$q_r = 14.15 \times \frac{299.99 - 279.99}{0.003} = 14.15 \times 6666.67 = 94{,}333.38$$

rectification coefficient

$$\mathrm{R} = \frac{94{,}333.38}{52{,}866.69} = 1.78$$

$c = 0.4$, $C = 1.1$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.
maximum value along x: 299.99
minimum value along x: 279.99
direct heat flux

$$q_d = 8.16 \times \frac{299.99 - 279.99}{0.003} = 8.16 \times 6666.67 = 54{,}400.03$$

$c = 0.4$, $C = 1.1$; reverse case $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.
maximum value along x: 299.99
minimum value along x: 279.99
reverse heat flux

$$q_r = 10.33 \times \frac{299.99 - 279.99}{0.003} = 10.33 \times 6666.67 = 68{,}866.70$$

rectification coefficient

$$\mathrm{R} = \frac{68{,}866.70}{54{,}400.03} = 1.26$$

$c = 0.5$, $C = 1.1$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.
maximum value along x: 299.99
minimum value along x: 279.99
direct heat flux

$$q_d = 8.97 \times \frac{299.99 - 279.99}{0.003} = 8.97 \times 6666.67 = 59{,}800.03$$

$c = 0.5$, $C = 1.1$; reverse case $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.
maximum value along x: 299.99
minimum value along x: 279.99

reverse heat flux

$$q_r = 8.97 \times \frac{299.99 - 279.99}{0.003} = 8.97 \times 6666.67 = 59{,}800.03$$

rectification coefficient

$$\mathrm{R} = \frac{59{,}800.03}{59{,}800.03} = 1$$

$c = 0.59$, $C = 1.1$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.
maximum value along x: 299.99
minimum value along x: 279.99
direct heat flux

$$q_d = 10.17 \times \frac{299.99 - 279.99}{0.003} = 10.17 \times 6666.67 = 67{,}800.03$$

$c = 0.59$, $C = 1.1$; reverse case $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.
maximum value along x: 299.99
minimum value along x: 279.99
reverse heat flux

$$q_r = 8.22 \times \frac{299.99 - 279.99}{0.003} = 8.22 \times 6666.67 = 54{,}800.03$$

rectification coefficient

$$\mathrm{R} = \frac{54{,}800.03}{67{,}800.03} = 0.81$$

$c = 0.65$, $C = 1.1$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.
maximum value along x: 299.99
minimum value along x: 279.99
direct heat flux

$$q_d = 11.30 \times \frac{299.99 - 279.99}{0.003} = 11.30 \times 6666.67 = 75{,}333.37$$

$c = 0.65$, $C = 1.1$; reverse case $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.
maximum value along x: 299.99
minimum value along x: 279.99
reverse heat flux

$$q_r = 7.94 \times \frac{299.99 - 279.99}{0.003} = 7.94 \times 6666.67 = 52{,}933.36$$

rectification coefficient

$$\mathrm{R} = \frac{52{,}933.36}{75{,}333.37} = 0.70$$

$c = 0.75$, $C = 1.1$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.
maximum value along x: 299.99
minimum value along x: 279.99
direct heat flux

$$q_d = 14.15 \times \frac{299.99 - 279.99}{0.003} = 14.15 \times 6666.67 = 94{,}333.38$$

$c = 0.75$, $C = 1.1$; reverse case $\mathrm{Si}_{1-c}\,\mathrm{Ge}_c$.
maximum value along x: 299.99
minimum value along x: 279.99

reverse heat flux

$$q_r = 8.76 \times \frac{299.99 - 279.99}{0.003} = 8.76 \times 6666.67 = 58{,}400.03$$

Rectification coefficient

$$\mathrm{R} = \frac{58{,}400.03}{94{,}333.38} = 0.62$$

It is worth noticing that the maximum value of the temperature (299.99 K) has been obtained at the edge $(x = 0, y = 0)$, while the minimum value of the temperature (279.99 K) has been obtained at the edge $x = 3\,\mathrm{mm}, y = 3\,\mathrm{mm}$. Such values fit the boundary condition (15) up to the error of 0.01 K due to the approximation of $\sqrt{300}$ and $\sqrt{280}$ in the boundary conditions (13) and (14).

Note that, for the computation above, we have used the average temperature gradient on the interval $[0, 0.003]$ m. Since a single global average gradient may significantly differ from local gradients at boundaries, given the variability in the temperature surfaces, one could wonder how much the mean value of the partial derivative with respect to x used in the computation above differs from the local value. For the sake of comparison we have calculated the values of $\left|\frac{\partial \mathrm{T}}{\partial x}\right|$ at the points $(0,0)$ and (L,L) for the compositions $0.05, 0.25, 0.5, 0.75$ and $C = 1.1$ in the direct and reverse cases. The values are shown in Tables A1 and A2. The mean values for the direct and reverse cases have been calculated as well. It is important to notice that the average of the mean values is 5840.89 in the direct case and 6244.97 in the reverse case. The values differ by less than 12.5% with respect to the average temperature gradient used in the calculations above, namely 6666.67.

Table A1. Values of the $\left|\frac{\partial \mathrm{T}}{\partial x}\right|$ (in $\mathrm{K\,m^{-1}}$) in the direct case calculated at the points $(0,0)$ and (L,L) for the indicated compositions and $C = 1.1$. The mean value is 5840.89.

c	0.05	0.25	0.5	0.75
$\left\|\frac{\partial \mathrm{T}}{\partial x}(0,0)\right\|$	60.49	1626.96	5614.74	12,379.90
$\left\|\frac{\partial \mathrm{T}}{\partial x}(\mathrm{L},\mathrm{L})\right\|$	18,998.00	5911.77	1787.14	345.21
mean value	9529.24	3770.86	3700.94	6362.55

Table A2. Values of the $\left|\frac{\partial \mathrm{T}}{\partial x}\right|$ (in $\mathrm{K\,m^{-1}}$) in the reverse case calculated at the points $(0,0)$ and (L,L) for the indicated compositions and $C = 1.1$. The average of the mean values is 6244.97.

c	0.05	0.25	0.5	0.75
$\left\|\frac{\partial \mathrm{T}}{\partial x}(0,0)\right\|$	22.69	356.80	1848.96	6117.96
$\left\|\frac{\partial \mathrm{T}}{\partial x}(\mathrm{L},\mathrm{L})\right\|$	22,753.60	11,962.00	5425.68	1572.10
mean value	11,338.14	6159.40	3637.32	3845.03

Appendix A.2. Rectification Coefficient for $C = 0.4$

Below we calculate the rectification coefficient R for different compositions and $C = 0.4$.

$c = 0.05$, $C = 0.4$; direct case $\mathrm{Si}_c\,\mathrm{Ge}_{1-c}$.

maximum value along x: 300

minimum value along x: 280

direct heat flux

$$q_d = 14.52 \times \frac{300 - 280}{0.003} = 14.52 \times 6666.67 = 96{,}800.05$$

$c = 0.05$, $C = 0.4$; reverse case $Si_{1-c}\,Ge_c$.
maximum value along x: 300
minimum value along x: 280
reverse heat flux

$$q_r = 37.15 \times \frac{300 - 280}{0.003} = 37.15 \times 6666.67 = 247{,}666.79$$

Rectification coefficient

$$\mathrm{R} = \frac{247{,}666.79}{96{,}800.05} = 2.56$$

$c = 0.10$, $C = 0.4$; direct case $Si_c\,Ge_{1-c}$.
maximum value along x: 300
minimum value along x: 280
direct heat flux

$$q_d = 10.46 \times \frac{300 - 280}{0.003} = 10.46 \times 6666.67 = 69{,}733.37$$

$c = 0.10$, $C = 0.4$; reverse case $Si_{1-c}\,Ge_c$.
maximum value along x: 300
minimum value along x: 280
reverse heat flux

$$q_r = 25.13 \times \frac{300 - 280}{0.003} = 25.13 \times 6666.67 = 167{,}533.42$$

rectification coefficient

$$\mathrm{R} = \frac{167{,}533.42}{69{,}733.37} = 2.40$$

$c = 0.15$, $C = 0.4$; direct case $Si_c\,Ge_{1-c}$.
maximum value along x: 300
minimum value along x: 280
direct heat flux

$$q_d = 8.94 \times \frac{300 - 280}{0.003} = 8.94 \times 6666.67 = 59{,}600.03$$

$c = 0.15$, $C = 0.4$; reverse case $Si_{1-c}\,Ge_c$.
maximum value along x: 300
minimum value along x: 280
reverse heat flux

$$q_r = 19.65 \times \frac{300 - 280}{0.003} = 19.65 \times 6666.67 = 131{,}000.07$$

Rectification coefficient

$$\mathrm{R} = \frac{131{,}000.07}{59{,}600.03} = 2.20$$

$c = 0.25$, $C = 0.4$; direct case $Si_c\,Ge_{1-c}$.
maximum value along x: 300
minimum value along x: 280

direct heat flux

$$q_d = 7.93 \times \frac{300 - 280}{0.003} = 7.93 \times 6666.67 = 52{,}866.69$$

$c = 0.25$, $C = 0.4$; reverse case $Si_{1-c}\,Ge_c$.
maximum value along x: 300
minimum value along x: 280
reverse heat flux

$$q_r = 14.15 \times \frac{300 - 280}{0.003} = 14.15 \times 6666.67 = 94{,}333.38$$

Rectification coefficient

$$\mathrm{R} = \frac{94{,}333.38}{52{,}866.69} = 1.78$$

$c = 0.4$, $C = 0.4$; direct case $Si_c\,Ge_{1-c}$.
maximum value along x: 300
minimum value along x: 280
direct heat flux

$$q_d = 8.16 \times \frac{300 - 280}{0.003} = 8.16 \times 6666.67 = 54{,}400$$

$c = 0.4$, $C = 0.4$; reverse case $Si_{1-c}\,Ge_c$.
maximum value along x: 300
minimum value along x: 280
reverse heat flux

$$q_r = 10.33 \times \frac{300 - 280}{0.003} = 10.33 \times 6666.67 = 68{,}866.70$$

rectification coefficient

$$\mathrm{R} = \frac{68{,}866.70}{54{,}400} = 1.26$$

$c = 0.5$, $C = 0.4$; direct case $Si_c\,Ge_{1-c}$.
maximum value along x: 300
minimum value along x: 280
direct heat flux

$$q_d = 8.97 \times \frac{300 - 280}{0.003} = 8.97 \times 6666.67 = 59{,}800.03$$

$c = 0.5$, $C = 0.4$; reverse case $Si_{1-c}\,Ge_c$.
maximum value along x: 300
minimum value along x: 280
reverse heat flux

$$q_r = 8.97 \times \frac{300 - 280}{0.003} = 8.97 \times 6666.67 = 59{,}800.03$$

rectification coefficient

$$\mathrm{R} = \frac{59{,}800.03}{59{,}800.03} = 1$$

$c = 0.59$, $C = 0.4$; direct case $Si_c\,Ge_{1-c}$
maximum value along x: 300
minimum value along x: 280

direct heat flux

$$q_d = 10.17 \times \frac{300 - 280}{0.003} = 10.17 \times 6666.67 = 67{,}800.03$$

$c = 0.59$, $C = 0.4$; reverse case $Si_{1-c}\,Ge_c$.
maximum value along x: 300
minimum value along x: 280
reverse heat flux

$$q_r = 8.22 \times \frac{300 - 280}{0.003} = 8.22 \times 6666.67 = 54{,}800.03$$

rectification coefficient

$$\mathrm{R} = \frac{54{,}800.03}{67{,}800.03} = 0.81$$

$c = 0.65$, $C = 0.4$; direct case $Si_c\,Ge_{1-c}$.
maximum value along x: 300
minimum value along x: 280
direct heat flux

$$q_d = 11.3 \times \frac{300 - 280}{0.003} = 11.3 \times 6666.67 = 75{,}333.37$$

$c = 0.65$, $C = 0.4$; reverse case $Si_{1-c}\,Ge_c$.
maximum value along x: 300
minimum value along x: 280
reverse heat flux

$$q_r = 7.94 \times \frac{300 - 280}{0.003} = 7.94 \times 6666.67 = 52{,}933.36$$

rectification coefficient

$$\mathrm{R} = \frac{52{,}933.36}{75{,}333.37} = 0.70$$

$c = 0.75$, $C = 0.4$; direct case $Si_c\,Ge_{1-c}$.
maximum value along x: 300
minimum value along x: 280
direct heat flux

$$q_d = 14.15 \times \frac{300 - 280}{0.003} = 14.15 \times 6666.67 = 94{,}333.38$$

$c = 0.75$, $C = 0.4$; reverse case $Si_{1-c}\,Ge_c$.
maximum value along x: 300
minimum value along x: 280
reverse heat flux

$$q_r = 8.76 \times \frac{300 - 280}{0.003} = 8.76 \times 6666.67 = 58{,}400.03$$

rectification coefficient

$$\mathrm{R} = \frac{58{,}400.03}{94{,}333.38} = 0.62$$

It is worth noticing that the maximum value of the temperature (300 K) has been obtained at the edge $(x = 0, y = 0)$, while the minimum value of the temperature (280 K) has been obtained at the edge $(x = 3\,\mathrm{mm}, y = 3\,\mathrm{mm})$. Such values fit the boundary condition (15); i.e., the error due to the approximation of $\sqrt{300}$ and $\sqrt{280}$ in the boundary conditions (13) and (14) is smaller than 0.01 K.

Also in this case, for the sake of comparison, we have calculated the values of $\left|\frac{\partial T}{\partial x}\right|$ at the points $(0,0)$ and (L,L) for the compositions $0.05, 0.25, 0.5, 0.75$ and $C = 0.4$ in the direct and reverse cases. The values are shown in Tables A3 and A4. The mean values for the direct and reverse cases have been calculated as well. It is important to notice that the average of the mean values is 5840.44 in the direct case and 6257.37 in the reverse case. The values differ by less than 12.5% with respect to the average temperature gradient used in the calculations above, namely 6666.67.

Table A3. Values of the $\left|\frac{\partial T}{\partial x}\right|$ (in $K\,m^{-1}$) in the direct case calculated at the points $(0,0)$ and (L,L) for the indicated compositions and $C = 0.4$. The average of the mean values is 5840.44.

c	0.05	0.25	0.5	0.75
$\left\|\frac{\partial T}{\partial x}(0,0)\right\|$	60.60	1626.71	5615.10	12,380.30
$\left\|\frac{\partial T}{\partial x}(L,L)\right\|$	18,997.50	5911.41	1786.89	345.07
mean value	9529.05	3769.06	3700.99	6362.68

Table A4. Values of the $\left|\frac{\partial T}{\partial x}\right|$ (in $K\,m^{-1}$) in the reverse case calculated at the points $(0,0)$ and (L,L) for the indicated compositions and $C = 0.4$. The average of the mean values is 6257.37.

c	0.05	0.25	0.5	0.75
$\left\|\frac{\partial T}{\partial x}(0,0)\right\|$	22.78	356.96	1849.21	6118.34
$\left\|\frac{\partial T}{\partial x}(L,L)\right\|$	22,753.00	11,961.50	5425.33	1571.87
mean value	11,387.89	6159.23	3637.27	3845.10

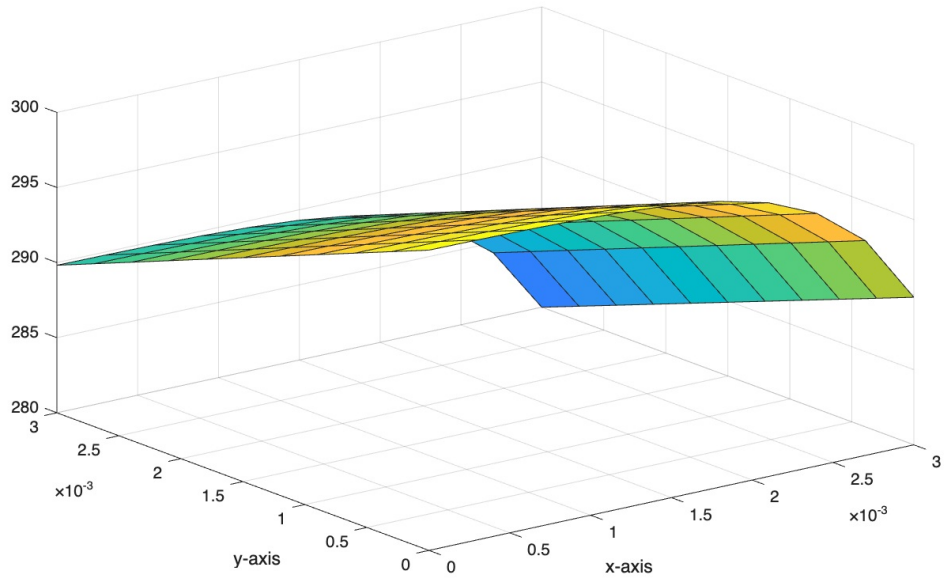


Figure A1. Direct case for $c = 0.05$ and $C = 1.1$.

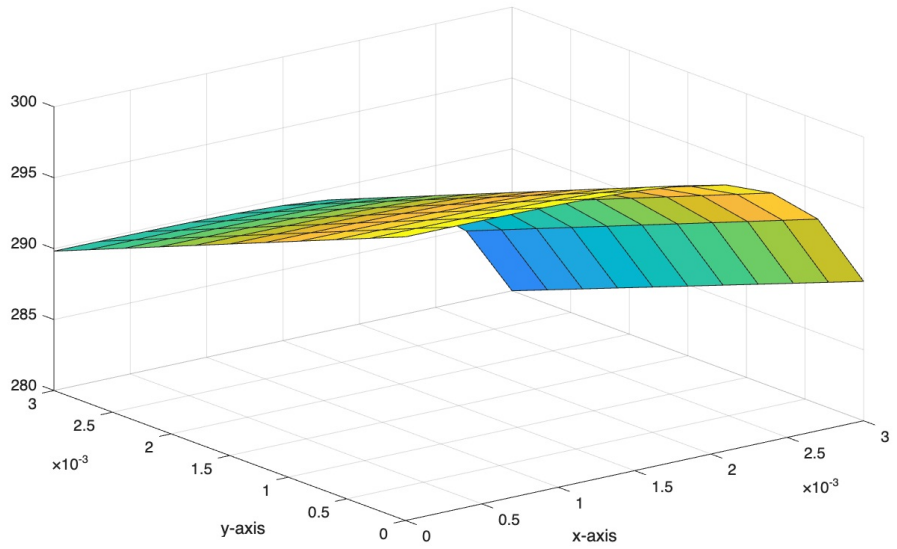


Figure A2. Reverse case for $c = 0.05$ and $C = 1.1$.

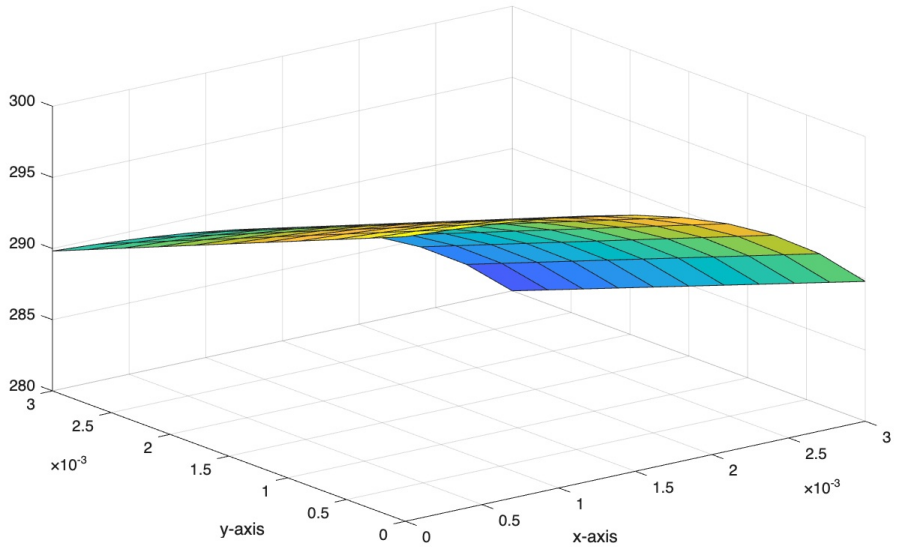


Figure A3. Direct case for $c = 0.1$ and $C = 1.1$.

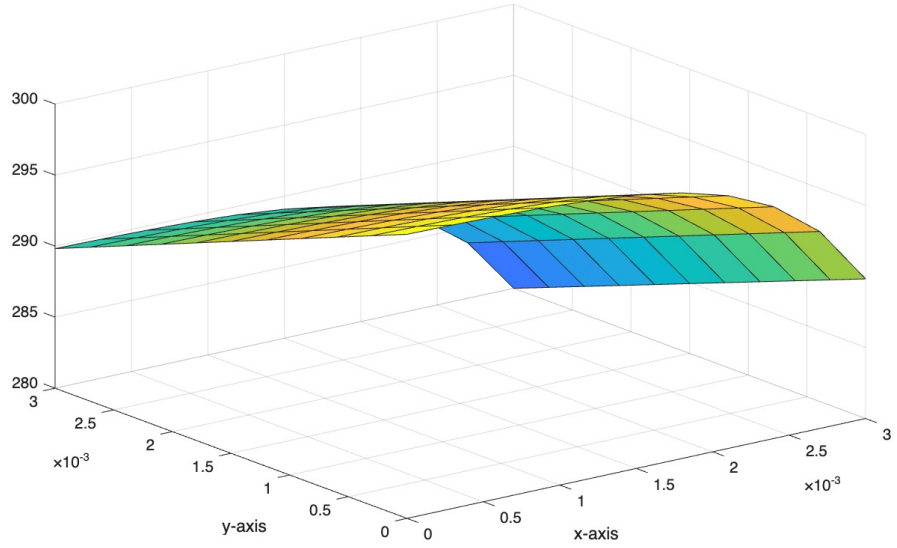


Figure A4. Reverse case for $c = 0.1$ and $C = 1.1$.

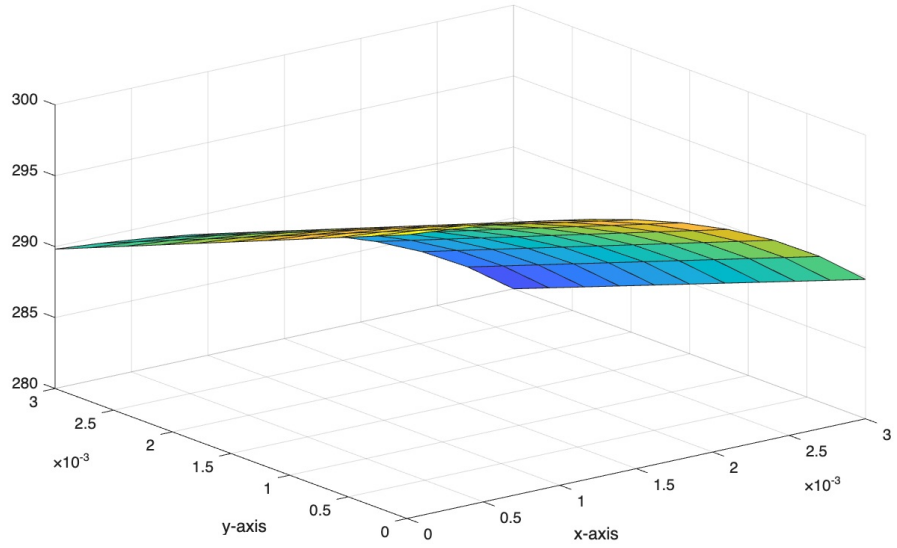


Figure A5. Direct case for $c = 0.15$ and $C = 1.1$.

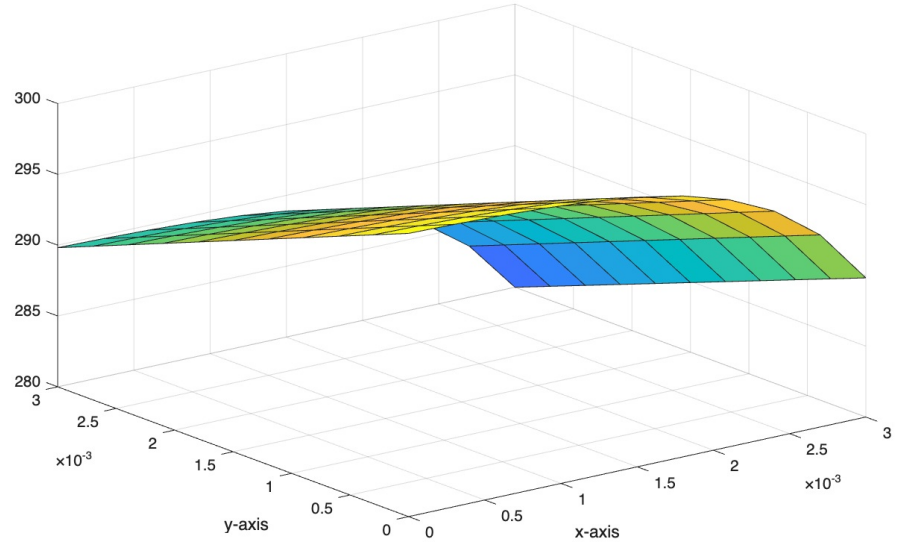


Figure A6. Reverse case for $c = 0.15$ and $C = 1.1$.

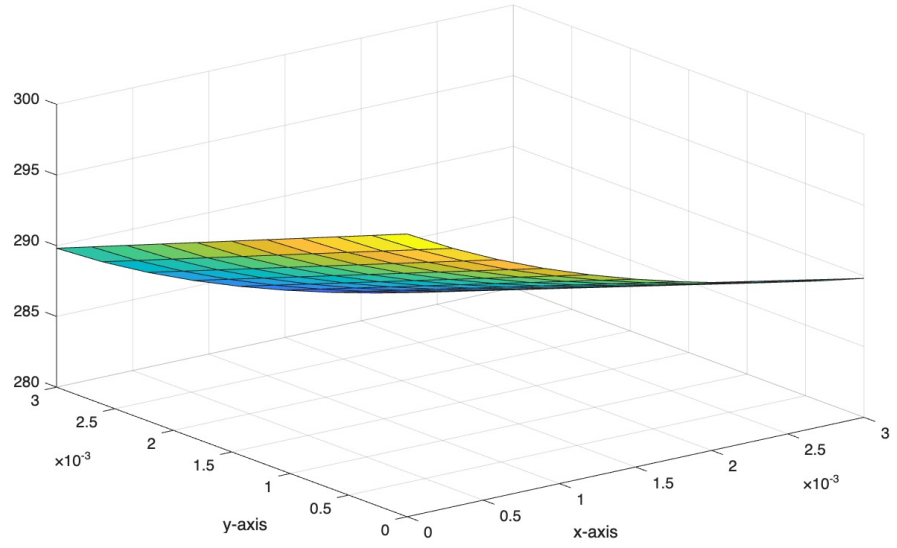


Figure A7. Direct case for $c = 0.25$ and $C = 1.1$.

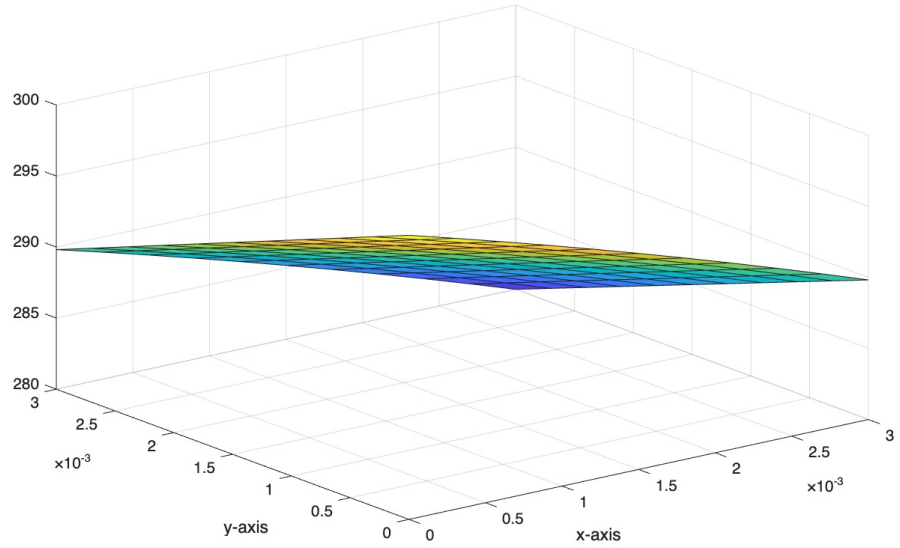


Figure A8. Reverse case for $c = 0.25$ and $C = 1.1$.

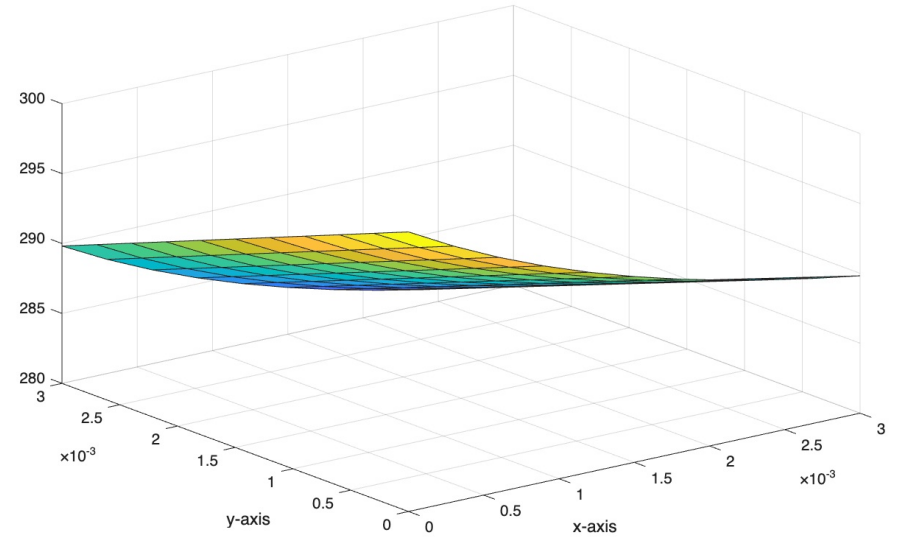


Figure A9. Direct case for $c = 0.4$ and $C = 1.1$.

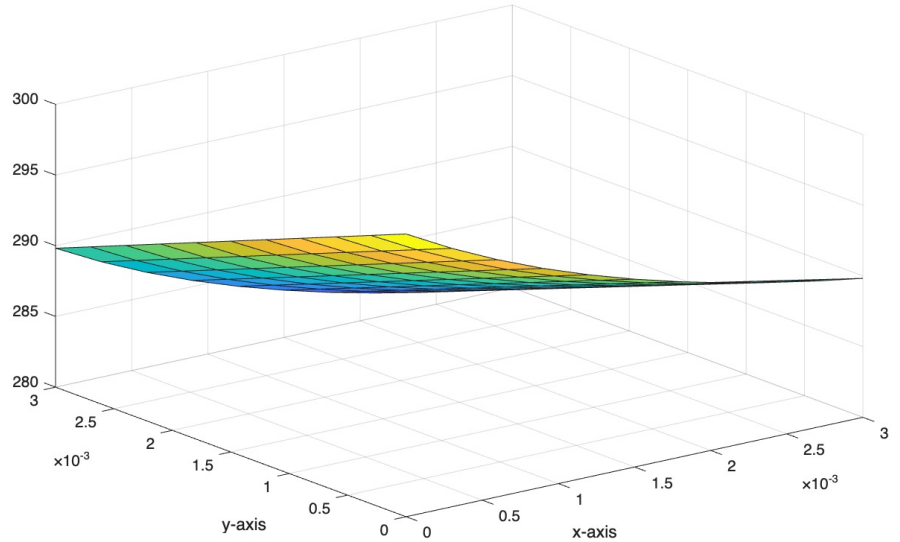


Figure A10. Reverse case for $c = 0.4$ and $C = 1.1$.

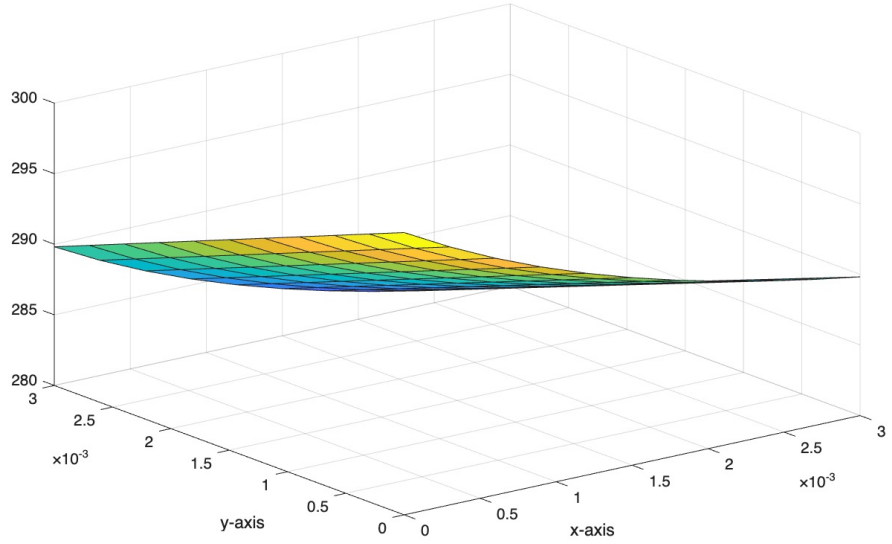


Figure A11. Direct case for $c = 0.5$ and $C = 1.1$.

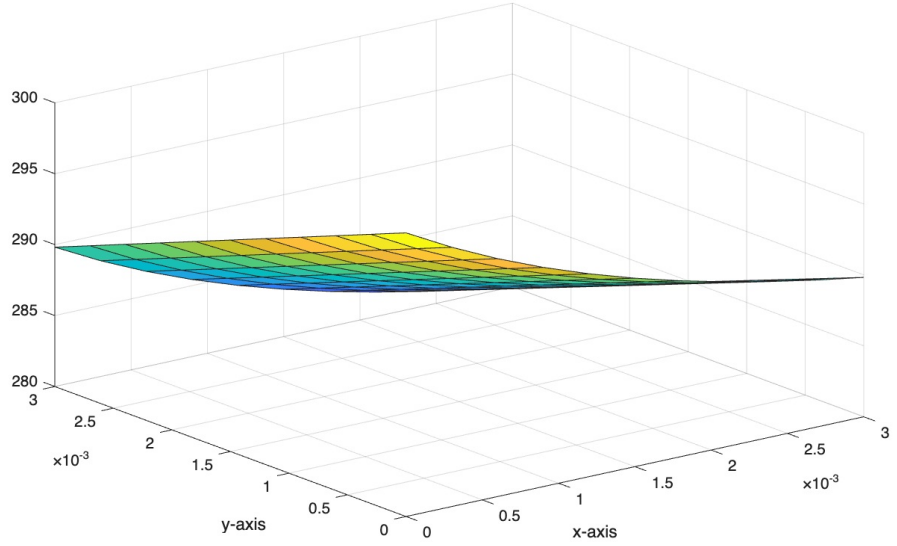


Figure A12. Reverse case for $c = 0.5$ and $C = 1.1$.

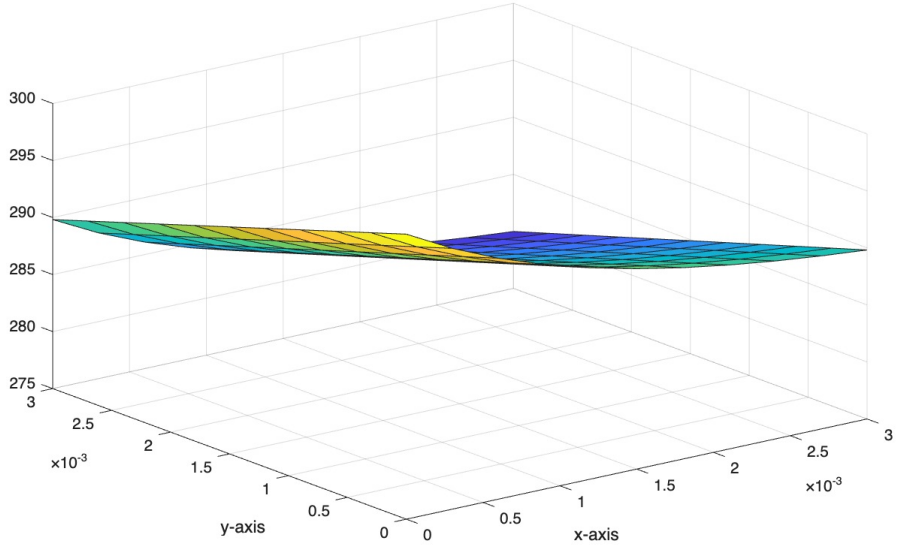


Figure A13. Direct case for $c = 0.59$ and $C = 1.1$.

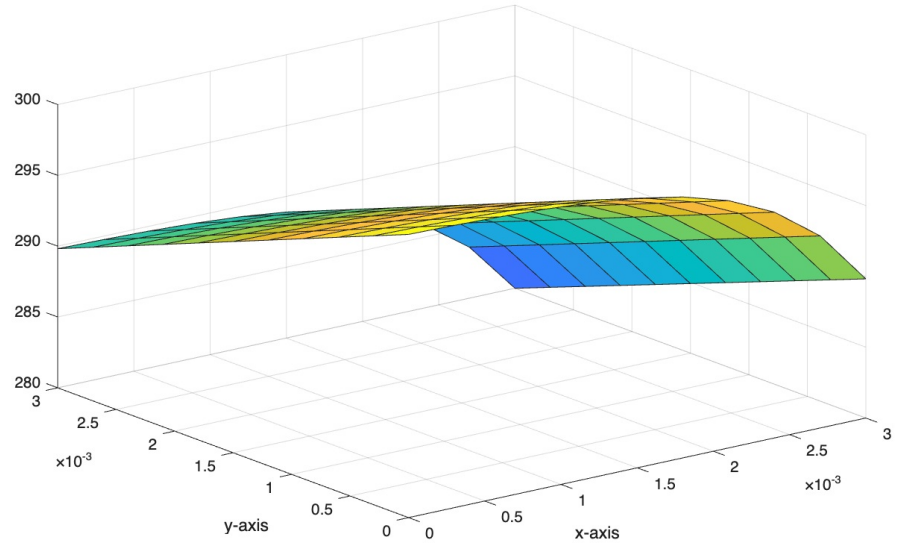


Figure A14. Reverse case for $c = 0.59$ and $C = 1.1$.

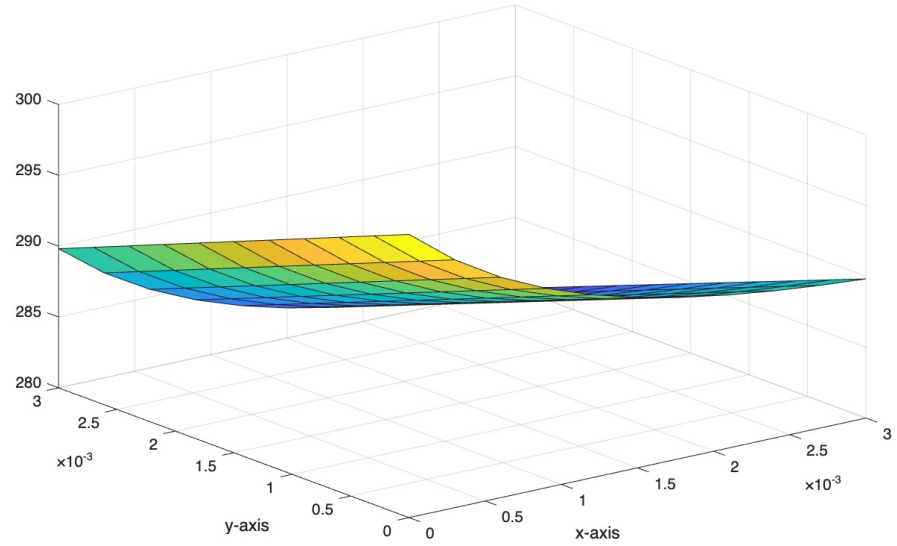


Figure A15. Direct case for $c = 0.65$ and $C = 1.1$.

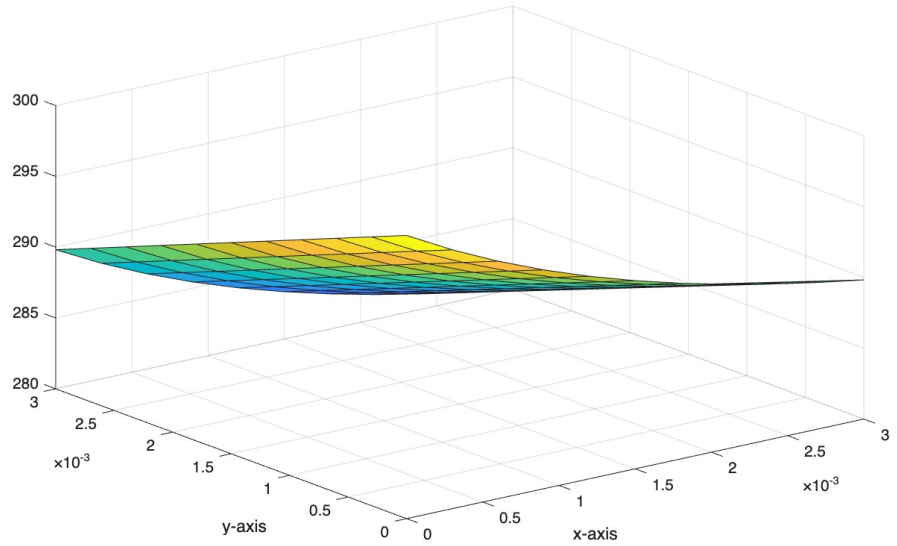


Figure A16. Reverse case for $c = 0.65$ and $C = 1.1$.

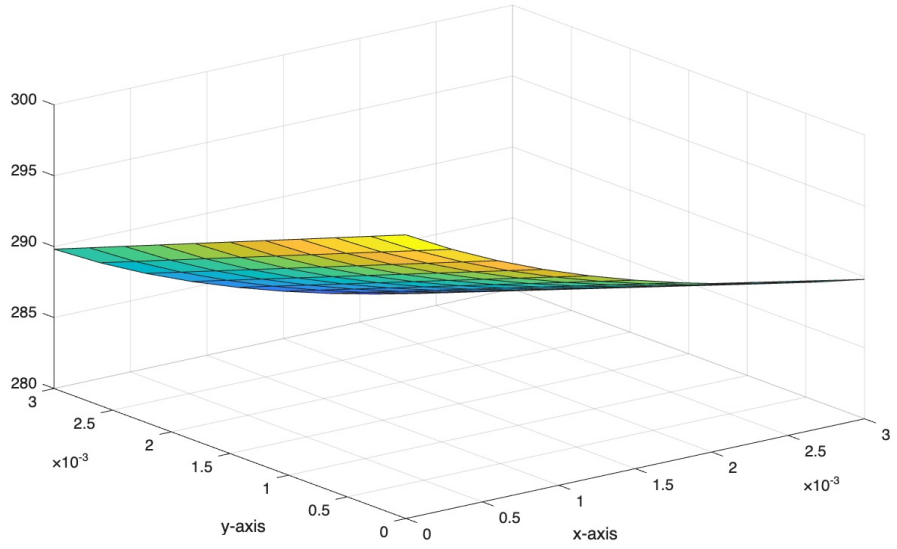


Figure A17. Direct case for $c = 0.75$ and $C = 1.1$.

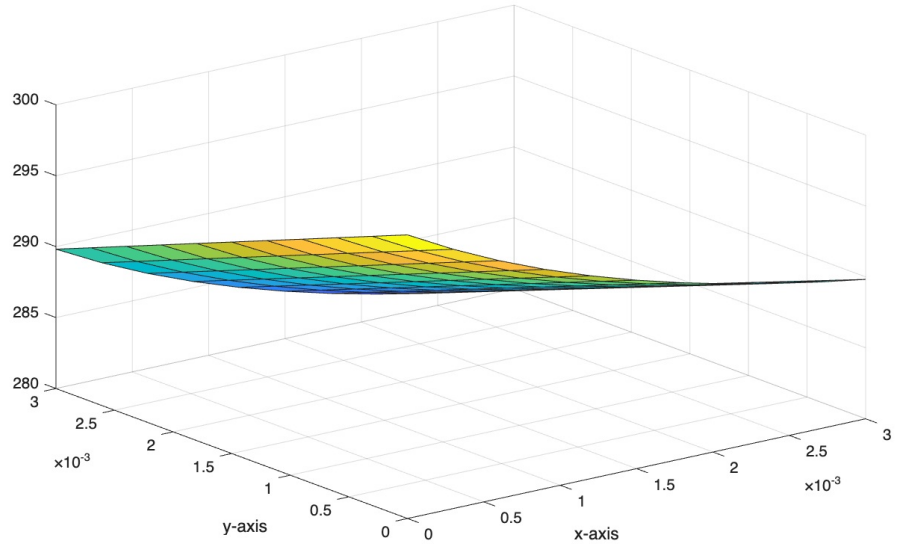


Figure A18. Reverse case for $c = 0.75$ and $C = 1.1$.

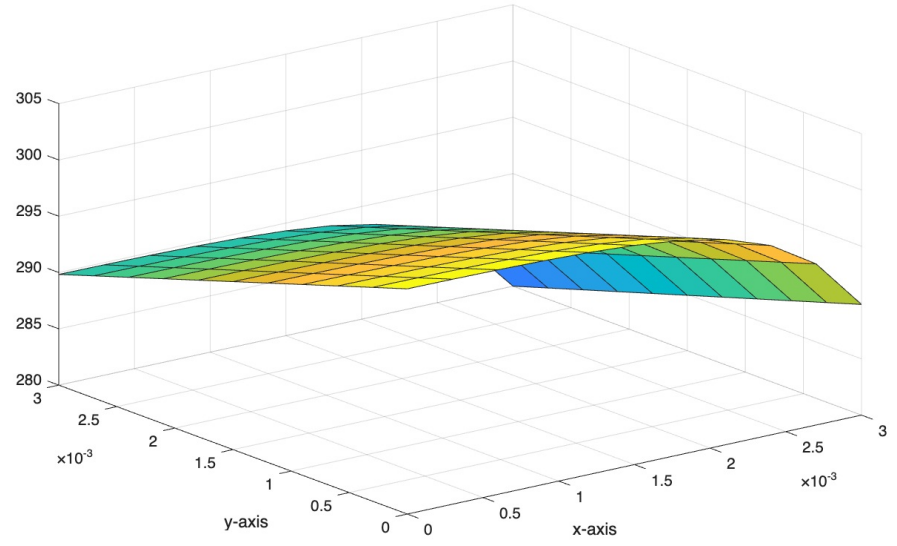


Figure A19. Direct case for $c = 0.05$ and $C = 0.4$.

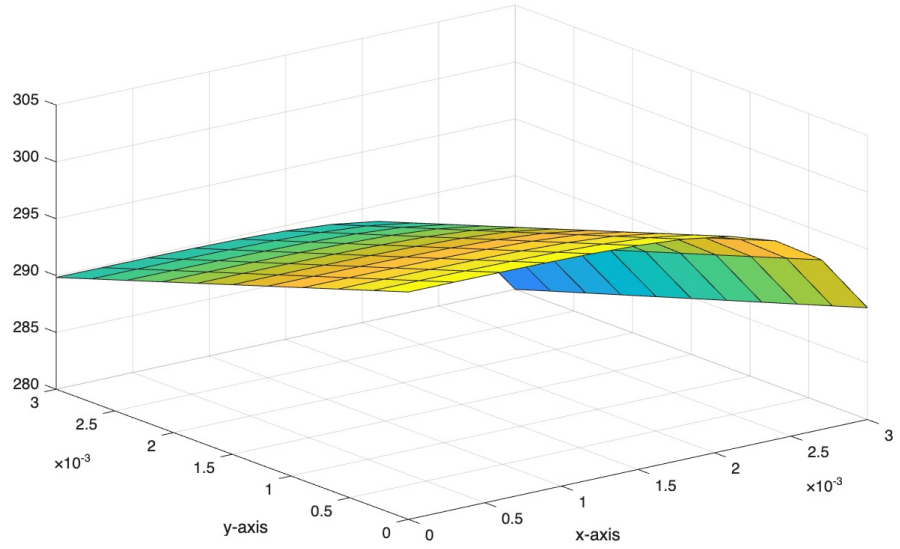


Figure A20. Reverse case for $c = 0.05$ and $C = 0.4$.

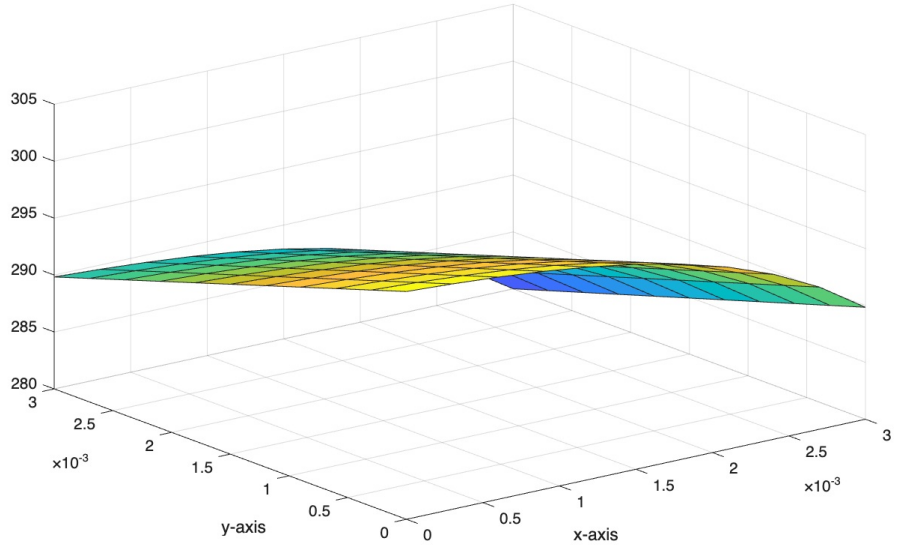


Figure A21. Direct case for $c = 0.1$ and $C = 0.4$.

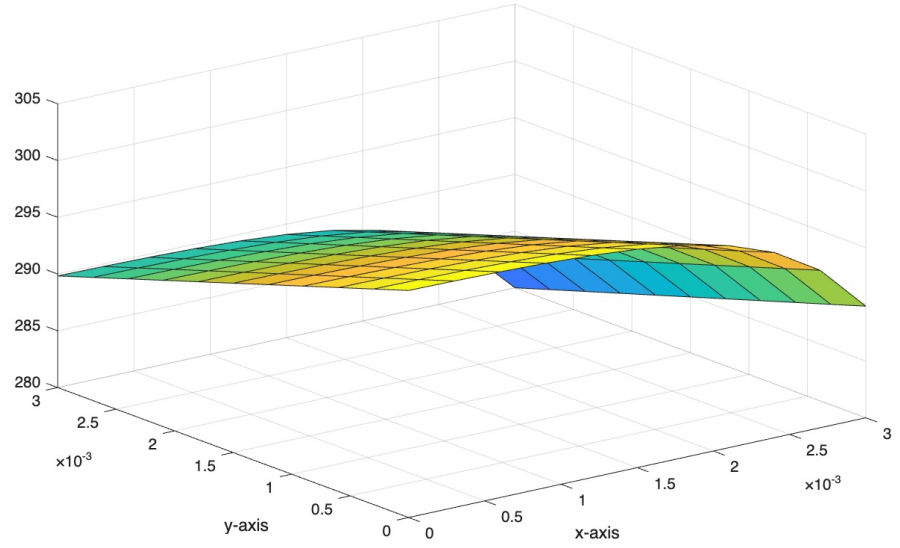


Figure A22. Reverse case for $c = 0.1$ and $C = 0.4$.

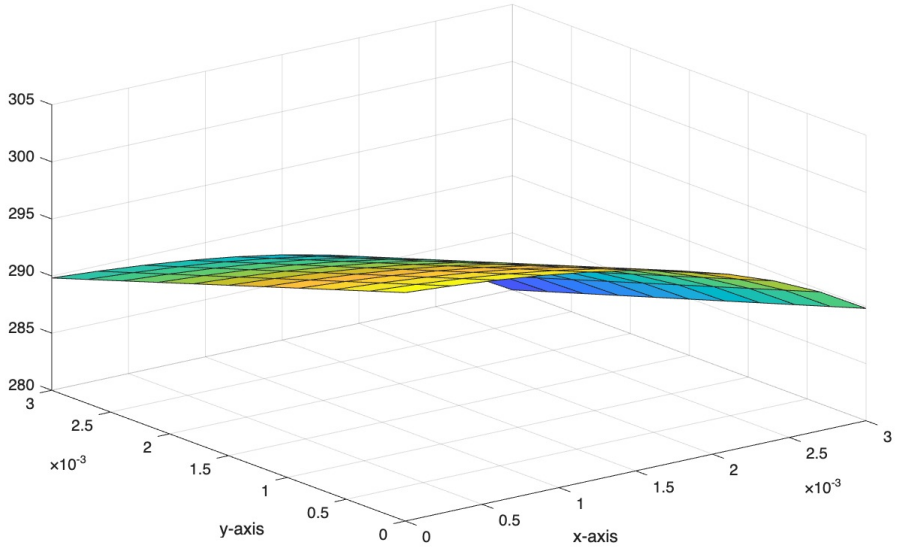


Figure A23. Direct case for $c = 0.15$ and $C = 0.4$.

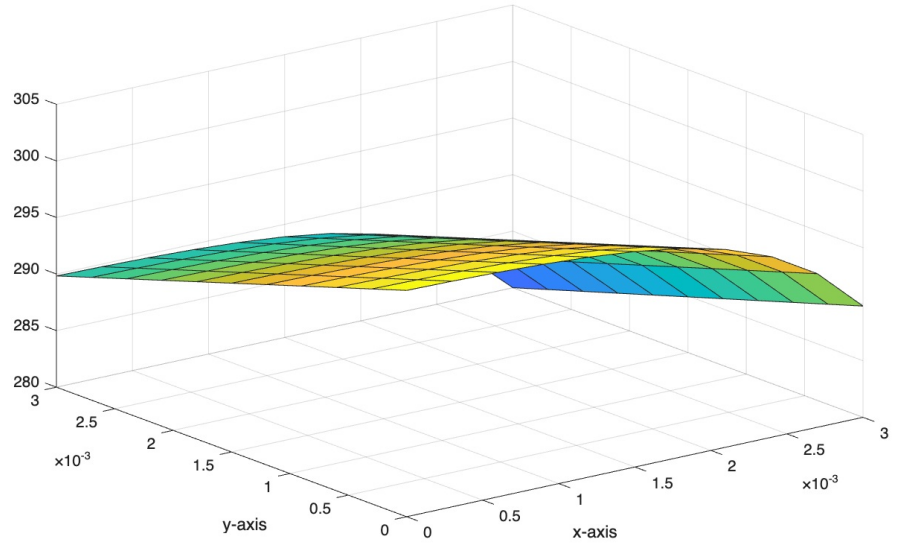


Figure A24. Reverse case for $c = 0.15$ and $C = 0.4$.

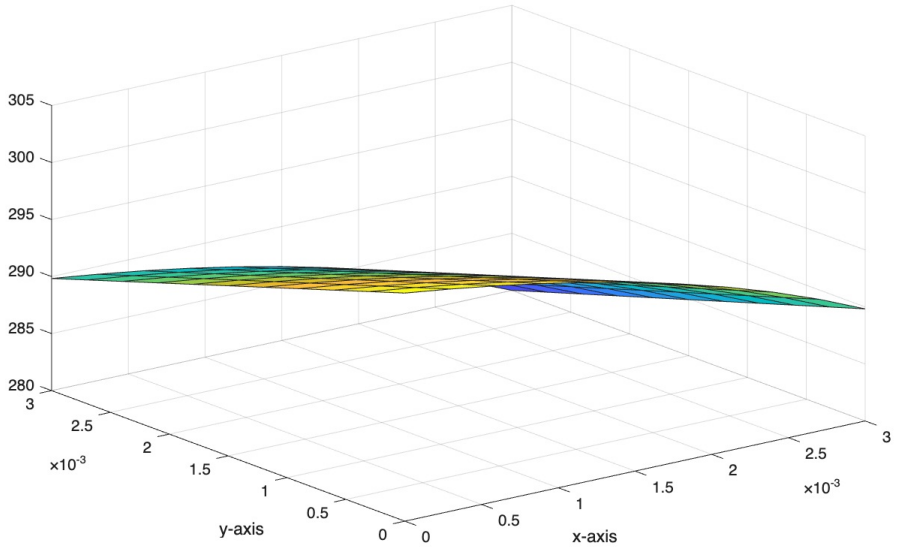


Figure A25. Direct case for $c = 0.25$ and $C = 0.4$.

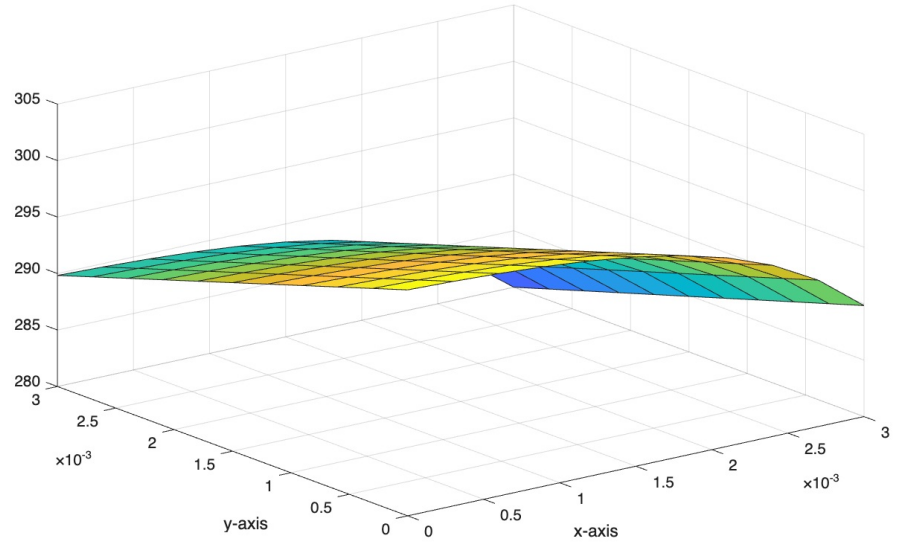


Figure A26. Reverse case for $c = 0.25$ and $C = 0.4$.

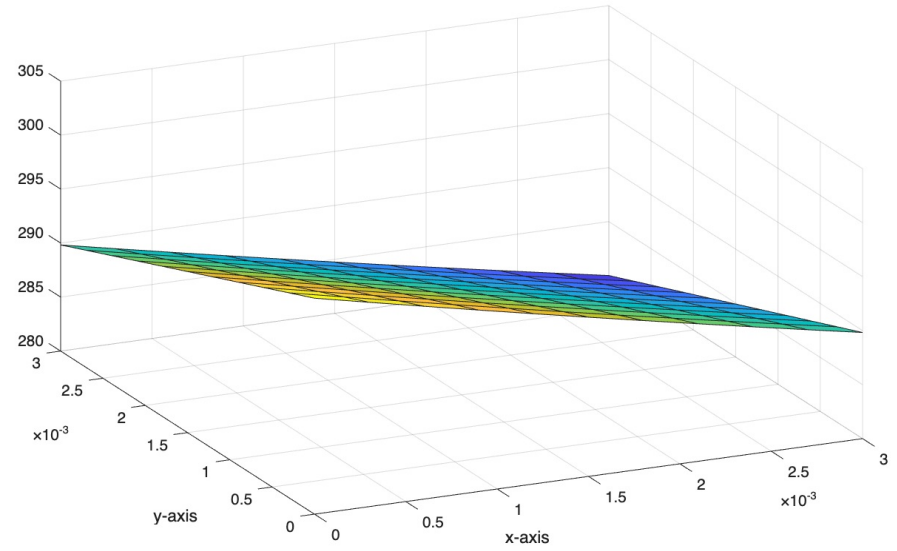


Figure A27. Direct case for $c = 0.4$ and $C = 0.4$.

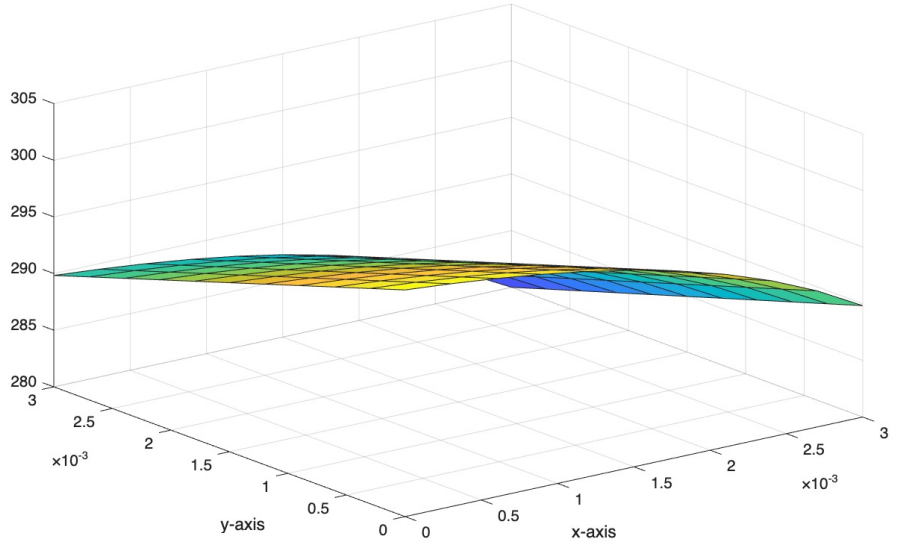


Figure A28. Reverse case for $c = 0.4$ and $C = 0.4$.

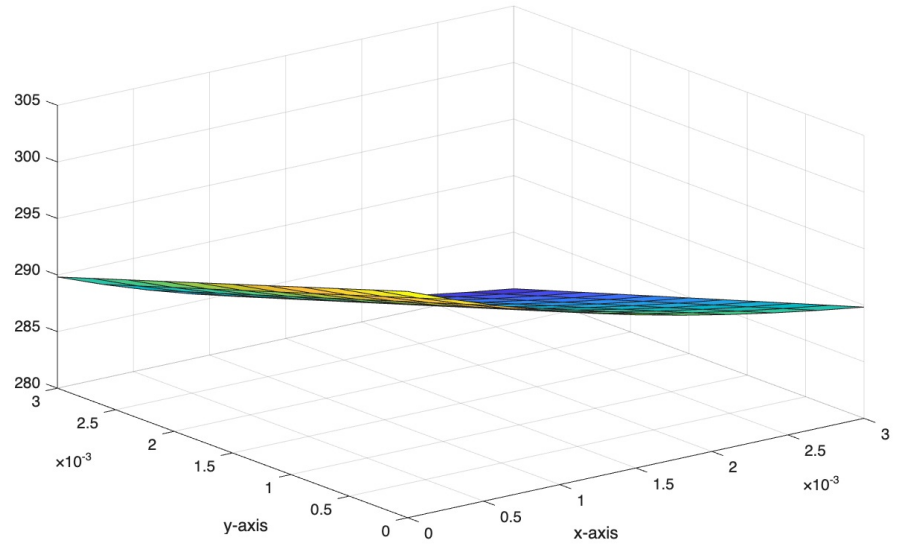


Figure A29. Direct case for $c = 0.5$ and $C = 0.4$.

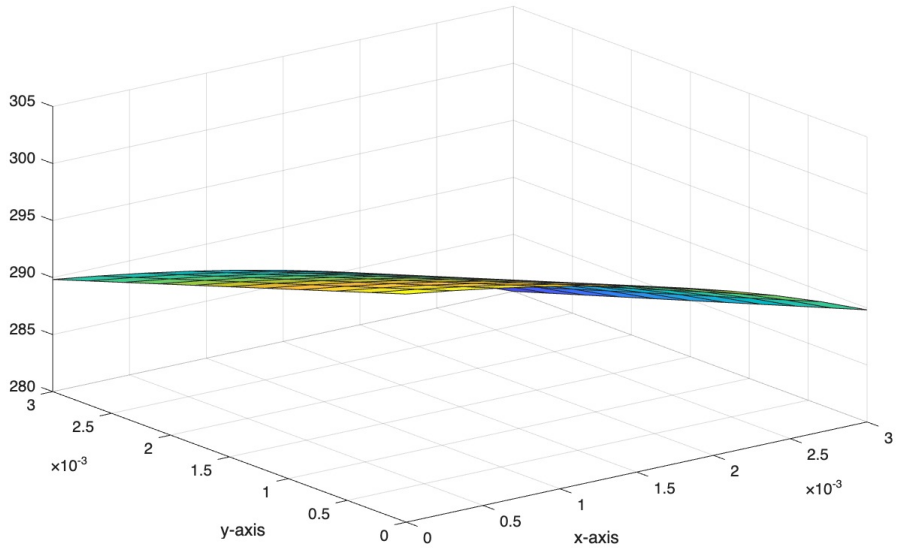


Figure A30. Reverse case for $c = 0.5$ and $C = 0.4$.

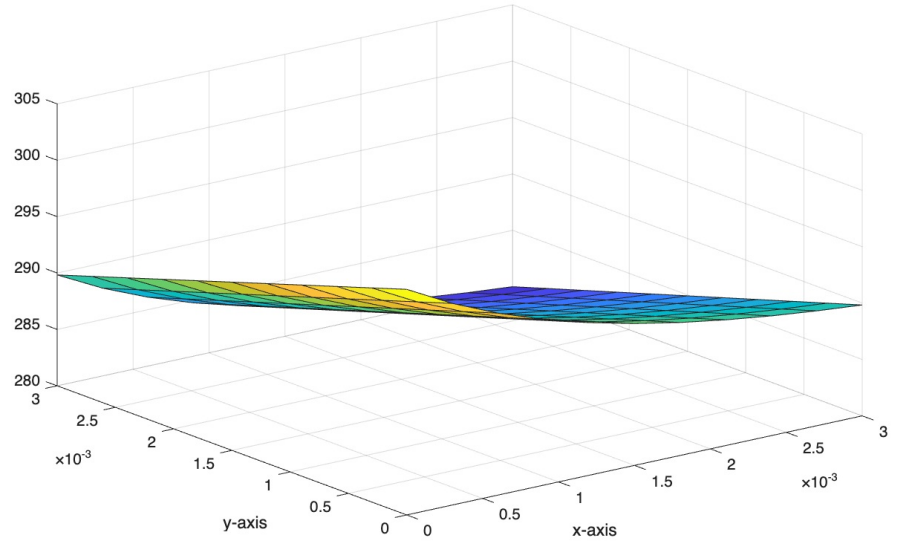


Figure A31. Direct case for $c = 0.59$ and $C = 0.4$.

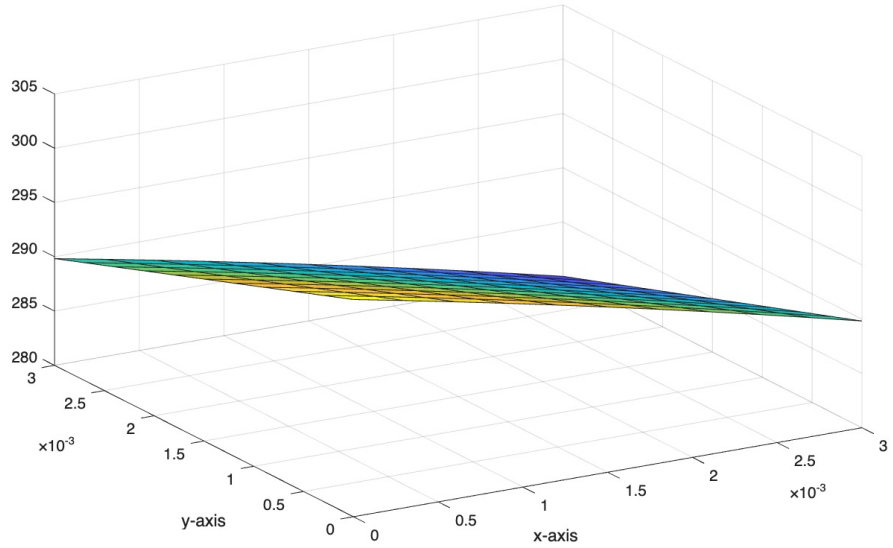


Figure A32. Reverse case for $c = 0.59$ and $C = 0.4$.

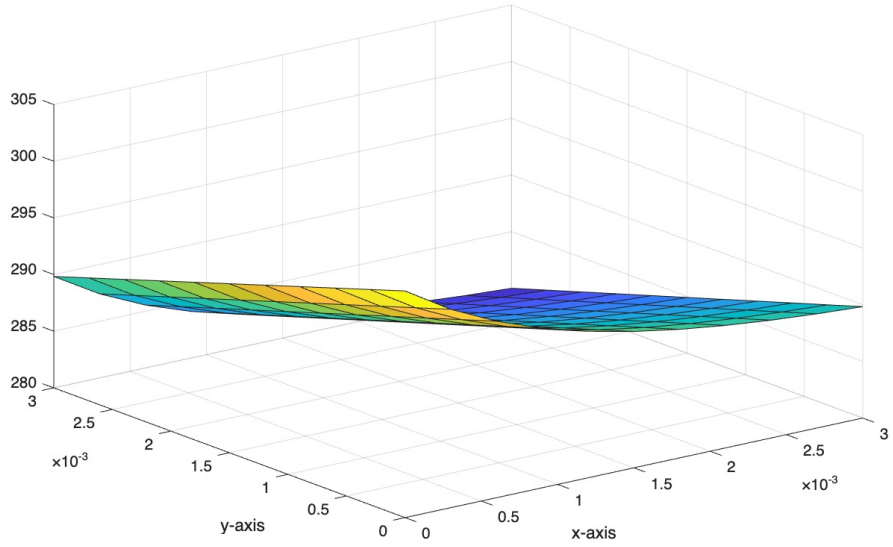


Figure A33. Direct case for $c = 0.65$ and $C = 0.4$.

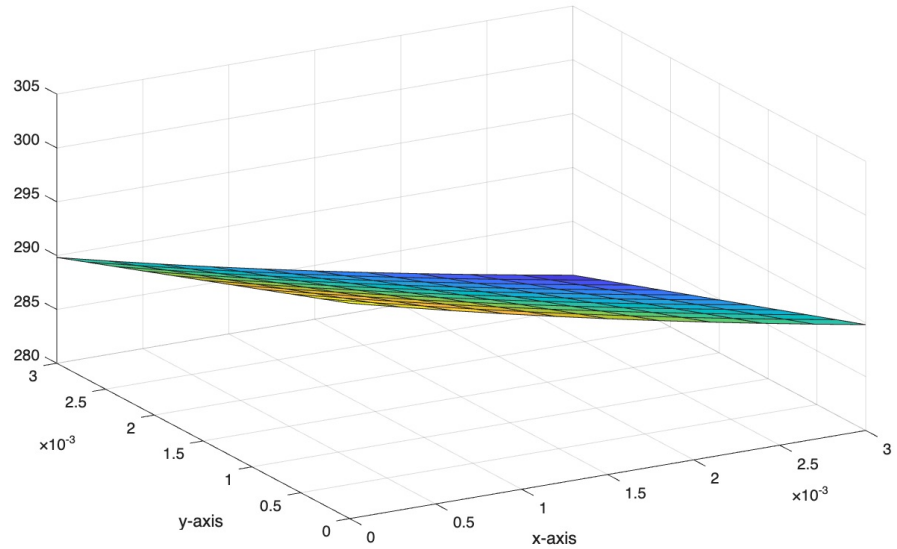


Figure A34. Reverse case for $c = 0.65$ and $C = 0.4$.

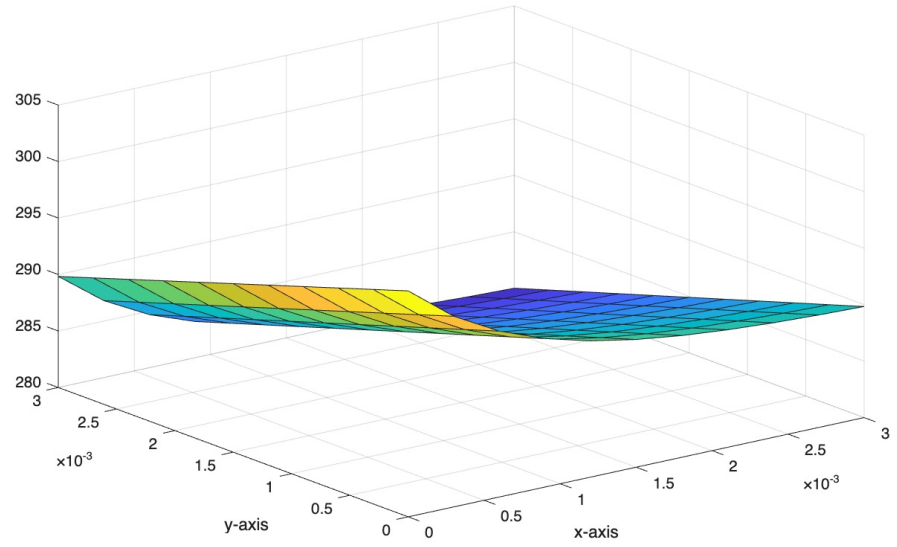


Figure A35. Direct case for $c = 0.75$ and $C = 0.4$.

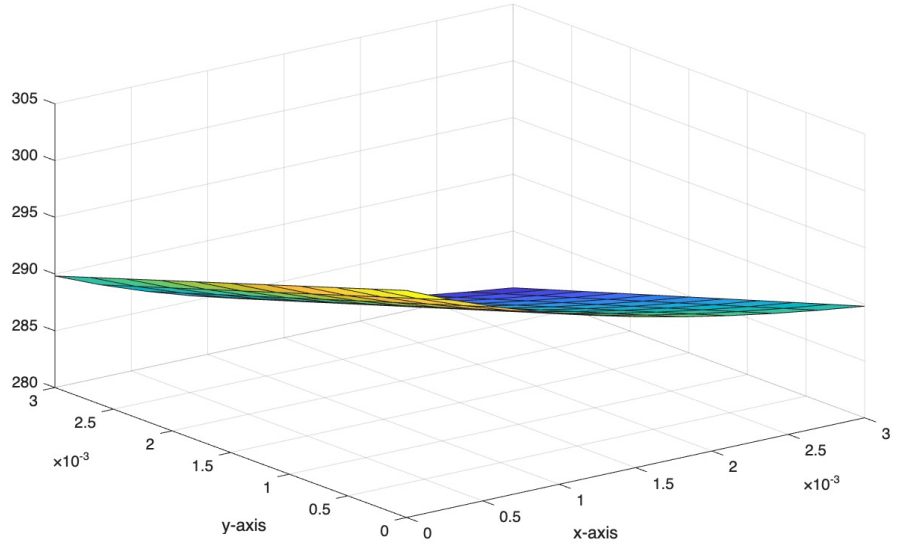


Figure A36. Reverse case for $c = 0.75$ and $C = 0.4$.

References

1. Jou, D.; Casas-Vázquez, J.; Lebon, G. *Extended Irreversible Thermodynamics*, 4th ed.; Springer: Berlin/Heidelberg, Germany, 2010. [CrossRef]
2. Dong, Y. *Dynamical Analysis of Non-Fourier Heat Conduction and Its Application in Nanosystems*; Springer: Berlin/Heidelberg, Germany, 2016. [CrossRef]
3. Lebon, G. Heat conduction at micro and nanoscales: A review through the prism of Extended Irreversible Thermodynamics. *J. Non-Equilib. Thermodyn.* **2014**, *39*, 35–59. [CrossRef]
4. Roberts, N.A.; Walker, D.G. A review of thermal rectification observations and models in solid materials. *Int. J. Therm. Sci.* **2011**, *50*, 648–662. [CrossRef]
5. Wong, M.Y.; Tso, C.Y.; Ho, T.C.; Lee, H.H. A review of state of the art thermal diodes and their potential applications. *Int. J. Heat Mass Transf.* **2021**, *164*, 120667. [CrossRef]
6. Wehmeyer, G.; Yabuki, T.; Monachon, C.; Wu, J.; Dames, C. Thermal diodes, regulators, and switches: Physical mechanisms and potential applications. *Appl. Phys. Rev.* **2017**, *2*, 041304. [CrossRef]
7. Maldovan, M. Sound and heat revolutions in phononics. *Nature* **2013**, *503*, 209–217. [CrossRef] [PubMed]
8. Machrafi, H.; Lebon, G.; Jou, D. Thermal rectifier efficiency of various bulk-nanoporous silicon devices. *Int. J. Heat Mass Transf.* **2016**, *97*, 603–610. [CrossRef]
9. Carlomagno, I.; Cimmelli, V.A.; Jou, D. Enhanced thermal rectification in graded $Si_c\,Ge_{1-c}$ alloys. *Mech. Res. Commun.* **2020**, *103*, 103472. [CrossRef]

10. Carlomagno, I.; Cimmelli, V.A.; Jou, D. Tunable Heat-Flux Rectification in Graded Nanowires in Non-Linear Guyer-Krumhansl Regime. *Nanomaterials* **2023**, *13*, 1442. [CrossRef] [PubMed]
11. Glassbrenner, C.; Slack, G. Thermal conductivity of Silicon and Germanium from 3° K to the melting point. *Phys. Rev.* **1964**, *134*, 1058–1069. [CrossRef]
12. Steele, M.; Rosi, F. Thermal conductivity and thermoelectric power of germanium-silicon alloys. *J. Appl. Phys.* **1958**, *29*, 1517–1520. [CrossRef]
13. Abeles, B.; Beers, D.; Cody, G.; Dismukes, J. Thermal conductivity of Ge-Si alloys at high temperatures. *Phys. Rev.* **1962**, *125*, 44–46. [CrossRef]
14. Rogolino, P.; Cimmelli, V.A. Fitting thermal conductivity and optimizing thermoelectric efficiency in Si_cGe_{1-c} nanowires. *Math. Comp. Simul.* **2020**, *176*, 279–291. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.