On some approximation processes generated by integrated means on noncompact real intervals

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Dedicated to Professor Mircea Ivan on the occasion of his 70th anniversary

Abstract

The purpose of the present paper is to carry out a detailed study of a sequence of positive linear operators acting on continuous function spaces on an arbitrary real interval and constructed by means of (Borel) integrated means with respect to two families of probability Borel measures on the underlying interval and a positive real parameter. The study is mainly focused on their approximation properties in weighted spaces of continuous functions with respect to wide classes of weights. Pointwise estimates as well as weighted norm estimates are also established. In the final section a weighted asymptotic formula is obtained.

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Introduction

In the papers [9], [10], [14] the authors introduced and studied a wide class of positive linear operators acting on spaces of continuous functions defined on a general convex compact subset of a locally convex space. Their construction is carried out in terms of (Borel) integrated means with respect to two families of probability Borel measures on the underlying domain and a positive real parameter. In the finite dimensional setting and for a special choice of the involved parameters, these operators turn into the classical Kantorovich operators and this was the reason why the authors referred to them as generalized Kantorovich operators. Furthermore, this class of operators extends the one of Bernstein-Schnabl operators, widely studied, e.g., in [3], [8] and the references therein, and, together with them, they share a nonmarginal relevance in facing, within such general domains, some approximation problems for continuous functions as well as for the solutions of special classes of initial-boundary value differential problems.

Very recently, in the paper [7] we used similar (Bochner) integrated means in Banach space settings in order to achieve some representation/approximation formulae for strongly continuous operator semigroups acting on such spaces. The method we employed led us to consider the following sequence of positive linear operators defined by

$$C_n(f)(x) := \int_J \cdots \int_J f\left(\frac{x_1 + \ldots x_n + r x_{n+1}}{n+r}\right) d\mu_x(x_1) \ldots d\mu_x(x_n) d\mu_n(x_{n+1}),$$

where J is an arbitrary interval, $x \in J$, $n \geq 1$, $r \geq 0$, μ_x and μ_n are probability Borel measures on J and f is a continuous real-valued function on J with at most quadratic growth. In the case where J is a compact interval, these operators are just the generalized Kantorovich operators we have mentioned previously. Moreover, for $r = 0$, they turn into the Bernstein-Schnabl operators on noncompact intervals already studied in [11].

Actually, in the paper [7] the operators C_n, $n \geq 1$, have an ancillary role and we limited ourselves to investigate very few approximation properties of them. In the present paper we carry out a more detailed study of such operators because they reveal to have an interest on their own in approximating wide classes of real-valued functions defined on noncompact intervals. Moreover, according to the particular examples we exhibit in Examples 2.2, the operators C_n, $n \geq 1$, generalize, by means of an unifying approach, several well known approximation processes on noncompact intervals, such as Szász-Mirakjan operators, Baskakov operators, Post-Widder operators, and Gauss-Weierstrass operators.

We mainly focus in studying their approximation properties in weighted function spaces of continuous functions on J with respect to wide classes of weights. We also establish pointwise estimates for uniformly continuous bounded functions as well as with respect to weighted norms.

Due to the generality of the parameters involved in the definition, the operators C_n, $n \geq 1$, can be also used for approximating p-fold integrable functions ($1 \leq p < +\infty$) and we reserve such possible development in a forthcoming paper.

In the final section we establish a weighed asymptotic formula, which could be possibly used in studying some classes of evolution equations on noncompact intervals along the directions shown, e.g., in [11] and the reference therein. This also will be subject of further investigations.

1 Notations and preliminary results

Throughout the paper we shall denote by J an arbitrary noncompact real interval with endpoints $r_1 := \inf J \in \mathbb{R} \cup \{-\infty\}$ and $r_2 := \sup J \in \mathbb{R} \cup \{+\infty\}$.

Let $\mathcal{F}(J, \mathbb{R})$ be the space of all real valued functions defined on J. As usual we shall denote by $C(J)$ (resp., $C_b(J)$) the space of all real valued continuous (resp., continuous and bounded) functions on J. The space $C_b(J)$, endowed with the natural (pointwise) order and the sup-norm $\|\cdot\|_\infty$, is a Banach lattice. We shall also consider the (closed) subspaces of $C_b(J)$

$$C_0(J) := \{f \in C(J) | \lim_{x \to r_i} f(x) = 0 \text{ whenever } r_i \notin J, i = 1, 2\},$$

$$C_*(J) := \{f \in C(J) | \lim_{x \to r_i} f(x) \in \mathbb{R} \text{ whenever } r_i \notin J, i = 1, 2\}.$$

Observe that a function $f \in C(J)$ belongs to $C_0(J)$ if for every $\varepsilon > 0$ there exists a compact subset K of J such that $|f(x)| \leq \varepsilon$ for every $x \in J \setminus K$.

Moreover we shall consider the space $\mathcal{K}(J)$ of all real valued continuous functions $f : J \to \mathbb{R}$ whose support $\mathrm{Supp}(f)$ is compact in J (here $\mathrm{Supp}(f) := \overline{\{x \in J | f(x) \neq 0\}}$). We observe that $\mathcal{K}(J)$ is dense in $C_0(J)$ and, if J is compact, $\mathcal{K}(J) = C(J)$.

We also recall that the symbol $UC_b(J)$ (resp., $UC_b^2(J)$) stands for the space of all uniformly continuous and bounded functions on J (resp., the space of all twice differentiable functions on J with uniformly continuous and bounded second-order derivative).

A bounded weight on J is a function $w \in C_b(J)$ such that $w(x) > 0$ for every $x \in J$. Then the symbol $C_b^w(J)$ (resp., $C_0^w(J)$) will stand for the Banach lattice of all functions $f \in C(J)$ such that $wf \in C_b(J)$ (resp.,

$wf \in C_0(J)$) endowed with the natural order and the weighted norm $\|\cdot\|_w$ defined by $\|f\|_w := \|wf\|_\infty$ $(f \in C_b^w(J))$.

Clearly, $C_b(J) \subset C_b^w(J)$ and $\|\cdot\|_w \le \|w\|_\infty \|\cdot\|_\infty$ on $C_b(J)$. In particular, if $w \in C_0(J)$, then $C_b(J) \subset C_0^w(J)$. Moreover, the space $C_0(J)$ is dense in $C_0^w(J)$ and, if $w \in C_0(J)$, then $C_*(J)$ is dense in $C_0^w(J)$ as well.

Now let $\mathcal{B}(J)$ be the σ-algebra of all Borel subsets of J and denote by $\mathcal{M}^+(J)$ (resp., $\mathcal{M}_b^+(J)$, $\mathcal{M}_1^+(J)$) the cone of all Borel (resp., bounded, probability Borel) measures on J. If $\mu \in \mathcal{M}^+(J)$, we shall denote by $\mathcal{L}^1(J,\mu)$ the space of all Borel measurable functions such that $\|f\|_1 := \int_J |f|\, d\mu < +\infty$. We also denote by λ_1 the Borel-Lebesgue measure on J and, for every $x \in J$, by ε_x the point-mass measure concentrated at x, i.e., for every $B \in \mathcal{B}(J)$,

$$\varepsilon_x(B) := \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

The symbol $\mathbf{1}$ denotes the constant function with constant value 1. Furthermore, for every $k \in \mathbb{N}$ and $x \in J$, we shall set

$$e_k(t) := t^k \quad \text{and} \quad \psi_x(t) := t - x \ \ (t \in J). \tag{1.1}$$

Finally note that, if $\mu \in \mathcal{M}_1^+(J)$ and $e_k \in \mathcal{L}^1(J,\mu)$ for some $k \ge 2$, then $e_h \in \mathcal{L}^1(J,\mu)$ for every $1 \le h \le k$ because $|e_h| \le 1 + |e_k|$.

2 Positive approximation processes generated by integrated means on noncompact real intervals

In this section we introduce the main object of study of the paper. We begin by presenting some preliminaries.

A *continuous selection of probability Borel measures on* J is a family $(\mu_x)_{x\in J}$ of probability Borel measures on J such that, for every $f \in C_b(J)$, the function $U(f)$ defined by

$$U(f)(x) := \int_J f\, d\mu_x \quad (x \in J) \tag{2.1}$$

is continuous on J. If such a selection is assigned, for every $n \ge 1$ and $x \in J$, the symbols μ_x^n and $\mu_{x,n}$ will stand, respectively, for the product measure on J^n of μ_x with itself n-times and for the image measure of μ_x^n under the mapping $\pi_n : J^n \to J$ defined by

$$\pi_n(x_1,\dots,x_n) := \frac{x_1 + \dots + x_n}{n} \quad ((x_1,\dots,x_n) \in J^n)\,. \tag{2.2}$$

Moreover, extending formula (2.1), if $f \in \bigcap_{x\in J} \mathcal{L}^1(J,\mu_x)$ we continue to denote by $U(f)$ the function

$$U(f)(x) := \int_J f\, d\mu_x \quad (x \in J). \tag{2.3}$$

From now on, we shall fix a real number $r \geq 0$, a continuous selection $(\mu_x)_{x\in J}$ of probability Borel measures on J satisfying, for each $x \in J$,

$$e_1 \in \mathcal{L}^1(J,\mu_x) \quad \text{and} \quad \int_J e_1\, d\mu_x = x,$$

and a sequence $(\mu_n)_{n\geq 1}$ of probability Borel measures on J.
For every $n \geq 1$ and $x \in J$, let $\lambda_{x,n,r}$ be the image measure of $\mu_x^n \otimes \mu_n$ under the mapping $\sigma_{n,r} : J^{n+1} \to J$ defined, for every $(x_1,\ldots,x_{n+1}) \in J^{n+1}$, by

$$\sigma_{n,r}(x_1,\ldots,x_{n+1}) = \frac{n}{n+r}\pi_n(x_1,\ldots,x_n)+\frac{r}{n+r}x_{n+1} = \frac{x_1+\ldots+x_n+rx_{n+1}}{n+r}$$

(see (2.2)). Then, the positive linear operators $(C_n)_{n\geq 1}$ we are interested in studying are defined by setting, for every $f \in \bigcap_{n\geq 1, x\in J} \mathcal{L}^1(J,\lambda_{x,n,r})$, $n \geq 1$, and $x \in J$,

$$\begin{aligned} C_n(f)(x) &:= \int_J f\, d\lambda_{x,n,r} \\ &= \int_J \int_{J^n} f\left(\frac{x_1+\ldots+x_n+rx_{n+1}}{n+r}\right) d\mu_x^n(x_1,\ldots,x_n)\, d\mu_n(x_{n+1}) \\ &= \int_J \cdots \int_J f\left(\frac{x_1+\ldots+x_n+rx_{n+1}}{n+r}\right) d\mu_x(x_1)\cdots d\mu_x(x_n)\, d\mu_n(x_{n+1}). \end{aligned} \tag{2.4}$$

For the sake of simplicity we shall restrict the operators C_n, $n \geq 1$, to the subspace of those functions $f \in C(J)$ such that $f \in \bigcap_{n\geq 1, x\in J} \mathcal{L}^1(J,\lambda_{x,n,r})$ and $C_n(f) \in C(J)$ for every $n \geq 1$. Such functions will be referred to as admissible functions with respect to the selection $(\mu_x)_{x\in J}$, the sequence $(\mu_n)_{n\geq 1}$ and $r \geq 0$. The linear subspace of all of them will be denoted by $L_a(J)$. From assumption (2.1) and the continuity property of the product measure (see [15, Proposition 13.12] and [13, Theorem 30.8]) it follows that $C_b(J) \subset L_a(J)$.

We note that for $r = 0$ the above operators turn into the Bernstein-Schnabl operators B_n associated with $(\mu_x)_{x\in J}$, introduced and studied in [11]. In this particular case the space of all admissible functions will be denoted by $C_a(J)$. More specifically, we have that $\lambda_{x,n,0} = \mu_{x,n}$ and, for any $f \in \bigcap_{n\geq 1, x\in J} \mathcal{L}^1(J, \mu_{x,n})$, $n \geq 1$ and $x \in J$,

$$\begin{aligned} B_n(f)(x) &:= \int_J f\, d\mu_{x,n} = \int_{J^n} f\left(\frac{x_1 + \ldots + x_n}{n}\right) d\mu_x^n(x_1, \ldots, x_n) \\ &= \int_J \cdots \int_J f\left(\frac{x_1 + \ldots + x_n}{n}\right) d\mu_x(x_1) \cdots d\mu_x(x_n). \end{aligned} \tag{2.5}$$

The operators C_n are, indeed, related to the operators B_n, as the next result shows.

Proposition 2.1. *Let $f \in L_a(J)$. Then, for every $n \geq 1$, $I_n(f) \in C_a(J)$ and*

$$C_n(f) = B_n(I_n(f)), \tag{2.6}$$

where

$$I_n(f)(x) := \int_J f\left(\frac{n}{n+r}x + \frac{r}{n+r}t\right) d\mu_n(t) \quad (x \in J).$$

Proof. Let $f \in L_a(J)$; then $f, C_n(f) \in C(J)$ and, for every $n \geq 1$ and $x \in J$, $f \circ \sigma_{n,r} \in \mathcal{L}^1(J, \mu_x^n \otimes \mu_n)$. Fix $n \geq 1$ and $x \in J$. As a consequence of Fubini-Tonelli Theorem,

(i) For every $x_1, \ldots, x_n \in J$, $f \circ \sigma_{n,r}(x_1, \ldots, x_n, \cdot) \in \mathcal{L}^1(\mu_n)$;

(ii) $\int_J f \circ \sigma_{n,r}(\cdot, \ldots, \cdot, t)\, d\mu_n(t) \in \mathcal{L}^1(\mu_x^n)$;

(iii) In (2.4) one can exchange the integration order.

From (i), by choosing $x_1 = x_2 = \ldots = x_n = x$, we get that $I_n(f)$ is well defined and, since f is continuous, it is obvious that $I_n(f) \in C(J)$. Moreover, (ii) means that $I_n(f) \circ \pi_n \in \mathcal{L}^1(\mu_x^n)$ and, hence, $I_n(f) \in \mathcal{L}^1(J, \mu_{x,n})$. From this, being n and x arbitrarily chosen, it follows that $I_n(f) \in C_a(J)$.

Finally, (iii) guarantees that (2.6) holds true. □

As explained in the Introduction, the operators C_n, $n \geq 1$, have been briefly studied in the setting of spaces of real-valued continuous functions with at most quadratic growth. In the current paper we attempt to study them in their full generality and in wider classes of weighted function spaces. A similar attempt has been carried out in [11] in the case $r = 0$, i.e., for Bernstein-Schnabl operators defined by (2.5).

Below we discuss some examples (see also [7, Section 5.2]). Due to the generality of the parameters involved in the definition, many other examples can be furnished. For additional ones in the compact framework we refer to [9, 10, 14].

Examples 2.2.

1. Let $(\alpha_p)_{p\geq 1}$ be a (finite or infinite) sequence of positive continuous functions on J such that $\sum_{p=1}^{\infty}\alpha_p = 1$ uniformly on compact subsets of J. Moreover, consider $(a_p)_{p\geq 1}$ in J and set, for every $x \in J$,

$$\mu_x := \sum_{p=1}^{\infty} \alpha_p(x)\varepsilon_{a_p}.$$

Then $(\mu_x)_{x\in J}$ is a continuous selection of probability Borel measures.

In this case, for every $f \in C_a(J)$, $n \geq 1$ and $x \in J$, we have

$$B_n(f)(x) = \sum_{p_1=1}^{\infty} \dots \sum_{p_n=1}^{\infty} \alpha_{p_1}(x)\dots\alpha_{p_n}(x) f\left(\frac{a_{p_1}+\ldots+a_{p_n}}{n}\right)$$

and hence, for every $f \in L_a(J)$,

$$C_n(f)(x) = \sum_{p_1=1}^{\infty} \dots \sum_{p_n=1}^{\infty} \alpha_{p_1}(x)\dots\alpha_{p_n}(x) \int_J f\left(\frac{a_{p_1}+\ldots+a_{p_n}}{n+r} + \frac{r}{n+r}t\right) d\mu_n(t).$$

In particular, set $J = [0,+\infty[$ and, for every $x \geq 0$, let μ_x be one of the following measures on $[0,+\infty[$:

(i) $\mu_x := \displaystyle\sum_{k=0}^{\infty} \frac{e^{-x}x^k}{k!}\varepsilon_k,$

(ii) $\mu_x := \displaystyle\frac{1}{1+x}\sum_{k=0}^{\infty}\left(\frac{x}{1+x}\right)^k \varepsilon_k.$

The corresponding operators C_n, $n \geq 1$, associated with $(\mu_x)_{x\geq 0}$ and $(\mu_n)_{n\geq 1}$ are, in case (i),

$$C_n(f)(x) := \sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\int_0^{+\infty} f\left(\frac{k+rs}{n+r}\right) d\mu_n(s), \tag{2.7}$$

for every $f \in L_a([0,+\infty[)$, $n \geq 1$ and $x \geq 0$ and, in the case (ii),

$$C_n(f)(x) := \sum_{k=0}^{\infty} \frac{1}{(1+x)^n}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k \int_0^{+\infty} f\left(\frac{k+rs}{n+r}\right) d\mu_n(s) \tag{2.8}$$

for every $f \in L_a([0,+\infty[)$, $n \geq 1$ and $x \geq 0$.

Specializing the measures μ_n, $n \geq 1$, *we obtain more specific examples. Let* $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 1}$ *be two sequences of real numbers such that* $0 \leq a_n < b_n \leq 1$ $(n \geq 1)$ *and set*

$$\mu_n = \frac{1}{b_n - a_n}\mathbf{1}_{[a_n,b_n]}\lambda_1, \tag{2.9}$$

where $\mathbf{1}_{[a_n,b_n]}$ *is the characteristic function of the interval* $[a_n, b_n]$.
Then, for $r > 0$, *operators* (2.7) *turn into*

$$\begin{aligned} C_n(f)(x) &= \frac{1}{b_n - a_n}\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\int_{a_n}^{b_n} f\left(\frac{k+rs}{n+r}\right) ds \\ &= \frac{1}{b_n - a_n}\frac{n+r}{r}e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_{\frac{k+ra_n}{n+r}}^{\frac{k+rb_n}{n+r}} f(\xi)\, d\xi. \end{aligned} \tag{2.10}$$

In particular, for $r = 1$,

$$C_n(f)(x) = \frac{n+1}{b_n - a_n}e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_{\frac{k+a_n}{n+1}}^{\frac{k+b_n}{n+1}} f(\xi)\, d\xi. \tag{2.11}$$

The above operators are strictly related to the generalization of the Szász-Mirakjan-Kantorovich operators we have introduced and studied in [6] and which are defined by

$$C_n^*(f)(x) = \frac{n}{b_n - a_n}e^{-nx}\sum_{n=0}^{\infty}\frac{(nx)^k}{k!}\int_{\frac{k+a_n}{n}}^{\frac{k+b_n}{n}} f(t)\, dt, \tag{2.12}$$

where f *ranges in a suitable function space on* $[0, +\infty[$.

Actually these last operators can be recovered by the operators (2.11) *by means of the formula* $C_n^*(f) = C_n(M_n(f))$ $(f \in L_a(J), n \geq 1)$, *where* $M_n(f)(s) = f\left(\frac{n+1}{n}s\right) \quad (s \geq 0)$.

Similarly, by choosing the sequence $(\mu_n)_{n\geq 1}$ *defined by* (2.9), *the operators* (2.8) *turn into*

$$\begin{aligned} C_n(f)(x) &= \frac{1}{b_n - a_n}\sum_{k=0}^{\infty}\frac{1}{(1+x)^n}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k\int_{a_n}^{b_n} f\left(\frac{k+rs}{n+r}\right) ds \\ &= \frac{1}{b_n - a_n}\frac{n+r}{r}\sum_{k=0}^{\infty}\frac{1}{(1+x)^n}\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k\int_{\frac{k+rb_n}{n+r}}^{\frac{k+ra_n}{n+r}} f(\xi)\, d\xi \end{aligned} \tag{2.13}$$

for every $f \in L_a([0, +\infty[)$, $n \geq 1$ *and* $x \geq 0$.

2. *For every* $x \geq 0$ *set*

$$\mu_x := \begin{cases} \varphi(x,\cdot)\lambda_1 & \text{if } x > 0; \\ \varepsilon_0 & \text{if } x = 0, \end{cases}$$

where the function $\varphi(x,\cdot)$ *is defined on* $[0,+\infty[$ *by*

$$\varphi(x,s) := \begin{cases} \frac{e^{-s/x}}{x} & \text{if } s > 0; \\ \varepsilon_0 & \text{if } s = 0. \end{cases}$$

Moreover, fix a family of measures $(\mu_n)_{n\geq 1}$ *in* $\mathcal{M}_1^+([0,+\infty[)$ *and* $r \geq 0$. *Then, for every* $f \in L_a(J)$,

$$C_n(f)(x) := \begin{cases} \frac{n^n}{x^n(n-1)!}\int_0^{+\infty} ds \int_0^{+\infty} s^{n-1}e^{-\frac{ns}{x}} f\left(\frac{ns+rt}{n+r}\right) d\mu_n(t) & \text{if } x > 0; \\ \\ \int_0^{+\infty} f\left(\frac{rt}{n+r}\right) d\mu_n(t) & \text{if } x = 0. \end{cases}$$

If, for instance, we choose $\mu_n = \varepsilon_{b_n}$, *where* $(b_n)_{n\geq 1}$ *is a sequence of positive real numbers, we have*

$$C_n(f)(x) := \begin{cases} \frac{n^n}{x^n(n-1)!}\int_0^{+\infty} s^{n-1}e^{-\frac{ns}{x}} f\left(\frac{ns+rb_n}{n+r}\right) ds & \text{if } x > 0; \\ \\ f\left(\frac{rb_n}{n+r}\right) & \text{if } x = 0. \end{cases} \tag{2.14}$$

3. *Fix* $\mu \in \mathcal{M}^+(J)$ *and consider a continuous positive function* $\varphi : J \times J \to \mathbb{R}$ *satisfying*

(a) $\displaystyle\int_J \varphi(x,y)\,d\mu(y) = 1$ *for every* $x \in \overset{\circ}{J}$;

(b) *for every compact subset* $K \subset J$ *there exists* $h \in \mathcal{L}^1(J,\mu)$ *such that* $\varphi(x,y) \leq h(y)$ *for every* $x \in K$ *and* $y \in J$.

Set $\mu_x := \varphi(x,\cdot)\mu$ $(x \in J)$. *Then* $(\mu_x)_{x\in J}$ *is a continuous selection of probability Borel measures.*

For instance, fix a strictly positive function $\alpha \in C(\mathbb{R})$ *and, for every* $x \in \mathbb{R}$, *let* μ_x *be the normal distribution of* $\mathbb{R}$ *with mean value* x *and variance* $2\alpha(x)$, *i.e.,* $\mu_x = \varphi(x,\cdot)\lambda_1$ *where, for* $x, y \in \mathbb{R}$,

$$\varphi(x,y) := \frac{1}{\sqrt{4\pi\alpha(x)}}\; e^{-\frac{1}{4\alpha(x)}(y-x)^2}.$$

Then $(\mu_x)_{x\in\mathbb{R}}$ *is a continuous selection of probability Borel measures on* $\mathbb{R}$.

The operators C_n, $n \geq 1$, *associated with* $(\mu_x)_{x\in\mathbb{R}}$ *and* $(\mu_n)_{n\geq 1}$ *are the operators defined by*

$$C_n(f)(x) = \sqrt{\frac{n}{4\pi\alpha(x)}}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} f\left(\frac{ns+rt}{n+r}\right) e^{-\frac{n}{4\alpha(x)}(\frac{ns+rt}{n+r}-x)^2} d\mu_n(t)$$

for every $f \in L_a(\mathbb{R})$, $x \in \mathbb{R}$ *and* $n \geq 1$ *(see [12]).*

3 Some functional-analytic properties

In this section we investigate the behaviour of the operators C_n, $n \geq 1$, in some function spaces and, especially, in weighted function spaces.

From now on we fix a bounded weight w on J and we assume that

$$w^{-1} \text{ is convex;} \tag{3.1}$$

moreover, we also suppose that

$$w^{-1} \in \bigcap_{x \in J} \mathcal{L}^1(J, \mu_x) \cap \bigcap_{n \geq 1} \mathcal{L}^1(J, \mu_n), \tag{3.2}$$

and

$$N_w := \sup_{x \in J, n \geq 1} \frac{w(x)}{n+r} \left(n \int_J w^{-1} \, d\mu_x + r \int_J w^{-1} \, d\mu_n \right) < +\infty. \tag{3.3}$$

From (3.1) and (3.2) it follows that

$$w^{-1} \in \bigcap_{n \geq 1, x \in J} \mathcal{L}^1(J, \lambda_{x,n,r}), \tag{3.4}$$

since, for each $n \geq 1$, $x \in J$ and $x_1, \ldots, x_n \in J$,

$$w^{-1} \left(\frac{x_1 + \ldots + x_n + r x_{n+1}}{n+r} \right) \leq \frac{w^{-1}(x_1) + \ldots + w^{-1}(x_n) + r w^{-1}(x_{n+1})}{n+r},$$

and hence

$$\int_J w^{-1} \, d\lambda_{x,n,r} \leq \frac{1}{n+r} \left(n \int_J w^{-1} \, d\mu_x + r \int_J w^{-1} \, d\mu_n \right) < +\infty.$$

From (3.4) it also follows that $w^{-1} \lambda_{x,n,r} \in \mathcal{M}_b^+(J)$ for every $x \in J$ and $n \geq 1$.

From now on, we shall assume that for every $n \geq 1$ the mapping

$$x \in J \mapsto w^{-1} \lambda_{x,n,r} \in \mathcal{M}_b^+(J) \text{ is continuous w.r.t. the weak topology,} \tag{3.5}$$

i.e., for every $\varphi \in C_b(J)$ the function $x \in J \mapsto \int_J \varphi w^{-1} \, d\lambda_{x,n,r}$ is continuous or, equivalently, the function $C_n(\varphi w^{-1})$ is continuous.

Remark 3.1. We observe that conditions (3.1), (3.2), (3.3), and (3.5) are satisfied for example by every $w \in C_b(J)$ such that $\inf_J w > 0$ (in particular by $w = 1$), by every polynomial weight $w_m(x) = (1 + x^m)^{-1}$ ($m \in \mathbb{N}, x \geq 0$) and by the families of measures $(\mu_x)_{x \in J}$ and $(\mu_n)_{n \geq 1}$ considered in Examples 2.2.

Proposition 3.2. *Assume that conditions* (3.1), (3.2), (3.3), *and* (3.5) *are fulfilled. Then*

(1) $C_b^w(J) \subset L_a(J)$. *In particular* $w^{-1} \in L_a(J)$*;*

(2) $C_n(w^{-1}) \in C_b^w(J)$ *and* $\|C_n(w^{-1})\|_w \leq N_w$ *for every* $n \geq 1$*;*

(3) for every $n \geq 1$, $C_n(C_b^w(J)) \subset C_b^w(J)$, C_n *is continuous from* $C_b^w(J)$ *into* $C_b^w(J)$ *and* $\|C_n\|_{C_b^w(J)} = \|C_n(w^{-1})\|_w \leq N_w$, *where* N_w *is defined by* (3.3).

(4) If $n \geq 1$, C_n *maps continuously* $C_b(J)$ *into* $C_b(J)$ *and* $\|C_n\|_{C_b(J)} = 1$.

Proof. If $f \in C_b^w(J)$, then $|f| \leq \|f\|_w w^{-1}$. Therefore, for any $n \geq 1$ and $x_1, \ldots, x_{n+1} \in J$,

$$\left| f\left(\frac{x_1 + \ldots + x_n + rx_{n+1}}{n+r}\right)\right| \leq \|f\|_w w^{-1}\left(\frac{x_1 + \ldots + x_n + rx_{n+1}}{n+r}\right)$$
$$\leq \|f\|_w \frac{w^{-1}(x_1) + \ldots + w^{-1}(x_n) + rw^{-1}(x_{n+1})}{n+r}.$$

Hence, for a given $x \in J$,

$$\int_J d\mu_x(x_1) \ldots \int_J \left| f\left(\frac{x_1 + \ldots + rx_{n+1}}{n+r}\right)\right| d\mu_n(x_{n+1})$$
$$\leq \|f\|_w \frac{1}{n+r}\left(n\int_J w^{-1}\, d\mu_x + r\int_J w^{-1}\, d\mu_n\right) < +\infty.$$

Thus $f \in \mathcal{L}^1(J, \lambda_{x,n,r})$. Finally, $C_n(f) = C_n(fww^{-1})$ is continuous by virtue of (3.5). This proves Statement (1).

In order to prove Statement (2), we first remark that, thanks to Part (1), for a given $n \geq 1$, $C_n(w^{-1}) \in C(J)$. On the other hand, for every $x \in J$,

$$w(x)C_n(w^{-1})(x)$$
$$\leq \int_J d\mu_x(x_1) \ldots \int_J w(x)w^{-1}\left(\frac{x_1 + \ldots + x_n + rx_{n+1}}{n+r}\right) d\mu_n(x_{n+1})$$
$$\leq \int_J d\mu_x(x_1) \ldots \int_J w(x)\frac{w^{-1}(x_1) + \ldots + w^{-1}(x_n) + rw^{-1}(x_{n+1})}{n+r}\, d\mu_n(x_{n+1})$$
$$= w(x)\frac{1}{n+r}\left(n\int_J w^{-1}\, d\mu_x + r\int_J w^{-1}\, d\mu_n\right) \leq N_w$$

and hence Part (2) follows.

As for Statement (3), let $n \geq 1$ and $f \in C_b^w(J)$. Then $|C_n(f)| \leq C_n(|f|) = C_n(w^{-1}(w|f|)) \leq \|f\|_w C_n(w^{-1})$; hence $C_n(f) \in C_b^w(J)$ and $\|C_n(f)\|_w \leq \|C_n(w^{-1})\|_w \|f\|_w$. Thus $\|C_n\|_{C_b^w(J)} \leq \|C_n(w^{-1})\|_w$. On the other hand, by definition of operator norm, $\|C_n\|_{C_b^w(J)} \geq \|C_n(w^{-1})\|_w$, being $\|w^{-1}\|_w = 1$.

Finally, Statement (4) is a consequence of Part (3), assuming $w = 1$. □

Proposition 3.3. *Assume that conditions* (3.1)*,* (3.2)*,* (3.3)*, and* (3.5) *are fulfilled and consider a weight $w \in C_0(J)$ Then, for every $n \geq 1$, C_n maps $C_0^w(J)$ into itself, it is continuous and $\|C_n\|_{C_0^w(J)} = \|C_n(w^{-1})\|_w$.*

Proof. Let $f \in C_0^w(J)$, so that $wf \in C_0(J)$; hence, for a fixed $\varepsilon > 0$, we can find a compact subset K of J such that, for every $x \in J \setminus K$, $|w(x)f(x)| \leq \frac{\varepsilon}{\|C_n(w^{-1})\|_w}$. Therefore, for every $x \in J$,

$$|C_n(f)(x)| \leq \left\{ \int_K + \int_{J \setminus K} \right\} |f| \, d\lambda_{x,n,r} \leq \max_K |f| + \frac{\varepsilon}{\|C_n(w^{-1})\|_w} C_n(w^{-1})(x) \, ,$$

from which it follows that

$$|w(x)C_n(f)(x)| \leq w(x) \max_K |f| + \varepsilon w(x) C_n(w^{-1})(x) \leq w(x) \max_K |f| + \varepsilon \, .$$

On the other hand, since $w \in C_0(J)$, there exists a compact subset K_1 of J such that $w(x) \max_K |f| \leq \varepsilon$ for every $x \in J \setminus K_1$; hence $|w(x)C_n(f)(x)| \leq 2\varepsilon$. Arguing as in the proof of Part (3) of Proposition 3.2, we get $\|C_n\|_{C_0^w(J)} \leq \|C_n(w^{-1})\|_w$.

To obtain the converse inequality, let $(\varphi_p)_{p \geq 1}$ be an increasing sequence in $\mathcal{K}(J)$ such that $0 \leq \varphi_p \leq 1$ for every $p \geq 1$ and $\sup_{p \geq 1} \varphi_p = 1$. Then, for every $x \in J$, by using Beppo Levi's theorem, we get

$$\begin{aligned}
& w(x)C_n(w^{-1})(x) = w(x) \int_J w^{-1} \, d\lambda_{x,n,r} = w(x) \int_J \sup_{p \geq 1} w^{-1} \varphi_p \, d\lambda_{x,n,r} \\
& = \sup_{p \geq 1} w(x) \int_J w^{-1} \varphi_p \, d\lambda_{x,n,r} = \sup_{p \geq 1} w(x) C_n(w^{-1}\varphi_p)(x) \leq \sup_{p \geq 1} \|C_n(w^{-1}\varphi_p)\|_w \\
& \leq \sup_{p \geq 1} \|C_n\|_{C_0^w(J)} \|w^{-1}\varphi_p\|_w \leq \sup_{p \geq 1} \|C_n\|_{C_0^w(J)} \|\varphi_p\|_\infty \leq \|C_n\|_{C_0^w(J)} \, ,
\end{aligned}$$

and this completes the proof. □

Remark 3.4. Assume that $J = [0, +\infty[$ and denote by $C_*^w([0, +\infty[)$ the space of all $f \in C_b^w(J)$ such that $\lim_{x \to +\infty} w(x)f(x) \in \mathbb{R}$. Under the same assumptions of Proposition 3.3, assume that

$$C_n(w^{-1}) \in C_*^w([0, +\infty[) \quad (n \geq 1). \tag{3.6}$$

Then, for every $n \geq 1$, $C_n(C_*^w([0,+\infty[)) \in C_*^w([0,+\infty[)$ and $\|C_n\|_{C_*^w([0,+\infty[)} = \|C_n(w^{-1})\|_w$.

In fact, fix $n \geq 1$ and $f \in C_*^w([0,+\infty[)$; moreover let $l := \lim_{x\to+\infty} w(x)f(x)$. Then $g = f - lw^{-1} \in C_0^w([0,+\infty[)$, so that, by means of Proposition 3.3, $C_n(g) \in C_0^w([0,+\infty[)$ and, hence $C_n(f) = C_n(g) + lC_n(w^{-1}) \in C_*^w([0,+\infty[)$.

For example, operators (2.12), and hence operators (2.11), satisfy (3.6) with respect to the weights $w_m(x) = (1+x^m)^{-1}$, for every $m \geq 1$ (see [6, Remark 3.2]).

In the special case $r = 0$, Propositions 3.2 and 3.3 hold true under simpler assumptions. More precisely, consider a bounded weight w on J verifying (3.1) and assume that

$$w^{-1} \in \bigcap_{x\in J} \mathcal{L}^1(J,\mu_x) \text{ and } M_w := \sup_{x\in J} w(x) \int_J w^{-1}\, d\mu_x < +\infty. \tag{3.7}$$

Clearly, condition (3.3) implies (3.7) for $r = 0$. Moreover, condition (3.5) turns into the following one: for every $n \geq 1$ the mapping

$$x \in J \mapsto w^{-1}\mu_{x,n} \in \mathcal{M}_b^+(J) \text{ is continuous w.r.t. the weak topology,} \tag{3.8}$$

i.e., for every $\varphi \in C_b(J)$ the function $B_n(\varphi w^{-1})$ is continuous.

Therefore we obtain the following result.

Corollary 3.5. *Assume that conditions* (3.1), (3.7) *and* (3.8) *are fulfilled. Then*

(1) $C_b^w(J) \subset C_a(J)$. *In particular* $w^{-1} \in C_a(J)$;

(2) $B_n(w^{-1}) \in C_b^w(J)$ *and* $\|B_n(w^{-1})\|_w \leq M_w$ *for every* $n \geq 1$;

(3) for every $n \geq 1$, $B_n(C_b^w(J)) \subset C_b^w(J)$, B_n *is continuous from* $C_b^w(J)$ *into* $C_b^w(J)$ *and* $\|B_n\|_{C_b^w(J)} = \|B_n(w^{-1})\|_w \leq M_w$;

(4) B_n $(n \geq 1)$ *maps continuously* $C_b(J)$ *into* $C_b(J)$ *and* $\|B_n\|_{C_b(J)} = 1$.

Moreover, if in addition $w \in C_0(J)$, *we have*

(5) For every $n \geq 1$, B_n *maps* $C_0^w(J)$ *into* $C_0^w(J)$, *it is continuous and* $\|B_n\|_{C_0^w(J)} = \|B_n(w^{-1})\|_w$.

(6) If $J = [0,+\infty[$, *assuming that, for any* $n \geq 1$, $B_n(w^{-1}) \in C_*^w([0,+\infty[)$, *then, for any* $n \geq 1$, $B_n(C_*^w([0,+\infty[)) \in C_*^w([0,+\infty[)$ *and* $\|B_n\|_{C_*^w([0,+\infty[)} = \|B_n(w^{-1})\|_w$.

Remark 3.6. As far as we know, Proposition 3.5 represents an improvement of the results in [11].

Coming back to the study of operators C_n, the following proposition holds.

Proposition 3.7. *Let $n \geq 1$. Assume that*

$$\lim_{x \to r_i} \lambda_{x,n,r}(I(b, r_j)) = 0 \quad \text{for every } b \in \overset{\circ}{J} \text{ and } i, j = 1, 2, \quad i \neq j, \tag{3.9}$$

where we recall that $r_1 = \inf J \in \mathbb{R} \cup \{-\infty\}$, $r_2 = \sup J \in \mathbb{R} \cup \{+\infty\}$ and $I(b, r_j)$ denotes the interval whose endpoints are b and r_j. Then, for every $f \in L_a(J)$ such that $\lim_{x \to r_i} f(x) \in \mathbb{R}$,

$$\lim_{x \to r_i} C_n(f)(x) = \lim_{x \to r_i} f(x).$$

In particular, if (3.9) is satisfied, then

(1) C_n maps $C_0(J)$ into itself, it is continuous and $\|C_n\|_{C_0(J)} = 1$;

(2) C_n maps $C_*(J)$ into itself, it is continuous and $\|C_n\|_{C_*(J)} = 1$.

Proof. Let $f \in L_a(J)$ such that $\lim_{x \to r_i} f(x) = l \in \mathbb{R}$. Without loss of generality, we may assume that $l = 0$. Otherwise, since $C_n(\mathbf{1}) = \mathbf{1}$, it should be enough to replace f with $f - l\mathbf{1}$.

Then, for a fixed $\varepsilon > 0$, there exists $b \in \overset{\circ}{J}$ such that $|f(x)| \leq \varepsilon/2$ for every $x \in I(b, r_i)$. If $\sup_{I(b,r_j)} |f| = 0$, then

$$|C_n(x)(f)| = \left| \left(\int_{I(b,r_i)} + \int_{I(b,r_j)} \right) f \, d\lambda_{x,n,r} \right| \leq \int_{I(b,r_i)} |f| \, d\lambda_{x,n,r} \leq \varepsilon/2.$$

Assume now that $\sup_{I(b,r_j)} |f| \neq 0$. From (3.9) it follows that there exists $c \in \overset{\circ}{J}$ such that, for every $x \in I(c, r_i)$ with $i \neq j$, $\lambda_{x,n,r}(I(b, r_j)) \leq \left(\sup_{I(b,r_j)} |f| \right)^{-1} \varepsilon/2$. Hence, for $x \in I(c, r_i)$,

$$|C_n(f)(x)| \leq \left(\int_{I(b,r_i)} + \int_{I(b,r_j)} \right) |f| \, d\lambda_{x,n,r} \leq \frac{\varepsilon}{2} + \lambda_{x,n,r}(I(b, r_j)) \sup_{I(b,r_j)} |f| \leq \varepsilon,$$

that is $\lim_{x \to r_i} C_n(f)(x) = 0$. □

4 Approximation properties

In this section we shall discuss the approximation properties of the sequence $(C_n)_{n\geq 1}$ in the setting of weighted function spaces.

In [7, Remark 3.2] it has been already proven that, for $J = [0, +\infty[$, if $e_1 \in L_a(J)$ and if there exists $C \geq 0$ such that $\int_J e_1 \, d\mu_n \leq C$ for every $n \geq 1$ and $\int_J e_1 \; d\mu_x = x$ for every $x \in J$, then, for every uniformly continuous and bounded function f, we have that

$$\lim_{n\to\infty} C_n(f)(x) = f(x) \tag{4.1}$$

for every $x \in J$. Moreover, under the additional assumptions that $e_2 \in L_a(J)$, $\sup_{x\in K} \int_J e_2 \, d\mu_x < +\infty$ for every compact subinterval K of J and $\sup_{n\geq 1} \int_J e_2 \, d\mu_n < +\infty$, then, for every $f \in C_b(J)$,

$$\lim_{n\to\infty} C_n(f) = f \tag{4.2}$$

uniformly on compact subintervals (see [7, Proposition 3.1]).

In order to deepen such approximation properties in weighted function spaces, from now on we assume that

$$e_1 \in \bigcap_{x\in J} \mathcal{L}^1(J, \mu_x) \cap \bigcap_{n\geq 1} \mathcal{L}^1(J, \mu_n) \quad \text{and} \quad \int_J e_1 \, d\mu_x = x \quad (x \in J). \tag{4.3}$$

From (4.3), by simple calculations, it follows that $e_1 \in L_a(J)$.

We preliminarily remark that, if for some $m \geq 1$,

$$e_m \in \bigcap_{x\in J} \mathcal{L}^1(J, \mu_x) \cap \bigcap_{n\geq 1} \mathcal{L}^1(J, \mu_n), \tag{4.4}$$

then $e_m \in \bigcap_{n\geq 1, x\in J} \mathcal{L}^1(J, \lambda_{x,n,r})$ for any $r \in [0, 1]$. In fact, for $r \in [0, 1]$ fixed,

$$\begin{aligned}
&\int_J |e_m| \; d\lambda_{x,n,r} \\
&\leq \int_J \cdots \int_J \left(\frac{|x_1| + \ldots + |x_n| + r|x_{n+1}|}{n+r} \right)^m \; d\mu_x(x_1) \cdots d\mu_x(x_n) \, d\mu_n(x_{n+1}) \\
&\leq \frac{1}{n+r} \left(n \int_J |e_m| \, d\mu_x + r \int_J |e_m| \, d\mu_n \right) < +\infty.
\end{aligned}$$

The following lemma concerning the operators B_n will be useful. For a detailed proof see [11, Lemma 4.1].

Lemma 4.1. *For a given $k \geq 1$ assume that $e_k \in \bigcap_{x\in J} L^1(J,\mu_x)$. Then $e_k \in \bigcap_{n\geq 1, x\in J} L^1(J,\mu_{x,n})$ and, for every $n \geq 1$,*

$$B_n(e_k) = \frac{1}{n^k} \sum_{k_1+\ldots+k_n=k} \binom{k}{k_1,\ldots,k_n} U(e_{k_1})\cdots U(e_{k_n}),$$

where $\binom{k}{k_1,\ldots,k_n} = \frac{k!}{k_1!\cdots k_n!}$ and the operator U is defined by (2.3).

Therefore, if $U(e_h)$ is continuous for every $1 \leq h \leq k$, then $e_k \in C_a(J)$. In particular, if $e_4 \in \bigcap_{x\in J} L^1(J,\mu_x)$ then, for every $n \geq 1$ and $x \in J$,

$$B_n(\mathbf{1}) = \mathbf{1}, \quad B_n(e_1) = e_1, \quad B_n(e_2) = \frac{n-1}{n} e_2 + \frac{U(e_2)}{n}, \tag{4.5}$$

$$B_n(\psi_x)(x) = 0, \qquad B_n(\psi_x^2)(x) = \frac{U(e_2)(x) - x^2}{n}, \tag{4.6}$$

$$\begin{aligned} B_n(\psi_x^4)(x) = \frac{1}{n^3}\Big(&U(e_4)(x) - 4xU(e_3)(x) + 3(n-1)U(e_2)^2(x) \\ &- 6(n-2)x^2 U(e_2)(x) + 3(n-1)x^4\Big), \end{aligned} \tag{4.7}$$

where the function ψ_x is defined in (1.1).

After these preliminaries we are ready to prove the following lemma.

Lemma 4.2. *Assume that, for a given $m \in \mathbb{N}$, (4.4) holds true. Then*
(1) For any $n \geq 1$,

$$\begin{aligned} C_n(e_m) = \frac{1}{(n+r)^m} \Bigg[&\left(r^m \int_J e_m \, d\mu_n\right) \mathbf{1} + n^m B_n(e_m) \\ &+ m \sum_{q=1}^{m-1} r^q n^{m-q} \left(\int_J e_q \, d\mu_n\right) B_n(e_{m-q}) \Bigg]. \end{aligned} \tag{4.8}$$

In particular, $C_n(\mathbf{1}) = \mathbf{1}$,

$$C_n(e_1) = \left(\frac{r}{n+r} \int_J e_1 \, d\mu_n\right) \mathbf{1} + \frac{n}{n+r} e_1 \tag{4.9}$$

and

$$C_n(e_2) = \left(\frac{r^2}{(n+r)^2} \int_J e_2 \, d\mu_n\right) \mathbf{1} + \left(\frac{2rn}{(n+r)^2} \int_J e_1 \, d\mu_n\right) e_1 + \frac{n^2}{(n+r)^2} B_n(e_2). \tag{4.10}$$

(2) If

$$U(e_i) \in C(J) \text{ for every } i = 1,\ldots,m, \tag{4.11}$$

then $e_m \in L_a(J) \cap C_a(J)$.
Conversely, if for every $i = 1, \dots, m$, $e_i \in L_a(J)$, *then* $e_i \in C_a(J)$.
(3) Under assumptions (4.4), $\psi_x^m \in \mathcal{L}^1(J, \lambda_{x,n,r})$; *in particular,*

$$C_n(\psi_x)(x) = \frac{r}{n+r}\left(\int_J e_1\, d\mu_n - x\right), \tag{4.12}$$

$$\begin{aligned} C_n(\psi_x^2)(x) &= \frac{1}{(n+r)^2}\left(r^2 \int_J \psi_x^2\, d\mu_n + n^2 B_n(\psi_x^2)(x)\right) \\ &\leq \frac{1}{(n+r)^2}\left(r^2 \int_J \psi_x^2\, d\mu_n + n\left(\int_J e_2\, d\mu_x - x^2\right)\right) \end{aligned} \tag{4.13}$$

and, for every $q \geq 2$, q *even,*

$$C_n(\psi_x^q)(x) \leq 2^{q-1}\left[\left(\frac{r}{n+r}\right)^q \int_J \psi_x^q\, d\mu_n + \left(\frac{n}{n+r}\right)^q B_n(\psi_x^q)(x)\right]. \tag{4.14}$$

(4) Under assumption (4.11), $\psi_x^m \in L_a(J) \cap C_a(J)$.

Proof. Part (1). Formula (4.8) has been proven in [10, Lemma 1.2] and from it, by simple calculations, we get (4.9)-(4.10).

Concerning Part (2), in order to show that $e_m \in L_a(J)$, it sufficies to prove that $C_n(e_m) \in C(J)$; this happens (see (4.8)) if $B_n(e_i) \in C(J)$, i.e., $e_i \in C_a(J)$, for every $i = 1, \dots, m$. By applying Lemma 4.1 this last condition is verified under (4.11). The converse follows directly from (4.8).

Formulae (4.12)-(4.13) in Part (3) are direct consequence of (4.9)-(4.10). The proof of (4.14) is based on the observation that, if q is an even number and $x_1, \dots, x_{n+1}, x \in J$, then

$$\begin{aligned} &\left(\frac{x_1 + \ldots + x_n + r x_{n+1}}{n+r} - x\right)^q \\ &\leq 2^{q-1}\left[\left(\frac{r}{n+r}\right)^q (x_{n+1} - x)^q + \left(\frac{n}{n+r}\right)^q \left(\frac{x_1 + \ldots + x_n}{n} - x\right)^q\right]. \end{aligned}$$

Finally, Part (4) follows from (4.8) and Lemma 4.1. □

In order to achieve the desired approximation properties, we shall appeal to the following Korovkin-type theorem which has been obtained in [5] (see [5, Example 4.9, 1] and [4, Example 2.3, 3] or, more directly, [1, Corollaries 6.13 and 6.14]).

Theorem 4.3. *Let* $w \in C_0(J)$ *be a weight such that* $e_2 \in C_0^w(J)$. *If* $(L_n)_{n \geq 1}$ *is a sequence of (bounded) positive linear operators from* $C_0^w(J)$ *into* $C_0^w(J)$ *satisfying* *(i)* $\sup_{n\geq 1} \|L_n\| < +\infty$,

(ii) $\lim_{n\to\infty} L_n(h) = h$ *in* $C_0^w(J)$ *for every* $h \in \{\mathbf{1}, e_1, e_2\}$,

then, for every $f \in C_0^w(J)$, $\lim_{n\to\infty} L_n(f) = f$ *in* $(C_0^w(J), \|\cdot\|_w)$.

From now on, we assume that

$$\sup_{n\geq 1} \int_J e_2 \, d\mu_n < +\infty, \tag{4.15}$$

so that $\sup_{n\geq 1} \int_J e_1 \, d\mu_n < +\infty$.

Theorem 4.4. *(1) Under assumption* (4.15), *let* $w \in C_0(J)$ *be a weight on* J *such that* (3.1), (3.2), (3.3), *and* (3.5) *are fulfilled and assume that* $e_2 \in C_0^w(J)$. *Then, for every* $f \in C_0^w(J)$,

$$\lim_{n\to\infty} C_n(f) = f \quad \textit{in } (C_0^w(J), \|\cdot\|_w) \tag{4.16}$$

and the convergence is uniform on compact subsets of J. *In particular, for every* $f \in C_b(J)$, $\lim_{n\to\infty} C_n(f) = f$ *uniformly on compact subsets of* J.

(2) Assume that $J = [0, +\infty[$ *and that* (3.6) *holds true. Furthermore suppose that*

$$\lim_{n\to\infty} \|C_n(w^{-1}) - w^{-1}\|_w = 0. \tag{4.17}$$

Then

$$\lim_{n\to\infty} C_n(f) = f \textit{ for every } f \in C_w^*([0, +\infty[). \tag{4.18}$$

Proof. (1) First of all we note that $e_2 \in L_a(J) \cap C_a(J)$ since $U(e_2) \in C(J)$ (see (4.11)). Furthermore, $(C_n)_{n\geq 1}$ is equibounded by virtue of Propositions 3.2 and 3.3. By Theorem 4.3, in order to get (4.16), it is sufficient to prove that $\lim_{n\to\infty} C_n(h) = h$ with respect to $\|\cdot\|_w$, for every $h \in \{\mathbf{1}, e_1, e_2\}$.

From (4.9) it follows that

$$\|C_n(e_1) - e_1\|_w \leq \|w\|_\infty \frac{rM_1}{r+n} + \frac{r}{n+r}\|e_1\|_w \to 0.$$

where $M_1 =: \sup_{n\geq 1} \int_J e_1 \, d\mu_n$. Moreover, by means of (4.10) and (4.5),

$$\|C_n(e_2) - e_2\|_w \leq \|w\|_\infty \frac{M_2 r^2}{(n+r)^2} + \frac{2nrM_1}{(n+r)^2}\|e_1\|_w + \frac{1}{n}(\|e_2\|_w + \|U(e_2)\|_w) \to 0,$$

where $M_2 = \sup_{n\geq 1} \int_J e_2\, d\mu_n$.

We remark that $U(e_2) \in C_0^w(J)$ because (4.10) implies that $B_n(e_2) \in C_0^w(J)$ and (4.5) holds true. This completes the proof of Statement (1).

As for Statement (2), fix $f \in C_*^w([0,+\infty[)$; then $g = f - lw^{-1} \in C_0^w([0,+\infty[)$, where $l := \lim_{x\to+\infty} w(x)f(x)$. Hence $\|C_n(f)-f\|_w \leq \|C_n(g)-g\|_w + l\|C_n(w^{-1}) - w^{-1}\|_w$, and so the result follows. □

Remarks 4.5.

1) According to Remark 3.1, Theorem 4.4 applies for every $w \in C_b(J)$ such that $\inf_J w > 0$ (in particular for $w = \mathbf{1}$) and, if $J = [0,+\infty[$, for every polynomial weight $w_m(x) = (1+x^m)^{-1}$ ($m \in \mathbb{N}, x \geq 0$) and for the measures in Examples 2.2.

2) We point out that operators (2.12), and hence operators (2.11), satisfy (4.17) with respect to the weights $w_m(x) = (1+x^m)^{-1}$, for every $m \geq 1$ (see [6, Proposition 2.1]).

We end this section by stating some estimates of the convergence in (4.1) and (4.2) as well as the one in Theorem 4.4. The estimates will be given in terms of the ordinary moduli of smoothness of the first and second order ω_1 and ω_2 (see, e.g., [3, Section 5.1]). Furthermore, they will be mainly stated in the special case where $J = [0,+\infty[$. Perhaps, for other kinds of noncompact intervals, other different techniques would be implemented.

Proposition 4.6. *Assume that $e_2 \in \bigcap_{x\in J} \mathcal{L}^1(J,\mu_x) \cap \bigcap_{n\geq 1} \mathcal{L}^1(J,\mu_n)$. Then*

(1) For every $f \in C_b(J)$, $n \geq 1$, $x \in J$,

$$|C_n(f)(x) - f(x)| \leq$$
$$\leq 2\left(1 + \sqrt{r^2 \int_J \psi_x^2\, d\mu_n + n\left(\int_J e_2\, d\mu_x - x^2\right)}\right)\omega_1\left(f, \frac{1}{\sqrt{n}}\right).$$

(2) If $J = [0,+\infty[$, for every $f \in UC_b(J)$, $n \geq 1$, $x \in J$,

$$|C_n(f)(x) - f(x)| \leq M_1\left(1 + \left(\int_J e_2\, d\mu_x + x^2\right)^{1/2}\right)^2 \omega_2\left(f, \frac{1}{\sqrt{n}}\right)$$
$$+ \left(2 + x + \int_J e_1\, d\mu_n\right)\omega_1\left(f, \frac{r}{n+r}\right),$$

where M_1 is a constant independent on f, $n \geq 1$ and $x \in J$.

Proof. Part (1) follows directly from [3, Theorem 5.1.2 and the subsequent remark] along with (4.6), (4.13), and [3, Lemma 5.1.1, Part (6)].

In order to prove Part (2), we shall appeal to [7, Theorem 4.3], where we have established a similar estimate for the approximation of operators semigroups on Banach spaces. More precisely, consider the Banach space $X := UC_b([0,+\infty[)$ endowed with the sup-norm and denote by $(T(t))_{t\geq 0}$ the translation semigroup defined on it, i.e., for every $x \geq 0$, $f \in X$ and $\xi \geq 0$, $T(x)(f)(\xi) := f(x+\xi)$. Clearly $\|T(x)\| \leq 1$ for every $x \geq 0$. Moreover, for every $n \geq 1$ and $x \geq 0$, consider the bounded linear operator $K_n(x) : X \to X$ defined by setting, for every $f \in X$ and $\xi \geq 0$,

$$\begin{aligned}&K_n(x)(f)(\xi) := \\ &= \int_J d\mu_x(s_1) \dots \int_J d\mu_x(s_n) \int_J T\left(\frac{s_1 + \dots + s_n + rs_{n+1}}{n+r}\right)(f)(\xi)\, d\mu_n(s_{n+1}) \,.\end{aligned}$$

Thus, for every $f \in X$ and $x \geq 0$, $T(x)(f)(0) = f(x)$ and $K_n(f)(0) = C_n(f)(x)$. Therefore, from formula (4.4) of [7, Theorem 4.3], we get

$$\begin{aligned}&|C_n(f)(x) - f(x)| = |K_n(x)(f)(0) - T(x)(f)(0)| \leq \|K_n(x)(f) - T(x)(f)\|_\infty \\ &\leq M_1\omega_2\left(f, \sqrt{\frac{\int_J e_2\, d\mu_x + x^2}{n}}\right) + \left(2 + x + \int_J e_1\, d\mu_n\right)\omega_1\left(f, \frac{r}{n+r}\right),\end{aligned}$$

and hence Statement (2) follows again from [3, Lemma 5.1.1, Part (6)]. □

In order to present some estimates of the rate of convergence with respect to the weighted norm (see (4.18)), we shall use a similarity technique.

Generally speaking, given an approximation process $(L_n)_{n\geq 1}$ on some Banach space X, if $R : X \to Y$ is an isometric isomorphism between X and another Banach space Y with inverse $S : Y \to X$, then it is possible to construct an approximation process $(L_n^*)_{n\geq 1}$ on Y by setting, for any $n \geq 1$, $L_n^* := R \circ L_n \circ S$. In such a case, $(L_n)_{n\geq 1}$ and $(L_n^*)_{n\geq 1}$ are said to be similar or isomorphic. Clearly, for every $u \in X$,

$$\|L_n(u) - u\|_X = \|L_n^*(R(u)) - R(u)\|_Y, \tag{4.19}$$

so that the problem of estimating the rate of convergence for $(L_n)_{n\geq 1}$ in X may be transferred to the (possibly easier to handle) sequence $(L_n^*)_{n\geq 1}$ in Y.

From now on, we shall assume that $J = [0,+\infty[$. As in Remark 3.4 we denote by $C_*^w([0,+\infty[)$ the linear subspace of all $f \in C_b^w([0,+\infty[)$ such that $\lim_{x\to+\infty} w(x)f(x) \in \mathbb{R}$. For the sake of simplicity, if $f \in C_*^w([0,+\infty[)$, we set $(wf)(\infty) := \lim_{x\to+\infty} w(x)f(x)$.

Consider the isometric isomorphism $R : C_*^w([0,+\infty[) \to C([0,1])$ defined by setting, for every $f \in C_*^w([0,+\infty[)$ and $t \in [0,1]$,

$$R(f)(t) = \begin{cases} (wf)(-\log t) & \text{if } 0 < t \le 1, \\ (wf)(\infty) & \text{if } t = 0. \end{cases}$$

Clearly, its inverse $S : C([0,1]) \to C_*^w([0,+\infty[)$ is defined by

$$S(g)(x) := w^{-1}(t) g(\sigma(x)) \quad (x \ge 0)$$

for every $g \in C([0,1])$, where

$$\sigma(x) := e^{-x} \quad (x \ge 0). \tag{4.20}$$

For every $n \ge 1$, we consider the similar positive linear operator $C_n^* : C([0,1]) \to C([0,1])$ defined by setting, for every $g \in C([0,1])$,

$$C_n^*(g) := R(C_n(S(g))).$$

By virtue of (4.19), we have that, for every $f \in C_*^w([0,+\infty[)$ and $n \ge 1$,

$$\|C_n(f) - f\|_w = \|C_n^*(R(f)) - R(f)\|_\infty. \tag{4.21}$$

Moreover $(C_n^*)_{n\ge1}$ is an approximation process in $C([0,1])$.

We are now ready to state some estimates of the rates of convergence with respect to the weighted norm $\|\cdot\|_w$.

Proposition 4.7. *Let $J = [0,+\infty]$. Under the same assumptions of Part 2) of Theorem 4.4, for every $f \in C_*^w([0,+\infty[)$ and $n \ge 1$, we have that*

$$\|C_n(f) - f\|_w \le \|C_n(w^{-1}) - w^{-1}\|_w \|f\|_w + \sqrt{N_w}\,\omega_1(R(f), \delta_n) + \left(N_w + \frac{1}{2}\right)\omega_2(R(f), \delta_n),$$

where N_w is defined by (3.3), *$\delta_n := \sup_{0\le t\le 1} \sqrt{\alpha_n(t)}$ with*

$$\alpha_n(t) = \begin{cases} w(-\log t)\, C_n(w^{-1}(\sigma - t\mathbf{1}))(-\log t) & \text{if } 0 < t \le 1, \\ 0 & \text{if } t = 0, \end{cases}$$

and σ is defined by (4.20).

Proof. For every $t \in [0,1]$ consider the function $\psi_t \in C([0,1])$ defined by $\psi_t(\eta) = \eta - t$ $(0 \le \eta \le 1)$. Thus $\psi_t = e_1 - t\mathbf{1}$.

We preliminarily observe that, for any given $n \geq 1$ and $t \in [0,1]$,

$$C_n^*(\mathbf{1})(t) = \begin{cases} (w\, C_n(w^{-1}))(-\log t) & \text{if } 0 < t \leq 1, \\ (wC_n(w^{-1}))(\infty) & \text{if } t = 0, \end{cases}$$

$$C_n^*(\psi_t)(t) = \begin{cases} (w\, C_n(w^{-1}(\sigma - t\mathbf{1})))(-\log t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } x = 0, \end{cases}$$

and

$$C_n^*(\psi_t^2)(t) = \begin{cases} (w\ C_n(w^{-1}(\sigma - t\mathbf{1})^2))(-\log t) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0, \end{cases}$$

i.e.,

$$C_n^*(\psi_t^2)(t) = \alpha_n(t) \quad (0 \leq t \leq 1). \tag{1}$$

For the sake of clarity, we specify that in the last two formulae we get 0 for $t = 0$ thanks to Proposition 3.3 and because $w^{-1}\sigma, w^{-1}\sigma^2 \in C_0^w([0,+\infty[)$.

From Part (3) of Proposition 3.2 we infer in particular that

$$C_n^*(\mathbf{1})(t) \leq N_w \quad (0 \leq t \leq 1). \tag{2}$$

Furthermore, since $R(w^{-1}) = \mathbf{1}$, from (4.21) it follows that

$$\|C_n^*(\mathbf{1}) - \mathbf{1}\|_\infty = \|C_n(w^{-1}) - w^{-1}\|_w. \tag{3}$$

Finally, by the Cauchy-Schwarz inequality for positive linear operators, for every $n \geq 1$ and $t \in [0,1]$, we have

$$|C_n^*(\psi_t)(t)| \leq \sqrt{C_n^*(\mathbf{1})(t)}\sqrt{C_n^*(\psi_t^2)(t)} \leq \sqrt{N_w}\sqrt{\alpha_n(t)} \leq \sqrt{N_w}\delta_n. \tag{4}$$

By applying Theorem 2.2.1 in [17] (see also [16, Theorem 10]), for every $n \geq 1$, $f \in C_*^w([0,+\infty[)$, $0 \leq t \leq 1$ and $\delta > 0$, we have

$$|C_n^*(R(f))(t) - R(f)(t)| \leq |C_n^*(\mathbf{1})(x) - 1||R(f)(t)|$$
$$+ \frac{1}{\delta}|C_n^*(\psi_t)(t)|\omega_1(R(f),\delta) + \left(C_n^*(\mathbf{1})(t) + \frac{1}{2\delta^2}C_n^*(\psi_t^2)(t)\right)\omega_2(R(f),\delta)$$

and hence, thanks to (3), (4), (2) and (1), we get

$$|C_n^*(R(f))(t) - R(f)(t)| \leq \|C_n(w^{-1}) - w^{-1}\|_w\|f\|_w + \sqrt{N_w}\frac{\delta_n}{\delta}\omega_1(R(f),\delta)$$
$$+ \left(N_w + \frac{\alpha_n(t)}{2\delta^2}\right)\omega_2(R(f),\delta).$$

Taking the supremum with respect to $t \in [0,1]$, as well as setting $\delta = \delta_n$ and recalling (4.21), we get the result. □

5 An asymptotic formula

In this last section, under suitable conditions, we shall establish an asymptotic formula for the operators C_n. To this end, from now on, we assume that $e_4 \in \bigcap_{x\in J} \mathcal{L}^1(J,\mu_x) \cap \bigcap_{n\geq 1} \mathcal{L}^1(J,\mu_n)$ and

$$\sup_{n\geq 1} \int_J e_4 \, d\mu_n < +\infty. \tag{5.1}$$

Moreover, we assume that there exists $b \geq 0$ such that

$$b = \lim_{n\to\infty} \int_J e_1 \, d\mu_n. \tag{5.2}$$

For every $x \in J$, set

$$\alpha(x) := \frac{1}{2}\left(\int_J e_2 d\mu_x - x^2\right) \tag{5.3}$$

and

$$\beta(x) = r(b - x)\,. \tag{5.4}$$

Finally, consider the second order differential operator

$$V(f) := \alpha f'' + \beta f' \quad (f \in C^2(J)).$$

Before stating the main result of this section, we note that, if $e_2 \in \bigcap_{x\in J} \mathcal{L}^1(J,\mu_x) \cap \bigcap_{n\geq 1} \mathcal{L}^1(J,\mu_n)$, then the subspace

$$E_2(J) := \left\{ f \in C(J) \;\mid\; \sup_{x\in J} \frac{|f(x)|}{1+x^2} < +\infty \right\}$$

is contained in $\bigcap_{x\in J} \mathcal{L}^1(J,\mu_x) \cap \bigcap_{n\geq 1} \mathcal{L}^1(J,\mu_n)$. In general $UC_b^2(J) \subset E_2$. Moreover, if $e_2 \in C_b^w(J)$, then $E_2(J) \subset C_b^w(J)$.

We are now in a position to state the following result which, in the case where J is a compact subset of $\mathbb{R}^p$, was shown in [10].

Theorem 5.1. *Let w be a bounded weight on J such that* (3.1), (3.2), (3.3), *and* (3.5) *are fulfilled and assume that $e_4 \in C_b^w(J)$. Moreover, suppose that* (5.1) *and* (5.2) *hold true. Then, for every $f \in UC_b^2(J)$,*

$$\lim_{n\to\infty} w\left[n\left(C_n(f) - f\right) - V(f)\right] = 0 \tag{5.5}$$

uniformly on J. In particular $\lim_{n\to\infty} n\left(C_n(f) - f\right) = V(f)$ uniformly on compact subsets of J.

Proof. We first point out that, given our assumptions, $U(e_i) \in C_b^w(J)$ for every $i = 1, \dots, 4$ (see (4.8)). According to [2, Theorem 1], to show (5.5), we have to prove that (see (5.3) and (5.4))

(a) $\lim_{n\to\infty} w(x)[nC_n(\psi_x^2)(x) - 2\alpha(x)] = 0$ uniformly on J;

(b) $\lim_{n\to\infty} w(x)x^k\,[nC_n(\psi_x)(x) - \beta(x)] = 0$ uniformly on J $(k = 0, 1)$;

(c) $\sup_{n\geq 1,\, x\in J} w(x)[nC_n(\psi_x^2)(x)] < +\infty$;

(d) $\lim_{n\to\infty} w(x)[nC_n(\psi_x^4)(x)] = 0$ uniformly on J.

To show (a), we remark that, taking (4.13) into account, we have

$$\begin{aligned} & w(x)|nC_n(\psi_x^2)(x) - 2\alpha(x)| \\ & \leq \left(1 - \frac{n^2}{(n+r)^2}\right) \|U(e_2) - e_2\|_w + w(x)\frac{r^2 n}{(n+r)^2}\left(\int_J \psi_x^2\, d\mu_n\right) \end{aligned}$$

and, by means of (5.1) and the fact that $e_4 \in C_b^w(J)$,

$$\sup_{x\in J, n\geq 1} w(x)\int_J \psi_x^2\, d\mu_n < +\infty. \tag{1}$$

From this we obtain Statement (a). Statement (b) is a consequence of (4.12) and (5.2). As for Statement (c), it follows from (4.14) and (1). Finally, Statement (d) is a consequence of (4.7), (4.14) and (5.1). □

Remark 5.2. From Theorem 5.1 it follows in particular that, if $f \in UC_b^2(J)$, then

$$\|C_n(f) - f\|_w = o\left(\frac{1}{n}\right) \quad \text{as } n \to \infty$$

if and only if $\alpha f'' + \beta f' = 0$ on J.

Examples 5.3. *1.* *If* $\lim_{n\to\infty}(a_n + b_n) \in \mathbb{R}$, *then the measures* μ_n *defined by* (2.9) *satisfy* (5.1) *and* (5.2), *with* $b = \lim_{n\to\infty} \frac{a_n+b_n}{2}$. *Hence, Theorem 5.1 applies, for example, to the operators* C_n *defined by* (2.10) *with*

$$Vf(x) := xf''(x) + r(b - x)f'(x) \quad (f \in UC_b^2([0, +\infty[),\ x \geq 0)$$

and to the operators defined in (2.13) *with*

$$Vf(x) := x(1 + x)f''(x) + r(b - x)f'(x) \quad (f \in UC_b^2([0, +\infty[),\ x \geq 0).$$

2. *If the sequence* $(b_n)_{n\geq 1}$ *satisfies the assumptions* $\lim_{n\to\infty} b_n = b \geq 0$ *and* $\sup_{b\geq 1} b_n^4 < +\infty$, *then the measures* $\mu_n = \varepsilon_{b_n}$ *satisfy* (5.1) *and* (5.2), *so Theorem 5.1 applies to the operators* C_n *defined by* (2.14) *with*

$$Vf(x) := \frac{x^2}{2} f''(x) + r(b-x)f'(x) \quad (f \in UC_b^2([0,+\infty[),\, x \geq 0).$$

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