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Filtered integration rules for finite weighted Hilbert transforms II

Donatella Occorsio[a,b,c] · Maria Grazia Russo[a,c] · Woula Themistoclakis[b,c]

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Abstract

This paper is the continuation of a previous work where the authors have introduced a new class of quadrature rules for evaluating the finite Hilbert transform. Such rules are product type formulae based on the filtered de la Vallée Poussin (shortly VP) type approximation. Here, we focus on some particular cases of interest in applications and show that further results can be obtained in such special cases. In particular, we consider an optimal choice of the quadrature nodes for which explicit formulae of the quadrature weights are given and sharper error estimates are stated.

1 Introduction

The Hilbert Transform (HT) appears in many mathematical models of applied sciences with different variants (see, e.g., [18, 19, 22, 27] and the references therein). Here we deal with the finite HT defined as follows

$$\mathcal{H}g(t) := \int_{-1}^{1} \frac{g(x)}{x-t}dx = \lim_{\epsilon\to 0}\int_{|x-t|\geq\epsilon} \frac{g(x)}{x-t}dx, \qquad -1<t<1. \tag{1}$$

In particular, we focus on the case in which the integrand g can be written as the following product

$$g(x) = f(x)u(x), \qquad |x|\leq 1,$$

where u is a Jacobi weight as follows

$$u(x) = v^{\alpha,\beta}(x) := (1-x)^{\alpha}(1+x)^{\beta}, \qquad \text{with} \quad \begin{cases} 0<|\alpha|,|\beta|<1 \quad \text{and} \\ \sigma := \alpha+\beta \in \{-1,0,+1\}, \end{cases} \tag{2}$$

and f is a continuous function on $(-1,1)$ such that

$$\lim_{x\to +1} f(x)u(x) = 0 \quad \text{if } \alpha>0, \qquad \text{and} \qquad \lim_{x\to -1} f(x)u(x) = 0 \quad \text{if } \beta>0.$$

We recall that such kind of integrand and, in particular, the conditions in (2) on the Jacobi exponents, are usual in Cauchy Singular Integral Equations (see, e.g. [2, 6, 17, 25, 26, 34]) where it is commonly set

$$u(x) = \frac{u_+(x)}{u_-(x)}, \qquad \text{with} \quad \begin{cases} u_+(x) := (1-x)^{\max\{\alpha,0\}}(1+x)^{\max\{\beta,0\}}, \\ u_-(x) := (1-x)^{\max\{-\alpha,0\}}(1+x)^{\max\{-\beta,0\}}. \end{cases} \tag{3}$$

Moreover, denoting by $E_n(f)_v$ the error of best uniform weighted approximation of f in the space $\mathbb{P}_n$ of algebraic polynomials of degree at most n, namely

$$E_n(f)_v := \inf_{P\in\mathbb{P}_n} \|(f-P)v\|_\infty, \tag{4}$$

it is well known that the previous assumptions on f ensure that

$$\lim_{n\to+\infty} E_n(f)_{u_+} = 0$$

and the rate of convergence is characterized by the degree of smoothness of f.

The first result we are going to state concerns the mapping properties of the HT operator $\mathcal{H}^u : f \to \mathcal{H}^u f$ where we set

$$\mathcal{H}^u f(t) := \int_{-1}^{1} \frac{f(x)}{x-t}u(x)dx, \qquad -1<t<1, \tag{5}$$

[a]Dipartimento di Matematica ed Informatica, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano 10, 85100 Potenza, Italy
[b]C.N.R. National Research Council of Italy, IAC Institute for Applied Computing "Mauro Picone",
Via P. Castellino, 111, 80131 Napoli, Italy
[c]INDAM-GNCS member, "Research ITalian network on Approximation (RITA)" member, TAA-UMI group member

and the integral is intended in the Cauchy principal value sense.

Improving previous results in [20, 21], we state the boundedness of $\mathcal{H}^u$ in a couple of Besov type spaces characterized by the same degree of smoothness. We refer the reader to the next section for all technical details. Here we point out that our result ensures for $\mathcal{H}^u f$ the same degree of smoothness of f, i.e. we get

$$E_n(f)_{u_+} = \mathcal{O}(n^{-r}) \implies E_n(\mathcal{H}^u f)_{u_-} = \mathcal{O}(n^{-r}), \qquad \forall r > 0. \tag{6}$$

As far as the numerical computation of the HT is concerned, there exists a wide literature on the quadrature rules for $\mathcal{H}^u f$ (see, e.g., [3, 4, 5, 8, 9, 13, 16, 28, 29]). In particular, in [30] the authors have recently proposed a new class of product integration rules (VP-rules) that offer some advantages with respect to other quadrature rules based on the same system of nodes. In general, given a Jacobi weight w and a pair of integers $0 < m < n$, the corresponding VP-rule takes the following form [30, Section 4]

$$\mathcal{H}^u_{n,m}(w, f, t) = \sum_{j=0}^{n+m-1} \rho^m_{n,j}(w, f) Q^u_j(w, t), \qquad -1 < t < 1, \tag{7}$$

with

$$Q^u_j(w,t) := \int_{-1}^{1} \frac{p_j(w,x)}{x-t} u(x) dx, \tag{8}$$

$$\rho^m_{n,j}(w, f) := \mu^m_{n,j} \sum_{k=1}^{n} \lambda^w_{n,k} p_j(w, x^w_{n,k}) f(x^w_{n,k}), \tag{9}$$

where $p_j(w,x)$, $j \in \mathbb{N}_0$, denotes the Jacobi orthonormal polynomial of degree j corresponding to w and having positive leading coefficient, $x^w_{n,k}$, $k = 1, ..n$, are the zeros of $p_n(w,x)$,

$$\lambda^w_{n,k} := \left(\sum_{j=0}^{n-1} [p_j(w, x^w_{n,k})]^2 \right)^{-1}, \qquad k = 1, 2, \dots, n, \tag{10}$$

are the Christoffel numbers related to w and

$$\mu^m_{n,j} := \begin{cases} 1, & \text{if} \quad j = 0, \dots, n-m, \\ \dfrac{n+m-j}{2m}, & \text{if} \quad n-m < j < n+m, \end{cases} \tag{11}$$

are filter coefficients of de la Vallée Poussin type (see, e.g. [38]).

Note that for a fixed number n of nodes, VP rules may differ by the choice of the Jacobi weight w and/or of the additional parameter m. As $n \to +\infty$ and the ratio m/n remains bounded in a compact subinterval of $(0, 1)$, the sufficient conditions on w are known (see [30, Th. 4.2 and Th. 3.1]) to ensure that the quadrature error

$$\mathcal{E}^u_{n,m}(w, f, t) := \left| \mathcal{H}^u f(t) - \mathcal{H}^u_{n,m}(w, f, t) \right|, \qquad -1 < t < 1, \tag{12}$$

converges to zero uniformly w.r.t. t belonging to any compact interval $I \subset (-1, 1)$. In particular, this is true choosing $w = u$ with u as in (2) and we focus on such case in the present paper.

From the computational point of view, thanks to a result in [34], in this case we have an analytic expression of the functions $Q^u_j(u,t)$ that hence can be computed more efficiently. Regarding the role of the parameter m and the numerical advantages of VP rules, we refer the reader to [30, Section 5] that also includes the case under consideration.

The main novelty of this paper concerns the theoretical estimates of the quadrature error. Regarding this, we recall that the results in [30] yield

$$E_n(f)_{u_+} = \mathcal{O}(n^{-r}) \implies \sup_{|t| \le 1} \mathcal{E}^u_{n,m}(w, f, t) u_-(t) = \mathcal{O}(n^{-r} \log n), \qquad \forall r > 0, \tag{13}$$

but no estimate is given in the case of less regular functions that satisfy a Dini type condition. Here we fill this gap and state that the quadrature error, similarly to $E_n(f)_{u_+}$, satisfies a Jackson type estimate involving the Ditzian–Totik weighted main part modulus of smoothness of f. As a consequence, we succeed in eliminating the $\log n$ factor in (13) and obtain the quadrature error converges to zero with the same rate of the error of best polynomial approximation of f, namely

$$E_n(f)_{u_+} = \mathcal{O}(n^{-r}) \implies \sup_{|t| \le 1} \mathcal{E}^u_{n,m}(w, f, t) u_-(t) = \mathcal{O}(n^{-r}), \qquad \forall r > 0. \tag{14}$$

The produced error estimates can be useful, for instance, in the numerical solution of Cauchy singular integral equations.

The outline of the paper is the following: Section 2 deals with mapping properties of Hilbert transform, Section 3 concerns the computational details of the proposed VP rule and Section 4 contains the new error estimates.

2 Mapping properties in Besov type spaces with uniform norms

For any Jacobi weight v with nonnegative exponents, we denote by C^0_v the space of all locally continuous functions on $[-1,1]$ (i.e. continuous in any compact interval $I \subseteq (-1,1)$) satisfying

$$\lim_{x\to\pm1} f(x)v(x) = 0, \qquad \text{if } v(\pm1) = 0.$$

Equipped with the norm

$$\|f\|_{C^0_v} := \|fv\|_\infty = \max_{x\in[-1,1]} |f(x)|v(x),$$

this is a Banach space characterized as follows (see for instance [23])

$$f \in C^0_v \iff \lim_{n\to\infty} E_n(f)_v = 0, \tag{15}$$

where $E_n(f)_v$ is the error of best approximation of $f \in C^0_v$ in the space $\mathbb{P}_n$ (cf. (4)).

We focus on the case that such error converges to zero sufficiently fast to have, for some $r \geq 0$, that

$$\sum_{n=1}^{\infty} (n+1)^{r-1} E_n(f)_v < \infty. \tag{16}$$

For any $r \geq 0$, functions f satisfying (16) belong to the Besov type space $B_r(v)$ that is a Banach space with norm

$$\|f\|_{B_r(v)} := \|fv\|_\infty + \sum_{n=1}^{\infty} (n+1)^{r-1} E_n(f)_v, \qquad r \geq 0. \tag{17}$$

These spaces have been firstly introduced in [12] for $r > 0$ and in [21] when $r = 0$. In particular, we recall that for all $r \geq 0$, $B_r(v)$ is compactly embedded in C^0_v. Moreover, $0 \leq r_1 < r_2$ implies that $B_{r_2}(v)$ is compactly embedded in $B_{r_1}(v)$ [17, 20].

Now, let us study the mapping properties of the operator $\mathcal{H}^u : f \to \mathcal{H}^u f$, with u as in (2), in the previous spaces with weight $v = u_\pm$ as defined in (3).

In the following, it is useful to observe that for any Jacobi weight u as in (2), only one of the cases below can occur

$$\begin{cases} \text{Case 1:} \;\; u = v^{a,-a} & \text{with } 0 < a < 1 \\ \text{Case 2:} \;\; u = v^{-a,a} & \text{with } 0 < a < 1 \\ \text{Case 3:} \;\; u = v^{a,1-a} & \text{with } 0 < a < 1 \\ \text{Case 4:} \;\; u = v^{-a,a-1} & \text{with } 0 < a < 1 \end{cases} \tag{18}$$

and, consequently, for any $0 < a < 1$, the weights $u_\pm$ defined in (3) are the following

$$u_+ = \begin{cases} v^{a,0} & \text{in Case 1} \\ v^{0,a} & \text{in Case 2} \\ u & \text{in Case 3} \\ 1 & \text{in Case 4} \end{cases} \qquad u_- = \begin{cases} v^{0,a} & \text{in Case 1} \\ v^{a,0} & \text{in Case 2} \\ 1 & \text{in Case 3} \\ u & \text{in Case 4} \end{cases} \tag{19}$$

Moreover, in the sequel we denote by $\mathcal{C}$ a positive constant that may have different values at different occurrences, and we write $\mathcal{C} \neq \mathcal{C}(n, f, \ldots)$ to mean that $\mathcal{C} > 0$ is independent of $n, f, \ldots$.

It is known that the map $\mathcal{H}^u : C^0_{u_+} \to C^0_{u_-}$ cannot be bounded, but it becomes bounded if we restrict the domain to Besov type subspaces. More precisely, we have the following (cf. [21, Th. 3.1] and [30, Th. 2.3]).

Theorem 2.1. *Let $u = v^{\alpha,\beta}$ be a Jacobi weight satisfying (2) and let $u_\pm$ be defined by (3). For all $t \in (-1,1)$ and any $f \in B_0(u_+)$, we have*

$$|\mathcal{H}^u f(t)| u_-(t) \leq \mathcal{C}\left(|f(t)|u_+(t) + \|f\|_{B_0(u_+)}\right), \qquad \mathcal{C} \neq \mathcal{C}(f,t). \tag{20}$$

In particular, this result implies the map $\mathcal{H}^u : B_r(u_+) \to C^0_{u_-}$ is a bounded map for all $r \geq 0$. Here we state the following stronger result

Theorem 2.2. *Let u and $u_\pm$ be Jacobi weights as in (2) and (3), respectively. For all $r > 0$, the map $\mathcal{H}^u : B_r(u_+) \to B_r(u_-)$ is bounded, i.e. for any $f \in B_r(u_+)$ we have*

$$\|\mathcal{H}^u f\|_{B_r(u_-)} \leq \mathcal{C}\|f\|_{B_r(u_+)}, \qquad \mathcal{C} \neq \mathcal{C}(f). \tag{21}$$

Proof of Theorem 2.2

Let us assume u is one of the weights in (18) and consider the operator $\mathcal{D}^u$ defined as follows

$$\mathcal{D}^u f(t) := A\mathcal{I}^u f(t) + B\mathcal{H}^u f(t), \qquad -1 < t < 1, \tag{22}$$

where $\mathcal{I}^u f(t) := f(t)u(t)$ and, according to (18), we set

$$A := \cos(\pi a) \qquad B := \frac{\sin(\pi a)}{\pi} \cdot \begin{cases} -1 & \text{in Cases 1 and 3,} \\ +1 & \text{in Cases 2 and 4.} \end{cases} \tag{23}$$

We recall that the boundedness of the map $\mathcal{D}^u : B_r(u_+) \to B_r(u_-)$, $r > 0$, has been already proved in [24] for the cases 1 and 2, and in [10] for the cases 3 and 4. More precisely, by [24, (3.8) and (4.28)] and [10, (3.17) and (4.7)] we have

$$\|\mathcal{D}^u f\|_{B_r(u_-)} \le \mathcal{C}\|f\|_{B_r(u_+)}, \qquad \mathcal{C} \neq \mathcal{C}(f), \qquad \forall r > 0.$$

On the other hand, $\mathcal{I}^u : B_r(u_+) \to B_r(u_-)$, $r > 0$, is a bounded map too, since by using [30, Lemma 2.1] and the identity $uu_- = u_+$, we have

$$\|\mathcal{I}^u f\|_{B_r(u_-)} \le \mathcal{C}\|(\mathcal{I}^u f)u_-\|_{B_r} = \|f u_+\|_{B_r} \le \mathcal{C}\|f\|_{B_r(u_+)}, \qquad \mathcal{C} \neq \mathcal{C}(f), \qquad \forall r > 0,$$

where we used B_r to denote $B_r(v^{0,0})$.

Hence the statement follows by taking into account that by (22) we have

$$\mathcal{H}^u f(t) = -\frac{A}{B}\mathcal{I}^u f(t) + \frac{1}{B}\mathcal{D}^u f(t), \qquad -1 < t < 1. \tag{24}$$

□

3 The proposed VP rule

For the numerical approximation of $\mathcal{H}^u f(t)$, we recall that in [30] the quadrature rule (7)–(9) has been obtained by replacing f with the following polynomial

$$V_n^m(w, f, x) := \sum_{k=1}^{n} f(x_{n,k}^w)\Phi_{n,k}^m(w, x), \qquad x \in [-1, 1], \tag{25}$$

where, using the notation previously introduced, we have

$$\Phi_{n,k}^m(w, x) := \lambda_{n,k}^w \sum_{j=0}^{n+m-1} \mu_{n,j}^m p_j(w, x)p_j(w, x_{n,k}^w), \qquad x \in [-1, 1]. \tag{26}$$

The polynomial $V_n^m(w, f, x)$ is known as filtered VP polynomial of f corresponding to the Jacobi weight w and the degree–parameters $n, m \in \mathbb{N}$ with $m < n$. It comes from a generalization of trigonometric de la Vallée Poussin means and is a near-best polynomial approximation of f w.r.t. uniform (suitably weighted) norms. For more details on its properties, we refer the reader to [7, 31, 32, 33, 36, 37].

The HT of the VP polynomial, namely $\mathcal{H}^u[V_n^m(w, f)](t)$ can be written in the form (7) that evidences the elements $\rho_{n,j}^m(w, f)$ (given by (9)) that are independent of t and therefore calculable only once for several t. The main difficulty lies in the computation of the functions $Q_j(w, t)$ defined in (8) as the HT of the Jacobi polynomial $p_j(w)$. Regarding these functions, we recall the following result has been deduced from the three-term recurrence relation (see, e.g.,[35])

$$b_{j+1}p_{j+1}(w, x) = (x - a_j)p_j(w, x) - b_j p_{j-1}(w, x), \qquad j = 0, 1, \dots, \tag{27}$$

where we set $p_k(w, x) = 0$ for any $k < 0$ and for $w = v^{\gamma,\delta}$ it is

$$p_0(w, x) = \left(\int_{-1}^{1} w(x)dx\right)^{-\frac{1}{2}} = \left(2^{\gamma+\delta+1}\frac{\Gamma(\gamma+1)\Gamma(\delta+1)}{\Gamma(\gamma+\delta+2)}\right)^{-\frac{1}{2}}, \tag{28}$$

and

$$\begin{aligned}
a_0 &= \frac{\delta-\gamma}{\gamma+\delta+2}, & b_0 &= 0, \\
a_1 &= \frac{\delta^2-\gamma^2}{(2+\gamma+\delta)(4+\gamma+\delta)}, & b_1 &= \sqrt{\frac{4(1+\gamma)(1+\delta)}{(2+\gamma+\delta)^2(3+\gamma+\delta)}}, \\
a_j &= \frac{\delta^2-\gamma^2}{(2j+\gamma+\delta)(2j+\gamma+\delta+2)}, & b_j &= \sqrt{\frac{4j(j+\gamma)(j+\delta)(j+\gamma+\delta)}{(2j+\gamma+\delta)^2(2j+\gamma+\delta+1)(2j+\gamma+\delta-1)}}, \quad j \ge 2.
\end{aligned} \tag{29}$$

Proposition 3.1. *[30, Proposition 4.1] For all Jacobi weights $u = v^{\alpha,\beta}$ and $w = v^{\gamma,\delta}$, and any $t \in (-1, 1)$, the functions $Q_j(t) := Q_j^u(w, t)$ defined in (8) satisfy the following three-term recurrence relation*

$$Q_{j+1}(t) = (A_j t + B_j)Q_j(t) - C_j Q_{j-1}(t) + D_j, \qquad j = 0, 1, \dots, \tag{30}$$

where the starting values are given by

$$Q_{-1}(t) = 0, \qquad Q_0(t) = p_0(w, t)\int_{-1}^{1}\frac{u(x)}{x-t}dt, \tag{31}$$

and, for all $j \ge 0$, the coefficients are defined by means of the coefficients in (29), as follows

$$A_j = \frac{1}{b_{j+1}}, \quad B_j = -\frac{a_j}{b_{j+1}}, \quad C_j = \frac{b_j}{b_{j+1}}, \quad D_j = \frac{1}{b_{j+1}}\int_{-1}^{1} p_j(w, x)u(x)dx.$$

Now let us focus on the case $w=u$ with u given by (2) or, equivalently, by (18).

The first aspect we underline concerns the computation of the starting value $Q_0(t)$ in Proposition 3.1 that, according to formula (31), involves the HT of the weight function u. We recall that for general Jacobi weights $u=v^{\alpha,\beta}$ with $\alpha,\beta>-1$ s.t. $\alpha+\beta\neq-1$ we have (see e.g. [15, p.290])

$$\int_{-1}^{1}\frac{u(x)}{x-t}dt=u(t)\pi\cot(\pi\alpha)-\frac{2^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\,{}_2F_1\left(-\alpha-\beta,1;1-\alpha;\frac{1-t}{2}\right), \tag{32}$$

where, as usual, Γ denotes the Gamma function, and ${}_2F_1$ the Hypergeometric function [15]. On the other hand, according to the analysis in [14], the computation by (32) becomes unstable for α "close" to 0 or 1. However, this problem can be overcome if $u=v^{\alpha,\beta}$ satisfies (2) since in this special case we have (see e.g. [5, p. 49])

$$\int_{-1}^{1}\frac{u(x)}{x-t}dt=u(t)\pi\cot(\pi\alpha)-\begin{cases}0 & \text{if } \alpha+\beta=-1,\\ \dfrac{\pi}{\sin(\pi\alpha)} & \text{if } \alpha+\beta=0,\\ \dfrac{\pi(1+t-2\alpha)}{\sin(\pi\alpha)} & \text{if } \alpha+\beta=+1.\end{cases} \tag{33}$$

Consequently, we can say that when α is close to 0 or 1 the starting value $Q_0(t)$ in Proposition 3.2 (and hence the whole quadrature) results to be more accurate and easier to compute if we can use formula (33) instead of the general formula (32). The different performance of these formulae is highlighted by the following experiment concerning with the VP rule for computing

$$\mathcal{H}^u f(t)=\int_{-1}^{1}\frac{\sin(x)}{x-t}u(x)dx,\qquad u=v^{\alpha,1-\alpha}$$

where $0<\alpha<1$ is closer and closer to 1. In Table 1, for increasing values of n, we report the maximum of the quadrature errors in (12) attained for $t\in\{-0.9,\ -0.8,\ -0.7,\ldots..0.9\}$ by fixing $w=u$ and computing Q_0 by means of formula (32) ($e_n^{VP\ gen}$ in the table) or by means of (33) ($e_n^{VP\ spec}$ in the table).

$\alpha=0.99$			$\alpha=0.999$			$\alpha=0.99999$		
n	$e_n^{VP\ gen}$	$e_n^{VP\ spec}$	n	$e_n^{VP\ gen}$	$e_n^{VP\ spec}$	n	$e_n^{VP\ gen}$	$e_n^{VP\ spec}$
11	7.73e-11	7.71e-11	11	2.66e-10	7.86e-11	11	4.13e-07	9.49e-11
31	1.86e-13	3.66e-14	31	1.87e-10	3.00e-13	31	4.13e-07	2.61e-11
51	1.50e-13	9.55e-14	51	1.87e-10	3.96e-13	51	3.44e-07	2.74e-11
71	1.90e-13	4.65e-14	71	1.20e-10	2.49e-13	71	4.13e-07	1.64e-11
91	5.16e-13	3.49e-13	91	1.87e-10	2.39e-13	91	4.13e-07	2.74e-11
111	2.44e-13	1.19e-13	111	1.87e-10	2.75e-13	111	4.13e-07	2.74e-11

Table 1: Test on the accurate computation of Q_0

For $\alpha=0.99$, eventhough the errors attained by means of (33) seem to show a little loss of accuracy as n increases, the results are little bit better than those attained by implementing the rule with (32). For $\alpha=0.999$ and $\alpha=0.99999$, both methods saturate, but the results attained by (33) are much better than those by (32), providing 3 more digits at least.

Regarding the computation of functions $Q_j(t)$ with $j>0$, we note that if we choose $w=u$ then also the recurrence relation (30) is simplified since we have $D_j=0,\ \forall j\geq 1$ due to the orthogonality of the polynomial system $\{p_n(u)\}_n$.

We remark that, in the general case, the coefficients $\{D_j\}_{j=1,..,n+m-1}$ can be exactly computed by means of the Gauss-Jacobi rule based on the weight u and $M:=\lfloor\frac{n+m+1}{2}\rfloor$ nodes. This requires at least $2(n+m)^2$ long operations leaving apart the additional cost in computing the needed quadrature nodes and weights. Hence the choice $w=u$ certainly implies a saving in the computation of the functions $\{Q_j\}_j$ using (30).

Moreover, whenever $w=u$ with u as in (2), we also have a different approach to computing these functions. Indeed, in such special cases, they are explicitly known [34] as specified by the following proposition.

Proposition 3.2. *If $w=u=v^{\alpha,\beta}$ holds with $0<|\alpha|,|\beta|<1$ and $\sigma=(\alpha+\beta)\in\{-1,0,1\}$, then the functions $Q_j^u(w,t)$ defined in (8) are given by the following formula*

$$Q_j^u(w,t)=\pi\cot(\pi\alpha)p_j(u,t)u(t)-\frac{\pi}{\sin(\pi\alpha)}p_{j+\sigma}(u^{-1},t),\qquad |t|<1,\qquad j\geq 0, \tag{34}$$

where it is agreed that $p_k(u^{-1},t)=0$ in the case $k<0$.

Proof of Proposition 3.2. Let $u=v^{\alpha,\beta}$ be one of the weights in (18), where we set

$$a=\begin{cases}\alpha & \text{in Cases 1 and 3},\\ -\alpha & \text{in Cases 2 and 4}.\end{cases}$$

By (8) and (22)–(24), we get

$$Q_j^u(u,t)=\mathcal{H}^u[p_j(u)](t)=-\frac{A}{B}u(t)p_j(u,t)+\frac{1}{B}\mathcal{D}^u[p_j(u)](t),$$

where, according to (23) we note that

$$-\frac{A}{B}=\pi\cot(\pi\alpha),\qquad \text{and}\qquad \frac{1}{B}=-\frac{\pi}{\sin(\pi\alpha)}.$$

Hence, the statement follows recalling that by [34, Th. 9.9, Th. 9.14] (see also [24, (4.7)] and [10, (3.9) and (3.10)]) we have

$$\mathcal{D}^u[p_j(u)]=p_{j+\sigma}(u^{-1}),\qquad \forall j\ge 0. \tag{35}$$

□

We conclude the section by giving a different expression of the VP rule (7) when the hypotheses of Proposition 3.2 are satisfied. More precisely, we observe that by collecting formulae (7), (9) and (34), and recalling the definition of VP polynomial given by (25)–(26), for all $u=v^{\alpha,\beta}$ as in (2), we get

$$\mathcal{H}^u_{n,m}(u,f,t)=\pi\cot(\pi\alpha)u(t)V^m_n(u,f,t)-\frac{\pi}{\sin(\pi\alpha)}\tilde{V}^m_n(u,f,t), \tag{36}$$

where we set

$$\tilde{V}^m_n(u,f,t):=\sum_{k=1}^{n}\lambda^u_{n,k}h(t,x^u_{n,k})f(x^u_{n,k}),\qquad n>m, \tag{37}$$

and

$$h(t,x):=\sum_{j=0}^{n+m-1}\mu^m_{n,j}p_{j+\sigma}(u^{-1},t)p_j(u,x),\quad -1\le t,x\le 1, \tag{38}$$

being $\sigma=\alpha+\beta$ and $\mu^m_{n,j}$ defined by (11).

Formula (36) will be used in the next section.

4 Quadrature error estimates

In this section we are going to analyze the error function in (12) when $w=u$ and (2) holds. In this case we adopt the following simplified notation:

$$\begin{aligned} H^m_n f(t)&:=\mathcal{H}^u_{n,m}(u,f,t), &\qquad \mathcal{E}^u_{n,m}f(t)&:=\mathcal{E}^u_{n,m}(u,f,t),\\ V^m_n f(t)&:=V^m_n(u,f,t), &\qquad \tilde{V}^m_n f(t)&:=\tilde{V}^m_n(u,f,t).\end{aligned}$$

Moreover, we use the notation $m\approx n$ to mean that there exist two fixed constants $c_2\ge c_1>1$ independent of $m,n\in\mathbb{N}$, such that

$$c_1 m\le n\le c_2 m. \tag{39}$$

Note that this condition ensures that as the number of nodes $n\to\infty$, also m and $n-m$ tend to ∞ with the same order. If $m\approx n$ in [30] the authors proved that for all $f\in B_0(u_+)$ one has

$$\lim_{n\to+\infty}\mathcal{E}^u_{n,m}f(t)u_-(t)=0,\qquad \text{uniformly w.r.t. } t\in[-1,1]. \tag{40}$$

Moreover, the convergence rate depends on two components: the pointwise approximation provided by the VP polynomial of f at the specific $t\in(-1,1)$, and the degree of smoothness of f. More precisely, we have the following

Theorem 4.1. *[30, Corollary 4.3] Let $u=u_+/u_-$ be Jacobi weights as in (2), (3). For all $m\approx n$, any $f\in B_r(u_+)$, $r>0$, and each $t\in(-1,1)$, we have*

$$\mathcal{E}^u_{n,m}f(t)u_-(t)\le\mathcal{C}\left|f(t)-V^m_n f(t)\right|u_+(t)+\mathcal{C}\,\frac{\log n}{n^r}\,\|f\|_{B_r(u_+)},\qquad \mathcal{C}\ne\mathcal{C}(n,f,t). \tag{41}$$

As regards the first addendum in (41), we recall that $V^m_n f$ is a near–best polynomial approximation of f in $C^0_{u_+}$, i.e. [36]

$$E_{n+m-1}(f)_{u_+}\le\|(f-V^m_n f)u_+\|_\infty\le\mathcal{C}E_{n-m}(f)_{u_+},\qquad \mathcal{C}\ne\mathcal{C}(n,m,f), \tag{42}$$

holds for all $f\in C^0_{u_+}$ and any $m\approx n$. In particular, since the condition (39) implies that, as $n\to+\infty$, also $(n\pm m)\to+\infty$, we get $\|(f-V^m_n f)u_+\|_\infty\to 0$ at the same rate of $E_n(f)_{u_+}$. The latter can be characterized by the following main part modulus of smoothness introduced in [11] by Z. Ditzian and V. Totik

$$\Omega^k_\varphi(f,\tau)_v:=\sup_{0<h\le\tau}\|v\Delta^k_{h\varphi}f\|_{L^\infty[-1+2h^2k^2,1-2h^2k^2]},$$

where

$$\Delta^k_{h\varphi}(f,x):=\sum_{i=0}^{k}(-1)^i\binom{k}{i}f\left(x+\frac{kh}{2}\varphi(x)-ih\varphi(x)\right),\qquad \varphi(x):=\sqrt{1-x^2}.$$

More precisely, we recall that for n sufficiently large and τ sufficiently small, the following Jackson and Stechkin type inequalities hold for all $f \in B_0(u_+)$ and any $k \in \mathbb{N}$ [11, Th. 8.2.1]

$$E_n(f)_{u_+} \le \mathcal{C}\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^k(f,\tau)_{u_+}}{\tau}d\tau, \qquad \mathcal{C} \neq \mathcal{C}(n,f), \tag{43}$$

$$\Omega_\varphi^k(f,\tau)_{u_+} \le \mathcal{C}\tau^k \sum_{n=0}^{[1/\tau]} (n+1)^{k-1} E_n(f)_{u_+}, \qquad \mathcal{C} \neq \mathcal{C}(f,\tau). \tag{44}$$

These inequalities imply the Besov norm can be equivalently expressed in the following form [12, Th. 3.1]

$$\|f\|_{B_r(u_+)} \sim \|f u_+\|_\infty + \int_0^1 \frac{\Omega_\varphi^k(f,\tau)_{u_+}}{\tau^{r+1}}d\tau, \qquad k > r > 0, \tag{45}$$

and in particular, for all (sufficiently large) $n \in \mathbb{N}$, we have

$$E_n(f)_{u_+} \le \mathcal{C}\frac{\|f\|_{B_r(u_+)}}{n^r}, \qquad \forall f \in B_r(u_+),\ r > 0, \quad \mathcal{C} \neq \mathcal{C}(n,f). \tag{46}$$

Hence, by (42), for any $t \in (-1,1)$ and $m \approx n$ sufficiently large, we deduce the first addendum in (41) can be bounded analogously to (43) and (46), namely

$$\left|f(t) - V_n^m f(t)\right| u_+(t) \le \mathcal{C}\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^k(f,\tau)_{u_+}}{\tau}d\tau, \qquad \forall f \in B_0(u_+), \tag{47}$$

$$\left|f(t) - V_n^m f(t)\right| u_+(t) \le \mathcal{C}\frac{\|f\|_{B_r(u_+)}}{n^r}, \qquad \forall f \in B_r(u_+), \quad r > 0, \tag{48}$$

hold with $\mathcal{C} \neq \mathcal{C}(n,f,t)$.

Nevertheless, in the case $f \in B_r(u_+)$ with $r > 0$, the estimate (41) does not ensure the same optimal behavior for the quadrature error $\mathcal{E}_{n,m}^u$, due to the presence of the logarithmic factor in the second addend to the right hand side of (41). Moreover, in the case we only know that $f \in B_0(u_+)$, Theorem 4.1 gives no input on the behavior of the quadrature error. Here we fill these gaps with the following

Theorem 4.2. *For all Jacobi weights $u = u_+/u_-$ as in (2), (3), for any $m \approx n$ with $n \in \mathbb{N}$ sufficiently large, for each $f \in B_0(u_+)$ and for all $t \in (-1,1)$, we have*

$$\mathcal{E}_{n,m}^u f(t) u_-(t) \le \mathcal{C}\left|f(t) - V_n^m f(t)\right| u_+(t) + \mathcal{C}\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^k(f,\tau)_{u_+}}{\tau}d\tau, \tag{49}$$

where $\mathcal{C} \neq \mathcal{C}(n,f,t)$.

Before giving the proof, let us analyze some consequences of this result.

We remark that, by virtue of (45), Theorem 4.2 ensures that (41) continues to hold if the $\log n$ factor is removed, namely if the hypotheses of Theorem 4.1 are satisfied then we have

$$\mathcal{E}_{n,m}^u f(t) u_-(t) \le \mathcal{C}\left|f(t) - V_n^m f(t)\right| u_+(t) + \mathcal{C}\,\frac{\|f\|_{B_r(u_+)}}{n^r}, \qquad \mathcal{C} \neq \mathcal{C}(n,f,t).$$

Moreover, recalling (47), (48), under the assumption of Theorem 4.2 we get the quadrature error satisfies

$$\mathcal{E}_{n,m}^u f(t) u_-(t) \le \mathcal{C}\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^k(f,\tau)_{u_+}}{\tau}d\tau, \qquad \forall f \in B_0(u_+), \qquad \mathcal{C} \neq \mathcal{C}(n,f,t), \tag{50}$$

$$\mathcal{E}_{n,m}^u f(t) u_-(t) \le \mathcal{C}\frac{\|f\|_{B_r(u_+)}}{n^r}, \qquad \forall f \in B_r(u_+),\ r > 0, \qquad \mathcal{C} \neq \mathcal{C}(n,f,t). \tag{51}$$

Taking into account that these estimates are analogous to (43) and (46), we conclude Theorem 4.2 guarantees that, as $m \approx n \to +\infty$, the quadrature error converges to zero with the same rate of the error of best approximation $E_n(f)_{u_+}$, i.e. (14) holds.

In order to prove Theorem 4.2, we premise the following Lemma:

Lemma 4.3. *Let $u = v^{\alpha,\beta}$ satisfies (2) and let $u_\pm$ be given by (3). For all $m \approx n$, any $t \in (-1,1)$ and each $f \in C_{u_+}^0$, we have*

$$\left\|u_- \tilde{V}_n^m f\right\|_\infty \le \mathcal{C}\max_{1\le k\le n} |f(x_k)| u_+(x_k), \qquad \mathcal{C} \neq \mathcal{C}(n,f), \tag{52}$$

where $\tilde{V}_n^m f(t)$ is defined by (37) and $\{x_k = x_{n,k}^u : k = 1,..,n\}$ is the set of the zeros of $p_n(u,x)$.

Proof of Lemma 4.3

First of all we note that $\tilde{V}_n^m f(t)$ is a polynomial of degree at most $(n+m-1)$ with $m \approx n$. Hence by Remez type inequality [11, (8.1.4)], we get

$$\|u_-\tilde{V}_n^m f\|_\infty \le \mathcal{C} \max_{|t|\le 1-\frac{\mathcal{C}}{n^2}} u_-(t)|\tilde{V}_n^m f(t)|, \qquad \mathcal{C} \ne \mathcal{C}(n,f).$$

On the other hand, using the definition (37) and setting for brevity $x_k := x_{n,k}^u$ and $\lambda_k := \lambda_{n,k}^u$, for any $|t| \le 1-\frac{\mathcal{C}}{n^2}$ we have

$$\begin{aligned}
\left|\tilde{V}_n^m f(t)\right| u_-(t) &= \left|\sum_{k=1}^n \lambda_k h(t,x_k) f(x_k)\right| u_-(t) \\
&\le \sum_{k=1}^n \lambda_k |h(t,x_k) f(x_k)| u_-(t) \\
&\le \left(\max_{1\le k\le n} |f(x_k)| u_+(x_k)\right) \sum_{k=1}^n \lambda_k \frac{|h(t,x_k)|}{u_+(x_k)} u_-(t).
\end{aligned}$$

Thus, in order to obtain (52) it remains to prove that

$$S(t) := u_-(t) \sum_{k=1}^n \lambda_k \frac{|h(t,x_k)|}{u_+(x_k)} \le \mathcal{C}, \qquad \forall |t| \le 1-\frac{\mathcal{C}}{n^2}, \tag{53}$$

where here and in the following, by $\mathcal{C}$ we always intend constants satisfying $\mathcal{C} \ne \mathcal{C}(n,t,k,f)$.

Now, we suppose that

$$x_0 := -1 < x_1 < x_2 < \ldots < x_n < x_{n+1} := 1,$$

and setting $\Delta x_k := x_{k+1} - x_k$, we recall that [1]

$$\lambda_k \sim u(x_k)\Delta x_k, \quad \text{and} \quad \Delta x_k \sim \frac{\sqrt{1-x_k^2}}{n}, \qquad k=1,\ldots,n,$$

where by $A \sim B$ we mean that $\mathcal{C}^{-1}B \le A \le \mathcal{C}B$ holds.

Consequently, since $u = u_+/u_-$, we have

$$S(t) \le \mathcal{C} u_-(t) \sum_{k=1}^n \frac{|h(t,x_k)|}{u_-(x_k)} \Delta x_k. \tag{54}$$

In order to estimate such sum, we consider different cases corresponding to the different positions of the nodes x_k w.r.t. $|t| \le 1-\frac{\mathcal{C}}{n^2}$ (arbitrarily fixed). As a consequence the sum at the right–hand side in (54) will be divided in several parts and the proof of (53) will be given at the end of all examined cases.

Case I: $|t-x_k| \le \mathcal{C}\frac{\sqrt{1-|t|}}{n}$

In this case, recalling that $|t| \le 1-\frac{\mathcal{C}}{n^2}$, it can be easily proved that

$$(1 \pm t) \sim (1 \pm x_k). \tag{55}$$

Moreover, we recall that for any Jacobi weight v the following estimate holds true [1]

$$|p_n(v,t)| \le \frac{\mathcal{C}}{\sqrt{v(t)}\sqrt[4]{1-t^2}}, \qquad \forall |t| \le 1-\frac{\mathcal{C}}{n^2}. \tag{56}$$

Hence, by using (56) in (38) and taking into account (55) we get

$$|h(t,x_k)| \le \mathcal{C}\frac{n}{\sqrt{u^{-1}(t)u(x_k)}\sqrt[4]{(1-t^2)(1-x_k^2)}} \le \mathcal{C}\frac{n}{\sqrt{1-t^2}}.$$

Consequently, setting

$$S_1(t) := u_-(t) \sum_{k:|t-x_k|\le\frac{\sqrt{1-|t|}}{n}} \frac{|h(t,x_k)|}{u_-(x_k)} \Delta x_k,$$

we have

$$S_1(t) \le \mathcal{C}\frac{n}{\sqrt{1-t^2}} \sum_{k:|t-x_k|\le\frac{\sqrt{1-|t|}}{n}} \Delta x_k \le \mathcal{C}. \tag{57}$$

Case II: $|t-x_k| \ge \mathcal{C}\frac{\sqrt{1-|t|}}{n}$

Starting from (38) and using the Abel transformation

$$\sum_{r=0}^N A_r B_r = A_N \sum_{r=0}^N B_r - \sum_{r=0}^{N-1} (A_{r+1}-A_r) \sum_{j=0}^r B_j,$$

with $N = n + m - 1$, $A_r = \mu_{n,r}^m$ and $B_r = p_{r+\sigma}(u^{-1}, t)p_r(u, x)$, by (11) we easily get

$$h(t,x) = \frac{1}{2m}\sum_{r=n-m}^{n+m-1}\left[\sum_{j=0}^{r} p_{j+\sigma}(u^{-1},t)p_j(u,x)\right]. \tag{58}$$

On the other hand, for all $t \neq x$, using the recurrence relation of Jacobi polynomials (27)–(29) and proceeding as in the proof of Darboux's formula, taking into account that (2) holds, by simple calculations (see, e.g., [24, pp. 83–84]) we get

$$\sum_{j=0}^{r} p_{j+\sigma}(u^{-1},t)p_j(u,x) = b_{r+1}\frac{p_{r+1}(u,x)p_{r+\sigma}(u^{-1},t) - p_r(u,x)p_{r+\sigma+1}(u^{-1},t)}{x-t} - \frac{\sin(\pi\alpha)}{\pi}\frac{1}{x-t},$$

where b_{r+1} is given by (29) taking $\gamma = \alpha$ and $\delta = \beta$.

Thus we have the following equivalent form of the kernel in (58)

$$h(t,x) = g(t,x) - \frac{\sin(\pi\alpha)}{\pi}\frac{1}{x-t}, \qquad t \neq x, \tag{59}$$

where, for all $t \neq x$, we set

$$g(t,x) := \frac{1}{2m}\sum_{r=n-m}^{n+m-1}\left[b_{r+1}\frac{p_{r+1}(u,x)p_{r+\sigma}(u^{-1},t) - p_r(u,x)p_{r+\sigma+1}(u^{-1},t)}{x-t}\right].$$

A pointwise estimate of such function can be obtained by using the asymptotic formula (see e.g. [35, Theorem 8.21.13])

$$p_r(v^{\gamma,\delta}, \cos\theta) = \gamma_r v^{-\frac{\gamma}{2}-\frac{1}{4}, -\frac{\delta}{2}-\frac{1}{4}}(\cos\theta)\left[\cos(r\theta + \rho) + \frac{\mathcal{O}(1)}{r\sin\theta}\right], \qquad \frac{\mathcal{C}}{r} \leq \theta \leq \pi - \frac{\mathcal{C}}{r}, \qquad r \in \mathbb{N}, \tag{60}$$

where $\rho := \frac{(2\gamma+2\delta+1)\theta}{2} - \frac{\pi}{2}(\gamma + \frac{1}{2})$, and $\gamma_r := \sqrt{\frac{(2r+\gamma+\delta+1)\Gamma(r+1)\Gamma(r+\gamma+\delta+1)}{\pi r\Gamma(r+\gamma+1)\Gamma(r+\delta+1)}}$.

For the sake of brevity we omit the details, but following the same lines of the proof of [24, Lemma 5.2] (see also [10, Proposition 5.3]), since $u = v^{\alpha,\beta}$, we get

$$|g(t,x)| \leq \mathcal{C}\frac{v^{-\frac{\alpha}{2}-\frac{1}{4}, -\frac{\beta}{2}-\frac{1}{4}}(x)v^{\frac{\alpha}{2}-\frac{1}{4}, \frac{\beta}{2}-\frac{1}{4}}(t)}{n|t-x|}\begin{cases}\frac{1-|t|}{|x-t|}, & \text{if } x \in A,\\ 1, & \text{if } x \in B,\end{cases} \tag{61}$$

where

$$A := \left\{x : \frac{\mathcal{C}\sqrt{1-|t|}}{n} \leq |t-x| \leq \frac{1-|t|}{2}\right\}, \qquad B := \left\{x : |t-x| \geq \frac{1-|t|}{2}\right\}.$$

In the sequel, we are going to use (61) in order to estimate the following sums

$$\begin{aligned} S_2(t) &:= u_-(t)\sum_{k:\ x_k \in A}\frac{|g(t,x_k)|}{u_-(x_k)}\Delta x_k,\\ S_3(t) &:= u_-(t)\sum_{k:\ x_k \in B}\frac{|g(t,x_k)|}{u_-(x_k)}\Delta x_k. \end{aligned}$$

Regarding the sum S_2, applying (61), taking into account that (55) holds for $x_k \in A$ too, and using $(1-t^2) \sim 1-|t|$, we get

$$\begin{aligned} S_2(t) &\leq \frac{\mathcal{C}}{n}\sum_{k:\ x_k \in A}\frac{1-|t|}{(t-x_k)^2\sqrt{1-t^2}}\Delta x_k \leq \mathcal{C}\frac{\sqrt{1-|t|}}{n}\sum_{k:\ x_k \in A}\frac{\Delta x_k}{(x_k-t)^2}\\ &\leq \mathcal{C}\frac{\sqrt{1-|t|}}{n}\int_{y\in A}\frac{dy}{(y-t)^2} \leq \mathcal{C}. \end{aligned} \tag{62}$$

Now let us estimate the sum S_3. Note that from (61) we deduce

$$S_3(t) \leq \frac{\mathcal{C}}{n}v^{\frac{|\alpha|}{2}-\frac{1}{4}, \frac{|\beta|}{2}-\frac{1}{4}}(t)\sum_{k:\ x_k \in B}\frac{v^{-\frac{|\alpha|}{2}-\frac{1}{4}, -\frac{|\beta|}{2}-\frac{1}{4}}(x_k)}{|t-x_k|}\Delta x_k. \tag{63}$$

On the other hand, denoted by d the index of the zero x_d closest to t, namely

$$|x_d - t| = \min_{1\leq k\leq n}|x_k - t|,$$

it is known that [23, Lemma 4.1.1]

$$\sum_{k\neq d}\frac{v^{\gamma,\delta}(x_k)}{|t-x_k|}\Delta x_k \leq \mathcal{C}v^{\gamma,\delta}(t), \qquad -1 < \gamma, \delta < 0. \tag{64}$$

Since the hypothesis $0<|\alpha|,|\beta|<1$ ensures $\gamma:=-\frac{|\alpha|}{2}-\frac{1}{4}$ and $\delta:=-\frac{|\beta|}{2}-\frac{1}{4}$ belong to $(-1,0)$, we note that (64) can be applied to (63). Hence, the previous estimate continues as follows

$$S_3(t)\leq \frac{\mathcal{C}}{n}\, v^{\frac{|\alpha|}{2}-\frac{1}{4},\frac{|\beta|}{2}-\frac{1}{4}}(t)\sum_{k:\ x_k\in B}\frac{v^{-\frac{|\alpha|}{2}-\frac{1}{4},-\frac{|\beta|}{2}-\frac{1}{4}}(x_k)}{|t-x_k|}\Delta x_k\leq \frac{\mathcal{C}}{n\sqrt{1-t^2}}\leq \mathcal{C}, \tag{65}$$

having used $(1-t^2)\sim 1-|t|\geq \frac{\mathcal{C}}{n^2}$ in the last inequality.

Proof of (53)

By using (59) and (64), if we start from (54) and apply the estimates (57), (62) and (65), then we obtain (53) as follows

$$\begin{aligned}S(t) &\leq S_1(t)+u_-(t)\sum_{k:|t-x_k|\geq\frac{\sqrt{1-|t|}}{n}}\frac{|h(t,x_k)|}{u_-(x_k)}\Delta x_k\\ &\leq \mathcal{C}+\mathcal{C}u_-(t)\sum_{k:|t-x_k|\geq\frac{\sqrt{1-|t|}}{n}}\frac{1}{u_-(x_k)}\left[|g(t,x_k)|+\frac{1}{|x_k-t|}\right]\Delta x_k\\ &\leq \mathcal{C}+\mathcal{C}S_2(t)+\ \mathcal{C}S_3(t)+\mathcal{C}u_-(t)\sum_{k\neq d}\frac{1}{u_-(x_k)|x_k-t|}\Delta x_k\leq \mathcal{C}.\end{aligned}$$

□

Now we are able to state the proof of Theorem 4.2.

Proof of Theorem 4.2.

First of all we note that, by virtue of (40), we have

$$u_-(t)\left[\mathcal{H}^u f(t)-H_n^m f(t)\right]=u_-(t)\sum_{j=0}^{\infty}\left[H_{2^{j+1}n}^{2^{j+1}m}f(t)-H_{2^jn}^{2^jm}f(t)\right],\quad |t|<1. \tag{66}$$

Throughout this proof, for brevity, corresponding to any $j\geq 0$ we set

$$N:=2^jn,\qquad M:=2^jm\qquad\text{and}\qquad F=f-P^*,$$

where $P^*\in\mathbb{P}_{N-M}$ satisfies

$$\max_{x\in I_N}|f(x)-P^*(x)|u_+(x)\leq \mathcal{C}\Omega_\varphi^s\left(f,\frac{1}{N}\right)_{u_+},\qquad \mathcal{C}\neq\mathcal{C}(N,f), \tag{67}$$

with $I_N:=[-1+\mathcal{C}N^{-2},1-\mathcal{C}N^{-2}]$, $\mathcal{C}\neq\mathcal{C}(N)$, and $s\in\mathbb{N}$ such that $s\leq n$. We remark the existence of such a polynomial P^* follows from [11, Eq. (8.2.4)] by taking into account that $m\approx n$.

Recalling that [36]

$$V_N^M P=P,\qquad \forall P\in\mathbb{P}_{N-M}, \tag{68}$$

we deduce

$$\mathcal{H}^u P(t)=\mathcal{H}^u[V_N^M P](t)=H_N^M P(t),\qquad |t|<1,\quad \forall P\in\mathbb{P}_{N-M}, \tag{69}$$

and consequently

$$H_{2N}^{2M}f(t)-H_N^M f(t)=H_{2N}^{2M}F(t)-H_N^M F(t),\quad |t|<1.$$

Hence, under the previous setting, we can write (66) as follows

$$u_-(t)\left[\mathcal{H}^u f(t)-H_n^m f(t)\right]=u_-(t)\sum_{j=0}^{\infty}\left[H_{2N}^{2M}F(t)-H_N^M F(t)\right],\quad |t|<1. \tag{70}$$

On the other hand, we recall that for all integers $n>m>0$, by (36), we have

$$H_n^m f(t)u_-(t)=\pi\cot(\pi\alpha)u_+(t)V_n^m f(t)-\frac{\pi}{\sin(\pi\alpha)}u_-(t)\tilde{V}_n^m f(t), \tag{71}$$

where the first addendum satisfies

$$\lim_{\substack{n\to+\infty\\ m\approx n}}\|(f-V_n^m f)u_+\|_\infty=0,\qquad \forall f\in C_{u_+}^0. \tag{72}$$

Summing up, by (70) and (71), we have

$$\begin{aligned}&u_-(t)\mathcal{E}_{n,m}^u f(t)=u_-(t)\left|\mathcal{H}^u f(t)-H_n^m f(t)\right|\\ =\ &u_-(t)\left|\sum_{j=0}^{\infty}\left[H_{2N}^{2M}F(t)-H_N^M F(t)\right]\right|\\ \leq\ &\mathcal{C}u_+(t)\left|\sum_{j=0}^{\infty}\left[V_{2N}^{2M}F(t)-V_N^M F(t)\right]\right|+\mathcal{C}\sum_{j=0}^{\infty}u_-(t)\left|\tilde{V}_{2N}^{2M}F(t)-\tilde{V}_N^M F(t)\right|,\end{aligned} \tag{73}$$

where $\mathcal{C} \neq \mathcal{C}(t,n,f)$.

On the other hand, from (68) and (72) we deduce

$$\begin{aligned} u_+(t)\left|\sum_{j=0}^{\infty}\left[V_{2N}^{2M}F(t)-V_N^M F(t)\right]\right| &= u_+(t)\left|\sum_{j=0}^{\infty}\left[V_{2N}^{2M}f(t)-V_N^M f(t)\right]\right| \\ &= u_+(t)\left|f(t)-V_n^m f(t)\right|. \end{aligned} \tag{74}$$

Moreover, applying Lemma 4.3 and (67), for any $j \geq 0$, we have

$$\begin{aligned} u_-(t)\left|\tilde{V}_{2N}^{2M}F(t)-\tilde{V}_N^M F(t)\right| &\leq u_-(t)\left|\tilde{V}_{2N}^{2M}F(t)\right|+u_-(t)\left|\tilde{V}_N^M F(t)\right| \\ &\leq \mathcal{C}\max_{x\in I_N}|F(x)|u_+(x) \\ &\leq \mathcal{C}\,\Omega_\varphi^s\left(f,\frac{1}{N}\right)_{u_+} = \mathcal{C}\,\Omega_\varphi^s\left(f,\frac{1}{2^j n}\right)_{u_+}, \quad \mathcal{C}\neq\mathcal{C}(n,j,f), \end{aligned}$$

and using well–known properties of the moduli of smoothness [11], this implies

$$\begin{aligned} \sum_{j=0}^{\infty}u_-(t)\left|\tilde{V}_{2N}^{2M}F(t)-\tilde{V}_N^M F(t)\right| &\leq \mathcal{C}\sum_{j=0}^{\infty}\Omega_\varphi^k\left(f,\frac{1}{2^j n}\right)_{u_+} \\ &\leq \mathcal{C}\sum_{j=0}^{\infty}\int_{\frac{1}{2^{j+1}n}}^{\frac{1}{2^j n}}\frac{\Omega_\varphi^k(f,\tau)_{u_+}}{\tau}\,d\tau \\ &\leq \mathcal{C}\int_0^{\frac{1}{n}}\frac{\Omega_\varphi^k(f,\tau)_{u_+}}{\tau}\,d\tau. \end{aligned} \tag{75}$$

Thus the statement follows from (73)–(75).

□

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