



Colourings of cubic graphs inducing isomorphic monochromatic subgraphs

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Colourings of cubic graphs inducing isomorphic monochromatic subgraphs

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In loving memory of Ella.

Abstract

A k -bisection of a bridgeless cubic graph G is a 2-colouring of its vertex set such that the colour classes have the same cardinality and all connected components in the two subgraphs induced by the colour classes (*monochromatic components* in what follows) have order at most k . Ban and Linial conjectured that *every bridgeless cubic graph admits a 2-bisection except for the Petersen graph*. A similar problem for the edge set of cubic graphs has been studied: Wormald conjectured that *every cubic graph G with $|E(G)| \equiv 0 \pmod{2}$ has a 2-edge colouring such that the two monochromatic subgraphs are isomorphic linear forests* (i.e. a forest whose components are paths). Finally, Ando conjectured that *every cubic graph admits a bisection such that the two induced monochromatic subgraphs are isomorphic*.

In this paper, we give a detailed insight into the conjectures of Ban–Linial and Wormald and provide evidence of a strong relation of both of them with Ando's conjecture. Furthermore, we also give computational and theoretical evidence in their support. As a result, we pose some open problems stronger than the above mentioned conjectures. Moreover, we prove Ban–Linial's conjecture for cubic cycle permutation graphs.

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As a by-product of studying 2-edge colourings of cubic graphs having linear forests as monochromatic components, we also give a negative answer to a problem posed by Jackson and Wormald about certain decompositions of cubic graphs into linear forests.

Keywords: colouring; bisection; linear forest; snark; cycle permutation graph; cubic graph; computation.

1 Introduction

All graphs considered in this paper are finite and simple (without loops or multiple edges). Most of our terminology is standard; for further definitions and notation not explicitly stated in the paper, please refer to [9].

A *bisection* of a cubic graph $G = (V, E)$ is a partition of its vertex set V into two disjoint subsets $(\mathcal{B}, \mathcal{W})$ of the same cardinality. Several times in the rest of the paper, we will identify a bisection of G with the vertex colouring of G with two colours B (black) and W (white), such that every vertex of \mathcal{B} and \mathcal{W} has colour B and W , respectively. Note that the colourings not necessarily have to be proper and in what follows we refer to a connected component of the subgraphs induced by a colour class as a *monochromatic component*. Following this terminology, we define a k -*bisection* of a graph G as a 2-colouring c of the vertex set $V(G)$ such that:

- (i) $|\mathcal{B}| = |\mathcal{W}|$ (i.e. it is a bisection), and
- (ii) each monochromatic component has at most k vertices.

Note that this is equivalent to the $(k + 2)$ -*weak bisections* introduced by Esperet, Tarsi and the fourth author in [16]. Indeed the existence of a k -bisection ($(k + 2)$ -weak bisection) is a necessary condition for the existence of a nowhere-zero $(k + 2)$ -flow, as it directly follows from a result by Jaeger [25] stating that *every cubic graph G with a circular nowhere-zero r -flow has a $\lfloor r \rfloor$ -weak bisection* (cf. [25] and [16] for more details).

There are several papers in literature considering 2-colourings of regular graphs which satisfy condition (ii), but not necessarily condition (i), see [4, 7, 22, 28]. In particular, it is easy to see that every cubic graph has a 2-colouring where all monochromatic connected components are of order at most 2, but, in general, such a colouring does not satisfy condition (i) and so it is not a 2-bisection. Thus, the existence of a 2-bisection in a cubic graph is not guaranteed. For instance, the Petersen graph does not admit a 2-bisection. However, the Petersen graph is an exception since it is the unique known bridgeless cubic graph without a 2-bisection. This led Ban and Linial to pose the following:

Conjecture 1.1 (Ban–Linial [6]) *Every bridgeless cubic graph admits a 2-bisection, except for the Petersen graph.*

Generally speaking, Ban–Linial’s Conjecture asks for a 2–vertex colouring of a cubic graph with certain restrictions on the induced monochromatic components. These restrictions deal with the order of the monochromatic components, hence they also have consequences on the structure of the admissible components. In particular, it is easy to prove that the two induced monochromatic subgraphs are isomorphic.

A similar problem for the edge set of a cubic graph has been studied. A *linear forest* is a forest whose components are paths and a *linear partition* of a graph G is a partition of its edge set into linear forests (cf. [24, 18]). Wormald made the following conjecture:

Conjecture 1.2 (Wormald [33]) *Let G be a cubic graph with $|E(G)| \equiv 0 \pmod{2}$ (or equivalently $|V(G)| \equiv 0 \pmod{4}$). Then there exists a linear partition of G in two isomorphic linear forests.*

Note that the statement of Wormald’s Conjecture is equivalent to the statement that *there exists a 2–edge colouring of $E(G)$ such that the two monochromatic subgraphs are isomorphic linear forests.* (In Section 3, we will define such a 2–edge colouring to be a *Wormald colouring*). Also in this case we are dealing with a colouring problem, an edge colouring one this time, with some restrictions on the admissible monochromatic components. In analogy with the Ban–Linial Conjecture, we will propose a stronger version of Conjecture 1.2 by adding a restriction on the order of the monochromatic components (see Problem 5.6).

The main aim of this paper is to give a detailed insight into the two previous conjectures and moreover to provide evidence for the strong relations of both of them with the following conjecture proposed by Ando [5] in the 90’s.

Conjecture 1.3 (Ando [5]) *Every cubic graph admits a bisection such that the two induced monochromatic subgraphs are isomorphic.*

The paper is organized as follows. In Section 2, we deal with the Ban–Linial’s Conjecture (cf. Conjecture 1.1). First of all, we observe that Ban–Linial’s Conjecture implies Ando’s Conjecture (cf. Proposition 2.3). Moreover, in Section 2.1, we prove Ban–Linial’s Conjecture for bridgeless cubic graphs in the special case of cycle permutation graphs. In Section 3, we show that Ando’s Conjecture is strongly related to a strengthened version of Wormald colourings (see Proposition 3.5).

We would also like to point out that we can give a positive answer for a lot of instances of Ando’s Conjecture using either 2–bisections or Wormald colourings. We find it interesting that the few possible *exceptions* for which the approach with 2–bisections is not feasible, are in the class of *not* 3–edge colourable cubic graphs. On the other hand, all known exceptions for which the approach with our strengthened version of Wormald colourings is not feasible, are in the class of 3–edge colourable cubic graphs. Note that, this indicates a promising possibility for a complete proof of Ando’s Conjecture by proving that a given cubic graph cannot be an exception for both of these two approaches (at least for a bridgeless cubic graph).

In Section 4 we present our computational results. It is known that there are infinitely many counterexamples to Ban–Linial’s Conjecture for 1–connected cubic graphs (see [17]) and in Section 4.1 we produce the complete list of all cubic graphs up to 32 vertices which do not admit a 2–bisection. In Section 4.2 we show that Ando’s conjecture does not have any counterexamples with less than 34 vertices and in Section 4.3 we present our computational results on Wormald’s conjecture.

Finally, in Section 5, as a by–product of our study of 2–edge colourings of cubic graphs having linear forests as monochromatic components, we obtain a negative answer (cf. Theorem 5.3) to a problem about *linear arboricity* posed by Jackson and Wormald in [24] (cf. also [32]). Recall that the linear arboricity of a graph G is the minimum number of forests whose connected components are paths of length at most k required to partition the edge set of a given graph G (cf. Section 5 for details). Moreover, we conclude the paper posing new problems and conjectures on this topic.

2 Ban–Linial Conjecture

We recall that a 2–bisection is a bisection (B, W) such that the connected components in the two induced subgraphs $G[B]$ and $G[W]$ have order at most two. That is: every induced subgraph is a union of isolated vertices and isolated edges.

In [6], Ban and Linial posed Conjecture 1.1 and proved Theorem 2.1 for cubic graphs. Note that the original formulation is in terms of *external bisections*, which are indeed equivalent to 2–bisections for cubic graphs.

Theorem 2.1 [6, Proposition 7] *Every 3–edge colourable cubic graph admits a 2–bisection.*

A stronger version of this last theorem is proven in [17] by Esperet, Tarsi and the fourth author in terms of $(k+2)$ –weak bisections (equivalent to k –bisections). In particular, they prove the following:

Theorem 2.2 [17] *If a bridgeless cubic graph G does not admit a 2–bisection, then it has circular flow number at least 5.*

Because of Theorem 2.2, a possible counterexample to Conjecture 1.1 has to be in the class of bridgeless cubic graphs with circular flow number at least 5 (exactly 5 if Tutte’s famous conjecture is true). An important subclass of bridgeless cubic graphs which are *not* 3–edge colourable is the class of *snarks* (i.e. a subclass of not 3–edge colourable graphs with additional restrictions on the girth and connectivity). There is a vast literature on snarks and their properties – see, e.g., [2, 3, 12, 26, 27, 30]. Snarks are especially interesting as they are often smallest possible counterexamples for conjectures: for many graph theory conjectures it can be proven that if the conjecture is false, the smallest counterexample must be a snark. The interested reader might find more information about snarks in,

e.g., [23] or [34]. Very few snarks with circular flow number 5 are known (see [30], [17] and [3]) and no snark without a 2-bisection, other than the Petersen graph, has been found until now.

Conjecture 1.1 is strongly related to Ando's Conjecture (i.e. Conjecture 1.3) and the following holds:

Proposition 2.3 *Let G be a bridgeless cubic graph. If G is a counterexample for Ando's Conjecture, then it is also a counterexample for the Ban-Linial Conjecture.*

Proof An easy counting argument shows that the number of isolated edges (vertices) in $G[B]$ and $G[W]$ is the same for every 2-bisection (B, W) of G . Thus every 2-bisection is a bisection with isomorphic parts. Moreover, it is not hard to prove that the Petersen graph P is not a counterexample for Ando's Conjecture (see Figure 1(b) and 1(c)). ■

The previous proposition implies that every cubic graph with a 2-bisection is not a counterexample for Ando's Conjecture. Not all cubic graphs admit a 2-bisection: next to the Petersen graph, an infinite family of 1-connected examples is known [16]. However, it is proven in [16] that every (not necessarily bridgeless) cubic graph admits a 3-bisection. It is not hard to prove that the Petersen graph has no 3-bisection with isomorphic parts (see Figure 1(a)), so there is no chance to prove in general that every cubic graph has a 3-bisection with isomorphic parts.

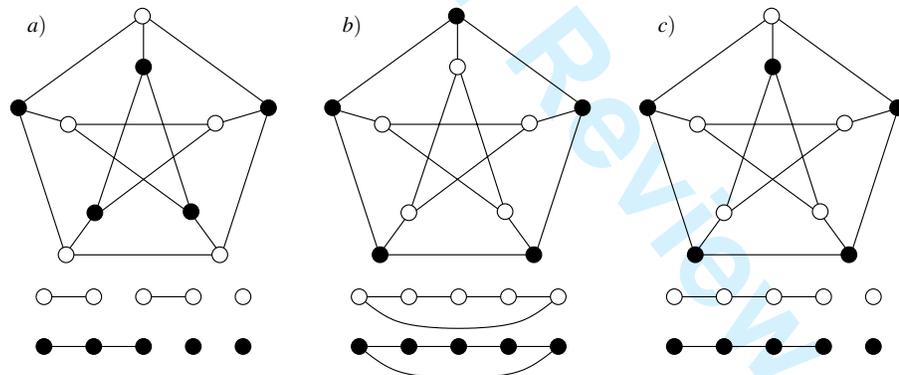


Figure 1: a) A 3-bisection of P ; b) and c) Bisections of P with isomorphic parts

In Section 2.1 we focus our attention on bridgeless cubic graphs. In particular, we consider a natural generalization of the Petersen graph, namely cycle permutation graphs, and we verify the Ban-Linial Conjecture (and by Proposition 2.3 also Ando's Conjecture) for all members of this infinite family. Note that infinitely many of them are not 3-edge colourable and several authors (see for instance [12] and [21]) refer to a member of this subclass as a permutation snark. Observe that we also prove the Ban-Linial Conjecture in the special case of claw-free cubic graphs in [1].

In Section 2.2 we consider the analogous relation between 2-bisections and bisections with isomorphic parts for the class of 1-connected cubic graphs. An infinite family of 1-connected cubic graphs which do not admit a 2-bisection is known [16] and it could

be that a member of this family might be a counterexample for Ando's Conjecture. In Proposition 2.13 we prove that this is not the case by constructing a bisection with isomorphic parts for all of these graphs. Using a computer program, we also construct all (1-connected) cubic graphs of order at most 32 without a 2-bisection and we verify that none of them is a counterexample for Ando's Conjecture. The reader should refer to Section 4 for more details about these computational results.

Finally, in Section 2.3 we propose a stronger version of Ando's Conjecture which we will relate to Conjecture 1.2 in Section 3.

2.1 2-bisections of cycle permutation graphs

In this section we will introduce cycle permutation graphs and prove that every such graph has a 2-bisection, except the Petersen graph.

Given a graph G with n vertices labelled $0, 1, \dots, n-1$, $n \geq 4$, and a permutation $\alpha \in S_n$, the symmetric group on the n symbols $\{0, 1, \dots, n-1\}$. The α -permutation graph of G , $P_\alpha(G)$ consists of two disjoint copies of G , G_1 and G_2 along with the n edges obtained by joining i in G_1 with $\alpha(i)$ in G_2 , for $i = 0, 1, \dots, n-1$. If the graph G is an n -cycle labeled $0, 1, 2, \dots, n-1$ where i is joined to $i+1$ and $i-1 \pmod{n}$ then $P_\alpha(G)$ is called a *cycle permutation graph* and will be denoted $C(n, \alpha)$. So, equivalently, a *cycle permutation graph* is a cubic graph consisting of two n -cycles C_1 and C_2 , labeled $0, \dots, n-1$ and such that i in C_1 is adjacent to $p(i)$ in C_2 , for some permutation $p \in S_n$ (where S_n denotes the symmetric group on $\{0, 1, \dots, n-1\}$).

Permutation graphs were first introduced by Chartrand and Harary in 1967 [15], and cycle permutation graphs were given this name in [29], but can also be found in [31] and other references. Clearly, the Petersen graph is the cycle permutation graph $P = C(5, (1, 4, 2, 3))$ and generalised Petersen graphs are cycle permutation graphs as well.

Let $G = C(n, p)$ be a cycle permutation graph, for some permutation $p \in S_n$, having cycles C_1 (external) and C_2 (internal). We will use the notation $p(i) = p_i$ for all $i \in \{0, \dots, n-1\}$, as long as subindices are clear, but sometimes we will keep $p(i)$ to avoid triple or quadruple subindices and/or use extra parenthesis to avoid confusion. Throughout the paper, we assume that $p_0 = 0$ and we fix the following labelling on the vertices of G . Here and along the rest of the paper all indices are taken modulo n .

- the vertices of the cycle C_1 are u_0, \dots, u_{n-1} with $u_i u_{i+1} \in E(G)$
- the vertices of the cycle C_2 are v_0, \dots, v_{n-1} with $v_i v_{i+1} \in E(G)$
- the edges given by the permutation p are $v_i u_{p_i}$ (see Figure 2).

Moreover, we always consider the class representatives in $\{0, \dots, n-1\}$. For every $p \in S_n$, we denote by p^{-1} the inverse of p and by \bar{p} the permutation such that $\bar{p}_i = n - p_i$ modulo n , for all i .

Proposition 2.4 *The graphs $C(n, p)$, $C(n, p^{-1})$ and $C(n, \bar{p})$ are isomorphic.*

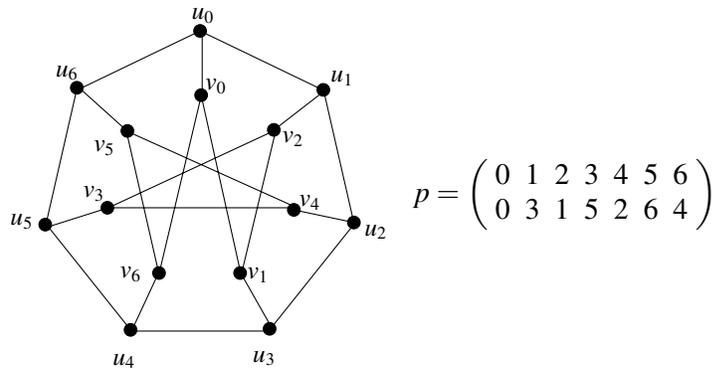


Figure 2: Labelling for a cycle permutation graph.

Proof The isomorphism between $C(n, p)$ and $C(n, p^{-1})$ is easily obtained by switching the roles of C_1 and C_2 . Moreover, $C(n, \bar{p})$ is obtained from $C(n, p)$ by labeling C_1 in counterclockwise order. ■

Proposition 2.5 *Let $G = C(n, p)$ be a cycle permutation graph. If n is even, then G admits a 2-bisection.*

Proof For an even n , we can alternately colour the edges of C_1 and C_2 with two colours and the edges not in C_1 or C_2 with a third colour, then the graph G is 3-edge colourable, and by Theorem 2.1 G admits a 2-bisection. ■

The main purpose of this section is to prove the existence of a 2-bisection for every cycle permutation graph, except for the Petersen graph (see Theorem 2.12). By Proposition 2.5, from now on, we can assume that n is odd. Furthermore, if $|p_i - p_{i+1}| = 1$ for some i , then G is Hamiltonian, and thus 3-edge colourable and hence admits a 2-bisection (by Theorem 2.1). Thus, we can also assume that $|p_i - p_{i+1}| > 1$ for all i . Finally, since n is odd, p_1 and \bar{p}_1 have different parity, hence we can select p such that p_1 is even. We summarize all previous observations in the following remark:

Remark 2.6 *To prove the existence of a 2-bisection for every cycle permutation graph, it is sufficient to prove it for all cycle permutation graphs $C(n, p)$ such that: n odd, p_1 even and $|p_i - p_{i+1}| > 1$ for all i .*

Let P be a path in a graph G , and let $x, y \in V(P)$. We denote by $P(x, y)$ the subpath of P between x and y . Let C be a cycle in a graph G , and let $x, y, z \in V(C)$. We denote by $C(x, z, y)$ and $C(x, \hat{z}, y)$ the (x, y) -path in C containing z and not containing z , respectively.

Now, we furnish a sufficient (but in general not necessary) condition to have a 2-bisection of $G = C(n, p)$. More precisely, we will prove that such a condition guarantees the existence of a particular 2-bisection for $G = C(n, p)$ which satisfies all of the following further properties:

- The cycle C_1 is alternately black and white, except for two consecutive white vertices, say a_r and a_{r+1} ;
- The cycle C_2 is alternately black and white, except for two consecutive black vertices, say b_s and b_{s+1} ;
- $a_{r+1}b_s \in E(G)$.

Note that the selection of $a_r a_{r+1}$ and $b_s b_{s+1}$ uniquely induces the entire 2-colouring, which is indeed a bisection. In order for such a 2-colouring to have monochromatic components of order at most two, it is enough to find the following *good* configuration in G .

Definition 2.7 Let $G = C(n, p)$ be a cycle permutation graph with external cycle C_1 and internal cycle C_2 . A good configuration in G is a set of six vertices $\{a_r, a_{r+1}, a, b_s, b_{s+1}, b\} \in V(G)$ such that: $a_r, a_{r+1}, a \in V(C_1)$; $b_s, b_{s+1}, b \in V(C_2)$; $a_r a_{r+1}, a_r b, a_{r+1} b_s, a b_{s+1}, b_s b_{s+1} \in E(G)$; and the paths $C_1(a, \widehat{a_r}, a_{r+1})$ (and so also $C_1(a, \widehat{a_{r+1}}, a_r)$) and $C_2(b, \widehat{b_{s+1}}, b_s)$ (and so also $C_2(b, \widehat{b_s}, b_{s+1})$) are both of even length (see Figure 3).

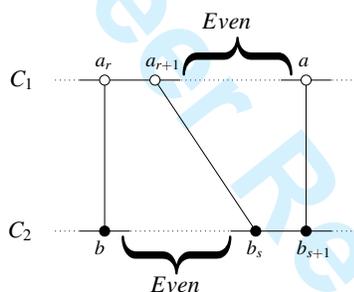


Figure 3: A good configuration

Proposition 2.8 Let $G = C(n, p)$ be a cycle permutation graph, with n odd. If G has a good configuration, then it admits a 2-bisection.

Proof Let $\{a_r, a_{r+1}, a, b_s, b_{s+1}, b\}$ be a good configuration of G . Set the vertex colour of a_r, a_{r+1} and a white; the vertex colour of b_s, b_{s+1} and b black; and all other vertices of G , alternately black and white in C_1 and C_2 . Note that such alternation is possible and uniquely determined by the previous choices, because the $C_1(a_{r+1}, \widehat{a_r}, a)$, $C_1(a, \widehat{a_{r+1}}, a_r)$, $C_2(b, \widehat{b_{s+1}}, b_s)$ and $C_2(b_{s+1}, \widehat{b_s}, b)$ are all of even length. All neighbours of the white vertices a_r, a_{r+1} are black, and those of the black vertices b_s, b_{s+1} are white. Moreover, if some edge between vertices of C_1 and C_2 is monochromatic, by the alternation of colours in those cycles, all of its neighbours will be of the opposite colour. Hence, the monochromatic components have order at most two, that is the 2-colouring is a 2-bisection of $C(n, p)$.

■

Lemma 2.9 *Let $G = C(n, p)$ be a cycle permutation graph, with n odd. If there exists an index i such that $p_{i+1} - p_i \equiv 1 \pmod{2}$ (lower indices taken modulo n), then G admits a 2-bisection.*

Proof Recall that we are assuming that p_1 is even, hence $p_1 - p_0 = p_1$ is even. First consider that there is an index i such that $p_{i+1} - p_i$ is odd. Let t be the smallest such index.

In order to find a good configuration in G , we consider the path $v_{t-1}v_tv_{t+1}$ in C_2 and their neighbours $u_{p_{t-1}}, u_{p_t}$ and $u_{p_{t+1}}$ in C_1 , i.e. $v_{t-1}u_{p_{t-1}}, v_tu_{p_t}, v_{t+1}u_{p_{t+1}} \in E(G)$, $u_{p_{t-1}}, u_{p_t}, u_{p_{t+1}} \in V(C_1)$.

Set $P = C_1(u_{p_{t-1}}, u_{p_t}, u_{p_{t+1}})$. We have that $|P(u_{p_{t-1}}, u_{p_t})|$ is even (because $p_t - p_{t-1}$ is even) and $|P(u_{p_t}, u_{p_{t+1}})|$ is odd (because $p_{t+1} - p_t$ is odd) and greater than 1 (because $|p_{i+1} - p_i| > 1$ for all $i \in \{0, \dots, n-1\}$). Let u be the neighbour of u_{p_t} in $P(u_{p_t}, u_{p_{t+1}})$, v be the neighbour of u in C_2 and denote by Q the path $C_2(v_{t+1}, v_t, v)$. There are two cases:

CASE 1 - Q ODD: Let $b_{s+1} = v_{t-1}$, $b_s = v_t$, $b = v$, $a_r = u$, $a_{r+1} = u_{p_t}$ and $a = u_{p_{t-1}}$. Then, $C_2(b, \widehat{b_{s+1}}, b_s)$ and $C_1(a, \widehat{a_{r+1}}, a_r)$ are both even and we have the desired good configuration.

CASE 2 - Q EVEN: Let $b_s = v_t$, $b_{s+1} = v_{t+1}$, $b = v$, $a_{r+1} = u_{p_t}$, $a_r = u$ and $a = u_{p_{t+1}}$. Then, $C_2(b, \widehat{b_{s+1}}, b_s)$ and $C_1(a, \widehat{a_r}, a_{r+1})$ are both even and we have the desired good configuration.

In both cases we have a good configuration, then, by Proposition 2.8, G admits a 2-bisection. ■

Now, we can focus our attention on the corresponding set of differences for the inverse permutation p^{-1} . But first, we collect a list of properties that we are going to implicitly use in the following proofs.

Remark 2.10 (i) *Let $G = C(n, p)$ be a cycle permutation graph with n odd and $p_{i+1} - p_i$ even for all i . Then, given two consecutive vertices of the inner cycle $v_i, v_{i+1} \in C_2$, we have that for $v_iu_{p_i}, v_{i+1}u_{p_{i+1}} \in E(G)$, the path $(u_{p_i}u_{(p_i)+1}u_{(p_i)+2} \dots u_{p_{(i+1)}})$ has even length.*

(ii) *Let $G = C(n, p)$ be a cycle permutation graph with n odd and $p_{i+1}^{-1} - p_i^{-1}$ odd for all i . Then, given two consecutive vertices of the outer cycle $u_{p_i}, u_{(p_i)+1} \in C_1$, we have that for $u_{p_i}v_i, u_{p_{i+1}}v_{p^{-1}((p_i)+1)} \in E(G)$, the path $(v_iv_{i+1} \dots v_{p^{-1}((p_i)+1)})$ has odd length.*

Recall that if two consecutive vertices of one of the two cycles C_1 and C_2 have the same colour, then we need that their neighbours in the other cycle have the opposite colour.

Lemma 2.11 *Let $G = C(n, p)$, n odd, be a cycle permutation graph. If $p_{i+1} - p_i$ is even for all i and there exists an index t such that $p_{t+1}^{-1} - p_t^{-1}$ is even, then G admits a 2-bisection.*

Proof Again, we make use of Proposition 2.8. Hence, we must only prove the existence of a good configuration. Consider the vertices u_t, u_{t+1} in C_1 and their neighbours $v_{p_t^{-1}}, v_{p_{t+1}^{-1}}$ in C_2 , respectively. Denote by v the vertex adjacent to $v_{p_t^{-1}}$ in C_2 but not in the $(v_{p_t^{-1}}, v_{p_{t+1}^{-1}})$ -path of C_2 of even length. Finally, denote by u the neighbour of v in C_1 . The (u, u_t) -path in C_2 of even length does not contain u_{t+1} because of the assumption that $p_{i+1} - p_i$ is even for all i . Let $b_s = v_{p_t^{-1}}, b_{s+1} = v, b = v_{p_{t+1}^{-1}}, a_r = u_t, a_{r+1} = u_{t+1}$ and $a = u$. Then, by our assumption on p and p^{-1} , $C_2(b, \widehat{b_{s+1}}, b_s)$ and $C_1(a, \widehat{a_{r+1}}, a_r)$ are both even and we have the desired good configuration. ■

In view of the previous lemma, from now on, we consider a cycle permutation graph $C(n, p)$ with n odd, $p_{i+1} - p_i$ even for all i and $p_{i+1}^{-1} - p_i^{-1}$ odd for all i . Note that the Petersen graph lies within this class.

Recall that a Generalised Petersen Graph $GP(n, k)$, for $2 \leq 2k < n$, is a simple cubic graph with vertex set $V(GP(n, k)) = \{u_0, \dots, u_{n-1}, v_0, \dots, v_{n-1}\}$ and edge set $E(GP(n, k)) = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i \in \{0, \dots, n-1\}\}$ (where all indices are taken modulo n).

Even though it is not hard to check that some cycle permutation graphs do not admit a good configuration (e.g. the Generalised Petersen Graph $GP(9, 2)$), we prove in the following theorem that all of them, except for the Petersen graph, admit a 2-bisection.

Theorem 2.12 *Let $G = C(n, p)$ be a cycle permutation graph, which is not the Petersen graph. Then G admits a 2-bisection.*

Proof By Lemma 2.5 we can assume that n is odd. By Lemma 2.9 we can assume that p satisfies $p_{i+1} - p_i$ even for all i . Finally, by Lemma 2.11 we can assume that p^{-1} satisfies $p_{i+1}^{-1} - p_i^{-1}$ odd for all i .

First of all, we consider the Generalised Petersen graph $G = GP(n, \frac{n-1}{2})$. If $n = 5$ we have the Petersen graph. If $n > 5$, G is 3-edge colourable (see [14]) and by Theorem 2.1, G admits a 2-bisection.

Hence, we can assume that p satisfies all previous conditions, but it is not $GP(n, \frac{n-1}{2})$.

Let $x, y \in V(G)$ be such that:

Without loss of generality, we may assume that $i = 0$, i.e. $x = v_{p^{-1}(1)}$ and $y = v_{p^{-1}(0)} = v_0$, so $u_0 y, u_1 x \in E(G)$. Now, we describe a 2-bisection of G . Let $c : V(G) \rightarrow \{B, W\}$ be a 2-colouring such that

- $c(u_0) = c(u_1) := W$;
- $c(u_i) := W$ if i is odd;
- $c(u_i) := B$ if i is even ($\neq 0$);
- $c(x) = c(y) := B$.

We need to describe the 2-colouring c on the rest of the inner cycle, in such a way that c is a 2-bisection.

Since the inner cycle C_2 is odd, the vertices x and y divide C_2 into two paths, one of even length, say P_1 , and one of odd length, say P_2 . Let x_0 and y_0 be the neighbours of x and y in P_1 ; and let x_1 and y_1 be the neighbours of x and y in P_2 , respectively.

Since $p_{i+1} - p_i$ is even and $p_{i+1}^{-1} - p_i^{-1}$ is odd, for all i , we have that the vertex $u_h \in C_1$, with $u_h y_1 \in E(G)$, is such that h is even. Then y_1 is adjacent to a black vertex of C_1 . Analogously, the vertex $u_k \in C_1$, with $u_k x_1 \in E(G)$, is such that k is even, hence also x_1 is adjacent to a black vertex in C_1 . Since x_1 has two black neighbours (x and u_k), we must set $c(x_1) := W$. Similarly, $c(y_1) := W$.

Now consider $x_0 u_l, y_0 u_m \in E(G)$, with $u_l, u_m \in C_1$. Then, again by $p_{i+1} - p_i$ even for all i , l and m are both odd, which implies that $c(u_l) = c(u_m) = W$ have been set.

Since the number of white vertices in C_1 is one more than the number of black vertices, we need the opposite situation in C_2 . We will obtain this condition in two different ways according to necessity: Type (I), where $c(x_0) = c(y_0) = W$, there are two consecutive black vertices somewhere in P_2 and the rest is alternating; and Type (II), where $c(x_0) = c(y_0) = B$, there are two consecutive white vertices somewhere in P_2 and the rest is alternating, except for the edges xx_0 and yy_0 which have both ends black (see Figure 4).

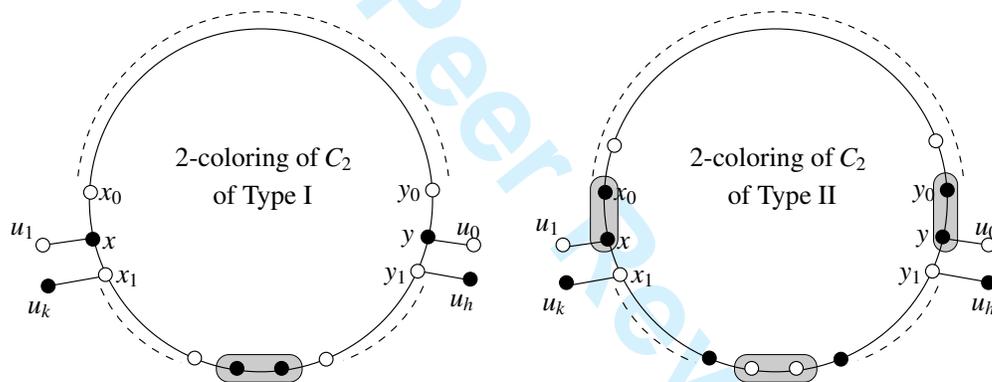


Figure 4: Type I and Type II colouring of C_2

Since the $C_2(x_1, \hat{x}, y_1)$ -path is odd, then there is an even number of vertices between x_1 and y_1 in this path, say $x_2, \dots, x_t, y_t, \dots, y_2$.

Now, we can complete the 2-colouring c in order to obtain a 2-bisection.

TRIVIAL CASE: there is no vertex between x_1 and y_1 . Then x_1, y_1 are two consecutive white vertices in C_2 , so we set $c(x_0) = c(y_0) = B$ and complete c as a Type II colouring.

GENERAL CASE: If the two neighbours of x_2 and y_2 in C_1 have both the same colour of x_1 and y_1 , i.e. white, then we must set $c(x_2) = c(y_2) = B$ to avoid monochromatic components of order larger than two. Otherwise, we can colour either $c(x_2) = W$ or $c(y_2) = W$, obtaining two consecutive white vertices (either $x_1 x_2$ or $y_1 y_2$) and we can complete c as a Type II colouring.

We repeat the process by looking at the neighbours of x_3, y_3 in C_1 , and so on. At each step, either the colouring of x_i and y_i is uniquely determined, or we can obtain two consecutive vertices of the same colour. If these two vertices are white, we complete c as

a Type II colouring; if these two consecutive vertices are black, we complete c as a Type I colouring.

Finally, if the colouring of all vertices $x_2, \dots, x_t, y_t, \dots, y_2$ is forced, then x_t and y_t share the same colour; we complete the colouring c as a Type I or a Type II colouring according to the colour of x_t and y_t . This completes the proof. ■

To conclude this section, we would like to remark that a direct proof of Ando's Conjecture for cycle permutation graphs is much easier. If we do not care about the order of connected components, then we can consider the bisection $B = V(C_1)$ and $W = V(C_2)$ and it is clear that such a bisection has isomorphic parts.

2.2 1-connected cubic graphs

In the previous sections, we have discussed the problem of constructing bisections of cubic graphs with isomorphic parts. In particular, we have observed that a 2-bisection is always a bisection with isomorphic parts (see Proposition 2.3). As mentioned before, only one example of bridgeless cubic graph without a 2-bisection is known (i.e. the Petersen graph). On the other hand, an infinite family of 1-connected cubic graphs with no 2-bisection is constructed in [16]. In this section, we prove that all members of this family (and also of a similar one) admit a 3-bisection with isomorphic parts.

First of all, we present the family of examples constructed in [16]: take the module L_h ($h \geq 0$) depicted in Figure 5. Note that L_0 is $K_{3,3}$ with one edge subdivided with a new vertex.

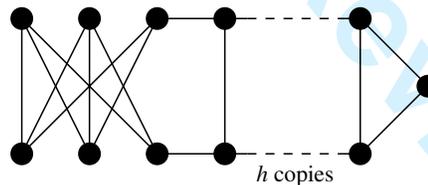


Figure 5: The module L_h

Let T_{ijk} be the graph obtained by taking L_i , L_j and L_k and adding a new vertex adjacent to the three vertices of degree 2.

It is proved in [16] that T_{ijk} does not admit a 2-bisection for all possible non-negative values of i, j and k .

Proposition 2.13 *For any $i, j, k \geq 0$, the graph T_{ijk} admits a 3-bisection with isomorphic parts.*

Proof The basic step is the 3-bisection of the graph T_{000} depicted in Figure 6. The proof proceeds by showing that if we replace the three copies of the module L_0 with copies of L_i , L_j and L_k , for arbitrary non-negative values of i, j and k , then we can still obtain a 3-bisection with isomorphic parts.

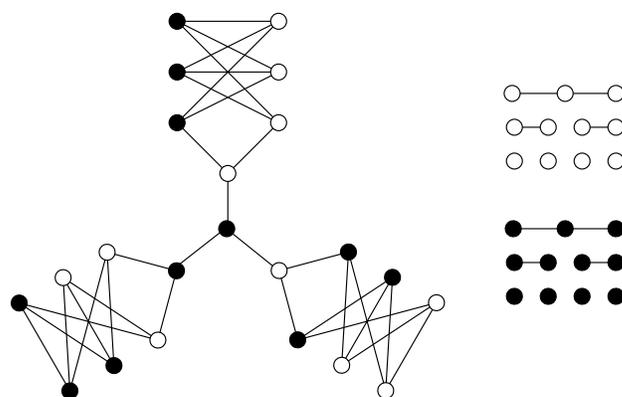


Figure 6: A 3-bisection of T_{000} with isomorphic parts

1) Replace the central copy of L_0 in T_{000} with a copy of L_i coloured as in Figure 7: the bisection of the graph obtained still has isomorphic parts since we only add isolated

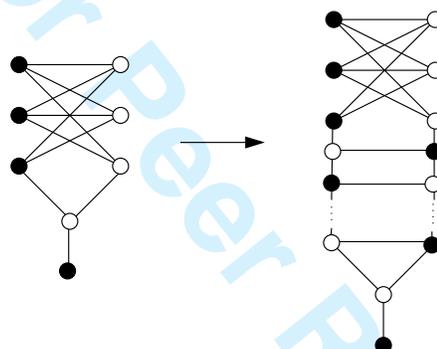


Figure 7

vertices in each monochromatic subgraph.

2) Replace the copy of L_0 on the right with a copy of L_j coloured as in Figure 8: the bisection of the graph obtained still has isomorphic parts since we replace an isolated

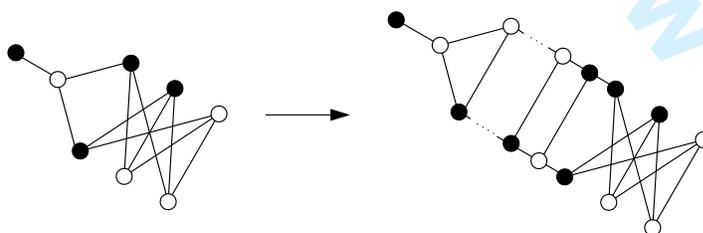


Figure 8

vertex with a path of length one in each part and we add the same number of isolated vertices in each of them.

3) Replace the copy of L_0 on the left with a copy of L_k coloured as in Figure 9 according to the parity of k : in both cases the bisection of the graph obtained still has isomorphic parts. If k is even, we add the same number of paths of length one in each part, if k is

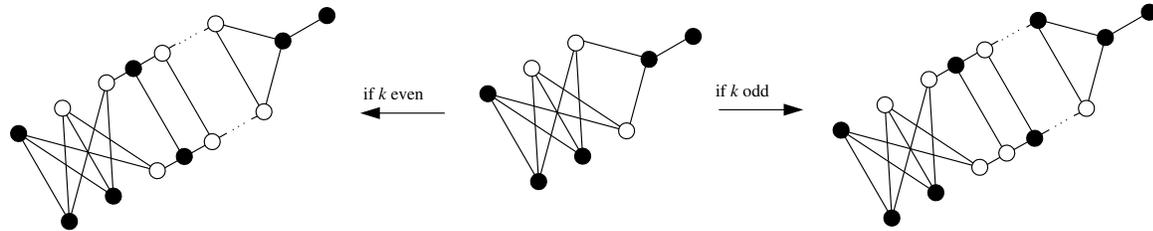


Figure 9

odd we replace a path of length one with a path of length two in each part and we add the same number of isolated vertices in each of them. ■

We would like to stress that if we substitute the central vertex of T_{ijk} with a copy of $K_{3,3}$ minus a vertex, then we have another infinite family of graphs without a 2-bisection. All previous arguments can be easily extended to prove that each member of this further family admits a 3-bisection with isomorphic parts.

Furthermore, all 1-connected graphs of order at most 32 without a 2-bisection are presented in Section 4. None of them is a counterexample for Ando's Conjecture.

2.3 Strong Ando Conjecture

We conclude this section with some properties of bisections with isomorphic parts that will allow us to state a strengthening of Ando's conjecture that will be discussed in Section 3.

Recall that, as was remarked at the beginning of this section, in all 2-bisections the two isomorphic parts are obviously linear forests, since the connected components are either isolated vertices or isolated edges. Moreover, we have verified that all known graphs without a 2-bisection admit a 3-bisection with isomorphic parts which are linear forests (this is not trivial since a connected component of order 3 could be a 3-cycle) except the Petersen graph (see Proposition 2.13 and Observation 4.3). Finally, note that the Petersen graph admits a 4-bisection with isomorphic parts which are linear forests (see Figure 1(c)).

In the following remark, we summarize all of these facts:

- Remark 2.14** (i) *In all 2-bisections the two monochromatic induced subgraphs are isomorphic and they are obviously linear forests;*
(ii) *All known graphs without a 2-bisection admit a 3-bisection with isomorphic parts which are linear forests, except the Petersen graph;*
(iii) *The Petersen graph admits a 4-bisection with isomorphic parts which are linear forests.*

This gives rise to the following natural question:

Problem 2.15 *Does every cubic graph admit a bisection with isomorphic parts and with the additional property that these parts are linear forests?*

To approach this problem, first of all, we observe that we can assume that both parts have maximum degree two.

Proposition 2.16 *Let G be a cubic graph admitting a bisection with isomorphic parts. Then G admits a bisection with isomorphic parts having maximum degree 2.*

Proof Take a bisection with isomorphic parts and with the minimum number of vertices of degree 3 in each induced monochromatic subgraph. Suppose that there exist two vertices of G of degree 3, each one from a monochromatic isomorphic part, say v_1 and v_2 (i.e. two corresponding vertices following the isomorphism between the black and the white monochromatic part). Then, we can obtain a bisection with isomorphic parts and a lower number of vertices of degree 3 by recolouring v_1 and v_2 with the opposite colour, which is a contradiction. ■

Proposition 2.16 and Remark 2.14 lead us to state the following strong version of Ando's Conjecture:

Conjecture 2.17 (Strong Ando Conjecture) *Every cubic graph admits a k -bisection such that the two induced subgraphs are isomorphic linear forests.*

We are curious about the optimal value of k in the previous conjecture and we leave it as an open problem. At this stage, we only remark that $k = 3$ is always sufficient in all known examples, except for the Petersen graph where we need $k = 4$.

Problem 2.18 *Does every cubic graph admit a 4-bisection such that the two induced subgraphs are isomorphic linear forests?*

3 Strong Wormald (edge) colourings vs Strong Ando (vertex) colourings

In this section, as mentioned in the Introduction, we develop a second and different approach introducing the concept of (*Strong*) *Wormald Colouring*, to present further evidence in support of Ando's Conjecture 1.3, and its stronger version (Conjecture 3.4).

In the Introduction, we have stated a Conjecture by Wormald (cf. Conjecture 1.2) in terms of isomorphic linear forests (i.e. *let G be a cubic graph with $|V(G)| \equiv 0 \pmod{4}$. Then, there exists a linear partition of G in two isomorphic linear forests*). Note that, in other words, Conjecture 1.2 states that *there exists a 2-edge colouring of G , $c_E : E(G) \rightarrow \{B, W\}$ such that both monochromatic subgraphs are linear forests and they are isomorphic*. We define such a colouring c_E to be a *Wormald colouring*.

Analogously, we can restate Ando's Conjecture (cf. Conjecture 1.3) in terms of colourings as follows:

Conjecture 3.1 (Restatement of Ando's conjecture [5]) *A cubic graph G admits a 2-vertex colouring of G , $c_V : V(G) \rightarrow \{B, W\}$, such that the monochromatic induced subgraphs are isomorphic.*

From now on, a 2-colouring of G , $c_V : V(G) \rightarrow \{B, W\}$, such that the monochromatic induced subgraphs are isomorphic will be called an *Ando colouring*. Trivially, in order to have isomorphic monochromatic induced subgraphs such a 2-colouring is a bisection.

It seems immediately clear that there should be a link between a Wormald Colouring of the edges of a cubic graph G and an Ando Colouring of the vertices of G . In order to establish a strong connection between the two problems, we need to strengthen some of the previous definitions. In particular, we need to exclude an induced path of length one (i.e. a path with one edge and two vertices) in a Wormald Colouring.

Definition 3.2 *Let G be a cubic graph with $|V(G)| \equiv 0 \pmod{4}$. A Strong Wormald Colouring is an edge colouring $c_E : E(G) \rightarrow \{B, W\}$ of G such that the monochromatic induced subgraphs are isomorphic linear forests with paths of length at least 2.*

Finally, we define a Strong Ando Colouring as an Ando Colouring such that the monochromatic induced subgraphs are not only isomorphic but also linear forests. More precisely:

Definition 3.3 *Let G be a cubic graph. A Strong Ando Colouring is a vertex colouring $c_V : V(G) \rightarrow \{B, W\}$ of G such that the monochromatic induced subgraphs are isomorphic linear forests.*

Obviously, a Strong Ando Colouring is an Ando Colouring. Moreover, we can restate Conjecture 2.17 in terms of colouring, as follows:

Conjecture 3.4 (Strong Ando Conjecture (Colouring Version)) *Every cubic graph admits a Strong Ando Colouring.*

The idea of introducing the concept of a Strong Wormald Colouring is mainly due to the following proposition.

Proposition 3.5 *Let G be a cubic graph with $|V(G)| \equiv 0 \pmod{4}$ admitting a Strong Wormald Colouring. Then, G admits a Strong Ando Colouring (and thus an Ando Colouring).*

Proof Let $c_E : E(G) \rightarrow \{B, W\}$ be a Strong Wormald Colouring. We construct a Strong Ando Colouring as follows:

$$c_V(v) := \begin{cases} B & \text{if } v \text{ is incident with two black edges of } c_E \\ W & \text{if } v \text{ is incident with two white edges of } c_E \end{cases}$$

The vertex colouring $c_V(v)$ is indeed a Strong Ando Colouring: every connected component of a monochromatic induced subgraph is a path, since it is obtained starting from a path of the induced linear forests of the Strong Wormald Colouring by removing the two final edges. Hence, the two monochromatic induced subgraphs are linear forests, and the previous argument implicitly proves that they are also isomorphic. ■

As far as we know, Wormald's Conjecture (i.e. every cubic graph of order $0 \pmod{4}$ has a Wormald Colouring) is open and we have no counterexamples for the Strong Ando Conjecture (i.e. every cubic graph has a Strong Ando Colouring, and so also an Ando Colouring). What happens if we consider Strong Wormald Colourings instead? Is it true that every cubic graphs of order congruent to $0 \pmod{4}$ has a Strong Wormald Colouring? The answer is negative in general, and we will construct an infinite family of graphs without a Strong Wormald Colouring. The family of these examples seems to represent a very thin subclass of the class of all cubic graphs and, even if they do not admit a Strong Wormald Colouring, all of them admit a 2-bisection and hence a Strong Ando Colouring.

Firstly, in order to study such a family and to have an efficient procedure to obtain graphs without a Strong Wormald Colouring, we characterize bisections which arise from a Strong Wormald Colouring.

Let G be a cubic graph such that $|V(G)| \equiv 0 \pmod{4}$. Let $c_E : E(G) \rightarrow \{B, W\}$ be a Strong Wormald Colouring. Let $c_V : V(G) \rightarrow \{B, W\}$ be the vertex colouring induced by c_E as described in the proof of Proposition 3.5. Let G' be the graph obtained from G by removing all edges with two ends of the same colour in c_V . In other words, $G' = (V(G'), E(G'))$ where $V(G') = V(G)$ and $E(G') = \{xy \in E(G) : c_V(x) \neq c_V(y)\} = \{xy \in E(G) : xy \text{ is a leaf in a monochromatic path of } c_E\}$.

Lemma 3.6 *Each connected component of G' is bipartite and isomorphic to a tree plus an edge.*

Proof Since all edges with ends of the same colour in c_V were removed from G , the graph G' is obviously bipartite with bipartition c_V . Consider the following orientation of G' : $e \in E(G')$ is oriented away from the vertex having in c_V the same colour as e in c_E .

This orientation cannot produce a vertex of in-degree larger than 1 in G' : take a vertex v , the colour in c_E of an inner edge in v is opposite to the colour of v in c_V , and we can have only one edge of such a colour incident to v by definition of c_V .

Since we do not have in-degree 2 at any vertex, every path of G' is oriented and every cycle C in G' is oriented (and even, since G' is bipartite). Furthermore, every edge incident to a vertex of C , and not in C , is directed away from C . Then, every path leaving C cannot come back to C . For the same reason, we cannot have two cycles in the same connected component of G' . Then, we have proved that if a connected component of G' has a cycle, then it is the unique cycle in such a component, in other words it is a tree plus an edge.

1
2
3 Finally, we observe that a connected component cannot be acyclic (i.e. a tree). Each
4 leaf is oriented towards its vertex of degree 1 by definition of c_V and c_E , then a path which
5 contains two leaves cannot be oriented, a contradiction.
6

7 Hence, a connected component of G' is bipartite and it contains exactly one cycle.
8
9 ■

10
11 Now, we prove that every bisection having the properties described in previous propo-
12 sitions arises from a Strong Wormald Colouring.
13
14

15 **Theorem 3.7** *Let G be a cubic graph with $|V(G)| \equiv 0 \pmod{4}$. The graph G*
16 *admits a Strong Wormald Colouring if and only if G admits a Strong Ando Colouring*
17 *such that each connected component of the bipartite subgraph G' of G , obtained by removing*
18 *all edges of G with ends of the same colour, is isomorphic to a tree plus an edge.*
19
20
21

22 **Proof** Let c_V be a Strong Ando Colouring of G with the additional property that each
23 connected component in the bipartite subgraph G' of G , obtained by removing all edges
24 of G with ends of the same colour, is isomorphic to a tree plus an edge.
25

26 We want to reconstruct a Strong Wormald Colouring c_E starting from c_V :
27 for all edges xy having both ends of the same colour in c_V . We set
28

$$29 \quad c_E(xy) := \begin{cases} B & \text{if } c_V(x) = c_V(y) = B \\ W & \text{if } c_V(x) = c_V(y) = W \end{cases}$$

30
31
32
33 Now we must define c_E on the remaining edges of G , which are exactly the edges of G' .
34 Consider a connected component of G' : it has a unique even cycle C , so colour the edges
35 of C alternately black and white. Now, colour the remaining edges following the rule that
36 a black vertex must have two black incident edges (and the same for white vertices). The
37 colouring c_E is uniquely determined up to the colour of each even cycle in G' . (Then we
38 have $2^{\#}$ even cycles in G' different edge colourings).
39
40
41

42 Finally, we have to prove that c_E is indeed a Strong Wormald Colouring.
43
44

$$45 \quad E(G) = E(G') \cup E(G'')$$

46
47 where $E(G')$ are the edges of G with ends of opposite colours in c_V , and $E(G'')$ are the
48 edges with ends of the same colour in c_V . The graph G'' is the union of two isomorphic
49 linear forests F_B and F_W , since c_V is a Strong Ando Colouring.
50

51 Now we show how a path (x_0, x_1, \dots, x_l) of length $\ell \geq 0$ in F_B (and an analogous
52 result holds for F_W) corresponds to a path of length $\ell + 2$ in G .
53

54 All vertices $x_i : i = 0, \dots, l$ are black in c_V , since they belong to F_B . Moreover, their
55 neighbours in G are all white vertices in c_V by definition of F_B . Possible inner vertices of
56 the path are all vertices of degree one in G' and so, by definition of c_E all edges incident
57 with the path in these vertices are white. Moreover, the two ends of the path, x_0 and x_l ,
58
59
60

are vertices of degree two in G' . Exactly one of the two edges incident with them in G' is black in c_E and these two black edges are isolated edges in G' because their ends not in the path are white. Hence, the linear forests induced by c_E are obtained by the linear forests induced by c_V by adding two edges at the ends of each path. This implies that the two linear forests are isomorphic and have length at least 2.

Therefore, c_E is a Strong Wormald Colouring and this proves the theorem. ■

Note that Theorem 3.7 not only has a relevance from a theoretical point of view, but as will be explained in Section 4.3, it also permits to design a more efficient algorithm to search for Strong Wormald Colourings.

We conclude this section with an example of a reconstruction of a Strong Wormald Colouring of the prism with 12 vertices starting from a suitable Strong Ando Colouring. Figure 10 shows a Strong Ando Colouring on the left, an edge colouring of the bipartite subgraph G' with all connected components isomorphic to a tree plus an edge in the center, and the Strong Wormald Colouring on the right.

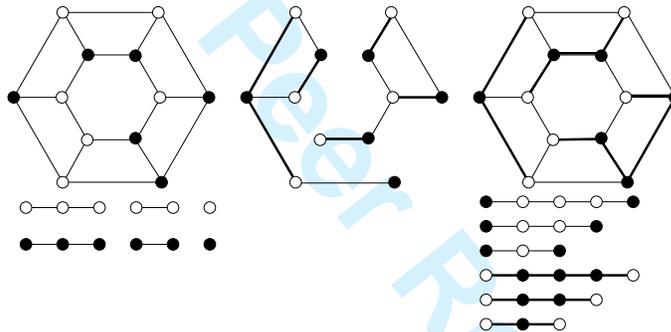


Figure 10: The reconstruction of a Strong Wormald Colouring obtained following the proof of Theorem 3.7

3.1 Cubic Graphs with no Strong Wormald Colouring

According to Conjecture 1.2 every cubic graph of order a multiple of four admits a Wormald Colouring. In previous section, we have defined a Strong Wormald Colouring as a Wormald Colouring of the edges of a cubic graph which avoids paths of length one. We have also observed that cubic graphs with a Strong Wormald Colouring cannot be counterexamples for Ando's Conjecture. Hence, the next natural step is the search of cubic graphs (of order a multiple of four) without a Strong Wormald Colouring. We have completed an exhaustive search for cubic graphs without a Strong Wormald Colouring up to 28 vertices (see Section 4.3 for a description of the algorithm and graph counts) and were able to find such graphs: the smallest one is the graph with 16 vertices at the top of Figure 11. However, at least for the 2-connected case, we have found very few examples of cubic graphs without a Strong Wormald Colouring and they share some common features. In particular, note that all 2-connected examples admit a 3-edge colouring and hence they admit a 2-bisection and an Ando Colouring as well.

In Figure 11, we propose a compact description of all 2-connected cubic graphs without a Strong Wormald Colouring up to 28 vertices. All of them represent several copies of the complete bipartite graph $K_{3,3}$ of order 6 with some edges/vertices removed. In particular, the subgraphs E_i and V_i ($i = 1, 2$) are obtained by removing i edges and vertices, respectively, from a copy of $K_{3,3}$.

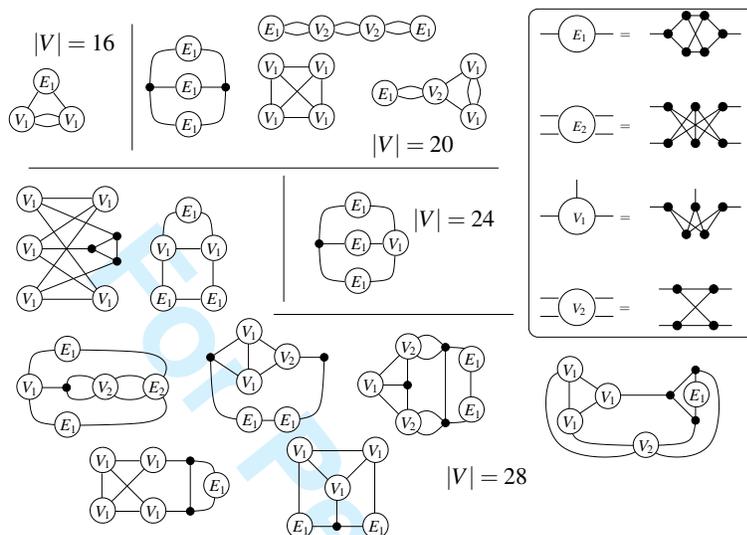


Figure 11: A complete list of all 2-connected cubic graphs without a Strong Wormald Colouring of order at most 28

To conclude this section, we show how we can generalize some of the examples from Figure 11 in order to obtain an infinite family of graphs without a Strong Wormald Colouring. First of all, we observe that a Strong Wormald Colouring induces, up to symmetry, only two different types of edge colourings of the subgraph E_1 . We will call these two induced colourings type α and type β colourings (see Figure 12). In particular, we represent in Figure 12 a colouring of type α and two symmetric colourings of type β . We denote by β_l and β_r a colouring of type β where the pending edge which belongs to a path having another edge in the copy of E_1 is on the left or on the right, respectively.

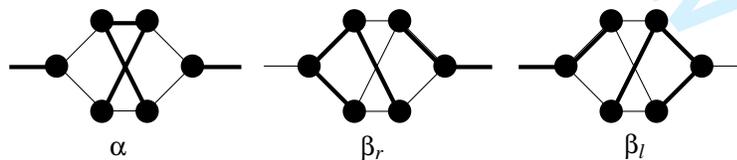


Figure 12: Different types of colourings of E_1

Now, we construct the infinite family of cubic graphs G_k , $k \geq 1$ (see Figure 13). The graph G_1 has 28 vertices and so it appears also in Figure 11. The graph G_k is obtained starting from G_1 by replacing the two copies of E_1 with $2k$ consecutive copies of E_1 .

Proposition 3.8 *For every $k \geq 1$, the graph G_k does not admit a Strong Wormald Colouring.*

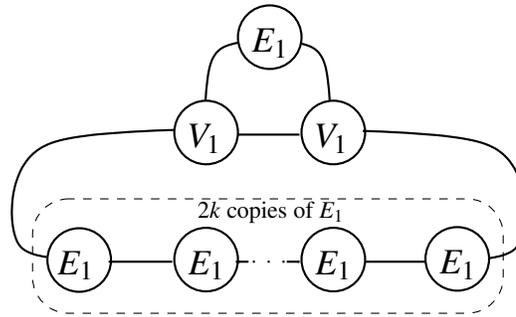


Figure 13: The graph G_k

Proof We argue by induction on k . We need to consider the second member of the family, G_2 , as base case. The graph G_2 has 40 vertices and we have directly checked it does not admit a Strong Wormald Colouring using a computer algorithm (see Section 4.3 for details of the algorithm). Now, we assume that G_k , $k \geq 2$, has no Strong Wormald Colouring and we prove it for G_{k+1} . By contradiction, suppose that G_{k+1} admits a Strong Wormald Colouring. Since $k \geq 2$, then $2(k+1) \geq 6$, that is: we have at least 6 consecutive copies of E_1 in G_{k+1} .

Assume that one of them, say E , is of type α . Since the two pending edges of E cannot be in a path of length one, the copy on the left of E is of type β_r and the one on the right of type β_l . More in general, repeating the same argument, all copies on the left of E are of type β_r and all copies on the right of E are of type β_l . Since we have at least six copies of E_1 , either we have three consecutive copies of β_r or three consecutive copies of β_l . Anyway, we can remove the last two of these three consecutive copies in order to obtain a graph isomorphic to G_k . Moreover, there is exactly one path of length two, one path of length three and one path of length five in each monochromatic component of the two removed copies of E_1 . Then, the two monochromatic subgraphs are still isomorphic in G_k and all paths are still of length at least two, hence the graph G_k inherits a Strong Wormald Colouring, a contradiction.

Therefore, assume no copy of E_1 is of type α . On the right of a copy of type β_r must be another copy of type β_r , and the same holds for copies of type β_l . Hence, since we have at least six copies of E_1 , we have three consecutive copies of type β_r or three consecutive copies of type β_l , anyway we obtain a contradiction by the same argument used above. ■

Finally, note that G_k is a 3-edge-colourable (bipartite) graph for every k . Hence, G_k admits a 2-bisection and by Proposition 2.3 is not a counterexample for Ando's Conjecture.

3.2 Strong Wormald Colouring of cubic graphs: $|V| \equiv 2 \pmod{4}$

The discussion of previous section creates an evident connection between Wormald's Conjecture and (Strong) Ando's Conjecture, but it also leaves a large gap between them.

Indeed, Wormald's Conjecture is only for cubic graphs of order a multiple of four while (Strong) Ando's Conjecture is stated for all cubic graphs. The purpose of this section is to fill this gap by defining a Strong Wormald Colouring for cubic graphs of order $2 \pmod{4}$.

The original Wormald Conjecture is not stated for cubic graphs of order not a multiple of four since they have an odd number of edges, thus the two monochromatic parts cannot have the same number of edges and hence they cannot be isomorphic.

We would like to give a definition of Strong Wormald Colouring for cubic graphs with an odd number of edges such that the existence of such a colouring implies the existence of a Strong Ando Colouring as we did previously. Moreover, we would like to have a definition that is satisfied by a large fraction of cubic graphs as it happens in the case of an even number of edges.

Our very simple idea is to consider an edge colouring of all edges of the graph except one, with an additional property on the uncoloured edge. The presence of this additional property will be exactly what we need to construct a Strong Ando Colouring starting from this edge colouring.

Definition 3.9 *Let G be a cubic graph of order $2 \pmod{4}$ and let $xy \in E(G)$ be an edge of G . A Strong Wormald Colouring of G is an edge colouring $c_E : E(G) - \{xy\} \rightarrow \{B, W\}$ of G such that the monochromatic induced subgraphs are isomorphic linear forests with paths of length at least 2, and, moreover, the vertex x is incident to two white edges and the vertex y is incident to two black edges.*

This definition of Strong Wormald Colouring allows to state the analogous of Proposition 3.5. We omit the proof since it proceeds exactly like the proof of Proposition 3.5 for order $0 \pmod{4}$.

Proposition 3.10 *Let G be a cubic graph with $|V(G)| \equiv 2 \pmod{4}$ admitting a Strong Wormald Colouring. Then, G admits a Strong Ando Colouring (and so an Ando Colouring).*

Moreover, we have also obtained the following characterization which is similar to the case $0 \pmod{4}$ of bisections arising from a Strong Wormald Colouring (cf. Theorem 3.7).

Theorem 3.11 *Let G be a cubic graph with $|V(G)| \equiv 2 \pmod{4}$. The graph G admits a Strong Wormald Colouring if and only if G admits a Strong Ando Colouring such that each connected component in the bipartite subgraph G' of G , obtained by removing all edges of G with ends of the same colour, is isomorphic to a tree plus an edge except for one that is a tree.*

The proof is analogous to the proof of Theorem 3.7. The unique difference is the connected component which is isomorphic to a tree, but this fact easily follows due to the presence of the uncoloured edge.

1
2
3 So far, we have followed exactly the same idea of the case of order $0 \pmod{4}$: we have
4 defined a Strong Wormald Colouring, we have proved that the existence of such an edge
5 colouring implies the existence of a (Strong) Ando Colouring and we have shown a char-
6 acterization in terms of bisections of the graphs admitting a Strong Wormald Colouring.
7

8 Note that a large amount of cubic graphs seem to have a Strong Wormald Colouring.
9 Indeed, the possibility of choosing the edge xy produces a better result than in the case
10 $0 \pmod{4}$. In particular, using a computer program, we have checked all cubic graphs
11 having at most 22 vertices of order $2 \pmod{4}$ and *all* of them admit a Strong Wormald
12 Colouring (see Section 4.3 for details).
13
14

15 We leave the existence of a cubic graph of order $2 \pmod{4}$ without a Strong Wormald
16 Colouring as an open problem.
17

18
19 **Problem 3.12** *Does every cubic graph of order $2 \pmod{4}$ have a Strong Wormald*
20 *Colouring?*
21

22
23 Note that, by Proposition 3.10, a positive answer to Problem 3.12 would imply (Strong)
24 Ando's conjecture for all cubic graphs of order $2 \pmod{4}$.
25
26
27

28 4 Computational Results

29
30

31 In this section we present our computational results. These results also give evidence to
32 support the correctness of the conjectures which were introduced in the previous sections.
33 More specifically, in Section 4.1 we present our computational results on the conjecture
34 of Ban and Linial, in Section 4.2 on Ando's conjecture and finally in Section 4.3 on
35 Wormald's conjecture.
36

37 We wrote programs for testing each of these conjectures or open problems. The source
38 code of these programs can be downloaded from [19].
39

40 In each case, we used the generator for cubic graphs *snarkhunter* [13, 11] to generate
41 all cubic graphs up to a given number of vertices and then applied our programs for testing
42 the conjectures on the generated graphs. In each case we went as far as computationally
43 feasible.
44
45
46

47 4.1 Computational results on the Ban–Linial Conjecture

48
49

50 We wrote a program for testing if a given graph has a 2-bisection. We generated all cubic
51 graphs up to 32 vertices and then tested which of these graphs do not admit a 2-bisection.
52 This yielded the following observation.
53
54

55 **Observation 4.1** *There are exactly 34 graphs among the cubic graphs with at most 32*
56 *vertices which do not admit a 2-bisection. All of these graphs, except the Petersen graph,*
57 *have connectivity 1.*
58
59
60

So this implies that the smallest counterexample to the Ban–Linial Conjecture must have at least 34 vertices. The counts of the graphs from Observation 4.1 can be found in Table 1.

The graphs from Observation 4.1 can also be downloaded and inspected in the database of interesting graphs from the *House of Graphs* [10] by searching for the keywords “no 2–bisection”.

Order	# no 2–bisection
0 – 8	0
10	1
12 – 20	0
22	1
24	1
26	3
28	5
30	9
32	14

Table 1: Number of cubic graphs which do not admit a 2–bisection.

Recall from Theorem 2.1 that Ban and Linial have proven that every 3–edge colourable cubic graph admits a 2–bisection. Recall that snarks (as mentioned earlier) are an important subclass of bridgeless not 3–edge colourable cubic graphs and in [12] Brinkmann, Hägglund, Markström and the second author have determined all snarks up to 36 vertices. We now also tested which snarks of order 34 and 36 admit a 2–bisection, which led to the following observation.

Observation 4.2 *The Petersen graph is the only snark up to 36 vertices which does not admit a 2–bisection.*

So this provides further evidence to support the correctness of Conjecture 1.1.

4.2 Computational results on Ando’s Conjecture

Recall that an easy counting argument shows that every 2–bisection is a bisection with isomorphic parts. So a possible counterexample to Ando’s conjecture must be a graph that does not have a 2–bisection.

We extended our program to test (the strong version of) Ando’s conjecture and applied it to the list of all cubic graphs up to 32 vertices of Observation 4.1 which do not have a 2–bisection. This led to the following observation.

Observation 4.3 *All 34 graphs of Observation 4.1 (i.e. all cubic graphs which do not admit a 2–bisection up to 32 vertices), except the Petersen graph, have a 3–bisection such that the two isomorphic induced monochromatic graphs are linear forests. The Petersen*

graph has no such bisection, but does admit a 4-bisection such that the two isomorphic induced monochromatic graphs are linear forests.

So this implies the following:

Corollary 4.4 *The Strong version of Ando's conjecture (i.e. Conjecture 2.17) (and thus also Ando's original conjecture) does not have any counterexamples with less than 34 vertices.*

4.3 Computational results on Wormald's Conjecture

In this section we describe our algorithm to test if a cubic graph admits a *Strong Wormald Colouring* and our results obtained by applying it to the complete lists of cubic graphs.

We implemented an algorithm which tries to find a Strong Wormald Colouring directly by edge colouring the graphs (see later), but it turns out that an algorithm which constructs bisections based on Theorem 3.7 and Theorem 3.11 is more efficient. The latter algorithm works as follows.

For every cubic graph G , we compute all bisections of G until a bisection is found which admits a Strong Wormald Colouring. A bisection leads to a Strong Wormald Colouring if and only if the following conditions (1),(3),(4) are fulfilled. We add a further condition, i.e. Condition (2), which is obviously weaker than Condition (3), but it is added for efficiency reasons. In particular, the four conditions are listed and executed in this order to obtain a better computational performance.

1. Both monochromatic induced subgraphs are linear forests;
2. Let denote by (l_0, \dots, l_t) the vector of lengths of the paths in one of the monochromatic subgraphs. Then for both monochromatic subgraphs, the following must hold: $\sum_i (l_i + 2) = \lfloor \frac{|E(G)|}{2} \rfloor$.
3. The two induced monochromatic subgraphs are isomorphic;
4. Consider the subgraph of G obtained by removing from G all edges with two ends in the same class of the bisection. If $|V(G)| \equiv 0 \pmod{4}$, then in each connected component of this subgraph, the number of vertices of this component needs to be equal to its number of edges (in other words it must be a tree + an edge). If $|V(G)| \equiv 2 \pmod{4}$, then in each connected component of this subgraph, the number of vertices of this component needs to be equal to its number of edges (in other words it must be a tree + an edge), except for one component one that must be a tree.

We tested which of the cubic graphs of order $0 \pmod{4}$ do not have a Strong Wormald Colouring. This led to the following observation.

Observation 4.5 *There are exactly 131 graphs without a Strong Wormald Colouring among the cubic graphs of order $0 \pmod{4}$ and at most 28 vertices.*

The graph counts and the connectivity statistics of the graphs from Observation 4.5 can be found in Table 2. All 2-connected examples are presented in Figure 11 from Section 3. The graphs from Observation 4.5 can also be downloaded and inspected in the database of interesting graphs from the *House of Graphs* [10] by searching for the keywords “no strong Wormald colouring”.

Order	Conn. 1	Conn. 2	Conn. 3	Total
4	0	0	0	0
8	0	0	0	0
12	0	0	0	0
16	1	1	1	3
20	20	3	1	24
24	18	1	0	19
28	72	12	1	85

Table 2: Number of cubic graphs of order $0 \pmod{4}$ up to 28 vertices which do not admit a strong Wormald Colouring. Conn. k lists the counts of the graphs with connectivity k .

We also implemented a second independent algorithm to test if a given graph has a Strong Wormald Colouring by edge colouring the graph. We also executed this edge colouring algorithm on all cubic graphs with $0 \pmod{4}$ vertices up to 24 vertices and this yielded exactly the same results as in Table 2.

We also verified that the 131 graphs from Observation 4.5 without a Strong Wormald Colouring are not counterexamples to the original conjecture of Wormald, that is: they admit a Wormald Colouring. So we can state the following:

Corollary 4.6 *Wormald’s Conjecture (i.e. Conjecture 1.2) does not have any counterexamples with less than 32 vertices.*

We also investigated Strong Wormald Colourings for graphs of order $2 \pmod{4}$. This led to the following observation.

Observation 4.7 *Every cubic graph of order $2 \pmod{4}$ and at most 22 vertices admits a Strong Wormald Colouring (and hence a (Strong) Ando Colouring).*

Also here the results from Observation 4.7 were independently obtained and confirmed by the bisections algorithm described above and by the algorithm which edge colours the graph.

5 A related problem on linear arboricity

As a by-product of our study of 2-edge colourings of cubic graphs having linear forests as monochromatic components, we obtain a negative answer to a problem about linear

arboricity posed by Jackson and Wormald in [24] (cf. also [32]). Furthermore, we also pose new problems and conjectures on this topic.

The k -linear arboricity of a graph G , introduced by Habib and Péroche [20], is the minimum number of k -linear forests (forests whose connected components are paths of length at most k) required to partition the edge set of G , and it is denoted by $la_k(G)$. Bermond et al. [8] conjectured that $la_5(G) = 2$ for every cubic graph G ; in other words that the edge set of a cubic graph can be partitioned into two k -linear forests for all $k \geq 5$. The conjecture is proven by Thomassen in [32].

In the same paper, Thomassen remarks that 5 cannot be replaced by 4 because of the two cubic graphs of order 6, but up until now it was unknown if there exists a graph with at least eight vertices for which 5 cannot be replaced by 4.

In fact, Jackson and Wormald explicitly ask in [24] if it is true that every cubic graph of order at least eight can be decomposed in two 4-linear forests.

Problem 5.1 (Jackson and Wormald [24]) *Is it true that $la_4(G) = 2$ for all cubic graphs G with at least eight vertices?*

We give in Theorem 5.3 a negative answer to this question by showing that the Heawood graph (see Figure 14) cannot be decomposed in two k -linear forests for $k < 5$. We also computationally show that the two cubic graphs of order 6 and the Heawood graph are the only graphs with this property up to at least 28 vertices (cf. Observation 5.4).

Lemma 5.2 *Let G be a cubic graph. Consider a decomposition of G in two linear forests. Then, the average length of a path in the decomposition is three.*

Proof An internal vertex of a path is a vertex of degree 2 in the path. Similarly, an external vertex is a vertex of degree 1 in the path. Observe that every vertex of G is internal in one linear forest and external in the other one. Since the number of paths in the decomposition is $|V(G)|/2$ (every path has exactly two external vertices), their average length is $\frac{2|E(G)|}{|V(G)|} = 3$. ■

Recall that the Heawood graph H is the point/line incidence graph of the Fano plane $PG(2, 2)$ which is bipartite, 3-regular, 3-arc transitive and has girth 6 and diameter 3.

Theorem 5.3 *Let H be the Heawood graph. Then, $la_4(H) > 2$.*

Proof First of all, note that H cannot have a decomposition in two 3-linear forests. Indeed, by Lemma 5.2, if the two linear forests have all paths of length at most three, then they have all paths of length exactly three. Moreover, since all paths have the same length, the number of paths in each forest is the same. A contradiction, since we have an even number of paths with three edges, but H has 21 edges, which is not a multiple of 6.

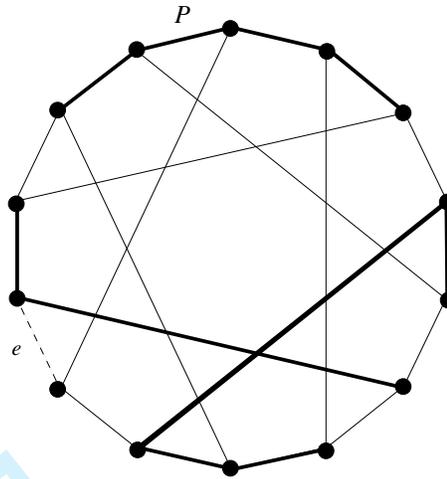


Figure 14: The partial 2-edge colouring of Heawood graph in the proof of Theorem 5.3

Now, suppose that H has a decomposition in two 4-linear forests, say F_1 and F_2 , and by the previous observation we can assume that at least one of them has a path of length 4, say $P \in F_1$. Consider the two external vertices of P , say u and v . The vertices u and v are both internal vertices in F_2 . Since P has even length, u and v are in the same bipartition class. Moreover, H has diameter 3, then u and v are at distance 2, and the four edges in F_2 which are incident to them are all distinct and form a path of length 4 (they cannot induce a 4-cycle since H has girth 6). Now, we can consider the external vertices of this new path of length 4 and we iterate the process. One can easily verify that we obtain a 2-edge colouring of $E(H)$ except for an edge, say e . Since H is 3-arc transitive, the edge colouring is, up to symmetries, independent by the choice of the first path P (see Figure 14). There is no way to colour the missing edge e so that we have two linear forests, so the assertion follows. ■

We also wrote a program to test if a cubic graph can be decomposed in two 4-linear forests. The source code of this program can be downloaded from [19]. Using the generator for cubic graphs *snarkhunter* [13, 11], we generated all cubic graphs up to 28 vertices and then tested which of these graphs cannot be decomposed in two 4-linear forests. This led to the following observation:

Observation 5.4 *The two cubic graphs of order 6 and the Heawood graph are the only cubic graphs up to at least 28 vertices which cannot be decomposed in two 4-linear forests.*

Following the flavour of what we have discussed in previous sections and, particularly, about Wormald's Conjecture, we wonder how we have to modify Thomassen's result if we require to have two linear forests which are also isomorphic. Obviously, this makes only sense if we consider cubic graphs of order $0 \pmod{4}$. We have extended our program to take this into account and surprisingly, it turns out that we do not need paths of length more than four for all graphs of order at most 24:

Observation 5.5 *Every cubic graph with $|V| \equiv 0 \pmod{4}$ with at most 24 vertices has*

1
2
3 a decomposition in two isomorphic 4-linear forests.
4

5 We leave the following strengthened version of Wormald's Conjecture as an open
6 problem:
7

8
9 **Problem 5.6** Does every cubic graph with $|V| \equiv 0 \pmod{4}$ have a decomposition in two
10 isomorphic 4-linear forests?
11

12
13 Another interesting and related open question posed by Jackson and Wormald in [24]
14 is the following:
15

16
17 **Problem 5.7 (Jackson and Wormald [24])** Does every cubic graph have a 2-edge
18 colouring such that each monochromatic connected component has at most four edges?
19

20
21 Note that a connected component is not necessarily a path here, so the Heawood
22 graph is not necessarily a counterexample in this case. Indeed, we can obtain such a valid
23 2-edge colouring of the Heawood graph by arbitrarily assigning a colour to the edge e in
24 Figure 14.
25

26
27 Furthermore, the two cubic graphs on 6 vertices admit a 2-edge colouring such that
28 each monochromatic connected component has at most four edges. In particular, the
29 bipartite cubic graph on 6 vertices admits, up to symmetry, only one 2-edge colouring
30 with all monochromatic connected component having at most four edges, and exactly one
31 of these connected components has precisely four edges.
32

33 Hence, it follows that we cannot have three as an upper bound in the previous prob-
34 lem and, from Observation 5.4, that the smallest counterexample must have at least 30
35 vertices.
36

37 38 39 Acknowledgements

40
41
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43 tructure at Ghent University.
44

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