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BOOK OF ABSTRACTS

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Graph signal processing and wavelet packets

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1 Introduction

Nowadays graphs became of significant importance given their use to describe complex system dynamics, with important applications to real world problems, e.g. graph representation of the brain, social networks, biological networks, spreading of a disease, etc., [1]-[5].

What is missing in graph signal processing is a general definition of Spectral Graph Wavelet Packets Transform in the same fashion as for the classical framework [6] (see also [7]), where the equivalent of frequency is represented by the eigenvalues of the Laplacian matrix. Bremer and coauthors in [8] introduced diffusion wavelet packets transforms starting from diffusion wavelet definition [9], based on a diffusion operator T on a manifold or a graph. Cloninger et al. in [10] defined the natural graph wavelet packet dictionaries by introducing a set of novel multiscale basis transforms by considering the distance between graph Laplacian eigenvectors.

In this paper we introduce a novel graph wavelet packets construction, to our knowledge different from the ones known in literature. Our work is inspired by the Spectral Graph Wavelet Transform (SGWT) defined by Hammond et al. in [11] and can be viewed as a generalization of their work. The result is a dictionary of frames particularly suitable for analyzing signals defined on graphs with a large number of nodes.

We will give some concrete examples on how the wavelet packets can be used for compressing, denoising and reconstruction by considering a signal, given by the fRMI (functional magnetic resonance imaging) data, on the nodes of voxel-wise brain graph $\mathscr{G}$ with 900.760 nodes (representing the brain voxels) defined in [1]-[2].

2 Main Results

2.1 Spectral wavelet operator and wavelet spaces

Let $\mathscr{G}$ denote an undirected connected weighted graph with N nodes, A the $N \times N$ (symmetric) adjacency matrix with non-negative real entries a_{ij}, where $a_{ij} > 0$ if there is an edge between vertices i and j. D denotes the diagonal $N \times N$ matrix with diagonal entries given by the degrees $d(i) = \sum_j a_{ij}$, and $\mathscr{L} = D - A$ is the non-normalized Laplacian. Alternatively one can consider the *normalized* Laplacian defined as in [1].

We shall denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$ the eigenvalues of $\mathscr{L}$ ordered in ascending order, including multiplicities, and by χ_ℓ, $\ell = 0, \ldots, N-1$ a set of corresponding orthonormal eigenvectors. Let $g : [0, +\infty) \to \mathbb{R}$ be a continuous function. In

[11] the authors call the operator $T_g : \mathbb{R}^N \to \mathbb{R}^N, \quad T_g = g(\mathscr{L})$, the *wavelet operator*, and for $s > 0$, $T_g^s = g(s\mathscr{L})$ is called the *wavelet operator at scale s*, i.e. the wavelet operator corresponding to the dilation of g by $s : g(s\cdot)$.

It follows from the definition that if χ_ℓ is an eingenvector of $\mathscr{L}$, corresponding to the eigenvalue λ_ℓ, then $T_g(\chi_\ell) = g(\lambda_\ell)\chi_\ell$. Also, for any $f \in \mathbb{R}^N$,

$$(T_g f)(m) = \sum_{\ell=0}^{N-1} g(\lambda_\ell)\hat{f}(\ell)\chi_\ell(m), \quad m \in \mathbb{R}^N. \tag{1}$$

In the classic case, wavelets are usually defined starting from a Multiresolution Analysis (MRA), $V_j \subset V_{j+1} \subset L^2(\mathbb{R})$. Translations of the scaling function φ span the core space V_0, and the translations of the wavelet ψ span the orthogonal complement W_0 of V_0 in V_1. In analogues way we define this spaces for the graph theory framework:

Definition 1. *Let $(s_j)_{j\geq 1}$ be a strictly increasing sequence of positive real numbers (thought as scales), set $s_0 = 1$. The space V_0 is spanned by the finite linear combinations of translates of the low-pass filter, h, $V_0 = \langle T_h\delta_n, n = 1,\ldots,N\rangle$. Hence*

$$f \in V_0 \Leftrightarrow \widehat{f}(\ell) = h(\lambda_\ell)\sum_{n=1}^{N} a_n \chi_\ell^*(n), \qquad a_n \in \mathbb{C}.$$

The space W_0 is spanned by the finite linear combinations of translates of the band-pass filter, g, $W_0 = \langle T_g\delta_n, n = 1,\ldots,N\rangle$.

Next, we define the spaces V_{-j} and W_{-j} for any $j \geq 1$. The space $V_{-j} = \langle T_h^{s_j} T_h^{s_{j-1}} \ldots T_h\delta_n, n = 1,\ldots,N\rangle$. The space $W_{-j} = \langle T_g^{s_j} T_h^{s_{j-1}} \ldots T_h\delta_n, n = 1,\ldots,N\rangle$.

Hence, following the same notation and the reasoning used for the wavelet spaces, we can define the *wavelet packets spaces*.

Definition 2. *We define $W_{0,0} = V_0$, $W_{1,0} = W_0$, and, for any $j \geq 1$, $W_{0,-j} = V_{-j}$, $W_{1,-j} = W_{-j}$. For any $j \geq 1$, we write $n \in \mathbb{N}$ in finite dyadic expansion $n = \sum_{i=0}^{j} \varepsilon_i 2^i, \quad \varepsilon_i = 0,1$ we define (set $s_0 = 1$) $W_{n,-j} = \langle T_{M_{\varepsilon_0}}^{s_0} T_{M_{\varepsilon_1}}^{s_1} \ldots T_{M_{\varepsilon_j}}^{s_j} \delta_k, k = 1,\ldots,N\rangle$, where*

$$M_{\varepsilon_i} = \begin{cases} h, \text{ if } \varepsilon_i = 0, \\ g, \text{ if } \varepsilon_i = 1. \end{cases}$$

and $T_{M_{\varepsilon_i}}^{s_i} = M_{\varepsilon_i}(s_i\mathscr{L})$.

2.2 Frames of spectral graph wavelet packets

The tuple $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_j)$ in the dyadic expansion of $n = 0,\ldots,2^{j+1}-1$ corresponds to the dyadic interval

$$I_{j,n} = \left[\frac{\varepsilon_j}{2} + \cdots + \frac{\varepsilon_0}{2^{j+1}}, \frac{\varepsilon_j}{2} + \cdots + \frac{\varepsilon_0}{2^{j+1}} + \frac{1}{2^{j+1}}\right) \subset [0,1)$$

Proposition 1. *Let $\mathscr{F}$ be a finite set of indices $(j,n) \in \mathbb{N} \times \{0,\ldots,2^{j+1}-1\}$, such that the collection $\{I_{j,n},\,(j,n) \in \mathscr{F}\}$ forms a partition of $[0,1)$: $[0,1) = \bigcup_{(j,n)\in\mathscr{F}} I_{j,n}$.*

Assume $G(\lambda) = h^2(\lambda) + g^2(\lambda) > 0$ for all $\lambda \in \mathbb{R}^+$. Let $L \geq 0$ be the maximum of the set $\{j \in \mathbb{N},\,(j,n) \in \mathscr{F},$ for some $n\}$. Then the system $\{T_{M_{(j,n)}}\delta_k,\,(j,n) \in \mathscr{F},\,k = 1,\ldots,N\}$ is a frame for $\mathbb{R}^N$ with lower and upper frame bound given respectively by

$$\min\{1, \min_{\lambda\in[0,\lambda_{max})} G^{L+1}(\lambda)\} \quad \textit{and} \quad \max\{1, \max_{\lambda\in[0,\lambda_{max})} G^{L+1}(\lambda)\}. \tag{2}$$

Hence if $\{T_{M_i}\delta_k,\,i = 0,1,\,k = 1,\ldots,N\}$ is a Parseval frame for $\mathbb{R}^N$, then the introduced system is a Parseval frame for $\mathbb{R}^N$, too.

3 Conclusion

As an application we have used a brain graph with 900.760 nodes, an fMRI signal and its noisy version, represented on the top row of Figure 1. After computing the wavelet packet coefficients we processed the signals, e.g. reconstruction, compressing and/or denoising. For lack of space, here, we report only the results for denoising using two different filters, bottom row of Figure 1.

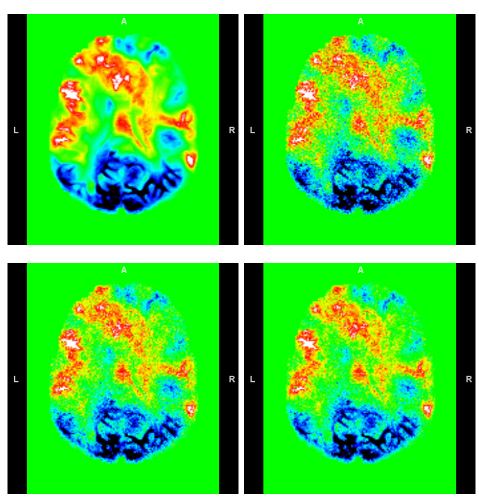

Fig. 1. Top row: (left panel) axial slices of the representation of the fRMI signal; (right panel) noisy fRMI signal on the brain graph. Bottom row: Reconstruction of the signal, from the left to the right after denoising at 33% for type-Meyer filter, and 54% spline filter, thresholding, respectively. The color spectrum assumes values from -0.002 black to 0.002 white.

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