

GENERALIZED SCHRÖDINGER-NEWTON SYSTEM IN DIMENSION $N \geq 3$: CRITICAL CASE

ANTONIO AZZOLLINI, PIETRO D'AVENIA, AND GIUSI VAIRA

ABSTRACT. In this paper we study a system which is equivalent to a nonlocal version of the well known Brezis Nirenberg problem. The difficulties related with the lack of compactness are here emphasized by the nonlocal nature of the critical nonlinear term.

We prove existence and nonexistence results of positive solutions when $N = 3$ and existence of solutions in both the resonance and the nonresonance case for higher dimensions.

CONTENTS

1. Introduction and statement of the main results	1
2. Preliminaries	3
2.1. The reduction method	3
2.2. The Palais-Smale condition	6
3. The dimension $N = 3$: positive solutions	8
3.1. An existence result	9
3.2. A nonexistence result	11
4. The dimensions $N \geq 4$	12
4.1. Positive solutions	12
4.2. The geometrical properties for the Linking Theorem	13
4.3. Sign changing solutions: the nonresonance case	15
4.4. Sign changing solutions: the resonance case	16
References	17

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In this paper we consider the following Schrödinger-Newton type system

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-3}u\phi & \text{in } \Omega \\ -\Delta \phi = |u|^{2^*-1} & \text{in } \Omega \\ u = \phi = 0 & \text{on } \partial\Omega \end{cases} \quad (\mathcal{SN})$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth and bounded domain and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent.

By applying a standard reduction, it is easy to see that (\mathcal{SN}) is equivalent to the following critical version of a Choquard type equation in a bounded domain, with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = \lambda u + \left(\int_{\Omega} G(x-y)|u(y)|^{2^*-1} dy \right) |u|^{2^*-3}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where G is the Green function of the Laplacian in the domain Ω with Dirichlet homogeneous boundary conditions (see e.g. [6]). As emphasized in [1], where (\mathcal{SN}) has been firstly introduced and positive solutions have been considered, the problem we are going to study is variational, and the particular choice of the powers in the first and the second equations brings as a

2010 *Mathematics Subject Classification.* 35J20, 35J57, 35J60.

Key words and phrases. Critical nonlinearity, Schrödinger-Newton system.

A. Azzollini, P. d'Avenia, and G. Vaira are partially supported by a grant of the group GNAMPA of INdAM. A. Azzollini is also partially supported by Fondo R.I.L. 2015, Project "Studio di equazioni differenziali alle derivate parziali nonlineari", Università degli Studi della Basilicata. G. Vaira is also partially supported by MIUR-PRIN project-201274FYK7 005.

consequence difficulties in questions related with compactness.

The analogy arising by a comparison with the classic problem of finding solutions to

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

leads to interpret (1.1) as a nonlocal version of the well known Brezis-Nirenberg problem, suggesting approaches similar to those in [3] and in [4, 7].

As observed by Brezis and Nirenberg in [3], the presence of a nonlinearity critically growing causes problems in obtaining compactness for the Palais-Smale sequences unless they lie on a certain sublevel of the functional associated to the problem. As a consequence, it is quite clear that large values of λ give a positive contribution in gaining compactness.

On the other hand, it was also observed that, if λ is greater than the first eigenvalue of the Laplacian with zero Dirichlet condition on Ω , we have no positive solution to (1.2). The question related to the existence of sign-changing solutions of (1.2) for $\lambda \geq \lambda_1$ was handled and solved in dimension $N \geq 4$ in [4], and successively reconsidered in [7] in presence of a general nonlinearity $g(x, u)$ in the place of the linear term λu .

In accordance with what is expected, the dimension of the space where we set the problem and the position of λ with respect to the eigenvalues in the spectrum of the Laplacian play a very important role also when we try to solve (SN). In particular, while for λ not belonging to the spectrum and $N \geq 4$ suitable estimates on the functional permit to achieve results similar to those already known for the Brezis-Nirenberg problem, the three-dimensional case and the so called “resonance case” are quite more delicate.

In order to present our results, let us first introduce some notations. Let λ_k 's, $k \in \mathbb{N}$, be the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary conditions on Ω . It is well known that

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots$$

with $\lambda_k \rightarrow +\infty$ as $k \rightarrow \infty$. We denote by $\sigma(-\Delta)$ the spectrum of $-\Delta$.

We recall the following results whose proofs are essentially contained in [1].

Proposition 1.1. *Problem (SN) has no positive solution for $\lambda \geq \lambda_1$ and, if Ω is starshaped, no solution for $\lambda \leq 0$.*

Moreover, if $N = 3$ and Ω corresponds to the ball B_R centered in 0 with radius $R > 0$, the problem (SN) has at least a positive radial ground state solution for any $\lambda \in]\frac{3}{10}\lambda_1, \lambda_1[$.

We are able to improve results in [1] as follows

Theorem 1.2. *Assume $N = 3$ and Ω corresponding to the ball B_R centered in 0 with radius $R > 0$. Then*

- (i) *problem (SN) has a positive radial ground state solution for any $\lambda \in](\frac{1}{4} + \frac{2}{5\pi^2})\lambda_1, \lambda_1[$;*
- (ii) *problem (SN) has no positive solution for $\lambda \in]0, \lambda^*]$, where $\lambda^* \in]\frac{\lambda_1}{16}, \frac{9}{64}\lambda_1[$ is explicitly determined as a solution of a suitable equation.*

We specify that when we say *positive solution* we mean a couple (u, ϕ) where both the functions are positive in Ω .

The existence result has been obtained by means of refined estimates on the explicit expression of the solution ϕ of the second equation, when u is chosen inside the well known one parameter family of functions introduced by Brezis and Nirenberg in [3]. We recall that the components of such a family, obtained cutting off the solutions of the critical problem (1.2) set in $\Omega = \mathbb{R}^N$, are used by Brezis and Nirenberg as test functions to prove that there exists at least a Palais-Smale sequence lying in the compactness sublevel.

As regards the nonexistence result, we are going to use an argument exploited in [3] for (1.2),

and based on the a priori information about the radial symmetry of any positive solution of (1.2). To this end, we preliminarily prove an analogous result on the symmetry of all positive solutions of (SN) by showing that, since the general assumptions of the maximum principle hold, the moving plane method applies in a quite natural way to the solutions of our system.

Higher dimensions permit to obtain more convenient estimates by which not only we enlarge the range of existence for what concerns positive solutions, but we are also able to study the presence of sign changing solutions for $\lambda \geq \lambda_1$. The situation changes considerably depending whether λ is in the spectrum or not.

We have the following two results.

Theorem 1.3. *Assume $N \geq 4$. Then problem (SN) has*

- (i) *a positive ground state solution for any $\lambda \in]0, \lambda_1[$;*
- (ii) *a sign changing solution for every $\lambda \in]\lambda_k, \lambda_{k+1}[$.*

Theorem 1.4. *Assume $N \geq 6$ and $\lambda \in \sigma(-\Delta)$. Then (SN) possesses a sign changing solution.*

Comparing these results with those in [7], we observe the loss of the dimension $N = 5$ when we are in the resonance case. The differences between the resonance and the nonresonance case, already pointed out by Gazzola and Ruf for the Brezis Nirenberg problem, here are emphasized by the behaviour of the nonlocal nonlinear term when we compute the functional on finite dimensional subspaces of $H_0^1(\Omega)$. In particular, it is quite interesting to observe that, differently from what happens for the critical term in the functional associated to the problem (1.2), in finite dimensional subspaces the integral of the nonlocal term does not have the growth of the norm to the critical power, but it behaves as the norm to the power $2(2^* - 1)$.

The paper is so organized: in Section 2, after having recalled the reduction method which classically applies to this kind of problems, we present some useful properties related with the reduced functional, and in particular we study the compactness of its Palais-Smale sequences. In Section 3 we are interested in positive solutions when $N = 3$ and we prove the existence and the nonexistence results contained in Theorem 1.2.

Finally in Section 4 we consider the case $N \geq 4$ looking for positive and sign changing solutions. In particular for the existence of the latter we take advantage of the Linking Theorem in [11] which we apply both in the resonance and in the nonresonance case in order to prove Theorem 1.3 and Theorem 1.4.

Notations: In what follows we let $H_0^1(\Omega)$ the usual Sobolev space equipped with the norm $\|u\| := (\int_{\Omega} |\nabla u|^2 dx)^{\frac{1}{2}}$ and, for any $u \in L^q(\Omega)$ we let $|u|_q := (\int_{\Omega} |u|^q dx)^{\frac{1}{q}}$. Moreover, with C, C_i we denote positive constants that can vary also from line to line and by B_r the ball in \mathbb{R}^N centered at zero with radius r .

2. PRELIMINARIES

2.1. The reduction method. As it is classical in the study of this type of systems, for every $u \in H_0^1(\Omega)$ there exists a unique $\phi_u \in H_0^1(\Omega)$ that solves the second equation of (SN). Hence we can reduce the system (SN) to the boundary value problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-3} u \phi_u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

In fact it can be easily proved that $(u, \phi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ is a solution of (SN) if and only if u solves (2.1) and $\phi = \phi_u$.

Moreover, solutions of (2.1) can be found as critical points of the C^1 one-variable functional

$I : H_0^1(\Omega) \rightarrow \mathbb{R}$

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{1}{2(2^* - 1)} \int_{\Omega} |\nabla \phi_u|^2 dx \\ &= \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} |u|^2 dx - \frac{1}{2(2^* - 1)} \int_{\Omega} \phi_u |u|^{2^* - 1} dx \end{aligned}$$

since for all $u, v \in H_0^1(\Omega)$ we have

$$I'(u)[v] = \int_{\Omega} \nabla u \nabla v dx - \lambda \int_{\Omega} uv dx - \int_{\Omega} |u|^{2^* - 3} u \phi_u v dx.$$

In the next lemma we summarize the properties of such a function ϕ_u that will be useful in the following.

Lemma 2.1. *For every fixed $u \in H_0^1(\Omega)$ we have:*

- (i) $\phi_u \geq 0$ a.e. in Ω ;
- (ii) for all $t > 0$, $\phi_{tu} = t^{2^* - 1} \phi_u$;
- (iii) $\|\phi_u\| \leq S^{-\frac{2^*}{2}} \|u\|^{2^* - 1}$, where $S = \inf_{v \in H^1(\mathbb{R}^N) \setminus \{0\}} \|\nabla v\|_2^2 / \|v\|_{2^*}^2$;
- (iv) $\|\phi_u\|^2 \geq 2\delta \|u\|_{2^*}^2 - \delta^2 \|u\|^2$ for any $\delta > 0$.

Moreover

- (v) for every $u, v \in H_0^1(\Omega)$,

$$\int_{\Omega} \phi_u |v|^{2^* - 1} dx = \int_{\Omega} \phi_v |u|^{2^* - 1} dx;$$

- (vi) for every $u, u_1, \dots, u_k \in H_0^1(\Omega)$,

$$\left| \phi_u - \sum_{i=1}^k \phi_{u_i} \right|_{2^*} \leq \frac{1}{S} \left| |u|^{2^* - 1} - \sum_{i=1}^k |u_i|^{2^* - 1} \right|_{\frac{2^*}{2^* - 1}};$$

- (vii) if (u_n) in $H_0^1(\Omega)$ and $u \in H_0^1(\Omega)$ are such that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then, up to subsequences, $\phi_{u_n} \rightharpoonup \phi_u$ in $H_0^1(\Omega)$ and strongly in $L^p(\Omega)$ for all $p \in [1, 2^*)$. Moreover

$$\int_{\Omega} \phi_{u_n} |u_n|^{2^* - 1} dx - \int_{\Omega} \phi_{u_n - u} |u_n - u|^{2^* - 1} dx = \int_{\Omega} \phi_u |u|^{2^* - 1} dx + o_n(1); \quad (2.2)$$

- (viii) if $W \subset H_0^1(\Omega)$ is a finite dimensional subspace, then there exists $C = C(W) > 0$ such that for any $w \in W$ we have

$$C^{-1} \|w\|^{2^* - 1} \leq \|\phi_w\| \leq C \|w\|^{2^* - 1}.$$

Proof. Property (i) is trivial (see e.g. [5]) and (ii) easily follows from

$$-\Delta \phi_{tu} = t^{2^* - 1} |u|^{2^* - 1} = -\Delta (t^{2^* - 1} \phi_u) \quad \text{in } \Omega$$

and $\phi_{tu} = \phi_u = 0$ on $\partial\Omega$.

Multiplying the second equation of (SN) by ϕ_u , integrating and using Hölder and Sobolev inequalities we have

$$\|\phi_u\|^2 = \int_{\Omega} \phi_u |u|^{2^* - 1} dx \leq |\phi_u|_{2^*} |u|_{2^*}^{2^* - 1} \leq S^{-\frac{2^*}{2}} \|u\|^{2^* - 1} \|\phi_u\|$$

and then (iii).

Moreover, multiplying the second equation of (SN) by $|u|$ and integrating we have

$$|u|_{2^*}^{2^*} = \int_{\Omega} \nabla \phi_u \nabla |u| dx \leq \frac{1}{2\delta} \|\phi_u\|^2 + \frac{\delta}{2} \|u\|^2 \quad \text{for any } \delta > 0$$

and so (iv).

To obtain (v) we observe that

$$\int_{\Omega} \phi_v |u|^{2^* - 1} dx = \int_{\Omega} \nabla \phi_u \nabla \phi_v dx = \int_{\Omega} \phi_u |v|^{2^* - 1} dx.$$

A further simple computation gives

$$\begin{aligned} \left| \phi_u - \sum_{i=1}^k \phi_{u_i} \right|_{2^*}^2 &\leq \frac{1}{S} \left\| \phi_u - \sum_{i=1}^k \phi_{u_i} \right\|^2 \\ &= \frac{1}{S} \int (\phi_u - \sum_{i=1}^k \phi_{u_i}) (|u|^{2^*-1} - \sum_{i=1}^k |u_i|^{2^*-1}) dx \\ &\leq \frac{1}{S} \left| \phi_u - \sum_{i=1}^k \phi_{u_i} \right|_{2^*} \left| |u|^{2^*-1} - \sum_{i=1}^k |u_i|^{2^*-1} \right|_{\frac{2^*}{2^*-1}} \end{aligned}$$

and (vi) follows.

To prove the first part of (vii) we can proceed as in [12, Proposition 2.4]. Furthermore by applying (v) we get

$$\begin{aligned} &\int_{\Omega} \phi_{u_n} |u_n|^{2^*-1} dx - \int_{\Omega} \phi_{u_n-u} |u_n - u|^{2^*-1} dx \\ &= \int_{\Omega} (\phi_{u_n} - \phi_{u_n-u}) (|u_n|^{2^*-1} - |u_n - u|^{2^*-1}) dx + 2 \int_{\Omega} (\phi_{u_n} - \phi_{u_n-u}) |u_n - u|^{2^*-1} dx. \end{aligned}$$

An easy variant of the classical Brezis-Lieb Lemma (see also [10, Lemma 2.5]) yields that

$$|u_n|^{2^*-1} - |u_n - u|^{2^*-1} \rightarrow |u|^{2^*-1} \quad \text{in } L^{\frac{2^*}{2^*-1}}(\Omega) \text{ as } n \rightarrow +\infty$$

and applying (vi) we get that

$$\phi_{u_n} - \phi_{u_n-u} \rightarrow \phi_u \quad \text{in } L^{2^*}(\Omega) \text{ as } n \rightarrow +\infty. \quad (2.3)$$

Hence

$$\int_{\Omega} (\phi_{u_n} - \phi_{u_n-u}) (|u_n|^{2^*-1} - |u_n - u|^{2^*-1}) dx \rightarrow \int_{\Omega} \phi_u |u|^{2^*-1} dx \text{ as } n \rightarrow +\infty.$$

Moreover, applying again [13, Proposition 5.4.7], we have $|u_n - u|^{2^*-1} \rightarrow 0$ in $L^{\frac{2^*}{2^*-1}}(\Omega)$. Hence, since $\phi_u \in L^{2^*}(\Omega)$ and using also (2.3),

$$\begin{aligned} &\int_{\Omega} (\phi_{u_n} - \phi_{u_n-u}) |u_n - u|^{2^*-1} dx \\ &= \int_{\Omega} (\phi_{u_n} - \phi_{u_n-u} - \phi_u) |u_n - u|^{2^*-1} dx + \int_{\Omega} \phi_u |u_n - u|^{2^*-1} dx \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$.

Finally, in order to see (viii), consider W a finite dimensional subspace of $H_0^1(\Omega)$. The second inequality is already known by (iii) and it actually does not depend on W .

Now, pick any $w \in W$ and consider the second equation of (SN).

Multiplying by $|w|$, integrating and applying the Holder inequality, we have

$$|w|_{2^*}^{2^*} = \int_{\Omega} \nabla \phi_w \nabla |w| dx \leq \|\phi_w\| \|w\|.$$

By equivalence of norms in finite dimensional spaces, we have that there exists C depending on W such that $C\|w\| \leq |w|_{2^*}$, and then we are done. \square

We conclude this section observing that, applying (iv) in Lemma 2.1 for $\delta = 1$, we have that for all $u \in H_0^1(\Omega)$

$$I(u) \leq \frac{N}{N+2} \|u\|^2 - \frac{\lambda}{2} |u|_2^2 - \frac{N-2}{N+2} |u|_{2^*}^{2^*}. \quad (2.4)$$

2.2. The Palais-Smale condition. As usual, in order to have compactness, a first crucial step consists in checking the Palais-Smale (PS for short) condition. We recall that a sequence $(u_n) \subset H_0^1(\Omega)$ is called a PS sequence for I at level c if $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$ in $[H_0^1(\Omega)]^{-1}$. The functional I satisfies the PS condition at level c , if every PS sequence at level c has a convergent subsequence in $H_0^1(\Omega)$.

Since we are in the critical case, we do not know if the functional I satisfies the PS condition at all levels. However there is a set of values in which it is preserved.

Lemma 2.2. *The functional I satisfies the Palais-Smale condition in $]-\infty, \frac{2}{N+2}S^{\frac{N}{2}}[$.*

Proof. Let $(u_n) \subset H_0^1(\Omega)$ be a PS sequence for I at level $c < \frac{2}{N+2}S^{\frac{N}{2}}$. First we show that (u_n) is bounded. In what follows all the convergences are meant up to a subsequence.

Indeed, using also (iv) of Lemma 2.1, we have

$$2c + o_n(1)\|u_n\| = 2I(u_n) - I'(u_n)[u_n] = \frac{2^* - 2}{2^* - 1}\|\phi_{u_n}\|^2 \geq C_1|u_n|_2^{2^*} - C_2\|u_n\|^2$$

so that

$$\|\phi_{u_n}\|^2 \leq C(1 + \|u_n\|)$$

and

$$|u_n|_2^{2^*} \leq C(1 + \|u_n\|^2).$$

Hence

$$\|u_n\|^2 = 2I(u_n) + \lambda|u_n|_2^2 + \frac{1}{2^* - 1}\|\phi_{u_n}\|^2 \leq C\left(1 + \|u_n\|^{\frac{4}{2^*}} + \|u_n\|\right)$$

and, since $2 < 2^*$, it follows that (u_n) is bounded.

Then we can assume that there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, $u_n \rightarrow u$ in $L^q(\Omega)$ for every $q \in [1, 2^*)$ and a.e. in Ω .

Now, let us set $f(s) := |s|^{2^*-3}s$. Since (u_n) is bounded in $L^{2^*}(\Omega)$, then $(f(u_n))$ is bounded in $L^{\frac{2^*}{2^*-2}}(\Omega)$ and so, in a standard way, it follows that $f(u_n) \rightharpoonup f(u)$ in $L^{\frac{2^*}{2^*-2}}(\Omega)$. Then, for all $\varphi \in C_0^\infty(\Omega)$, using also (vii) of Lemma 2.1, Hölder and Sobolev inequalities, and since $\phi_u \varphi \in L^{\frac{2^*}{2}}(\Omega)$,

$$\begin{aligned} \left| \int_{\Omega} f(u_n)\phi_{u_n}\varphi \, dx - \int_{\Omega} f(u)\phi_u\varphi \, dx \right| &\leq \left| \int_{\Omega} (\phi_{u_n} - \phi_u) f(u_n)\varphi \, dx \right| + \left| \int_{\Omega} (f(u_n) - f(u))\phi_u\varphi \, dx \right| \\ &\leq C|\varphi|_\infty \|u_n\|^{2^*-2} \|\phi_{u_n} - \phi_u\|_{\frac{2^*}{2}} + o_n(1) \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$. Hence

$$I'(u_n)[\varphi] \rightarrow I'(u)[\varphi]$$

and, by density, we get

$$0 = I'(u)[u] = \|u\|^2 - \lambda|u|_2^2 - \|\phi_u\|^2,$$

from which

$$I(u) = \frac{2^* - 2}{2(2^* - 1)}\|\phi_u\|^2 \geq 0. \quad (2.5)$$

Since $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ we get

$$\|u_n\|^2 = \|u_n - u\|^2 + \|u\|^2 + o_n(1).$$

Then, by using the strong convergence $u_n \rightarrow u$ in $L^2(\Omega)$ and (2.2) we get

$$I(u_n) = I(u) + I_0(u_n - u) + o_n(1) \quad (2.6)$$

with

$$I_0(u) = \frac{1}{2}\|u\|^2 - \frac{1}{2(2^* - 1)}\|\phi_u\|^2.$$

Furthermore

$$\begin{aligned} o_n(1) &= I'(u_n)[u_n - u] = (I'(u_n) - I'(u))[u_n - u] \\ &= \|u_n - u\|^2 - \lambda|u_n - u|_2^2 - \int_{\Omega} \phi_{u_n} f(u_n)(u_n - u) dx + \int_{\Omega} \phi_u f(u)(u_n - u) dx. \end{aligned} \quad (2.7)$$

Since $u_n \rightharpoonup u$ in $L^{2^*}(\Omega)$ and $\phi_u f(u) \in L^{\frac{2^*}{2^*-1}}(\Omega)$,

$$\int_{\Omega} \phi_u f(u)(u_n - u) dx = o_n(1).$$

Moreover

$$\begin{aligned} \int_{\Omega} \phi_{u_n} f(u_n)(u_n - u) dx &= \int_{\Omega} \phi_{u_n} |u_n|^{2^*-1} dx - \int_{\Omega} \phi_u |u|^{2^*-1} dx \\ &\quad - \int_{\Omega} (\phi_{u_n} - \phi_u) f(u_n) u dx - \int_{\Omega} \phi_u u (f(u_n) - f(u)) dx. \end{aligned} \quad (2.8)$$

Since the sequence $((\phi_{u_n} - \phi_u) f(u_n))$ is bounded in $L^{\frac{2^*}{2^*-1}}(\Omega)$, $\phi_{u_n} \rightarrow \phi_u$ and $f(u_n) \rightarrow f(u)$ a.e. in Ω , by [13, Proposition 5.4.7] we have

$$\int_{\Omega} (\phi_{u_n} - \phi_u) f(u_n) u dx = o_n(1) \quad (2.9)$$

and, analogously, we can prove that

$$\int_{\Omega} \phi_u u (f(u_n) - f(u)) dx = o_n(1). \quad (2.10)$$

Then, using (2.2), (2.9) and (2.10) in (2.8), we obtain

$$\int_{\Omega} \phi_{u_n} f(u_n)(u_n - u) dx = \int_{\Omega} \phi_{u_n - u} |u_n - u|^{2^*-1} dx + o_n(1).$$

Moreover, by (2.7) we get

$$\|u_n - u\|^2 - \int_{\Omega} \phi_{u_n - u} |u_n - u|^{2^*-1} dx = o_n(1) \quad (2.11)$$

and so

$$I_0(u_n - u) = \frac{1}{2} \|u_n - u\|^2 - \frac{1}{2(2^* - 1)} \|u_n - u\|^{2^*} + o_n(1) = \frac{2}{N + 2} \|u_n - u\|^2 + o_n(1).$$

On the other hand, from (2.6) we get, using also (2.5),

$$I_0(u_n - u) = I(u_n) - I(u) + o_n(1) \leq c + o_n(1) < \frac{2}{N + 2} S^{\frac{N}{2}}.$$

Then it follows that

$$\limsup_n \|u_n - u\|^2 < S^{\frac{N}{2}}$$

and so, by (2.11) and (iii) of Lemma 2.1,

$$o_n(1) = \|u_n - u\|^2 - \int_{\Omega} \phi_{u_n - u} |u_n - u|^{2^*-1} dx \geq \|u_n - u\|^2 \left[1 - \left(\frac{\|u_n - u\|^2}{S^{\frac{N}{2}}} \right)^{2^*-2} \right] \geq C \|u_n - u\|^2.$$

Hence $u_n \rightarrow u$ in $H_0^1(\Omega)$. \square

3. THE DIMENSION $N = 3$: POSITIVE SOLUTIONS

In dimension 3 this type of problems turns out to be rather delicate even in the Brezis-Nirenberg problem (1.2) (see [3]) and so we consider only the case in which Ω is a ball. Moreover, for simplicity but without losing generality, we take $\Omega = B_1$ so that $\lambda_1 = \pi^2$.

Before we present our existence and nonexistence results, we introduce the following important symmetry property of all positive solutions, essentially based on the moving plane technique due to Gidas, Ni and Nirenberg [8]. To prove it we use a standard argument suitably adapted for our purposes. We point out that the dimension N does not have any role in the proof, even if we are here interested only in the case $N = 3$.

Lemma 3.1. *If $\Omega = B_1$, any positive solution (u, ϕ) of (SN), with $u, \phi \in C^2(B_1) \cap C(\bar{B}_1)$, is such that both u and ϕ are radially symmetric.*

Proof. Assume $u \in C^1(\bar{B}_1)$ in order to compute the derivatives in ∂B_1 (this further assumption can be removed generalizing the meaning of the derivatives on the boundary) and $\lambda > 0$ since we have no solution for $\lambda \leq 0$.

Define for any $\mu \in [0, 1]$ the sets

$$P_\mu = \{x \in \mathbb{R}^3 \mid x_3 = \mu\}$$

and

$$E_\mu = \{x \in B_1 \mid \mu < x_3 < 1\}.$$

Moreover, for any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, let $x_\mu = (x_1, x_2, 2\mu - x_3)$ be the reflection of x with respect to P_μ and assume the following notation for any $x \in E_\mu$:

$$u_\mu(x) = u(x_\mu).$$

By the Hopf Lemma (see for example [9, Lemma 3.4]), we have that for all $\mu \in]0, 1[$ the outward derivative $\frac{\partial u}{\partial \nu}$ is negative in $\partial E_\mu \cap \partial B_1$. Of course, since ∇u and ν have the same direction, this means that the verse of ∇u is inward, and then we deduce that $\frac{\partial u}{\partial x_3} < 0$ in $\partial E_\mu \cap \partial B_1$.

By continuity, we deduce that for values of μ close to 1, we have $\frac{\partial u}{\partial x_3} < 0$ everywhere in $E_\mu \cup \tilde{E}_\mu$, where $\tilde{E}_\mu = \{x \in B_1 \mid x_\mu \in E_\mu\}$. Of course, this fact implies that values of μ close to 1 are in $\mathcal{A} := \{\mu \in]0, 1] \mid \forall x \in E_\mu : u(x) < u_\mu(x)\}$.

Now, define $\mu_0 = \inf\{\mu \in]0, 1] \mid \eta \in \mathcal{A}, \forall \eta \in [\mu, 1]\}$.

We want to prove by contradiction that $\mu_0 = 0$ so that we assume $\mu_0 > 0$.

Observe that in E_μ we have $-\Delta u_\mu = (-\Delta u)_\mu$, $\frac{\partial u_\mu}{\partial x_3} = -\left(\frac{\partial u}{\partial x_3}\right)_\mu$ and $\phi_{u_\mu} = (\phi_u)_\mu$.

Denote $\phi = \phi_u$, $\phi_\mu = (\phi_u)_\mu$ and $w = u_{\mu_0} - u$.

Of course we have $w \geq 0$ in $\overline{E_{\mu_0}}$ and

$$-\Delta w = \lambda w + u_{\mu_0}^{2^*-2} \phi_{\mu_0} - u^{2^*-2} \phi \text{ in } E_{\mu_0}. \quad (3.1)$$

Since we have

$$\begin{aligned} -\Delta(\phi_{\mu_0} - \phi) &= u_{\mu_0}^{2^*-1} - u^{2^*-1} \geq 0 \text{ in } E_{\mu_0}, \\ \phi_{\mu_0} - \phi &\geq 0 \text{ on } \partial E_{\mu_0}, \end{aligned}$$

by the weak maximum principle we deduce that $\phi_{\mu_0} \geq \phi$ in E_{μ_0} .

Now, by (3.1), we have that $-\Delta w \geq 0$ in E_{μ_0} and then, by the strong maximum principle and since $w = 0$ in P_{μ_0} , we have that either $w > 0$ in E_{μ_0} or w is constant. Of course w is not constant since $w > 0$ on $\partial E_{\mu_0} \setminus P_{\mu_0}$ and then w is positive in E_{μ_0} .

Moreover, by the Hopf Lemma, in P_{μ_0} we have $\frac{\partial w}{\partial x_3} > 0$ and then, since $x = x_{\mu_0}$, we get $\frac{\partial u}{\partial x_3} = -\frac{1}{2} \frac{\partial w}{\partial x_3} < 0$.

Then, again by continuity, there exists V a neighborhood of P_{μ_0} such that $\frac{\partial u}{\partial x_3} < 0$ in $V \cap B_1$.

In conclusion, since $u < u_{\mu_0}$ in E_{μ_0} and $\frac{\partial u}{\partial x_3} < 0$ in $V \cap B_1$, we deduce that actually there exists

$\varepsilon > 0$ such that $\mu_0 - \varepsilon > 0$ and $[\mu_0 - \varepsilon, 1] \subset \mathcal{A}$: a contradiction.
We deduce that $\mu_0 = 0$ which implies

$$u \leq u_0 \text{ in } E_0. \quad (3.2)$$

Since the same arguments can be repeated replacing the interval $]0, 1]$ with $[-1, 0[$, E_μ with $F_\mu = \{x \in B_1 \mid -1 < x_3 < \mu\}$ and defining $\mu_0 = \sup\{\mu \in [-1, 0[\mid \eta \in \mathcal{A}, \forall \eta \in [-1, \mu]\}$, we can prove that

$$u \leq u_0 \text{ in } F_0. \quad (3.3)$$

It is easy to see that (3.2) and (3.3) together imply that $u = u_0$ in B_1 . Since this conclusion can be obtained in the same way for any reflection plane containing the origin, we deduce that u is radial. The symmetry of ϕ comes easily from that of u . \square

3.1. An existence result. This section is devoted to the proof of (i) of Theorem 1.2. Since the geometrical assumptions of the Mountain Pass Theorem are satisfied, we set

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0, 1], H_0^1(\Omega)) : \gamma(0) = 0, I(\gamma(1)) < 0\}$ and, by Lemma 2.2, we only have to prove that

$$c < \frac{2}{5} S^{\frac{3}{2}} = \frac{3\sqrt{3}}{10} \pi^2.$$

So we consider a smooth positive function $\varphi = \varphi(r)$ such that $\varphi(0) = 1$, $\varphi'(0) = 0$ and $\varphi(1) = 0$. Following [3] we set $r = |x|$,

$$u_\varepsilon(r) = \frac{\varphi(r)}{(\varepsilon + r^2)^{\frac{1}{2}}}$$

and we get the estimates

$$\|u_\varepsilon\|^2 = K_1 \varepsilon^{-\frac{1}{2}} + 4\pi \int_0^1 |\varphi'(r)|^2 dr + O(\varepsilon^{\frac{1}{2}}), \quad |u_\varepsilon|_2^2 = 4\pi \int_0^1 \varphi^2(r) dr + O(\varepsilon^{\frac{1}{2}}),$$

where

$$K_1 := \int_{\mathbb{R}^3} \frac{|x|^2}{(1 + |x|^2)^3} dx = \frac{3}{4} \pi^2.$$

We have that

$$\sup_{t > 0} I(tu_\varepsilon) = I(t_\varepsilon u_\varepsilon) = \frac{2}{5} \frac{(\|u_\varepsilon\|^2 - \lambda |u_\varepsilon|_2^2)^{\frac{5}{4}}}{\|\phi_{u_\varepsilon}\|^{\frac{1}{2}}} \quad (3.4)$$

where

$$t_\varepsilon := \sqrt[8]{\frac{\|u_\varepsilon\|^2 - \lambda |u_\varepsilon|_2^2}{\|\phi_{u_\varepsilon}\|^2}}.$$

Moreover

$$(\|u_\varepsilon\|^2 - \lambda |u_\varepsilon|_2^2)^{\frac{5}{4}} = K_1^{\frac{5}{4}} \varepsilon^{-\frac{5}{8}} \left(1 + \frac{5}{4} \frac{A(\varphi)}{K_1} \sqrt{\varepsilon} + o(\sqrt{\varepsilon})\right) \quad (3.5)$$

where

$$A(\varphi) := 4\pi \left(\int_0^1 |\varphi'(r)|^2 dr - \lambda \int_0^1 \varphi^2(r) dr \right).$$

Now, since u_ε is radial and smooth, also ϕ_{u_ε} is radial and smooth. Moreover, for $s \in (0, 1)$

$$\phi_{u_\varepsilon}''(s) + 2 \frac{\phi_{u_\varepsilon}'(s)}{s} = -u_\varepsilon^5(s),$$

namely,

$$(s^2 \phi_{u_\varepsilon}'(s))' = -s^2 u_\varepsilon^5(s).$$

Integrating in $(0, r)$, for $r \in (0, 1)$, and using the smoothness of ϕ_{u_ε} ($\phi'_{u_\varepsilon}(0) = 0$), we get

$$\begin{aligned}\phi'_{u_\varepsilon}(r) &= -\frac{1}{r^2} \int_0^r s^2 |u_\varepsilon(s)|^5 ds \\ &= -\frac{1}{r^2} \int_0^r \frac{\varphi^5(s) s^2}{(\varepsilon + s^2)^{\frac{5}{2}}} ds \\ &= -\frac{1}{r^2} \left(\int_0^r s^2 \frac{\varphi^5(s) - 1}{(\varepsilon + s^2)^{\frac{5}{2}}} ds + \int_0^r \frac{s^2}{(\varepsilon + s^2)^{\frac{5}{2}}} ds \right)\end{aligned}$$

and so, using the fact that $\varphi(0) = 1$ and $\varphi'(0) = 0$,

$$\begin{aligned}\|\phi_{u_\varepsilon}\|^2 &= 4\pi \int_0^1 \frac{1}{r^2} \left(\int_0^r s^2 \frac{\varphi^5(s) - 1}{(\varepsilon + s^2)^{\frac{5}{2}}} ds + \int_0^r \frac{s^2}{(\varepsilon + s^2)^{\frac{5}{2}}} ds \right)^2 dr \\ &\geq 4\pi \left[\int_0^1 \frac{1}{r^2} \left(\int_0^r \frac{s^2}{(\varepsilon + s^2)^{\frac{5}{2}}} ds \right)^2 dr \right. \\ &\quad \left. + 2 \int_0^1 \frac{1}{r^2} \left(\int_0^r s^2 \frac{\varphi^5(s) - 1}{(\varepsilon + s^2)^{\frac{5}{2}}} ds \right) \left(\int_0^r \frac{s^2}{(\varepsilon + s^2)^{\frac{5}{2}}} ds \right) dr \right] \\ &\geq 4\pi \left[\int_0^1 \frac{1}{r^2} \left(\int_0^r \frac{s^2}{(\varepsilon + s^2)^{\frac{5}{2}}} ds \right)^2 dr \right. \\ &\quad \left. - C \int_0^1 \frac{1}{r^2} \left(\int_0^r \frac{s^4}{(\varepsilon + s^2)^{\frac{5}{2}}} ds \right) \left(\int_0^r \frac{s^2}{(\varepsilon + s^2)^{\frac{5}{2}}} ds \right) dr \right] \\ &= 4\pi \left[\frac{1}{9\varepsilon^2} \int_0^1 \frac{r^4}{(\varepsilon + r^2)^3} dr \right. \\ &\quad \left. - \frac{C}{3\varepsilon} \int_0^1 \left(\log \left(\frac{r + \sqrt{\varepsilon + r^2}}{\sqrt{\varepsilon}} \right) - \frac{r(4r^2 + 3\varepsilon)}{3(\varepsilon + r^2)^{\frac{3}{2}}} \right) \frac{r}{(\varepsilon + r^2)^{\frac{3}{2}}} dr \right] \\ &= \frac{\pi^2}{12} \varepsilon^{-\frac{5}{2}} \left(1 - \frac{16}{3\pi} \sqrt{\varepsilon} + o(\sqrt{\varepsilon}) \right)\end{aligned}$$

Hence

$$\|\phi_{u_\varepsilon}\|^{\frac{1}{2}} \geq \sqrt[4]{\frac{\pi^2}{12}} \varepsilon^{-\frac{5}{8}} \left(1 - \frac{16}{3\pi} \sqrt{\varepsilon} + o(\sqrt{\varepsilon}) \right)^{\frac{1}{4}} = \sqrt[4]{\frac{\pi^2}{12}} \varepsilon^{-\frac{5}{8}} \left(1 - \frac{4}{3\pi} \sqrt{\varepsilon} + o(\sqrt{\varepsilon}) \right). \quad (3.6)$$

Then, using (3.5) and (3.6) in (3.4), we get

$$\sup_{t>0} I(tu_\varepsilon) \leq \frac{2}{5} S^{\frac{3}{2}} \left(1 + \frac{5A(\varphi) + 4\pi}{3\pi^2} \sqrt{\varepsilon} + o(\sqrt{\varepsilon}) \right).$$

If we take $\varphi(r) = \cos\left(\frac{\pi r}{2}\right)$, then $A(\varphi) = 2\pi\left(\frac{\pi^2}{4} - \lambda\right)$ and we conclude assuming that

$$\lambda > \lambda_1 \left(\frac{1}{4} + \frac{2}{5\pi^2} \right).$$

It can be showed that, applying usual arguments, we are allowed to assume u nonnegative. As a trivial consequence of the strong maximum principle applied to both the equations, we actually deduce that the solution is positive and, by Lemma 3.1, radial.

Finally we prove it is a ground state arguing as in of [1, Step 2 of the Proof of Theorem 1.1].

3.2. A nonexistence result. In this section we prove the (ii) of Theorem 1.2. We reason as in [3], taking into account that, by Lemma 3.1, we are allowed to consider the ODE radial formulation of our problem. So, we write $u(x) = u(r)$ and $\phi(x) = \phi(r)$ where $r = |x|$ and we assume by contradiction (u, ϕ) is a positive solution of

$$-u'' - \frac{2}{r}u' = \lambda u + \phi u^4 \quad \text{on } (0, 1) \quad (3.7)$$

$$-\phi'' - \frac{2}{r}\phi' = u^5 \quad \text{on } (0, 1) \quad (3.8)$$

with the following boundary conditions

$$u'(0) = u(1) = 0, \quad \phi'(0) = \phi(1) = 0.$$

Let ψ be any smooth function such that $\psi(0) = 0$.

We multiply (3.7) by $r^2\psi u'$, we integrate on $(0, 1)$ and we get

$$\begin{aligned} -\frac{\psi(1)(u'(1))^2}{2} + \int_0^1 \left(\frac{r^2}{2}\psi' - r\psi \right) (u')^2 dr &= -\frac{\lambda}{2} \int_0^1 (r^2\psi' + 2r\psi)u^2 dr \\ &\quad - \frac{1}{5} \int_0^1 (r^2\psi' + 2r\psi)\phi u^5 dr - \frac{1}{5} \int_0^1 r^2 u^5 \psi \phi' dr. \end{aligned} \quad (3.9)$$

We multiply (3.8) by $r^2\psi\phi'$, we integrate by parts on $(0, 1)$ and we get

$$-\frac{\psi(1)(\phi'(1))^2}{2} + \int_0^1 \left(\frac{r^2}{2}\psi' - r\psi \right) (\phi')^2 dr = \int_0^1 r^2 u^5 \psi \phi' dr. \quad (3.10)$$

Then, by (3.9) and (3.10) we obtain

$$\begin{aligned} -\frac{\psi(1)(u'(1))^2}{2} - \frac{\psi(1)(\phi'(1))^2}{10} \\ + \int_0^1 \left(\frac{r^2}{2}\psi' - r\psi \right) (u')^2 dr + \frac{1}{5} \int_0^1 \left(\frac{r^2}{2}\psi' - r\psi \right) (\phi')^2 dr \\ = -\frac{\lambda}{2} \int_0^1 (r^2\psi' + 2r\psi)u^2 dr - \frac{1}{5} \int_0^1 (r^2\psi' + 2r\psi)\phi u^5 dr. \end{aligned} \quad (3.11)$$

Now, multiplying (3.7) by $\left(\frac{r^2}{2}\psi' - r\psi\right)u$ and integrating in $(0, 1)$ we have

$$-\frac{1}{4} \int_0^1 r^2 \psi''' u^2 dr + \int_0^1 \left(\frac{r^2}{2}\psi' - r\psi \right) (u')^2 dr = \int_0^1 \left(\frac{r^2}{2}\psi' - r\psi \right) (\lambda u^2 + \phi u^5) dr \quad (3.12)$$

and, multiplying (3.8) by $\left(\frac{r^2}{2}\psi' - r\psi\right)\phi$ and integrating in $(0, 1)$, we have

$$\int_0^1 \left(\frac{r^2}{2}\psi' - r\psi \right) (\phi')^2 dr - \frac{1}{4} \int_0^1 \phi^2 r^2 \psi''' dr = \int_0^1 \left(\frac{r^2}{2}\psi' - r\psi \right) \phi u^5 dr. \quad (3.13)$$

Combining (3.11), (3.12) and (3.13) we get

$$\begin{aligned} \int_0^1 r^2 \left(\frac{1}{4}\psi''' + \lambda\psi' \right) u^2 dr + \frac{1}{20} \int_0^1 r^2 \phi^2 \psi''' dr \\ = \frac{\psi(1)(u'(1))^2}{2} + \frac{\psi(1)(\phi'(1))^2}{10} + \frac{4}{5} \int_0^1 u^5 \phi r (\psi - r\psi') dr. \end{aligned} \quad (3.14)$$

Our aim is to find those $\lambda > 0$ for which there exists a smooth function ψ_λ such that $\psi_\lambda(0) = 0$ and (3.14) does not hold.

As a first step, consider $0 < \lambda \leq \frac{\lambda_1}{16}$ and set $\psi_\lambda(r) = \sin(\sqrt{4\lambda}r)$ as in [3]. We have $\psi_\lambda(1) \geq 0$ and $\frac{1}{4}\psi_\lambda''' + \lambda\psi_\lambda' = 0$. Moreover, for any $r \in]0, 1[$,

$$\psi_\lambda'''(r) \leq 0 < \psi_\lambda(r) - r\psi_\lambda'(r)$$

and then equality (3.14) is violated.

Now, for $k > 0$ and $\lambda \in J :=]\frac{\lambda_1}{16}, \frac{\lambda_1}{4}[$, consider

$$\psi_{k,\lambda}(r) = 1 - 2\lambda r^2 + k \sin(\sqrt{4\lambda}r) - \cos(\sqrt{4\lambda}r)$$

and observe that, for all $r \in]0, 1]$,

$$\frac{1}{4}\psi_{k,\lambda}'''(r) + \lambda\psi_{k,\lambda}'(r) < 0 < \psi_{k,\lambda}(r) - r\psi_{k,\lambda}'(r).$$

Moreover, for $\lambda \in J$, we have

$$\sup_{r \in]0,1]} \psi_{k,\lambda}'''(r) \leq 0 \iff \psi_{k,\lambda}'''(1) \leq 0 \iff k \leq -\tan(\sqrt{4\lambda})$$

and

$$\psi_{k,\lambda}(1) \geq 0 \iff k \geq \frac{\cos(\sqrt{4\lambda}) - 1 + 2\lambda}{\sin(\sqrt{4\lambda})}.$$

Then

$$\frac{\cos(\sqrt{4\lambda}) - 1 + 2\lambda}{\sin(\sqrt{4\lambda})} \leq -\tan(\sqrt{4\lambda}),$$

namely

$$1 + (2\lambda - 1) \cos(\sqrt{4\lambda}) \geq 0,$$

which holds if $\lambda \in]\frac{\lambda_1}{16}, \lambda^*]$, where λ^* is the unique solution of $1 + (2\lambda - 1) \cos(\sqrt{4\lambda}) = 0$ in J .

4. THE DIMENSIONS $N \geq 4$

In this section we are interested in studying (\mathcal{SN}) for $N \geq 4$ in the resonance and in the nonresonance case in order to provide the proofs of Theorem 1.3 and Theorem 1.4.

In particular, first we use similar arguments as those in Section 3.1 to prove the existence of a positive solution at the mountain pass level when $0 < \lambda < \lambda_1$. Then, after checking the geometrical assumptions of the Linking Theorem in [11], we show that sign changing solutions exist for $\lambda \geq \lambda_1$ both in the resonance and, provided $N \geq 6$, in the nonresonance case.

4.1. Positive solutions. We are reduced to prove that $c < \frac{2}{N+2}S^{\frac{N}{2}}$. As in [3] we define

$$u_\varepsilon(x) = \frac{\varphi(|x|)}{(\varepsilon + |x|^2)^{\frac{N-2}{2}}},$$

where φ is a smooth positive function compactly supported in Ω such that $\varphi(x) = 1$ in some neighborhood of 0. We have

$$|\nabla u_\varepsilon|_2^2 = \frac{K_1}{\varepsilon^{\frac{N-2}{2}}} + O(1), \quad |u_\varepsilon|_{2^*}^2 = \frac{K_2}{\varepsilon^{\frac{N-2}{2}}} + O(\varepsilon)$$

and

$$|u_\varepsilon|_2^2 = \begin{cases} K_3 \varepsilon^{-\frac{N-4}{2}} + O(1) & \text{if } N \geq 5 \\ K_3 |\log \varepsilon| + O(1) & \text{if } N = 4 \end{cases}$$

where K_1, K_2, K_3 are positive constants and $K_1/K_2 = S$ (see [3]).

Then, using (2.4), we have the following estimate of the mountain pass level

$$c \leq \max_{t>0} I(tu_\varepsilon) \leq \frac{2}{N+2} \left(\frac{|\nabla u_\varepsilon|_2^2 - \frac{N+2}{2N} \lambda |u_\varepsilon|_2^2}{|u_\varepsilon|_{2^*}^2} \right)^{\frac{N}{2}} \leq \frac{2}{N+2} \left(S + \varepsilon^{\frac{N-2}{2}} (O(1) - \psi(\varepsilon)) \right)^{\frac{N}{2}}$$

where ψ is a function such that $\lim_{\varepsilon \rightarrow 0} \psi(\varepsilon) = +\infty$.

Of course, if ε is sufficiently small, we conclude.

4.2. The geometrical properties for the Linking Theorem. In this section we investigate the geometrical properties of the functional I in order to verify the assumptions of the Linking Theorem. In what follows, we recall the geometrical construction developed in [7] to find sign changing solutions for $\lambda \geq \lambda_1$.

Let us denote by e_j the eigenfunctions relative to $\lambda_j \in \sigma(-\Delta)$ such that $\|e_j\|_2 = 1$. Let $k \in \mathbb{N}$ and let us define

$$H^- := \text{span}\{e_j : j = 1, \dots, k\}; \quad H^+ := (H^-)^\perp.$$

Without loss of generality let $0 \in \Omega$ and let us take m so large such that $B_{\frac{2}{m}} \subset \Omega$. Consider the function $\zeta_m : \Omega \rightarrow \mathbb{R}$ defined by

$$\zeta_m(x) := \begin{cases} 0 & \text{if } x \in B_{\frac{1}{m}} \\ m|x| - 1 & \text{if } x \in A_m := B_{\frac{2}{m}} \setminus B_{\frac{1}{m}} \\ 1 & \text{if } x \in \Omega \setminus B_{\frac{2}{m}}. \end{cases}$$

Let $e_j^m := \zeta_m e_j$ be the approximating eigenfunctions and let

$$H_m^- := \text{span}\{e_j^m : j = 1, \dots, k\}.$$

We have the following

Lemma 4.1. *As $m \rightarrow \infty$ we have*

$$e_j^m \rightarrow e_j \text{ in } H_0^1(\Omega) \quad \text{and} \quad \max_{w \in H_m^-; \|w\|_2=1} \|w\|^2 \leq \lambda_k + c_k m^{2-N} \quad (4.1)$$

where $c_k > 0$.

Moreover, if $\lambda \geq \lambda_k$,

$$\sup_{w \in H_m^-} I(w) = O(m^{-\frac{N^2-4}{4}}). \quad (4.2)$$

Proof. The first part is proved in [7, Lemma 2].

Moreover for all $m \in \mathbb{N}^*$ and $w \in H_m^-$, by (4.1) and (viii) of Lemma 2.1,

$$I(w) \leq \frac{c_k m^{2-N}}{2(\lambda_k + c_k m^{2-N})} \|w\|^2 - C \|w\|^{2(2^*-1)} \leq C m^{-\frac{N^2-4}{4}}.$$

□

We consider the family of functions

$$u_\varepsilon^*(x) := \alpha_N \frac{\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + |x|^2)^{\frac{N-2}{2}}}, \quad \varepsilon > 0, \quad (4.3)$$

where $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$, which solve

$$\begin{cases} -\Delta u = u^{2^*-1} & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

and satisfy

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon^*|^2 = \int_{\mathbb{R}^N} |u_\varepsilon^*|^{2^*} = S^{\frac{N}{2}}$$

for all $\varepsilon > 0$.

Let $\eta \in C_c^\infty(B_{\frac{1}{m}})$ be a positive cut-off function such that $\eta \equiv 1$ in $B_{\frac{1}{2m}}$, $\eta \leq 1$ in $B_{\frac{1}{m}}$, $|\nabla \eta|_\infty \leq 4m$ and consider

$$u_\varepsilon(x) := \eta(x) u_\varepsilon^*(x). \quad (4.4)$$

As $\varepsilon \rightarrow 0$ we have the following estimates due to Brezis and Nirenberg [3]

$$\|u_\varepsilon\|^2 = S^{\frac{N}{2}} + O(\varepsilon^{N-2}), \quad \|u_\varepsilon\|_{2^*}^{2^*} = S^{\frac{N}{2}} + O(\varepsilon^N). \quad (4.5)$$

For $v \in H_m^- \oplus \mathbb{R}^+ \{u_\varepsilon\}$ we write $v = w + tu_\varepsilon$. By definition

$$\text{supp}(u_\varepsilon) \cap \text{supp}(w) = \emptyset.$$

We have

Lemma 4.2. *If $u, w \in H_0^1(\Omega)$ are such that $\text{supp}(u) \cap \text{supp}(w) = \emptyset$, then for $v = u + w$ we have*

$$I(v) = I(u) + I(w) - \frac{1}{2^* - 1} \int_{\Omega} \phi_w |u|^{2^* - 1} dx. \quad (4.6)$$

Proof. Of course, $\|v\|^2 = \|w\|^2 + \|u\|^2$ and $|v|_2^2 = |w|_2^2 + |u|_2^2$.

Moreover, if $u, w \in H_0^1(\Omega)$ have disjoint supports, then

$$-\Delta \phi_v = |u|^{2^* - 1} + |w|^{2^* - 1} = -\Delta(\phi_u + \phi_w) \quad \text{in } \Omega$$

and $\phi_v = 0 = \phi_u + \phi_w$ on $\partial\Omega$.

By uniqueness,

$$\phi_v = \phi_u + \phi_w.$$

So we have

$$\begin{aligned} I(v) &= \frac{1}{2} \|u\|^2 + \frac{1}{2} \|w\|_2^2 - \frac{\lambda}{2} |u|_2^2 - \frac{\lambda}{2} |w|_2^2 - \frac{1}{2(2^* - 1)} \|\phi_u + \phi_w\|^2 \\ &= I(u) + I(w) - \frac{1}{2^* - 1} \int_{\Omega} \nabla \phi_u \nabla \phi_w dx \\ &= I(u) + I(w) - \frac{1}{2^* - 1} \int_{\Omega} \phi_w |u|^{2^* - 1} dx \end{aligned}$$

and then we conclude. \square

Let $P_k : H_0^1(\Omega) \rightarrow H^-$ the orthogonal projection. In view of Lemma 4.1 if m is large enough then (see [7])

$$P_k H_m^- = H^- \quad \text{and} \quad H_m^- \oplus H^+ = H_0^1(\Omega).$$

Moreover, let us define

$$\mathcal{H} = \{h \in C(\bar{Q}_\varepsilon^m, H_0^1(\Omega)) : h|_{\partial Q_\varepsilon^m} = \text{id}_{\partial Q_\varepsilon^m}\}$$

where

$$Q_\varepsilon^m := \{w \in H_m^- : \|w\| < R\} \oplus [0, R] \{u_\varepsilon\}.$$

Now we verify that the geometrical assumptions of the Linking Theorem hold in our case.

Lemma 4.3. *Let $\lambda_k \leq \lambda < \lambda_{k+1}$. We have that there exist $0 < \rho < R$ such that, uniformly for ε sufficiently small,*

- (a) $\{u \in H^+ : \|u\| = \rho\}$ and ∂Q_ε^m link;
- (b) the functional I is bounded from below by a positive constant on $\{u \in H^+ : \|u\| = \rho\}$;
- (c) $\sup_{u \in \partial Q_\varepsilon^m} I(u) \leq \omega_m$ with $\omega_m \rightarrow 0$ as $m \rightarrow \infty$.

Proof. Property (a) is standard (see [2, Lemma 1.3]).

Moreover, by (iii) of Lemma 2.1, we have

$$I(u) \geq \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} |u|_2^2 - C \|u\|^{2(2^* - 1)}.$$

Since $\lambda < \lambda_{k+1}$, if $u \in H^+$ we find

$$I(u) \geq C(\|u\|^2 - \|u\|^{2(2^* - 1)})$$

for any $u \in H^+$. Therefore, if ρ is small enough, we get that

$$I(u) \geq C > 0$$

for all $u \in \{u \in H^+ : \|u\| = \rho\}$ and so (b) is proved.

To prove (c) we observe that, by (4.2), $I(w) \leq \omega_m$.

Moreover, by (2.4) and (4.5), we have

$$\begin{aligned} I(Ru_\varepsilon) &\leq \frac{NR^2}{N+2} \|u_\varepsilon\|^2 - \frac{N-2}{N+2} R^{2^*} |u_\varepsilon|_{2^*}^{2^*} \\ &= \frac{S^{\frac{N}{2}} R^2}{N+2} \left[N + O(\varepsilon^{N-2}) - (N-2 + O(\varepsilon^N)) R^{2^*-2} \right] \end{aligned}$$

and this becomes negative for R large enough and uniformly for ε small. Then, by (4.2) and (4.6), we have $I(w + Ru_\varepsilon) \leq \omega_m$.

Finally, since $\max_{0 \leq r \leq R} I(ru_\varepsilon) < +\infty$ and for all $w \in H_m^-$ with $\|w\| = R$ we have

$$I(w) \leq \frac{N}{N+2} R^2 - CR^{2^*},$$

by (4.6) we conclude that $I(v) \leq 0$ in $\{w \in H_m^- : \|w\| = R\} \oplus [0, R]\{u_\varepsilon\}$ if R is sufficiently large. \square

Let us set

$$c := \inf_{h \in \mathcal{H}} \sup_{v \in Q_\varepsilon^m} I(h(v)).$$

Since $\text{id}_{\partial Q_\varepsilon^m} \in \mathcal{H}$ we have

$$c \leq \sup_{v \in Q_\varepsilon^m} I(v)$$

and, of course, $c > 0$ by (a) and (b) of Lemma 4.3.

We complete our proofs if we show that for ε small enough

$$\max_{v \in Q_\varepsilon^m} I(v) < \frac{2}{N+2} S^{\frac{N}{2}}. \quad (4.7)$$

This is the aim of the last part of the section. We distinguish two cases:

- $\lambda \notin \sigma(-\Delta)$, and $\lambda > \lambda_1$;
- $\lambda \in \sigma(-\Delta)$ and $N \geq 6$.

4.3. Sign changing solutions: the nonresonance case. Here we assume that there exist $k \in \mathbb{N}^*$ and $\theta > 0$ such that

$$\lambda_k + \theta \leq \lambda < \lambda_{k+1}. \quad (4.8)$$

We prove (4.7) under condition (4.8).

Let us choose m so large such that

$$c_k m^{2-N} < \theta \quad (4.9)$$

where c_k is as in Lemma 4.1.

By contradiction let us assume that for all $\varepsilon > 0$

$$\sup_{v \in Q_\varepsilon^m} I(v) \geq \frac{2}{N+2} S^{\frac{N}{2}}.$$

It is easy to see that the supremum is attained and then, for all $\varepsilon > 0$, there exist $w_\varepsilon \in H_m^-$ and $t_\varepsilon \geq 0$ such that

$$I(v_\varepsilon) = \max_{v \in Q_\varepsilon^m} I(v) \geq \frac{2}{N+2} S^{\frac{N}{2}}, \quad (4.10)$$

where $v_\varepsilon := w_\varepsilon + t_\varepsilon u_\varepsilon$.

Observe that by Lemma 4.1, (4.8) and (4.9) we have

$$I(w_\varepsilon) \leq \frac{c_k m^{2-N} - \theta}{2(\lambda_k + c_k m^{2-N})} \|w_\varepsilon\|^2 - \frac{1}{2(2^* - 1)} \int_\Omega \phi_{w_\varepsilon} |w_\varepsilon|^{2^*-1} dx \leq 0, \quad (4.11)$$

which improves the estimate (4.2).

Moreover we have the following property of the family (t_ε) .

Lemma 4.4. *There exists $C > 0$ such that $t_\varepsilon \geq C$ for all $\varepsilon > 0$. In particular we have that there exists a sequence $(\varepsilon_n) \subset \mathbb{R}_+^*$ such that $\varepsilon_n \rightarrow 0$ and $t_{\varepsilon_n} \rightarrow 1$ as $n \rightarrow +\infty$.*

Proof. By (4.6), (4.10) and (4.11) we deduce that $t_\varepsilon \geq C > 0$ for all $\varepsilon > 0$.

Since (t_ε) is bounded, there exists $(\varepsilon_n) \subset \mathbb{R}_+^*$ such that $\varepsilon_n \rightarrow 0$ and $t_{\varepsilon_n} \rightarrow t_0$ as $n \rightarrow +\infty$. For simplicity we label $\varepsilon = \varepsilon_n$ and so $\varepsilon \rightarrow 0$.

By (4.3) we have that $|t_\varepsilon u_\varepsilon|_2^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Hence, by (2.4) and (4.5) we have

$$I(t_\varepsilon u_\varepsilon) \leq \left(\frac{N}{N+2} t_0^2 - \frac{N-2}{N+2} t_0^{2^*} \right) S^{\frac{N}{2}} + o(1).$$

Then, by Lemma 4.2,

$$I(v_\varepsilon) \leq I(w_\varepsilon) + I(t_\varepsilon u_\varepsilon) \leq I(w_\varepsilon) + \varphi(t_0) S^{\frac{N}{2}} + o(1) \quad (4.12)$$

where

$$\varphi(t) = \frac{N}{N+2} t^2 - \frac{N-2}{N+2} t^{2^*}.$$

We note that

$$\max_{t \geq 0} \varphi(t) = \varphi(1) = \frac{2}{N+2} \quad \text{and} \quad \varphi(t) < \frac{2}{N+2} \quad \forall t \geq 0, t \neq 1.$$

So, taking into account (4.11), if it was $t_0 \neq 1$, then for ε small inequality (4.12) would contradict (4.10). \square

Moreover we recall the following result.

Lemma 4.5 ([7, Lemma 5]). *There exists a function $\tau = \tau(\varepsilon)$ such that $\tau(\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$ and*

$$\int_{\Omega} |t_\varepsilon u_\varepsilon|^2 dx \geq \tau(\varepsilon) \varepsilon^{N-2}$$

for ε small.

Hence we can conclude the proof in this case.

Proof of (ii) of Theorem 1.3 completed. First we observe that, by (2.4), (4.5), Lemma 4.5

$$\begin{aligned} I(t_\varepsilon u_\varepsilon) &\leq \frac{N}{N+2} t_\varepsilon^2 \|u_\varepsilon\|^2 - \frac{\lambda}{2} |t_\varepsilon^2 u_\varepsilon|_2^2 - \frac{N-2}{N+2} t_\varepsilon^{2^*} |u_\varepsilon|_{2^*}^{2^*} \\ &\leq \left(\frac{N}{N+2} t_\varepsilon^2 - \frac{N-2}{N+2} t_\varepsilon^{2^*} \right) S^{\frac{N}{2}} + C(1 - \tau(\varepsilon)) \varepsilon^{N-2} \\ &< \frac{2}{N+2} S^{\frac{N}{2}} \end{aligned}$$

for ε small enough. Then by (4.6) and (4.11) we get

$$I(v_\varepsilon) < \frac{2}{N+2} S^{\frac{N}{2}}$$

which contradicts (4.10).

Hence, the Linking Theorem can be applied using Lemma 2.2 and Lemma 4.3. \square

4.4. Sign changing solutions: the resonance case. Let $\lambda = \lambda_k$ and $\mu \in (\lambda_k, \lambda_{k+1})$. As in the previous section we want to show that (4.7) holds.

By contradiction assume that for all m and all $\varepsilon > 0$ there exists $v_\varepsilon^m = w_\varepsilon^m + t_\varepsilon^m u_\varepsilon^m \in Q_\varepsilon^m$ such that

$$I(v_\varepsilon^m) \geq \frac{2}{N+2} S^{\frac{N}{2}} \quad (4.13)$$

where u_ε^m is defined in (4.4).

Then, as in Lemma 4.4, but using (4.2) instead of (4.11), the sequence (t_ε^m) satisfies again

$$t_\varepsilon^m \geq C > 0 \quad (4.14)$$

uniformly for m large enough and $\varepsilon > 0$.

As in [7, Lemma 6] one can prove the following result.

Lemma 4.6. *Let $m \rightarrow +\infty$ and assume that $\varepsilon = \varepsilon(m) = o(\frac{1}{m})$; then*

$$\|u_\varepsilon^m\|^2 = S^{\frac{N}{2}} + O[(\varepsilon m)^{N-2}], \quad |u_\varepsilon^m|_{2^*}^{2^*} = S^{\frac{N}{2}} + O[(\varepsilon m)^N].$$

Moreover,

$$\int_{\Omega} |t_\varepsilon^m u_\varepsilon^m|^2 dx \geq C\varepsilon^2.$$

Proof. We only prove the last estimate: for any m large enough such that (4.14) holds and ε small enough such that $B_\varepsilon \subset B_{\frac{1}{2m}}$, we have

$$\int_{\Omega} |t_\varepsilon^m u_\varepsilon^m|^2 dx \geq C \int_0^\varepsilon \frac{\varepsilon^{N-2}}{(\varepsilon^2 + r^2)^{N-2}} r^{N-1} dr = C\varepsilon^2. \quad \square$$

Hence we can conclude as follows.

Proof of Theorem 1.4 completed. We choose $\varepsilon = \varepsilon(m)$ in order to deal only with the parameter m and to have a contradiction to (4.13) for m large enough. We take

$$\varepsilon(m) = m^{-\frac{N-2}{N-4}} \log^{-\frac{1}{2}}(m).$$

Therefore, as $m \rightarrow +\infty$ then $\varepsilon(m) = o(\frac{1}{m})$ and Lemma 4.6 applies.

From now on, we express only the dependence on m .

By (2.4), Lemma 4.6, if $N \geq 5$ and m is large enough we have

$$\begin{aligned} I(t_m u_m) &\leq \frac{N}{N+2} t_m^2 \|u_m\|^2 - \frac{\lambda_k}{2} |t_m^2 u_m|_2^2 - \frac{N-2}{N+2} t_m^{2^*} |u_m|_{2^*}^{2^*} \\ &\leq \left(\frac{N}{N+2} t_m^2 - \frac{N-2}{N+2} t_m^{2^*} \right) S^{\frac{N}{2}} - (C_1 - C_2 \log^{-\frac{N-4}{2}}(m)) m^{-\frac{2(N-2)}{N-4}} \log^{-1}(m) \\ &\leq \frac{2}{N+2} S^{\frac{N}{2}} - C m^{-\frac{2(N-2)}{N-4}} \log^{-1}(m). \end{aligned}$$

Then, using also (4.6) and (4.2), we have

$$\begin{aligned} I(v^m) &\leq I(t_m u^m) + I(w^m) \\ &\leq \frac{2}{N+2} S^{\frac{N}{2}} - C_1 m^{-\frac{2(N-2)}{N-4}} \log^{-1}(m) + C_2 m^{-\frac{N^2-4}{4}} \\ &= \frac{2}{N+2} S^{\frac{N}{2}} - \left(C_1 \log^{-1}(m) - C_2 m^{-\frac{(N-2)(N^2-2N-16)}{4(N-4)}} \right) m^{-\frac{2(N-2)}{N-4}} \end{aligned}$$

which, for m large enough and $N \geq 6$, contradicts (4.13).

Hence, the Linking Theorem, through Lemma 2.2 and Lemma 4.3, allows us to conclude. \square

REFERENCES

- [1] A. Azzollini, P. d'Avenia *On a system involving a critically growing nonlinearity*, J. Math. Anal. Appl. **387** (2012), 433–438. [1](#), [2](#), [10](#)
- [2] V. Benci, P.H. Rabinowitz, *Critical point theorems for indefinite functionals*, Invent. Math. **52** (1979), 241–273. [14](#)
- [3] H. Brezis, L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477. [2](#), [8](#), [9](#), [11](#), [12](#), [13](#)
- [4] A. Capozzi, D. Fortunato, G. Palmieri, *An existence result for nonlinear elliptic problems involving critical Sobolev exponent*, Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1985), 463–470. [2](#)
- [5] T. D'Aprile, D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), 893–906. [4](#)
- [6] L.C. Evans, *Partial differential equations*, Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, RI, 2010. [1](#)

- [7] F. Gazzola, B. Ruf, *Lower-order perturbations of critical growth nonlinearities*, Adv. Differential Equations **2** (1997), 555–572. [2](#), [3](#), [13](#), [14](#), [16](#), [17](#)
- [8] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243. [8](#)
- [9] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin, 2001. [8](#)
- [10] V. Moroz, J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*. J. Funct. Anal. **265** (2013), 153–184. [5](#)
- [11] P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics **65**, American Mathematical Society, Providence, RI, 1986. [3](#), [12](#)
- [12] D. Ruiz, *On the Schrödinger-Poisson-Slater system: behavior of minimizers, radial and nonradial cases*, Arch. Ration. Mech. Anal. **198** (2010), 349–368. [5](#)
- [13] M. Willem, *Functional analysis. Fundamentals and applications*, Cornerstones, Birkhäuser/Springer, New York, 2013. [5](#), [7](#)

(A. Azzollini)

DIPARTIMENTO DI MATEMATICA, INFORMATICA ED ECONOMIA
UNIVERSITÀ DEGLI STUDI DELLA BASILICATA
VIA DELL'ATENEIO LUCANO 10, 85100 POTENZA, ITALY
E-mail address: antonio.azzollini@unibas.it

(P. d'Avenia)

DIPARTIMENTO DI MECCANICA, MATEMATICA E MANAGEMENT
POLITECNICO DI BARI
VIA ORABONA 4, 70125 BARI, ITALY
E-mail address: pietro.davenia@poliba.it

(G. Vaira)

DIPARTIMENTO DI MATEMATICA "G. CASTELNUOVO"
UNIVERSITÀ DEGLI STUDI DI ROMA SAPIENZA
PIAZZALE ALDO MORO 5, 00185 ROMA, ITALY
E-mail address: giusi.vaira@uniroma1.it