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Extended interpolation on the square and a mixed interpolating sequence

Ahmed Nasri[1,2] · Domenico Mezzanotte[3,4] · Donatella Occorsio[3,4]

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Abstract
This paper investigates an extended interpolation process based on the zeros of Jacobi polynomials to approximate functions on $[-1, 1]^2$. By combining a classical Lagrange interpolating polynomial sequence with its extended counterpart, a new mixed polynomial sequence is introduced, which significantly reduces the number of required function evaluations. Convergence conditions in suitable weighted function spaces are rigorously analyzed and some numerical tests are presented to support the efficiency of the proposed scheme.

Keywords Lagrange interpolation · Orthogonal polynomials · Approximation by polynomials

Mathematics Subject Classification 33C45 · 41A10 · 65D05

1 Introduction

Let $w := v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta},\ \alpha, \beta > -1$, be a Jacobi weight and $\{P_n(w)\}_n$, the corresponding sequence of orthonormal polynomials. Denoting by $\overline{w}(x) := (1-x^2)w(x)$, $\mathbf{w}(x, y) := v^{\alpha,\beta}(x)v^{\alpha,\beta}(y)$ and $\bar{\mathbf{w}}(x, y) := (1-x^2)(1-y^2)\mathbf{w}(x, y)$, we introduce the extended Lagrange polynomial $\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\boldsymbol{w}}, f, x, y)$ interpolating a bivariate

Ahmed Nasri, Domenico Mezzanotte and Donatella Occorsio have contributed equally to this work.

✉ Domenico Mezzanotte
domenico.mezzanotte@unibas.it

Ahmed Nasri
aahmednasri@gmail.com

Donatella Occorsio
donatella.occorsio@unibas.it

1 Department of Fundamental Studies of Science and Technology, University of Mohamed Seddik Ben Yahia, Ouled Aissa, Bp 98, 18000 Jijel, Algeria

2 LANOS Laboratory, University of Badji-Mokhtar, P.O. Box 12, 23000 Annaba, Algeria

3 Department of Basic and Applied Sciences, University of Basilicata, Via dell'Ateneo Lucano 10, 85100 Potenza, Italy

4 Istituto per le Applicazioni del Calcolo "Mauro Picone", Naples Branch, C.N.R. National Research Council of Italy, Via Pietro Castellino 111, 80131 Naples, Italy

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function $f(x, y)$ at the $(2m+1)^2$ zeros of $Q_{2m+1,2m+1}(x, y) := Q_{2m+1}(x)Q_{2m+1}(y)$, where $Q_{2m+1}(z) = P_{m+1}(w, z)P_m(\overline{w}, z)$. The bivariate extended Lagrange operator $\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\boldsymbol{w}})$ is defined as the tensor product of the two extended univariate operators $L_{m+1,m}(w, \overline{w})$.

Extended interpolation processes can be constructed using sequences of orthogonal polynomials concerning the same or different weight functions, provided that the interpolation nodes have a "fine" distribution. It has been shown that interpolation points with *arc sine* distribution are ideal candidates for ensuring the optimal growth of the Lebesgue constants associated with the interpolation process based on them Criscuolo et al. (1992); Szabados and Vértesi (1990); Mastroianni and Prössdorf (1994); Mastroianni and Milovanović (2008). In the univariate case, these methods have been extensively studied by various authors on both bounded (Criscuolo et al. 1990; Criscuolo and Della Vecchia 1993; Criscuolo et al. 1993) and unbounded intervals (Occorsio 2009; Occorsio and Russo 2014b, a), with error estimates given in different norms.

An important feature of extended interpolation is its ability to approximate functions using $2m$-degree interpolating polynomials, constructed from the zeros of polynomials of degrees m and $m+1$. This approach enables the development of high-degree interpolation processes while avoiding the challenges associated with computing the zeros of high-degree orthogonal polynomials. Moreover, in all the cases that both the sequences of ordinary and extended interpolating polynomials approximate a function f with the same rate of convergence, once the ordinary polynomial interpolating f at $m+1$ nodes has been constructed, the extended $2m$-degree polynomial is obtained by reusing the $m+1$ samples. In the univariate case, this behavior has motivated the use of mixed interpolating sequences (both on $(-1, 1)$ and $(0, +\infty)$) obtained by suitably mixing ordinary and extended interpolating sequences (Occorsio and Russo 2021; Mezzanotte et al. 2021). Mixed interpolating sequences of this type have played a significant role in designing "fast" and accurate methods in various contexts, including the numerical evaluation of ordinary and hypersingular integrals (Occorsio and Russo 2020; Mezzanotte and Occorsio 2022, 2024; Diogo et al. 2020) and the development of numerical methods for approximating solutions of integral equations (Occorsio and Russo 2021; Mezzanotte et al. 2021). In the latter case, the size of the resulting linear systems is reduced compared to classical methods, thus decreasing the overall computational cost.

After introducing and studying the main properties of the bivariate extended polynomial $\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\boldsymbol{w}}, f, x, y)$, the main purpose of this paper is the construction of a suitable "mixed" interpolating sequence to approximate functions f defined on the square $\mathrm{S} := [-1, 1]^2$. The functions under consideration may exhibit algebraic singularities along the sides of S. To obtain the mixed scheme, for a fixed m, the ordinary polynomial $\mathcal{L}_{m+1}(\mathbf{w}, f)$ is first computed, and then the extended polynomial $\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\boldsymbol{w}}, f)$ is obtained by reusing the samples of f computed in the first step. In this way, while preserving the precision of the process, we reduce the number of function evaluations by approximately 44% compared to the case where only the ordinary bivariate Lagrange polynomial is used. Of course, the global process makes sense since we prove that under certain assumptions, both sequences (and consequently the mixed one) have optimal Lebesgue constants and hence are uniformly convergent to any function f belonging to suitable weighted Sobolev-type spaces of functions.

The outline of the paper is as follows. Section 2 introduces the notations and the weighted function spaces considered in this paper. Section 3 provides some preliminary results and recalls univariate extended interpolation. In Sect. 4, we present the main results of the paper, whose proofs are collected in Sect. 6 to avoid interrupting the methodological discussion.

Section 5 presents numerical tests that confirm the theoretical estimates and demonstrate the efficiency of the mixed interpolation scheme. Finally, Sect. 7 concludes the paper.

2 Notations and spaces of functions

Throughout this manuscript the constant $\mathcal{C}$ will be used several times, having different meanings in different formulas. Moreover, we will write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is a positive constant independent of the parameters $a, b, \ldots,$ and $\mathcal{C} = \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on $a, b, \ldots$. Moreover, if $A, B \geq 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a constant $0 < \mathcal{C} \neq (A, B)$ such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C} B.$$

In what follows, $\mathbf{P}_m$ denotes the set of the univariate polynomials of degree at most m, and $\mathbf{P}_{m,m}$ the space of the bivariate polynomials of degree at most m in each variable.

For any bivariate function $g(x, y)$ defined in square $\mathrm{S} := [-1, 1]^2$, the notation g_x (g_y) will be adopted to consider g as a function of the only variable y (or x). Finally, we set

$$v^{\rho,\sigma}(z) := (1-z)^{\rho}(1+z)^{\sigma},$$

where $\rho, \sigma \in \mathbf{R},\ z \in (-1, 1)$.

2.1 Spaces of functions

In this section, we define the spaces of functions we will deal with. Setting $\mathbf{u}(x, y) = v^{\gamma_1,\delta_1}(x)\, v^{\gamma_2,\delta_2}(y),\ \gamma_1, \delta_1, \gamma_2, \delta_2 \geq 0$, we define the space $C_{\mathbf{u}}$,

$$C_{\mathbf{u}} = \left\{ f \in C^0\left((-1,1)^2\right) : \begin{array}{l} -1 \leq y \leq +1 \ \lim_{x\to\pm 1} f(x, y)\,\mathbf{u}(x, y) = 0 \\ -1 \leq x \leq +1 \ \lim_{y\to\pm 1} f(x, y)\,\mathbf{u}(x, y) = 0 \end{array} \right\},$$

where the limit conditions hold uniformly $w.r.t.$ the free variable. Whenever one or more of the parameters $\gamma_1, \delta_1, \gamma_2, \delta_2$ is greater than 0, the functions in $C_{\mathbf{u}}$ can be singular along one or more sides of the square S. In the case $\gamma_1 = \delta_1 = \gamma_2 = \delta_2 = 0$ the definition reduces to the case of continuous functions, and we set $C_{\mathbf{u}} = C(\mathrm{S})$. The space $C_{\mathbf{u}}$ is equipped with the weighted uniform norm on the square

$$\|f\|_{C_{\mathbf{u}}} = \|f\mathbf{u}\|_{\infty} = \sup_{(x,y)\in \mathrm{S}} |f(x, y)|\,\mathbf{u}(x, y).$$

For smoother functions, i.e., for functions having some derivatives which can be discontinuous on the boundaries of S, we introduce the following Sobolev-type space of order $r \in \mathbf{N},\ r \geq 1$

$$W_r(\mathbf{u}) = \left\{ f \in C_{\mathbf{u}} : M_r(f, \mathbf{u}) := \max\left\{ \left\| f_y^{(r)} \varphi_y^r \mathbf{u} \right\|_{\infty}, \left\| f_x^{(r)} \varphi_x^r \mathbf{u} \right\|_{\infty} \right\} < \infty \right\},$$

where the superscript (r) denotes the $r-th$ derivative of the one-dimensional function f_x or f_y, and $\varphi(x, y) = \sqrt{1-x^2}\sqrt{1-y^2}$. We equip $W_r(\mathbf{u})$ with the norm

$$\|f\|_{W_r(\mathbf{u})} = \|f\mathbf{u}\|_{\infty} + M_r(f, \mathbf{u}).$$

In the case $\mathbf{u}(x, y) = 1$ we will write W_r.

The error of best polynomial approximation in $C_{\mathbf{u}}$ by means of bivariate polynomials in $\mathbf{P}_{m,m}$ is defined as

$$E_{m,m}(f)_{\mathbf{u}} = \inf_{p\in\mathbf{P}_{m,m}} \|[f-p]\,\mathbf{u}\|_{\infty}.$$

If h is a continuous univariate function on $(-1,1)$ and $u_1 = v^{\gamma_1,\delta_1}$, $\gamma_1,\delta_1 \geq 0$, let $C_{u_1} = \{f \in C((-1,1)) : \lim_{x\to\pm 1} f(x)u_1(x) = 0\}$, and $E_m(h)_{u_1}$ be the weighted best approximation error of h. We recall that the following Favard estimate holds

$$E_m(h)_{u_1} \leq \mathcal{C}\frac{\|h\|_{W_r(u_1)}}{m^r}, \quad \forall h \in W_r(u_1),$$

being $W_r(u_1)$ the univariate Sobolev space defined as

$$W_r(u_1) = \left\{ f \in C_{u_1} : \|fu_1\| + \|f^{(r)}\varphi_1^r u_1\|_{\infty} < \infty \right\}, \quad \varphi_1(x) = \sqrt{1-x^2}.$$

In Occorsio and Russo (2018) it was proved that the bivariate best approximation error can be estimated through the univariate best approximation errors $w.r.t.$ each of the two variables of f, i.e.

$$E_{m,m}(f)_u \leq \mathcal{C}\left[\sup_{x\in[-1,1]} u_1(x)E_m(f_x)_{u_2} + \sup_{y\in[-1,1]} u_2(y)E_m\left(f_y\right)_{u_1} \right],$$

where $0 < \mathcal{C} \neq \mathcal{C}(m,f)$. Hence, it follows that if $f \in W_r(\mathbf{u})$ then

$$E_{m,m}(f)_{\mathbf{u}} \leq \mathcal{C}\frac{M_r(f,\mathbf{u})}{m^r},$$

where the constant $0 < \mathcal{C} \neq \mathcal{C}(m,f)$.

3 Orthogonal polynomials and Lagrange interpolation processes

Let $w = v^{\alpha,\beta}$, $\alpha,\beta > -1$, be a Jacobi weight and $\{P_m(w)\}_m$ the corresponding sequence of orthonormal polynomials with positive leading coefficients, i.e.

$$P_m(w,z) = c_m z^m + \text{terms of lower degree}\ldots, \quad c_m > 0.$$

Denoting by $z^w_{m,i}$, $i = 1,\ldots,m$, the zeros of $P_m(w)$, we consider the Lagrange polynomial interpolating the univariate function G at $\left\{z^w_{m,i}\right\}_{i=1}^m$,i.e. the polynomial $L^w_m(G) \in \mathbf{P}_{m-1}$ s.t. $L^w_m(G, z^w_{m,i}) = G(z^w_{m,i})$, $i = 1,2,\ldots,m$. The polynomial $L^w_m(G)$ can be represented in the following form

$$L^w_m(G,z) = \sum_{i=1}^m l^w_{m,i}(z)\,G\left(z^w_{m,i}\right), \tag{1}$$

where

$$l^w_{m,i}(z) = \frac{P_m(w,z)}{P'_m\left(w, z^w_{m,i}\right)\left(z - z^w_{m,i}\right)}.$$

Denoting by $K_m(w,z,y)$ the Darboux kernel defined as

$$K(w,z,y) = \sum_{j=0}^{m-1} P_j(w,z)P_j(w,y),$$

and recalling that the Christoffel numbers are defined as

$$\lambda_{m,k}(w) = \frac{1}{K_m(w, z^w_{m,k}, z^w_{m,k})}, \quad k = 1, 2, \dots, m,$$

the fundamental interpolating polynomials take the form

$$l^w_{m,i}(z) = \frac{K_m(w, z, z^w_{m,i})}{K_m(w, z^w_{m,i}, z^w_{m,i})} = \lambda_{m,i}(w) \sum_{j=0}^{m-1} P_j(w, z) P_j(w, z^w_{m,i}),$$

which allows us to represent the interpolating polynomial $L^w_m(G)$ in the orthonormal basis

$$L^w_m(G, z) = \sum_{j=0}^{m-1} P_j(w, z) \sum_{i=1}^{m} \lambda_{m,i}(w) P_j(w, z^w_{m,i}) G\left(z^w_{m,i}\right). \tag{2}$$

About the Lebesgue constants, which are defined as

$$\|L^w_m \mathbf{u}\|_\infty := \sup_{f \in C_{\mathbf{u}}} \frac{\|L^w_m(f)\mathbf{u}\|}{\|f\|_{C_{\mathbf{u}}}}$$

and the rate of convergence of the interpolating process, in Mastroianni and Russo (1997) it was proved that under suitable assumptions on the weights w, u, the polynomial $L^w_m(f)$ behaves almost like the $(m-1)-$th best approximation polynomial of $f \in C_u$, except the extra factor $\log m$. This is stated in the following

Theorem 3.1 *Let* $u(x) = v^{\gamma,\delta}(x)$ *with* $\gamma, \delta \geq 0$ *and* $w(x) = v^{\alpha,\beta}(x)$, $\alpha, \beta > -1$, *be such that*

$$\max\left\{0, \frac{\alpha}{2} + \frac{1}{4}\right\} \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4},$$
$$\max\left\{0, \frac{\beta}{2} + \frac{1}{4}\right\} \leq \delta \leq \frac{\beta}{2} + \frac{5}{4},$$

then

$$\|L^w_m \mathbf{u}\|_\infty \sim \log m,$$

and for any $f \in C_u$ *one has*

$$\|[f - L_m(w, f)]\, u\|_\infty \leq \mathcal{C}\,(1 + \log m)\, E_{m-1}(f)_u\,,$$

where $\mathcal{C} \neq \mathcal{C}(f, m)$.

3.1 Ordinary Lagrange interpolation on S

Let $w_1(x) := v^{\alpha_1,\beta_1}(x)$, $w_2(y) := v^{\alpha_2,\beta_2}(y)$ be two Jacobi weights, with $\alpha_1, \beta_1, \alpha_2, \beta_2 > -1$ and let $\{z^{w_1}_{m,i}\}_{i=1}^m, \{z^{w_2}_{m,i}\}_{i=1}^m$, $i = 1, \dots, m$, be the zeros of $P_m(w_1)$ and $P_m(w_2)$, respectively.

Setting $\mathbf{w}(x, y) = w_1(x) w_2(y)$, the bivariate polynomial $\mathcal{L}_m(\mathbf{w}, f) \in \mathbf{P}_{m-1,m-1}$ interpolating $f \in C_{\mathbf{u}}$ at the pairs $\left(z^{w_1}_{m,i}, z^{w_2}_{m,j}\right)$, $i, j = 1, \dots, m$, i.e. $\mathcal{L}_m\left(\mathbf{w}, f, z^{w_1}_{m,i}, z^{w_2}_{m,j}\right) = f(z^{w_1}_{m,i}, z^{w_2}_{m,j})$, $i, j = 1, \dots, m$, takes the form

$$\mathcal{L}_m(\mathbf{w}, f, x, y) = \sum_{i=1}^{m} \sum_{j=1}^{m} l^{w_1}_{m,i}(x)\, l^{w_2}_{m,j}(y)\, f\left(z^{w_1}_{m,i}, z^{w_2}_{m,j}\right).$$

The Lebesgue constants are defined as

$$\|\mathcal{L}_m(\mathbf{w}, f)\mathbf{u}\|_\infty := \sup_{f \in C_\mathbf{u}} \frac{\|\mathcal{L}_m(\mathbf{w}, f)\mathbf{u}\|}{\|f\|_{C_\mathbf{u}}}$$

and according to Occorsio and Russo (2011), under suitable assumptions on the parameters γ_1, δ_1 and γ_2, δ_2, they behave as $\mathcal{O}(\log^2 m)$. In fact, the following result was proved.

Proposition 3.1 *Let* $\mathbf{u}(x, y) = v^{\gamma_1,\delta_1}(x)\, v^{\gamma_2,\delta_2}(y)$ *with* $\gamma_1, \delta_1, \gamma_2, \delta_2 \geq 0$ *and* $\alpha_1, \beta_1, \alpha_2, \beta_2 > -1$, *be such that*

$$\max\left\{0, \frac{\alpha_1}{2} + \frac{1}{4}\right\} \leq \gamma_1 \leq \frac{\alpha_1}{2} + \frac{5}{4}, \quad \max\left\{0, \frac{\beta_1}{2} + \frac{1}{4}\right\} \leq \delta_1 \leq \frac{\beta_1}{2} + \frac{5}{4}, \tag{3}$$

$$\max\left\{0, \frac{\alpha_2}{2} + \frac{1}{4}\right\} \leq \gamma_2 \leq \frac{\alpha_2}{2} + \frac{5}{4}, \quad \max\left\{0, \frac{\beta_2}{2} + \frac{1}{4}\right\} \leq \delta_2 \leq \frac{\beta_2}{2} + \frac{5}{4}, \tag{4}$$

then for any $f \in C_\mathbf{u}$ *it follows*

$$\|[f - \mathcal{L}_m(\mathbf{w}, f)]\,\mathbf{u}\|_\infty \leq \mathcal{C}\left(1 + \log^2 m\right) E_{m-1,m-1}(f)_\mathbf{u}\,.$$

where $\mathcal{C} \neq \mathcal{C}(f, m)$.

Remark 3.1 In the case $\alpha_1 = \alpha_2 =: \alpha$, $\beta_1 = \beta_2 =: \beta$, $w_1 = w_2 = w$, and setting $\gamma_1 = \gamma_2 =: \gamma$, $\delta_1 = \delta_2 =: \delta$, i.e. $u_1 = u_2 = u$ we have

$$\mathcal{L}^*_m(\mathbf{w}, f, x, y) = \sum_{i=1}^{m}\sum_{j=1}^{m} l^w_{m,i}(x)\, l^w_{m,j}(y)\, f\left(z^w_{m,i}, z^w_{m,j}\right), \tag{5}$$

and under the assumptions (3) with $\alpha_1 =: \alpha$, $\beta_1 =: \beta$, for any $f \in C_\mathbf{u}$ it follows

$$\left\|\left[f - \mathcal{L}^*_m(\mathbf{w}, f)\right]\mathbf{u}\right\|_\infty \leq \mathcal{C}\left(1 + \log^2 m\right) E_{m-1,m-1}(f)_\mathbf{u}\,, \quad \mathcal{C} \neq \mathcal{C}(f, m)\,.$$

3.2 Extended interpolation in [−1, 1]

Given the weight $w(z) = v^{\alpha,\beta}(z)$, let $\overline{w}(z) = (1 - z^2)w(z)$, and $\{P_m(\overline{w})\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. Since it is known that the zeros $\{z^{\overline{w}}_{m,k}\}_{k=1}^m$ of $P_m(\overline{w})$ interlace the zeros $\{z^w_{m+1,k}\}_{k=1}^{m+1}$ of $P_{m+1}(w)$, it is possible to consider the Lagrange polynomial interpolating f at the zeros $\{t_i\}_{i=1}^{2m+1}$ of $Q_{2m+1}(z) := P_{m+1}(w, z)P_m(\overline{w}, z)$, where

$$t_{2i-1} = z^w_{m+1,i}, \quad i = 1, 2, \ldots, m+1,$$
$$t_{2i} = z^{\overline{w}}_{m,i}, \quad i = 1, 2, \ldots, m,$$

which can be written as

$$L_{m+1,m}(w, \overline{w}, f, z) = \sum_{k=1}^{2m+1} \frac{Q_{2m+1}(z)}{Q'_{2m+1}(t_k)(z - t_k)} f(t_k), \tag{6}$$

or also as

$$L_{m+1,m}(w, \overline{w}, f, z) = P_m(\overline{w}, z)\sum_{k=1}^{m+1} l^w_{m+1,k}(z)\frac{f(z^w_{m+1,k})}{P_m(\overline{w}, z^w_{m+1,k})} + P_{m+1}(w, z)\sum_{k=1}^{m} l^{\overline{w}}_{m,k}(z)\frac{f(z^{\overline{w}}_{m,k})}{P_{m+1}(w, z^{\overline{w}}_{m,k})}. \tag{7}$$

One of the advantages evidenced by the representation in (7) is that the extended polynomial "doubles" the degree of the approximating polynomials $L_{m+1}(w, f)$ by reusing the samples of f needed to express it.

About the extended process (7), we recall the following result (Occorsio and Russo 2021, Theorem 2.2)

Theorem 3.2 *Let $w = v^{\alpha,\beta}$, $\alpha, \beta > -1$ and $u = v^{\gamma,\delta}$, $\gamma, \delta \geq 0$. The assumptions*

$$\alpha + 1 \leq \gamma \leq \alpha + 2, \qquad \beta + 1 \leq \delta \leq \beta + 2, \tag{8}$$

are necessary and sufficient to get

$$\|L_{m+1,m}(w, \overline{w})u\|_\infty := \sup_{\|fu\|_\infty=1} \|L_{m+1,m}(w, \overline{w}, f)u\|_\infty \sim \log m. \tag{9}$$

Moreover, for any $f \in C_u$, it follows that

$$\|[f - L_{m+1,m}(w, \overline{w}, f)]u\|_\infty \leq \mathcal{C}(1 + \log m)E_{2m}(f)_u, \tag{10}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

4 Main results

4.1 Extended interpolation on the square S

For a given function $f \in C_{\mathbf{u}}$, now we introduce the extended polynomial interpolating a given function f on the square. To be more precise, setting $\mathbf{w}(x, y) = v^{\alpha,\beta}(x)v^{\alpha,\beta}(y)$ and $\bar{\boldsymbol{w}}(x, y) = (1 - x^2)(1 - y^2)\mathbf{w}(x, y)$, with $\alpha, \beta > -1$, let us denote by $\{t_i\}_{i=1}^{2m+1}$ the zeros of $Q_{2m+1}(z) = P_{m+1}(w, z)P_m(\overline{w}, z)$.

Then, the interpolating polynomial $\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\boldsymbol{w}}, f, x, y)$ such that

$$\mathcal{L}^*_{m+1,m}\left(\mathbf{w}, \bar{\boldsymbol{w}}, f, t_i, t_j\right) = f(t_i, t_j), \qquad \begin{array}{l} i = 1, 2, \ldots, 2m+1, \\ j = 1, 2, \ldots, 2m+1, \end{array}$$

can take the following form

$$\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\boldsymbol{w}}, f, x, y) = \sum_{i=1}^{2m+1}\sum_{j=1}^{2m+1} \frac{Q_{2m+1}(x)}{Q'_{2m+1}(t_i)(x - t_i)} \frac{Q_{2m+1}(y)}{Q'_{2m+1}(t_j)(y - t_j)} f\left(t_i, t_j\right).$$

By using the explicit representation of the extended polynomial in each variable, it takes the following expression

$$
\begin{aligned}
\mathcal{L}^*_{m+1,m}\,(\mathbf{w},\bar{\mathbf{w}},f,x,y)
&=\sum_{r=1}^{m+1} l^{w}_{m+1,r}(x)\frac{P_m(\overline{w},x)}{P_m(\overline{w},z^{w}_{m+1,r})}\sum_{k=1}^{m+1} l^{w}_{m+1,k}(x)\frac{P_m(\overline{w},y)}{P_m(\overline{w},z^{w}_{m+1,k})}f(z^{w}_{m+1,r},z^{w}_{m+1,k})\\
&+\sum_{r=1}^{m+1} l^{w}_{m+1,r}(x)\frac{P_m(\overline{w},x)}{P_m(\overline{w},z^{w}_{m+1,r})}\sum_{k=1}^{m} l^{\overline{w}}_{m,k}(y)\frac{P_{m+1}(w,y)}{P_{m+1}(w,z^{\overline{w}}_{m,k})}f(z^{w}_{m+1,r},z^{\overline{w}}_{m,k})\\
&+\sum_{r=1}^{m} l^{\overline{w}}_{m,r}(x)\frac{P_{m+1}(w,x)}{P_{m+1}(w,z^{\overline{w}}_{m,r})}\sum_{k=1}^{m+1} l^{w}_{m+1,k}(y)\frac{P_m(\overline{w},y)}{P_m(\overline{w},z^{w}_{m+1,k})}f(z^{\overline{w}}_{m,r},z^{w}_{m+1,k})\\
&+\sum_{r=1}^{m} l^{\overline{w}}_{m,r}(x)\frac{P_{m+1}(w,x)}{P_{m+1}(w,z^{\overline{w}}_{m,r})}\sum_{k=1}^{m} l^{\overline{w}}_{m,k}(y)\frac{P_m(\overline{w},y)}{P_m(\overline{w},z^{\overline{w}}_{m,k})}f(z^{\overline{w}}_{m,r},z^{\overline{w}}_{m,k}). \qquad (11)
\end{aligned}
$$

About the behavior of the corresponding Lebesgue constants and the error estimate, we can prove the following

Theorem 4.1 *Let* $\mathbf{w}(x,y)=v^{\alpha,\beta}(x)v^{\alpha,\beta}(y)$ *and* $\bar{\mathbf{w}}(x,y)=(1-x^2)(1-y^2)\mathbf{w}(x,y)$, *with* $\alpha,\beta>-1$, *and the weight* $\mathbf{u}(x,y)=v^{\gamma,\delta}(x)v^{\gamma,\delta}(y)$, $\gamma,\delta\ge 0$ *be such that*

$$\alpha+1\le\gamma\le\alpha+2,\quad \beta+1\le\delta\le\beta+2.$$

Then, for any $f\in C_{\mathbf{u}}$ *it follows*

$$\left\|\mathcal{L}^*_{m+1,m}\,(\mathbf{w},\bar{\mathbf{w}};f)\,\mathbf{u}\right\|_\infty\le\mathcal{C}\log^2 m\,\|f\mathbf{u}\|_\infty, \qquad (12)$$

and hence

$$\left\|\left[f-\mathcal{L}^*_{m+1,m}\,(\mathbf{w},\bar{\mathbf{w}};f)\right]\mathbf{u}\right\|_\infty\le\mathcal{C}\left(1+\log^2 m\right)\,E_{2m,2m}(f)_{\mathbf{u}},\quad \mathcal{C}\ne\mathcal{C}(m,f). \qquad (13)$$

Remark 4.1 Unlike the just-described interpolation process, the one simply based on the zeros of the polynomial $\overline{Q}_{2m+1}(z)=P_{m+1}(w,z)P_m(w,z)$ has Lebesgue constants that always diverge algebraically. This behavior is explained by the fact that the zeros of $\overline{Q}_{2m+1}$ are not *arc sine* distributed (Mastroianni and Milovanović 2008).

4.2 The mixed sequence

Given Proposition 3.1 and Theorem 4.1, we can conclude that under certain common hypotheses, both sequences $\{\|\mathcal{L}^*_{m+1}(\mathbf{w})\|_\infty\}_m$ and $\{\|\mathcal{L}^*_{m+1,m}(\mathbf{w},\bar{\mathbf{w}})\|\}_m$ diverge with optimal order $\log^2 m$. To be more precise, the following Corollary holds true

Corollary 4.1 *With* $\gamma,\delta\ge 0$ *and* $\alpha,\beta>-1$, *be such that*

$$\alpha+1\le\gamma\le\frac{\alpha}{2}+\frac{5}{4},\quad \beta+1\le\delta\le\frac{\beta}{2}+\frac{5}{4}.$$

then

$$\left\|\mathcal{L}^*_{m+1}\,(\mathbf{w})\,\mathbf{u}\right\|_\infty\sim\log^2 m,$$

and

$$\left\|\mathcal{L}^*_{m+1,m}\,(\mathbf{w},\bar{\mathbf{w}})\,\mathbf{u}\right\|_\infty\sim\log^2 m,$$

where the constants in $'\sim'$ *are independent of m.*

By the representation in (11), the set of f samples involved to construct the extended polynomial $\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\boldsymbol{w}}, f)$ includes the $(m+1)^2$ values $\{f(z^w_{m+1,r}, z^w_{m+1,k})\}_{r,k}$ required for the ordinary polynomial $\mathcal{L}^*_{m+1}(\mathbf{w}, f)$ in (5). By this remark, and taking into account Corollary 4.1, we introduce a "mixed" polynomial sequence, organized in such a way that once the polynomial $\mathcal{L}^*_{m+1}(\mathbf{w}, f)$ has been constructed, for the computation of $\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\boldsymbol{w}}, f)$ the samples $\{f(z^w_{m+1,r}, z^w_{m+1,k})\}_{r,k}$ required for $\mathcal{L}^*_{m+1}(\mathbf{w}, f)$ can be reused.

Following an idea used in the univariate case in Occorsio and Russo (2021), for a given m, we consider the sequence

$$\mathcal{L}^*_{m+1}(\mathbf{w}, f, x, y), \mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\boldsymbol{w}}, f, x, y), \mathcal{L}^*_{4m+1}(\mathbf{w}, f, x, y), \mathcal{L}^*_{4m+1,4m}(\mathbf{w}, \bar{\boldsymbol{w}}, f, x, y), \ldots,$$

and carrying out the construction of each couple of the type $\left\{\mathcal{L}^*_{s+1}(\mathbf{w}, f, x, y), \mathcal{L}^*_{s+1,s}(\mathbf{w}, \bar{\boldsymbol{w}}, f, x, y)\right\}_s$ by a fair algorithm, the degree of the interpolant in each variable will be doubled, saving a quarter of the computations of f. So, the mixed sequence we consider is here defined as

$$\mathbf{L}_{2^j m}(f, x, y) := \begin{cases} \mathcal{L}^*_{2^j m+1}(\mathbf{w}, f, x, y), & j = 0, 2, 4, \ldots, \\ \mathcal{L}^*_{2^{j-1}m+1, 2^{j-1}m}(\mathbf{w}, \bar{\boldsymbol{w}}, f, x, y), & j = 1, 3, 5, \ldots. \end{cases} \tag{14}$$

Based on the previous theorems, we can state the following proposition.

Proposition 4.1 *Let* $\mathbf{u}(x, y) = v^{\gamma,\delta}(x)\, v^{\gamma,\delta}(y)$ *with* $\gamma, \delta \geq 0$ *and* $\mathbf{w}(x, y) = v^{\alpha,\beta}(x) v^{\alpha,\beta}(y)$, *with* $\alpha, \beta > -1$. *Under the assumptions*

$$\alpha + 1 \leq \gamma \leq \frac{\alpha}{2} + \frac{5}{4}, \quad \beta + 1 \leq \delta \leq \frac{\beta}{2} + \frac{5}{4}, \tag{15}$$

for any $f \in C_{\mathbf{u}}$ *it follows*

$$\left\| \left[f - \mathbf{L}_{2^k m}(f, x, y) \right] \mathbf{u} \right\|_\infty \leq \mathcal{C} \left(1 + \log^2 m\right) E_{2^k m, 2^k m}(f)_{\mathbf{u}}.$$

where $\mathcal{C} \neq \mathcal{C}(f, m)$.

Observe that if we approximate f by the subsequence of ordinary interpolating polynomials defined as

$$\left\{ \mathcal{L}^*_{2^k m+1}(\mathbf{w}, f) \right\}_{k=0}^{2q-1}, \tag{16}$$

$(2q + m(2^{2q} - 1))^2 \sim m^2 (2^{2q} - 1)^2$ samples of f are needed. Meanwhile, for the compounded couples of ordinary and extended polynomials defined as

$$\left\{ \mathcal{L}^*_{2^k m+1}(\mathbf{w}, f, x, y), \mathcal{L}^*_{2^k m+1, 2^k m}(\mathbf{w}, \bar{\boldsymbol{w}}, f, x, y) \right\}_{k=0,2,\ldots,2q-2} \tag{17}$$

the required samples of f are

$$\begin{aligned} \left(\sum_{k=0}^{q-1} 2^{2k+1} m + q \right) \left(\sum_{k=0}^{q-1} 2^{2k+1} m + q \right) &= \left(\sum_{k=0}^{q-1} 2^{2k+1} m + q \right)^2 \\ &= \left(q + \frac{2m}{3} (2^{2q} - 1) \right)^2 \sim \frac{4}{9} m^2 (2^{2q} - 1)^2 \end{aligned}$$

Thus, by suitably implementing the algorithm, if we use (17), we need $\frac{4}{9} m^2 (2^{2q} - 1)^2$ instead of $m^2 (2^{2q} - 1)^2$ function evaluations, which allows us to perform the global subsequence computation with around 44% of that required by the sequence (16).

Table 1 Example 1

$m+1$	$\mathcal{E}^{Ord}_{m+1}(f)$	# f eval	$(m+1, m)$	$\mathcal{E}^{Ext}_{m+1,m}(f)$	# f new eval
5	1.96e−3	25			
9	5.77e−6	81	(5, 4)	8.28e−7	16
17	8.34e−12	289			
33	1.11e−16	1089	(17, 16)	1.52e−16	256

5 Numerical examples

In this section we propose some examples to show the performance of the mixed interpolation scheme introduced in this paper. We will compare the results achieved by the mixed sequence with those attained by the corresponding ordinary sequence of the same degrees. Denoting by $\mathbf{I}_n$ a sufficiently dense grid of $n \times n$ uniform points $(\xi_i, \xi_j)_{i=1,2,\ldots,n,j=1,2,\ldots,n}$, we define the errors

$$\mathcal{E}^{Ord}_{m+1}(f) = \max_{(\xi_i,\xi_j)\in\mathbf{I}_n} \left| f(\xi_i, \xi_j) - \mathcal{L}^*_{m+1}\left(\mathbf{w}, f, \xi_i, \xi_j\right)\right| \mathbf{u}(\xi_i, \xi_j), \tag{18}$$

$$\mathcal{E}^{Ext}_{m+1,m}(f) = \max_{(\xi_i,\xi_j)\in\mathbf{I}_n} \left| f(\xi_i, \xi_j) - \mathcal{L}^*_{m+1,m}\left(\mathbf{w}, \bar{\mathbf{w}}, f, \xi_i, \xi_j\right)\right| \mathbf{u}(\xi_i, \xi_j). \tag{19}$$

In each test, we compare the maximum absolute errors achieved by the mixed scheme (14), with the corresponding sequence based on the ordinary Lagrange interpolating polynomials in (16), for the same degrees of the interpolating polynomial we employ. In Example 2, we have also compared our results with those obtained by interpolation at Padua points. The Padua interpolant is a polynomial of total degree N, interpolating f at $(N+1)(N+2)/2$ points, which are intersections of N Lissajous curves. We point out that the Padua set includes nodes lying on the boundary of S and the interpolant based on them provides an optimal approximation of f, since the corresponding Lebesgue constants grow logarithmically.

Each table contains:

- the degree of the interpolating polynomials;
- the number of function evaluations used for the corresponding interpolant polynomial;
- the maximum errors $\mathcal{E}^{Ord}_{m+1}(f)$, $\mathcal{E}^{Ext}_{m+1,m}(f)$ attained in $\mathbf{I}_n$.

In all the tests, the parameters of the weights $\mathbf{w}$, $\mathbf{u}$ satisfy the assumptions in (15). We point out that all computations were performed with double machine precision (2.2204×10^{-16}).

Example 1

$$f_1(x, y) = \sin(x^2 y^2),$$
$$\mathbf{w}(x, y) \equiv 1, \quad \text{and} \quad \mathbf{u}(x, y) = v^{1,1}(x)v^{1,1}(y).$$

The function $f_1 \in W_r(\mathbf{u})$, $\forall r \geq 1$, and Table 1 shows that both the sequences converge rapidly.

Example 2

$$f_2(x, y) = \frac{3}{4}\exp\left(\frac{-(9x-2)^2 + (9y-2)^2}{4}\right) + \frac{3}{4}\exp\left(-\frac{(9x+1)^2}{49} + \frac{9y+1}{10}\right)$$
$$+ \frac{1}{2}\exp\left(\frac{-(9x-7)^2 + (9y-3)^2}{4}\right) - \frac{1}{5}\exp\left(-(9x-4)^2 - (9y-7)^2\right),$$

Table 2 Example 2

$m+1$	$\mathcal{E}^{Ord}_{m+1}(f)$	# f eval	$(m+1, m)$	$\mathcal{E}^{Ext}_{m+1,m}(f)$	# f new eval
5	3.60e−1	25			
9	2.25e−1	81	(5, 4)	7.05e−2	16
17	4.47e−2	289			
33	1.14e−3	1089	(17, 16)	3.17e−3	256
65	2.12e−8	4225			
129	6.66e−15	16,641	(65, 64)	5.51e−15	4096

Table 3 Example 3

$m+1$	$\mathcal{E}^{Ord}_{m+1}(f)$	# f eval	$(m+1, m)$	$\mathcal{E}^{Ext}_{m+1,m}(f)$	# f new eval
5	5.06e−3	25			
9	5.85e−4	81	(5,4)	1.09e−3	16
17	7.77e−5	289			
33	9.46e−6	1089	(17,16)	1.26e−6	256
65	1.18e−6	4225			
129	1.48e−7	16,641	(65,64)	1.55e−7	4096

$$\mathbf{w}(x, y) = \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}, \quad \text{and} \quad \mathbf{u}(x, y)v^{0.1,0.1}(x)v^{0.1,0.1}(y).$$

This function belongs to the family tests of *Franke functions* in Renka and Brown (1999). $f_2 \in W_r(\mathbf{u})$ for any $r \geq 1$. For this test, we want to compare our results with those achieved by the interpolation of f_2 on the so-called Padua points and tested in Caliari et al. (2008). In this case, with $N = 60$ and $(N+1)(N+2)/2 = 1891$ function evaluations, the maximum error is $4e-11$(see Table 1 in (Caliari et al. 2008, p. 291)). As shown in Table 2, the application of the mixed interpolation scheme enables us to reach machine precision with a reasonable number of function evaluations.

Example 3

$$f_3(x, y) = \frac{(1+x+y)\sin\left(\sqrt[4]{(1+y)(1+x)}\right)}{\sqrt[4]{(1+y)(1+x)}},$$

$$\mathbf{w}(x, y) = \sqrt{1-x^2}\sqrt{1-y^2}, \quad \text{and} \quad \mathbf{u}(x, y) = (1+y)^{\frac{5}{4}}(1+x)^{\frac{5}{4}}.$$

The function f_3 belongs to the space $W_2(\mathbf{u})$. This means that the theoretical error behaves as $\mathcal{O}\left(\frac{\log^2 m}{m^2}\right)$. However, as Table 3 shows, the numerical results are better than the expected ones.

Example 4

$$f_4(x, y) = |x - 0.3|^{\frac{7}{2}} \exp(y),$$

$$\mathbf{w}(x, y) = \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}, \quad \text{and} \quad \mathbf{u}(x, y) \equiv 1.$$

Table 4 Example 4

$m+1$	$\mathcal{E}^{Ord}_{m+1}(f)$	# f eval	$(m+1, m)$	$\mathcal{E}^{Ext}_{m+1,m}(f)$	# f new eval
5	2.83e−1	25			
9	3.80e−3	81	(5, 4)	4.35e−2	16
17	2.06e−4	289			
33	3.47e−5	1089	(17, 16)	2.17e−5	256
65	9.43e−7	4225			
129	7.72e−8	16,641	(65, 64)	6.21e−8	4096

In this test $f_4 \in W_3(\mathbf{u})$, therefore the theoretical error behaves as $\mathcal{O}\left(\frac{\log^2 m}{m^3}\right)$. Table 4 confirms our theoretical expectations.

6 Proofs

Proof of Theorem 4.1 Starting from the definition of $\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\mathbf{w}}; f, x, y)$, we have

$$
\begin{aligned}
&|\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\mathbf{w}}; f, x, y)\mathbf{u}(x, y)| \\
&\quad = \left| \sum_{i=1}^{2m+1} \sum_{j=1}^{2m+1} \frac{Q_{2m+1}(x)}{Q'_{2m+1}(t_i)(x-t_i)} \frac{Q_{2m+1}(y)}{Q'_{2m+1}(t_i)(y-t_i)} f\left(t_i, t_j\right) \mathbf{u}(x, y) \right| \\
&\quad \le \sum_{i=1}^{2m+1} \sum_{j=1}^{2m+1} \left| \frac{Q_{2m+1}(x)}{Q'_{2m+1}(t_i)(x-t_i)} \frac{Q_{2m+1}(y)}{Q'_{2m+1}(t_i)(y-t_i)} f\left(t_i, t_j\right) \frac{\mathbf{u}(t_i, t_j)}{\mathbf{u}(t_i, t_j)} \mathbf{u}(x, y) \right| \\
&\quad \le \|f\|_{C_u} \\
&\qquad \times \sup_{(x,y)\in S} \sum_{i=1}^{2m+1} \left| \frac{Q_{2m+1}(x)}{Q'_{2m+1}(t_i)(x-t_i)} \right| \frac{u(x)}{u(t_i)} \sum_{j=1}^{2m+1} \left| \frac{Q_{2m+1}(y)}{Q'_{2m+1}(t_i)(y-t_i)} \right| \frac{u(y)}{u(t_j)}
\end{aligned}
$$

and taking into account Theorem 3.2 whose assumptions are satisfied, it follows

$$
\left\| \mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\mathbf{w}}; f)\mathbf{u} \right\|_\infty \le \mathcal{C} \|f\|_{C_u} \log^2 m,
$$

and (12) has been proved. Let $P^*_{2m,2m}$ be the polynomial of the best approximation of $f \in C_u$. Then

$$
\begin{aligned}
\|[f - \mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\mathbf{w}}, f)]\mathbf{u}\|_\infty &\le \|\left[f - P^*_{2m,2m}\right]\mathbf{u}\|_\infty + \|\mathcal{L}^*_{m+1,m}(\mathbf{w}, \bar{\mathbf{w}}; f - P^*_{2m,2m})\mathbf{u}\|_\infty \\
&\le \mathcal{C}(1 + \log^2 m) E_{2m,2m}(f)_{\mathbf{u}}
\end{aligned}
$$

(13) follows, and the theorem is completely proved. □

Proof of Proposition 4.1 We observe that under the assumptions (15), Remark 3.1 and Theorem 4.1 are both satisfied. Hence, the thesis follows. □

7 Conclusions and future works

In this paper, we introduced a bivariate extended Lagrange polynomial and employed it in the context of a mixed interpolation process to approximate a function f on the square $\mathrm{S} := [-1, 1]^2$. This approach enabled the development of accurate high-degree interpolation schemes while mitigating the instability issues typically associated with computing the zeros of high-degree orthogonal polynomials. Moreover, it allowed a reduction in the number of function evaluations compared to the standard bivariate Lagrange interpolation process. Motivated by the promising results obtained in the numerical experiments, we plan to apply this new interpolation process to the development of novel mixed quadrature rules and the numerical treatment of bivariate integral equations.Indeed, many problems in different fields are described by bivariate integral equations. Some well-known cases of study are the radiance and rendering equation (Keller 1997; Kajiya 1986). In accordance with the framework outlined in Mezzanotte et al. (2021), we plan to propose a Nyström method that incorporates the bivariate extended interpolation process introduced herein to approximate the solution of a bivariate Fredholm integral equation, with the primary objective of achieving a substantial reduction in the computational complexity of the problem under investigation. This reduction is attained by virtue of the fact that the dimension of the linear system arising from the proposed method is considerably smaller than that of the corresponding system obtained when employing the same method based on the classical bivariate Lagrange interpolant.

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Data Availability No data was used for the research described in the article.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

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