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Voronovskaya type results for Bernstein-Chlodovsky operators preserving e^{-2x}

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ABSTRACT

In this paper we continue the study of certain Bernstein-Chlodovsky operators B_n^* preserving the exponential function e^{-2x} $(x \geq 0)$, recently introduced in [4]. In particular, we prove some Voronovskaya type theorems and we deduce some properties of the B_n^*'s, such as saturation results. We also compare this new class of operators with the classical Bernstein-Chlodovsky ones, proving that the operators B_n^* provide better approximation results for certain functions.

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1. Introduction

In order to approximate functions defined on unbounded intervals, in 1937 Chlodovsky introduced and studied the following Bernstein-type operators

$$B_{n,h_n}(f)(x) = \sum_{k=0}^{n} f\left(\frac{h_n k}{n}\right)\binom{n}{k}\left(\frac{x}{h_n}\right)^k\left(1-\frac{x}{h_n}\right)^{n-k}$$

for $n \geq 1$, $x \geq 0$, f belonging to a suitable space, and $(h_n)_{n\geq 1}$ being a sequence of strictly positive real numbers such that $\lim_{n\to\infty} h_n = +\infty$ (see [13]; see also [17, pp. 36–37]). Observe that the B_{n,h_n}'s are not positive operators, hence many authors worked with a positive modification of theirs, that for an abuse of notation we continue to denote by B_{n,h_n}. Such operators are known as Bernstein-Chlodovsky operators and they are defined by

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$$B_{n,h_n}(f)(x) = \begin{cases} \sum_{k=0}^{n} f\left(\frac{h_n k}{n}\right)\binom{n}{k}\left(\frac{x}{h_n}\right)^k\left(1-\frac{x}{h_n}\right)^{n-k} & \text{if } 0 \le x \le h_n\,, \\ f(x) & \text{if } x > h_n \end{cases} \tag{1}$$

(see, e.g., [8,9]). For more recent developments on Bernstein-Chlodovsky operators and their variants we refer the interested readers to, e.g., [2,15,16]. In particular, Voronovskaya theorems for Bernstein-Chlodovsky operators and their generalizations have been deeply studied (see, for example, [1,5,9–11]).

Note that the operators B_{n,h_n} fix constant and linear functions. Motivated by the increasing interest in operators which preserve different functions in order to get better properties (for a survey on these topics see, e.g., [3]), in [4] we introduced a particular modification B_n^* of the operators (1) that allows to reproduce constants and the exponential function f_2, where $f_2(x) = e^{-2x}$ $(x \ge 0)$.

Approximation properties of the sequence $(B_n^*)_{n\ge1}$, both in spaces of continuous functions and in some weighted function spaces, can be found in [4], where we generally focused on uniform convergence behaviors. However, pointwise convergence has as a crucial role as uniform one. This fact directed us to describe pointwise approximation properties of the sequence $(B_n^*)_{n\ge1}$ in terms of pointwise Voronovskaya type results. We also obtain an asymptotic formula with respect to a weighted norm.

More precisely, we prove that the operators B_n^* are involved in an asymptotic formula with respect to a certain second-order degenerate differential operator. It can be shown, by using some results in [7], that this differential operator is the generator of a positive C_0-semigroup in certain weighted spaces. It might be interesting to investigate the eventuality that such a semigroup can be represented in terms of iterates of the operators B_n^*, as an application of the classical Trotter representation theorem (see [8, Proposition 1.6.7]).

In the paper, as a consequence of the asymptotic formula, we also deduce several properties of the operators B_n^*; in particular, by comparing the relevant asymptotic formulas, we prove that the B_n^*'s perform better than the operators B_{n,h_n} in approximating certain decreasing convex functions. Moreover, we obtain some saturation results, in the same spirit of [14].

The paper is organized as follows: after a section collecting the basic preliminaries, we proceed, in Section 3, to present some Voronovskaya type theorems for the operators B_n^*. The paper ends with a section devoted to several applications of the results contained in Section 3.

2. Preliminaries

Throughout the paper, $C([0,+\infty[)$ stands for the space of all continuous real valued functions on $[0,+\infty[$ and $C_b([0,+\infty[)$ for the space consisting of all functions in $C([0,+\infty[)$ which are also bounded. $C_b([0,+\infty[)$ is a Banach lattice if endowed with the sup-norm $\|\cdot\|_\infty$ and the natural pointwise ordering. Moreover, we shall denote by $C_*([0,+\infty[)$ the Banach sublattice of $C_b([0,+\infty[)$ defined as

$$C_*([0,+\infty[) = \{f \in C([0,+\infty[) \,:\, \exists \lim_{x\to+\infty} f(x) \in \mathbb{R}\}\,.$$

The symbol $UC_b([0,+\infty[)$ indicates the space of all uniformly continuous functions on $[0,+\infty[$ that are also bounded.

Further, for every $m \ge 1$, we define the weighted space

$$E_m := \{f \in C([0,+\infty[) \,:\, \sup_{x\ge0} w_m(x)|f(x)| \in \mathbb{R}\},$$

endowed with the norm

$$\|f\|_m := \sup_{x\geq 0} w_m(x)|f(x)| \qquad (f \in E_m), \tag{2}$$

where

$$w_m(x) = \frac{1}{1+x^m} \quad (x \geq 0)\,.$$

We also need its natural subspace

$$E_m^* = \{f \in E_m \,:\, \exists \lim_{x\to+\infty} w_m(x)f(x) \in \mathbb{R}\}\,.$$

In [4] we introduced the modified Bernstein-Chlodovsky operators B_n^* defined as follows: for every $n \geq 1$, $f \in C_*([0,+\infty[)$ and $x \geq 0$,

$$B_n^*(f)(x) = \begin{cases} \displaystyle\sum_{k=0}^{n} f\left(\frac{h_n k}{n}\right)\binom{n}{k}\left(\frac{1-e^{-(2x)/n}}{1-e^{-(2h_n)/n}}\right)^k\left(1-\frac{1-e^{-(2x)/n}}{1-e^{-(2h_n)/n}}\right)^{n-k} & \text{if } 0 \leq x \leq h_n \\ f(x) & \text{if } x > h_n\,. \end{cases} \tag{3}$$

The B_n^*'s preserve constants and $B_n^*(f_2) = f_2$, where $f_2(x) = e^{-2x}$ $(x \geq 0)$. Further, for every $n \geq 1$, B_n^* is a positive linear operator from $C_*([0,+\infty[)$ into itself and $\|B_n^*\|_{C_*([0,+\infty[)} = 1$.

Moreover, under the assumption

$$\lim_{n\to\infty} \frac{h_n}{n} = 0\,, \tag{4}$$

if $f \in C_*([0,+\infty[)$, then $\lim_{n\to\infty} B_n^*(f) = f$ uniformly on $[0,+\infty[$.

For sufficiently large n, $B_n^*(E_m^*) \subset E_m^*$ and, if $f \in E_m^*$, then $\lim_{n\to\infty} B_n^*(f) = f$ with respect to $\|\cdot\|_m$.

We point out that, if $f \in C_*([0,+\infty[)$ and $0 \leq x \leq h_n$,

$$B_n^*(f)(x) := (B_{n,h_n}(f) \circ r_n)(x) \tag{5}$$

for every $n \geq 1$, where

$$r_n(x) = h_n \frac{1-e^{-(2x)/n}}{1-e^{-(2h_n)/n}} \qquad \text{for } 0 \leq x \leq h_n. \tag{6}$$

It can be useful to recall some properties of $(r_n)_{n\geq 1}$. First,

$$r_n(0) = 0\,, r_n(h_n) = h_n\,, \text{ and } 0 < r_n(x) \leq M_n x \quad \text{for every } x > 0\,, \tag{7}$$

where

$$M_n := \frac{2h_n/n}{1-e^{-2h_n/n}} \quad \text{for every } n \geq 1. \tag{8}$$

Note that $M_n \geq 1$ and, under the assumption (4),

$$\lim_{n\to\infty} M_n = 1\,. \tag{9}$$

Moreover, for every $n \geq 1$,

$$r_n(x) \geq x \quad \text{for any } 0 \leq x \leq h_n, \tag{10}$$

and, under hypothesis (4),

$$\lim_{n\to\infty} r_n = e_1 \tag{11}$$

pointwise on $[0, +\infty[$, and uniformly on compact sub-intervals of $[0, +\infty[$.

Coming back to the operators B_n^*, we now compute them in the functions e_i^x, where for a given $x \geq 0$ and $i \in \mathbb{N}$,

$$e_i^x(t) = (t - x)^i \quad (t \geq 0).$$

In particular, we briefly write e_i for the power functions e_i^0.

Combining (5) with [11, Lemma 2.1], we get the following formulas: for every $n \geq 1$,

$$B_n^*(e_0) = e_0 \tag{12}$$

and, for $0 \leq x \leq h_n$,

$$B_n^*(e_1)(x) = r_n(x)\,, \tag{13}$$

$$B_n^*(e_2)(x) = \frac{n-1}{n} r_n^2(x) + \frac{h_n}{n} r_n(x)\,, \tag{14}$$

$$B_n^*(e_3)(x) = \frac{(n-1)(n-2)}{n^2} r_n^3(x) + 3\frac{h_n}{n}\frac{n-1}{n} r_n^2(x) + \frac{h_n^2}{n^2} r_n(x)\,, \tag{15}$$

and

$$\begin{aligned} B_n^*(e_4)(x) = {} & \frac{(n-1)(n-2)(n-3)}{n^3} r_n^4(x) + 6\frac{h_n}{n}\frac{(n-1)(n-2)}{n^2} r_n^3(x) \\ & + 7\frac{h_n^2}{n^2}\frac{n-1}{n} r_n^2(x) + \frac{h_n^3}{n^3} r_n(x)\,. \end{aligned} \tag{16}$$

Using (12)-(14) it is easy to see that, for any $n \geq 1$ and $x \in [0, h_n]$,

$$B_n^*(e_1^x)(x) = r_n(x) - x \tag{17}$$

and

$$B_n^*(e_2^x)(x) = (r_n(x) - x)^2 - \frac{1}{n} r_n^2(x) + \frac{h_n}{n} r_n(x)\,. \tag{18}$$

Moreover, from (12)-(16) it follows that, for any $n \geq 1$ and $x \in [0, h_n]$,

$$\begin{aligned} B_n^*(e_4^x)(x) = {} & (r_n(x) - x)^4 + \frac{h_n^3}{n^3} r_n(x) + \frac{h_n^2}{n^2} r_n(x) \left[7\frac{n-1}{n} r_n(x) - 4x \right] \\ & + 6\frac{h_n}{n} r_n(x) \left[\frac{(n-1)(n-2)}{n^2} r_n^2(x) - 2\frac{n-1}{n} x r_n(x) + x^2 \right] \\ & + r_n^2(x) \left[\frac{-6n^2 + 11n - 6}{n^3} r_n^2(x) + 4x\frac{3n-2}{n^2} r_n(x) - x^2\frac{6}{n} \right]. \end{aligned} \tag{19}$$

Finally note that, from definition (3), it follows that $B_n^*(e_i^x)(x) = 0$ for $i \geq 1$ and $x > h_n$.

Lemma 1. *For $x \geq 0$ and n sufficiently large,*

$$\left| \frac{n}{h_n} B_n^*(e_1^x)(x) - x \right| \leq x^2 \alpha_n + x \beta_n \, , \tag{20}$$

$$\left| \frac{n}{h_n} B_n^*(e_2^x)(x) - x \right| \leq x^3 \alpha_n \left(M_n - 1 \right) + x^2 \left((M_n - 1)(\beta_n + 1) + \frac{M_n^2}{h_n} \right) + x \left(M_n - 1 \right) \, , \tag{21}$$

and

$$\frac{n}{h_n} B_n^*(e_4^x)(x) \leq x^5 \alpha_n (M_n - 1)^3 + x^4 \gamma_n + x^3 \sigma_n + x^2 \tau_n + x \frac{h_n^2}{n^2} M_n \, , \tag{22}$$

where $(\alpha_n)_{n \geq 1}$, $(\beta_n)_{n \geq 1}$, $(\gamma_n)_{n \geq 1}$, $(\sigma_n)_{n \geq 1}$, $(\tau_n)_{n \geq 1}$ are suitable sequences of real numbers.

Proof. First note that for $x = 0$ the above formulas are easily verified thanks to (17)-(19), and the fact that $r_n(0) = 0$ for every $n \geq 1$.

Now fix $x > 0$; since $h_n \to +\infty$ as $n \to \infty$, for n large enough we have that $h_n \geq x$. Keeping (6) and (17) in mind, we may write

$$\begin{aligned} \frac{n}{h_n} B_n^*(e_1^x)(x) - x &= x \frac{n}{h_n} \left[\frac{1 - e^{-2x/n}}{2x/n} \frac{2h_n/n}{1 - e^{-2h_n/n}} - 1 - \frac{h_n}{n} \right] \\ &= x \frac{n}{h_n} \left[\left(\frac{1 - e^{-2x/n}}{2x/n} - 1 \right) \frac{2h_n/n}{1 - e^{-2h_n/n}} + \frac{2h_n/n}{1 - e^{-2h_n/n}} - 1 - \frac{h_n}{n} \right] . \end{aligned}$$

As a consequence,

$$\left| \frac{n}{h_n} B_n^*(e_1^x)(x) - x \right| \leq x^2 \frac{C}{h_n} \frac{2h_n/n}{1 - e^{-2h_n/n}} + x \frac{n}{h_n} \left| \frac{2h_n/n}{1 - e^{-2h_n/n}} - 1 - \frac{h_n}{n} \right| ,$$

where we have used the inequality $\left| \frac{1 - e^{-2x/n}}{2x/n} - 1 \right| \leq C \frac{x}{n}$ which holds for n large enough, where C does not depend on x and n.

Setting, for every $n \geq 1$,

$$\alpha_n = \frac{C}{h_n} \frac{2h_n/n}{1 - e^{-2h_n/n}} \tag{23}$$

and

$$\beta_n = \frac{n}{h_n} \frac{2h_n/n - (1 - e^{-2h_n/n}) - h_n/n(1 - e^{-2h_n/n})}{1 - e^{-2h_n/n}} \tag{24}$$

(note that $\beta_n \geq 0$) we get (20).

In order to achieve estimate (21), we observe that, thanks to (17) and (18),

$$\frac{n}{h_n} B_n^*(e_2^x)(x) - x = \left(\frac{n}{h_n} B_n^*(e_1^x)(x) - x \right) (r_n(x) - x) + x(r_n(x) - x) - \frac{1}{h_n} r_n^2(x) + (r_n(x) - x).$$

Then, keeping (7)-(10) and (20) in mind,

$$
\begin{aligned}
&\left|\frac{n}{h_n}B_n^*(e_2^x)(x)-x\right| \\
&\le \left|\frac{n}{h_n}B_n^*(e_1^x)(x)-x\right|(r_n(x)-x)+x(r_n(x)-x)+\frac{1}{h_n}r_n^2(x)+(r_n(x)-x) \\
&\le (x^3\alpha_n+x^2\beta_n)(M_n-1)+x^2\left(M_n-1+\frac{M_n^2}{h_n}\right)+x(M_n-1)\,.
\end{aligned}
$$

Finally, by means of (17), (19) and (20), we get

$$
\begin{aligned}
&\frac{n}{h_n}B_n^*(e_4^x)(x)=\left(\frac{n}{h_n}B_n^*(e_1^x)(x)-x\right)(r_n(x)-x)^3+x(r_n(x)-x)^3 \\
&+\frac{h_n^2}{n^2}r_n(x)+\frac{h_n}{n}r_n(x)\left[7\frac{n-1}{n}r_n(x)-4x\right] \\
&+6r_n(x)\left[\frac{(n-1)(n-2)}{n^2}r_n^2(x)-2\frac{n-1}{n}xr_n(x)+x^2\right] \\
&+\frac{n}{h_n}r_n^2(x)\left[\frac{-6n^2+11n-6}{n^3}r_n^2(x)+4x\frac{3n-2}{n^2}r_n(x)-x^2\frac{6}{n}\right] \\
&\le (x^2\alpha_n+x\beta_n)(M_n-1)^3x^3+x^4(M_n-1)^3 \\
&+\frac{h_n^2}{n^2}M_nx+\frac{h_n}{n}M_nx^2\left[7\frac{n-1}{n}M_n-4\right] \\
&+6M_nx\left[\frac{(n-1)(n-2)}{n^2}(r_n^2(x)-x^2)-2\frac{n-1}{n}x(r_n(x)-x)+x^2\frac{2-n}{n^2}\right] \\
&+\frac{n}{h_n}M_n^2x^2\left[\frac{-6n^2+11n-6}{n^3}(r_n^2(x)-x^2)+4x\frac{3n-2}{n^2}(r_n(x)-x)+x^2\frac{3n-6}{n^3}\right] \\
&\le (x^5\alpha_n+x^4\beta_n)(M_n-1)^3+x^4(M_n-1)^3 \\
&+\frac{h_n^2}{n^2}M_nx+\frac{h_n}{n}M_nx^2\left[7\frac{n-1}{n}M_n-4\right] \\
&+6M_nx^3\left[\frac{(n-1)(n-2)}{n^2}(M_n^2-1)+2\frac{n-1}{n}(M_n-1)+\frac{2-n}{n^2}\right] \\
&+\frac{M_n^2x^4}{h_n}\left(\frac{-6n^2+11n-6}{n^2}(M_n^2-1)+4\frac{3n-2}{n}(M_n-1)+\frac{3n-6}{n^2}\right) \\
&=x^5\alpha_n(M_n-1)^3+x^4\gamma_n+x^3\sigma_n+x^2\tau_n+x\frac{h_n^2}{n^2}M_n\,,
\end{aligned}
$$

where, for every $n\ge 1$,

$$
\begin{aligned}
\gamma_n&=(\beta_n+1)(M_n-1)^3 \\
&+\frac{M_n^2}{h_n}\left(\frac{-6n^2+11n-6}{n^2}(M_n^2-1)+4\frac{3n-2}{n}(M_n-1)+\frac{3n-6}{n^2}\right), \qquad (25)\\
\sigma_n&=6M_n\left[\frac{(n-1)(n-2)}{n^2}(M_n^2-1)+2\frac{n-1}{n}(M_n-1)+\frac{2-n}{n^2}\right], \qquad (26)
\end{aligned}
$$

and

$$
\tau_n=\frac{h_n}{n}M_n\left[7\frac{n-1}{n}M_n-4\right]. \quad \square \qquad (27)
$$

Corollary 2. *Assume that* (4) *holds. Then, for every* $x \geq 0$,

$$\lim_{n\to\infty} \frac{n}{h_n} B_n^*(e_1^x)(x) = x = \lim_{n\to\infty} \frac{n}{h_n} B_n^*(e_2^x)(x) \tag{28}$$

and

$$\lim_{n\to\infty} \frac{n}{h_n} B_n^*(e_4^x)(x) = 0. \tag{29}$$

Proof. It is sufficient to apply inequalities (20)-(22) and to observe that $M_n \to 1$ (see (9)) and, thanks to (23)-(27), $\alpha_n, \beta_n, \gamma_n, \sigma_n, \tau_n \to 0$ as $n \to \infty$. □

We end these preliminaries with the following lemma.

Lemma 3. *Under hypothesis* (4), *for every* $x \geq 0$ *one has*

$$\lim_{n\to\infty} \frac{n^2}{h_n^2} B_n^*\left(e_4^x\right)(x) = 3x^2.$$

Proof. If $x = 0$ the above formula is trivial thanks to (19) and the fact that $r_n(0) = 0$ for every $n \geq 1$.

Let $x > 0$ and assume that $h_n \geq x$ (this is always true for n large enough). Taking (17) and (19) into account, we have

$$\begin{aligned}
&\frac{n^2}{h_n^2} B_n^*(e_4^x)(x) = \left(\frac{n}{h_n} B_n^*(e_1^x)(x)\right)^2 (r_n(x) - x)^2 + \frac{h_n}{n} r_n(x) \\
&+ r_n(x)\left[7\frac{n-1}{n} r_n(x) - 4x\right] + 6\frac{n}{h_n} r_n(x)\left[\frac{(n-1)(n-2)}{n^2} r_n^2(x)\right. \\
&\left.-2\frac{n-1}{n} x r_n(x) + x^2\right] + \frac{n^2}{h_n^2} r_n^2(x)\left[\frac{-6n^2+11n-6}{n^3} r_n^2(x) + 4x\frac{3n-2}{n^2} r_n(x) - x^2\frac{6}{n}\right] \\
&\sim 3x^2 + 6r_n(x)\left[\frac{(n-1)(n-2)}{n^2}(r_n(x)+x)\frac{n}{h_n} B_n^*(e_1^x)(x) - 2\frac{n-1}{n} x \frac{n}{h_n} B_n^*(e_1^x)(x)\right] \\
&+ r_n^2(x)\left[\frac{n}{h_n}\frac{-6n^2+11n-6}{n^3}(r_n(x)+x)\frac{n}{h_n} B_n^*(e_1^x)(x) + 4x\frac{n}{h_n}\frac{3n-2}{n^2}\frac{n}{h_n} B_n^*(e_1^x)(x)\right] \\
&\sim 3x^2,
\end{aligned}$$

by virtue of (4), (11), and (28). □

3. Voronovskaya type results

In this section, we are interested in determining Voronovskaya type results for the sequence $(B_n^*)_{n\geq 1}$. We begin with a pointwise asymptotic formula.

Theorem 4. *Consider a function* $f \in C_*([0, +\infty[)$ *such that* f'' *exists at a point* $x \geq 0$. *Then*

$$\lim_{n\to\infty} \frac{n}{h_n}\left[B_n^*(f)(x) - f(x)\right] = x\left(f'(x) + \frac{1}{2} f''(x)\right). \tag{30}$$

Proof. If $x = 0$, (30) holds because of $B_n^*(f)(0) = f(0)$. For $x > 0$ fixed, by virtue of Taylor's expansion of f at the point x we get

$$f(t)=f(x)+f'(x)(t-x)+\frac{f''(x)}{2}(t-x)^2+h_x(t)(t-x)^2,$$

where $h_x \in C_b([0,+\infty[)$ (since $f \in C_b([0,+\infty[)$) and $h_x(x)=0$.

Accordingly,

$$\frac{n}{h_n}(B_n^*(f)(x)-f(x))=\frac{n}{h_n}f'(x)\,B_n^*(e_1^x)(x)$$
$$+\frac{n}{h_n}\frac{f''(x)}{2}B_n^*(e_2^x)(x)+\frac{n}{h_n}B_n^*(h_x e_2^x)(x).$$

Therefore,

$$\frac{n}{h_n}\left[B_n^*(f)(x)-f(x)\right]-xf'(x)-\frac{xf''(x)}{2}$$
$$=\left(\frac{n}{h_n}B_n^*\left(e_1^x\right)(x)-x\right)f'(x)+\frac{1}{2}\left(\frac{n}{h_n}B_n^*\left(e_2^x\right)(x)-x\right)f''(x)$$
$$+\frac{n}{h_n}B_n^*\left(he_x^2\right)(x)=p_n(x)f'(x)+q_n(x)f''(x)+\frac{n}{h_n}B_n^*\left(he_2^x\right)(x),$$

p_n and q_n being defined by

$$p_n(x):=\frac{n}{h_n}B_n^*\left(e_1^x\right)(x)-x, \tag{31}$$
$$q_n(x):=\frac{n}{2h_n}B_n^*\left(e_2^x\right)(x)-\frac{x}{2}. \tag{32}$$

Then, taking (28) into account, $p_n(x)\to 0$, $q_n(x)\to 0$ as $n\to\infty$, hence the statement is proven once we show that

$$\lim_{n\to\infty}\frac{n}{h_n}B_n^*(h_x e_2^x)(x)=0.$$

Set $\varepsilon>0$ and $M_x:=\sup_{n\geq 1}\dfrac{n}{h_n}B_n^*(e_2^x)(x)<+\infty$ since (28) holds true; then there exists $\delta>0$ such that, if $y>0$ is such that $|x-y|<\delta$, then $|h_x(y)|\leq\dfrac{\varepsilon}{M_x}$.

From this it follows that, for all $y>0$,

$$|h_x(y)e_2^x(y)|\leq\frac{\varepsilon}{M_x}e_2^x(y)+\frac{M}{\delta^2}e_4^x(y),$$

where $M:=\|h_x\|_\infty$.

Since (29) holds true, for sufficiently large n, we have that

$$\frac{n}{h_n}B_n^*(e_4^x)(x)\leq\frac{\delta^2}{M}\varepsilon.$$

Hence, for sufficiently large n,

$$\frac{n}{h_n}|B_n^*(h_x e_2^x)(x)|\leq\frac{n}{h_n}B_n^*(|h_x e_2^x|)(x)$$
$$\leq\frac{\varepsilon}{M_x}\frac{n}{h_n}B_n^*(e_2^x)(x)+\frac{n}{h_n}\frac{M}{\delta^2}B_n^*(e_4^x)(x)\leq 2\varepsilon,$$

and this completes the proof. $\square$

To determine the rate of pointwise convergence of the operators B_n^* and to present an upper bound for the error of pointwise approximation, we study, under some additional conditions, a quantitative asymptotic formula.

Theorem 5. *Consider a function $f \in C_*([0,+\infty[)$, which is twice differentiable in $[0,+\infty[$ with $f'' \in C_*([0,+\infty[)$. Then, for any $x \in [0,+\infty[$,*

$$\left|\frac{n}{h_n}\left[B_n^*(f)(x) - f(x)\right] - xf'(x) - \frac{xf''(x)}{2}\right| \le |p_n(x)|\,|f'(x)|$$
$$+ |q_n(x)|\,|f''(x)| + \frac{1}{2}\left(2|q_n(x)| + x + 2s_n(x)\right)\omega\left(f''; \sqrt{h_n/n}\right),$$

where, for every $\delta > 0$, $\omega(f'', \delta)$ denotes the usual modulus of continuity of f'', $p_n(x)$ and $q_n(x)$ are defined, respectively, by (31) and (32), and

$$s_n(x) := \frac{n^2}{2h_n^2} B_n^*(e_4^x)(x)\,. \tag{33}$$

Proof. Arguing as in the proof of Theorem 4, for $x \in [0, h_n]$ we have

$$\left|\frac{n}{h_n}\left[B_n^*(f,x) - f(x)\right] - xf'(x) - \frac{xf''(x)}{2}\right|$$
$$\le |p_n(x)||f'(x)| + |q_n(x)||f''(x)| + \frac{n}{h_n}\left|B_n^*\left(h_x e_2^x\right)(x)\right|,$$

where p_n and q_n are defined by (31) and (32), respectively. By using Taylor's formula with Lagrange remainder, we can take

$$h_x(t) := \frac{f''(\eta) - f''(x)}{2}$$

with η between x and t.

To complete the proof, we must estimate the last term $\frac{n}{h_n}\left|B_n^*\left(h_x e_x^2\right)(x)\right|$. Using the fact that, for every $\delta > 0$,

$$|f''(\eta) - f''(t)| \le \left(1 + \frac{(\eta - t)^2}{\delta^2}\right)\omega(f''; \delta) \le \left(1 + \frac{(x - t)^2}{\delta^2}\right)\omega(f''; \delta),$$

we get

$$|h_x(t)| \le \frac{1}{2}\left(1 + \frac{e_2^x(t)}{\delta^2}\right)\omega(f''; \delta) \quad (\delta > 0).$$

Accordingly, by means of the Cauchy-Schwarz inequality,

$$\frac{n}{h_n}\left|B_n^*\left(h_x e_2^x\right)(x)\right| \le \frac{n}{h_n} B_n^*\left(|h_x|\, e_2^x\right)(x)$$
$$\le \frac{n}{2h_n} B_n^*(e_2^x)(x)\omega(f''; \delta) + \frac{n}{2\delta^2 h_n} B_n^*(e_4^x)(x)\,\omega(f''; \delta)\,.$$

Choosing $\delta = \sqrt{\frac{h_n}{n}}$ and using (33), we obtain

$$\left|\frac{n}{h_n}\left[B_n^*(f)(x)-f(x)\right]-xf'(x)-\frac{xf''(x)}{2}\right|\le |p_n(x)|\,|f'(x)|$$
$$+|q_n(x)|\,|f''(x)|+\frac{1}{2}\left(2|q_n(x)|+x+2s_n(x)\right)\omega\left(f'';\sqrt{h_n/n}\right),$$

which is desired for $x\in[0,h_n]$. Note that, when $x>h_n$, $B_n^*(f)(x)=f(x)$ by definition and hence the claim is obvious, since $p_n(x)=-x$, $q_n(x)=-x/2$ and $s_n(x)=0$. □

Remark 1. Note that, according to Corollary 2 and Lemma 3, the sequences $(p_n(x))_{n\ge1}$ and $(q_n(x))_{n\ge1}$ vanish at infinity and $(s_n(x))_{n\ge1}$ has a finite limit for all $x\in[0,+\infty[$.

Now we proceed to establish a Voronovskaya type result with respect to the weighted norm $\|\cdot\|_m$ (see (2)).

Theorem 6. *Let $m\ge5$. For every $f\in C^2([0,+\infty[)\cap E_m^*$ such that $f''\in UC_b([0,+\infty[)$, we obtain*

$$\lim_{n\to\infty}\frac{n}{h_n}\left(B_n^*(f)-f\right)=e_1\left(f'+\frac{1}{2}f''\right)\quad \textit{with respect to } \|\cdot\|_m\,.$$

Proof. In order to get the statement we apply [6, Theorem 1]. In particular, we need to verify the following conditions:

(i) $\lim_{n\to\infty}w_m(x)x^k\left(\frac{n}{h_n}B_n^*(e_1^x)(x)-x\right)=0\ (k=0,1)$,
(ii) $\lim_{n\to\infty}w_m(x)\left(\frac{n}{h_n}B_n^*(e_2^x)(x)-x\right)=0$,
(iii) $\lim_{n\to\infty}w_m(x)\frac{n}{h_n}B_n^*(e_4^x)(x)=0$,

uniformly with respect to $x\ge0$. Moreover,

(iv) $\sup_{x\ge0,n\ge1}w_m(x)\frac{n}{h_n}B_n^*(e_2^x)(x)<+\infty$.

First of all, observe that all conditions are trivial for $x=0$. For a fixed $x>0$, let N such that $x\le h_n$ for every $n\ge N$. Inequality (20) yields

$$w_m(x)\left|\frac{n}{h_n}B_n^*(e_1^x)(x)-x\right|\le\frac{x^2}{1+x^m}\alpha_n+\frac{x}{1+x^m}\beta_n\le\alpha_n+\beta_n\,,$$

and

$$w_m(x)x\left|\frac{n}{h_n}B_n^*(e_1^x)(x)-x\right|\le\frac{x^3}{1+x^m}\alpha_n+\frac{x^2}{1+x^m}\beta_n\le\alpha_n+\beta_n\,,$$

so that condition (i) follows, as $\alpha_n+\beta_n\to0$ as $n\to\infty$ (see (23) and (24)).

Condition (ii) is a consequence of inequality (21). Indeed,

$$w_m(x)\left|\frac{n}{h_n}B_n^*(e_2^x)(x)-x\right|\le\frac{x^3}{1+x^m}\alpha_n(M_n-1)$$
$$+\frac{x^2}{1+x^m}\left((M_n-1)(\beta_n+1)+\frac{M_n^2}{h_n}\right)+\frac{x}{1+x^m}(M_n-1)$$
$$\le\alpha_n(M_n-1)+\left((M_n-1)(\beta_n+1)+\frac{M_n^2}{h_n}\right)+(M_n-1)\to0$$

thanks to (9), $\alpha_n, \beta_n \to 0$, and $h_n \to +\infty$ as $n \to \infty$.

We pass now to show condition (iii). Taking (22) into account,

$$
\begin{aligned}
& w_m(x)\frac{n}{h_n}B_n^*(e_4^x)(x)\frac{x^5}{1+x^m}\alpha_n(M_n-1)^3 \\
& + \frac{x^4}{1+x^m}\gamma_n + \frac{x^3}{1+x^m}\sigma_n + \frac{x^2}{1+x^m}\tau_n + \frac{x}{1+x^m}\frac{h_n^2}{n^2}M_n \\
& \leq \alpha_n(M_n-1)^3 + \gamma_n + \sigma_n + \tau_n + \frac{h_n^2}{n^2}M_n
\end{aligned}
$$

and the last quantity tends to 0 as $n \to \infty$ (see (25)-(27)). For the last condition we argue as follows. From (18),

$$
\frac{n}{h_n}B_n^*(e_2^x)(x) \leq \frac{n}{h_n}(M_n^2-1)x^2 + M_n x\,.
$$

Therefore,

$$
w_m(x)\frac{n}{h_n}B_n^*(e_2^x)(x) \leq \frac{n}{h_n}(M_n^2-1)\frac{x^2}{1+x^m} + M_n\frac{x}{1+x^m} \leq \frac{n}{h_n}(M_n^2-1) + M_n\,,
$$

the last sequence being convergent as $n \to \infty$. Indeed, $M_n \to 1$ and, on account of (8),

$$
\begin{aligned}
& \lim_{n\to\infty}\frac{n}{h_n}(M_n^2-1) = 2\lim_{n\to\infty}\frac{n}{h_n}(M_n-1) = 2\lim_{n\to\infty}\frac{2-n/h_n(1-e^{-2h_n/n})}{1-e^{-2h_n/n}} \\
& = 2\lim_{n\to\infty}\frac{2-n/h_n(2h_n/n-2h_n^2/n^2)}{2h_n/n} = 2\,.
\end{aligned}
$$

This completes the proof. □

4. Some consequences of the asymptotic formula

The asymptotic formula in Theorem 4 can be employed in order to get other properties of the operators B_n^*.

First, we show that, under suitable conditions, the operators B_n^* perform better than classical Bernstein-Chlodowsky operators B_{n,h_n} in approximating certain functions.

To this end, we recall the asymptotic formula for the operators B_{n,h_n} (see [5]), i.e., for every $f \in C_*([0,+\infty[)$ such that f'' exists at a certain point $x \geq 0$,

$$
\lim_{n\to\infty}\frac{n}{h_n}\left(B_{n,h_n}(f)(x) - f(x)\right) = \frac{1}{2}xf''(x).
$$

By using the comparison theorem and the same methods in [12, Theorem 9]), we have the following result.

Proposition 7. *Consider a function $f \in C_*([0,+\infty[)$ such that f'' exists at a certain point $x \geq 0$. Moreover, assume that there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$,*

$$
f(x) \leq B_n^*(f)(x) \leq B_{n,h_n}(f)(x)\,. \tag{34}
$$

Then

$$
-\frac{1}{2}f''(x) \leq f'(x) \leq 0. \tag{35}
$$

In particular, if (34) holds true on $[0,+\infty[$*, f is decreasing and convex.*

Conversely, assume that at a given point $x_0 \in]0,+\infty[$*, (35) holds with strict inequalities. Then there exists* $n_0 \in \mathbb{N}$ *such that, for every* $n \geq n_0$*,*

$$f(x_0) < B_n^*(f)(x_0) < B_{n,h_n}(f)(x_0)\,.$$

Remark 2. We recall that the solutions to the differential equation

$$\frac{1}{2}f''(x) + f'(x) = 0$$

belong to the linear space $\langle e_0, f_2\rangle$ generated by e_0 and f_2. The functions f satisfying (35) strictly for all $x \in [0,+\infty[$ might be viewed as super-solutions of it.

Another consequence of the existence of an asymptotic formula is a localization form of the positivity.

Proposition 8. *Consider* $h \in C_*([0,+\infty[)$ *and* $x \geq 0$*. If* $h \geq 0$ *on a neighborhood of* N_x *of* x*, then*

$$B_n^*(h)(x) \geq 0 + o\left(\frac{h_n}{n}\right).$$

Proof. Fix $x_1, x_2 \in N_x$ with $x_1 < x < x_2$ and consider the function $\widetilde{h} \in C_*([0,+\infty[)$ defined as

$$\widetilde{h}(t) = \begin{cases} h(x_1) & 0 \leq t < x_1 \\ h(t) & x_1 \leq t \leq x_2 \\ h(x_2) & x_2 < t. \end{cases}$$

It is clear that $\widetilde{h} \in C_*([0,+\infty[)$ and it is positive on $[0,+\infty[$. Besides, $(\widetilde{h}-h)''$ exists on $]x_1,x_2[$ and $(\widetilde{h}-h)' = (\widetilde{h}-h)'' = 0$ on $]x_1,x_2[$. Finally, the asymptotic formula (30) yields

$$\lim_{n\to\infty} \frac{n}{h_n} B_n^*(\widetilde{h}-h)(x) = 0$$

and, from the linearity and the positivity of the operators B_n^*, we get

$$0 \leq B_n^*(\widetilde{h})(x) = B_n^*(h)(x) + o\left(\frac{h_n}{n}\right),$$

which is our claim. □

Other topics related to the asymptotic formula are saturation-type results, that we obtain as a special case of the ones in [14]. The next proposition is known as a pointwise saturation result and it is in some sense an inverse of the asymptotic formula (see [14, Lemma 3 and Theorem 1]).

Proposition 9. *Let* $]a,b[$ *be a bounded sub-interval of* $[0,+\infty[$ *and consider* $f \in C_*([0,+\infty[)$*. Let* ψ *be a finitely-valued Lebesgue-integrable function on* $]a,b[$ *such that, for each* $x \in]a,b[$*,*

$$\liminf_{n\to\infty} \frac{n}{h_n}\left[B_n^*(f)(x) - f(x)\right] \leq \psi(x) \leq \limsup_{n\to\infty} \frac{n}{h_n}\left[B_n^*(f)(x) - f(x)\right].$$

Then f is twice differentiable a.e. on $]a,b[$ and, a.e. on $]a,b[$,

$$\psi = e_1\left(f' + \frac{1}{2}f''\right).$$

In the next results, the trivial class and the saturation class for the operators B_n^* are characterized (see [14, Proposition 1 and Proposition 2]).

Proposition 10. *Let* $]a, b[$ *be a bounded sub-interval of* $[0, +\infty[$ *and consider* $f \in C_*([0, +\infty[)$. *Then, for each* $x \in]a, b[$,

$$\frac{n}{h_n}\left(B_n^*(f)(x) - f(x)\right) = o(1)$$

if and only if $f \in \langle e_0, f_2 \rangle$ *in* $]a, b[$.

Proposition 11. *Let* $]a, b[$ *be a bounded sub-interval of* $[0, +\infty[$ *and consider* $f \in C_*([0, +\infty[)$. *Moreover, fix* $M \geq 0$. *Then, for each* $x \in]a, b[$,

$$\frac{n}{h_n}\left|B_n^*(f)(x) - f(x)\right| \leq M + o(1)$$

if and only if

$$\left|e_1\left(f' + \frac{f''}{2}\right)\right| \leq M \, a.e. \text{ in }]a, b[.$$

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