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ON THE DIRICHLET PROBLEM FOR THE MOISIL-THEODORESCU SYSTEM

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Dedicated to our dear friend Prof. Dr. George Jaiani on his 80th birthday

Abstract. In the present paper we consider the Dirichlet problem for the Moisil-Theodorescu system. Using the theory of self-conjugate differential forms, we prove an existence theorem with data in L^1 and refine a related version of the Brothers Riesz theorem. Completeness theorems on the boundary are also proved for this BVP. These theorems are for systems obtained by means of harmonic polynomials.

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1 Introduction

In the last years, there has been a renewed interest for boundary value problems for the Moisil-Theodorescu system (see, e.g., [1, 2, 3, 4, 21, 23, 24, 26, 27, 28, 29]):

$$\nabla u = \operatorname{curl} v, \quad \operatorname{div} v = 0. \tag{1}$$

It is well known that this system provides an example of multidimensional generalized Cauchy–Riemann system.

In [6, 7] a new approach hinging on the concept of self-conjugate differential form was proposed to study such systems. A non homogeneous differential form U is said to be self-conjugate if $dU = \delta U$, where d e δ are the differential and the codifferential, respectively. Such forms provide a generalization in $\mathbb{R}^n$ of holomorphic functions. For instance, solutions of Moisil-Theodorescu system in $\mathbb{R}^3$ and of Fueter or Cimmino system in $\mathbb{R}^4$, harmonic vectors (i.e. vectors u such that $\operatorname{div} u = 0, \operatorname{curl} u = 0$), harmonic forms (i.e. homogeneous differential forms u such that $du = 0$, $\delta u = 0$), conjugate forms (i.e. differential forms u, v such that $du = \delta v$, $\delta u = 0$, $dv = 0$) can be seen as self-conjugate differential forms.

In particular, the non homogeneous form $U = u_0 + u_2$ is self conjugate in a domain $\Omega \subset \mathbb{R}^3$ if and only if Moisil-Theodorescu system (1) is satisfied in Ω, where the scalar function u is u_0 and the components of the vector v are the coefficients of the form u_2.

A comprehensive treatment of self-conjugate forms is given in [7].

The aim of the present paper is twofold.

Firstly, we present an existence theorem for the Dirichlet problem for the Moisil-Theodorescu system. We also observe that a Brothers Riesz theorem holds for this system. These results are essentially contained in [6] and [7]. However, we believe it is worthwhile to write them down explicitly, also because - thanks to the Lemma 3 we give here - we are able to simplify the general result in the particular case of Moisil-Theodorescu system. We remark that the Brothers Riesz theorem for the Moisil-Theodorescu system states that, if the boundary values of a solution of this system are measures, then these measures have to be absolutely continuous. We prove also that, if λ_0, λ_1, λ_2, λ_3 are Borel measures defined on the boundary Σ of a domain such that

$$\int_\Sigma (\operatorname{div} \omega \, d\lambda_0 + (\operatorname{curl} \omega)_j \, d\lambda_j) = 0$$

for any harmonic vector polynomial $\omega = (\omega_1, \omega_2, \omega_3)$, then λ_0, λ_1, λ_2, λ_3 are absolutely continuous with respect to the two-dimensional Lebesgue measure on Σ.

Secondly, we prove the completeness property on the boundary for polynomial solutions of the Moisil-Theodorescu system. This result falls within the field of boundary completeness problems. For an introduction to this topic and a review of several known results, see Section 2 of [9]. The latter contains a fairly complete list of references, to which we add the more recent references [10, 11, 12, 13]. It should be noted that in all previous results, the completeness theorems are proved for BVPs that are index problems. This implies that the completeness property holds in spaces with finite codimension. In the present paper the completeness property on the boundary is considered for a problem with infinite compatibility conditions, which leads to results that hold in spaces of infinite codimension.

The paper is organized as follows

After some preliminaries given in Section 2, we prove existence and uniqueness theorem for the Dirichlet problem in Section 3, by rewriting and completing some general results obtained in [7].

Section 4 is devoted to completeness theorems. In particular it is proved that, if Ω is a bounded domain such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected, then the system $\{(\delta\omega^i_j, - * d\omega^i_j, -d\omega^i_j)\}$ is complete in the L^p norm in a certain subspace of $L^p_0(\Sigma) \times L^p_1(\Sigma) \times L^p_2(\Sigma)$ $(1 \leq p < \infty)$. Here $\{\omega^i_j\}$ is a sequence of harmonic polynomial 1-forms such that any harmonic polynomial 1-form can be written as a finite linear combination of forms in $\{\omega^i_j\}$.

In Section 5, the Brothers Riesz theorem proved in [6] for general self-conjugate differential forms has been refined in the particular case of forms corresponding to the Moisil-Theodorescu system. As a consequence, the completeness in the uniform norm has been obtained.

In the last section all the results obtained in the present paper have been reformulated without using differential forms.

2 Preliminaries

Throughout the paper, unless stated otherwise, we will assume that Ω is a bounded domain in $\mathbb{R}^3$ with a C^1 boundary Σ.

We recall here some definitions and properties of differential forms and self-conjugate differential forms. For the reader's convenience, we are going to consider them only in $\mathbb{R}^3$, but the theory can be developed in $\mathbb{R}^n$ (see [7]).

Let $0 \le k \le 3$; a differential form of degree k, briefly a k-form, defined in a domain $\Omega \subset \mathbb{R}^3$ is represented as

$$u_k = \frac{1}{k!}\, u_{j_1 \cdots j_k} dx^{j_1} \cdots dx^{j_k}\,, \tag{2}$$

where $u_{j_1\cdots j_k}$ are the components of a skew-symmetric covariant tensor defined in Ω. We adopt the standard summation convention on repeated indices. We shall use the generalized Kronecker delta

$$\delta^{j_1\cdots j_p}_{i_1\cdots i_p} = \begin{cases} +1 & \text{if } j_1,\dots,j_p \text{ is an even permutation of } i_1,\dots,i_p \\ -1 & \text{if } j_1,\dots,j_p \text{ is an odd permutation of } i_1,\dots,i_p \\ 0 & \text{otherwise.} \end{cases}$$

and the Levi-Civita symbol $\varepsilon_{j_1 j_2 j_3} = \delta^{1\,2\,3}_{j_1 j_2 j_3}$.

Let $x = \xi(t)$ be a local parametric equation of Σ and let (2) be a k-form defined and continuous in an open set $A \supset \Sigma$. We write $u_k\big|_\Sigma$ for the k-form on Σ which locally is given by

$$\xi_t^* u = \frac{1}{k!} u_{j_1\cdots j_k}[\xi(t)] \frac{\partial \xi_{j_1}}{\partial t_{s_1}} \cdots \frac{\partial \xi_{j_k}}{\partial t_{s_k}} dt^{s_1} \cdots dt^{s_k}\,. \tag{3}$$

If u is a 0 -form, we put $\xi_t^* u = u[\xi(t)]$. We call $u_k\big|_\Sigma$ the restriction of u_k on Σ.

More generally, if u_k is defined in A and its coefficients $u_{j_1\cdots j_k}$ admit a trace on Σ in some generalized sense (trace of Sobolev space, non tangential trace, etc.) by writing $u_k\big|_\Sigma$ we mean the form (3) in which $u_{j_1\cdots j_k}[\xi(t)]$ is the trace of $u_{j_1\cdots j_k}$ in this generalized sense.

We remark that

$$dx^{s_1} dx^{s_2}\big|_\Sigma = \varepsilon_{p\, s_1 s_2} \nu_p \, d\sigma\,, \tag{4}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ is the outward unit normal on Σ and $d\sigma$ is the standard surface element.

By $*u_k$ we denote the dual Hodge form, i.e. the $(3-k)$-form

$$*u_k = \frac{1}{k!\,(3-k)!}\, \varepsilon_{s_1\cdots s_k i_1 \cdots i_{3-k}}\, u_{s_1\cdots s_k} dx^{i_1} \cdots dx^{i_{3-k}}\,.$$

A direct computation shows that

$$**\, u_k = u_k. \tag{5}$$

As usual, by du_k and δu_k we denote the differential and the co-differential of u_k, respectively:

$$du_k = \frac{1}{k!}\frac{\partial}{\partial x_h} u_{j_1\cdots j_k} dx^h dx^{j_1} \cdots dx^{j_k} \,,$$

$$\delta u_k = (-1)^k * d * u_k \,. \tag{6}$$

We note that (5) and (6) lead to

$$*\delta u_k = (-1)^k d * u_k. \tag{7}$$

It is well known that $d^2 u_k = 0$ and $\delta^2 u_k = 0$ for any sufficiently smooth k-form. If u_k and v_h are a k-form and a h-form, respectively, we have

$$u_k \wedge v_h = (-1)^{kh} v_h \wedge u_k \,, \tag{8}$$

$$d(u_k \wedge v_h) = du_k \wedge v_h + (-1)^k u_k \wedge dv_h \,. \tag{9}$$

We have also

$$u_k \wedge *v_k = v_k \wedge *u_k \,. \tag{10}$$

The Laplacian of the k-form u_k is defined as

$$\Delta u_k = -(d\delta + \delta d) u_k$$

and we say that u_k is harmonic in Ω if $\Delta u_k = 0$ in Ω. We note that

$$\Delta u_k = \frac{1}{k!} \, \Delta u_{s_1\cdots s_k} dx^{s_1} \cdots dx^{s_k} \tag{11}$$

and then the form u_k is harmonic if and only if all the coefficients $u_{s_1\cdots s_k}$ are harmonic functions.

A non homogeneous differential form $U = u_0 + u_1 + u_2 + u_3$ is said to be self-conjugate if

$$dU = \delta U \,. \tag{12}$$

In particular, if $U = u_0 + u_2$, where $u_2 = v_1 dx^2 dx^3 + v_2 dx^3 dx^1 + v_3 dx^1 dx^2$, equation (12) is satisfied if, and only if,

$$du_0 = \delta u_2, \quad du_2 = 0.$$

We can write these conditions as

$$\nabla u_0 = \mathrm{curl}(v_1, v_2, v_3), \quad \mathrm{div}(v_1, v_2, v_3) = 0.$$

This means that the non homogeneous form $U = u_0 + u_2$ is self-conjugate if and only if the vector (u_0, v_1, v_2, v_3) satisfies the Moisil-Theodorescu system.

Let us consider now the double form introduced by Hodge

$$s_k(x,y) = \sum_{s_1 < \cdots < s_k} s_0(x,y) \, dx^{s_1} \cdots dx^{s_k} dy^{s_1} \cdots dy^{s_k} \,,$$

where $s_0(x,y)$ is the fundamental solution of the Laplace equation

$$s_0(x,y) = -\frac{1}{\omega_3}\,|x-y|^{-1},$$

ω_3 being the hypersurface measure of the unit sphere of $\mathbb{R}^3$. The Hodge double form satisfies the following identities (for the proof, see COLAUTTI [15, p.309])

$$\begin{aligned}
\underset{x}{*}s_k(x,y) &= \underset{y}{*}s_{3-k}(x,y),\\
\delta_x s_{k+1}(x,y) &= d_y s_k(x,y),\\
\underset{x}{*}d_x s_k(x,y) &= (-1)^{k+1}\underset{y}{*}\,d_y s_{3-k-1}(x,y),\\
\underset{y}{*}d_y \underset{x}{*}\,d_x s_k(x,y) &= -d_y d_x s_{3-k-2}(x,y).
\end{aligned}$$

We will use the following notations:

$L^p(\Omega)$ ($L^p(\Sigma)$) is the vector space of all measurable real valued functions u such that $|u|^p$ is integrable over Ω (over Σ).

$L^p_k(\Omega)$ is the vector space of k-forms defined in Ω such that their coefficients belong to $L^p(\Omega)$.

$L^p_k(\Sigma)$ is the vector space of k-forms defined on Σ such that their coefficients are L^p functions in a coordinate system of class C^1 (and then in every coordinate system of class C^1).

If $m \in \mathbb{N}$, by $C^m_k(\Omega)$ ($C^m_k(\Sigma)$) we denote the space of k-forms such that their coefficients belong to $C^m(\Omega)$ ($C^m(\Sigma)$ in a coordinate system of class C^{m+1} and then in every coordinate system of class C^{m+1}).

$\mathring{C}^\infty(\Omega)$ is the space of C^∞ functions with compact support contained in Ω and $\mathring{C}^\infty_k(\Omega)$ is the space of k-forms whose coefficients belong to $\mathring{C}^\infty(\Omega)$.

By means of the double Hodge form we have that if u_k is a k-form in $C^1_k(\overline{\Omega})\cap C^2_k(\Omega)$ with $\Delta u_k \in L^1_k(\Omega)$, the following "Stokes formula" holds (see [15, p.308])

$$\int_\Omega s_k(x,y)\wedge *\Delta u_k(y) + \int_{+\Sigma}[u_k(y)\wedge \underset{y}{*}\,d_y s_k(x,y) - \delta_y s_k(x,y)\wedge *u_k(y)$$
$$+\delta u_k(y)\wedge \underset{y}{*}s_k(x,y) - s_k(x,y)\wedge *du_k(y)] = \begin{cases} u_k(x) & \text{if } x\in\Omega,\\ 0 & \text{if } x\notin\overline{\Omega}.\end{cases}$$

As a consequence of this representation formula, we have that, if $\psi_k \in \mathring{C}^\infty_k(A)$, A being a domain in $\mathbb{R}^3$, then

$$\psi_k(x) = \int_A s_k(x,y)\wedge *\Delta\psi_k(y), \quad \forall x\in\mathbb{R}^3\,.$$

In the present paper we are going to consider also k-measures on Σ. Roughly speaking, a k-measure is a k-form whose coefficients are Borel measures. We denote by $M_k(\Sigma)$ the space of k-measures defined on Σ. It can be shown that $M_k(\Sigma)$ can be identified with the dual of $C^0_k(\Sigma)$. We refer to [17] for a complete treatment of this subject.

3 Existence and uniqueness theorem for the Dirichlet problem

In [7] a very general existence theorem for the Dirichlet problem for self-conjugate differential forms is given. Here we describe this result in the particular case of forms corresponding to the Moisil-Theodorescu system in $\mathbb{R}^3$ and refer to [7] for the general theory.

Let $\mathcal{U}$ be the space defined as

$$\Big\{u_0+u_2 \in L^1_0(\Omega)\oplus L^1_2(\Omega) : \exists\, \Phi = \varphi_0+\varphi_2,\ \widetilde{\Phi} = \widetilde{\varphi}_2 \in L^1_0(\Sigma)\oplus L^1_1(\Sigma)\oplus L^1_2(\Sigma)$$

such that

$$\begin{aligned}
&\int_\Omega (dv_1\wedge *u_2 - \delta v_1\wedge *u_0 - \delta v_3 \wedge *u_2) \qquad (13)\\
&= \int_{+\Sigma} (v_1\wedge \widetilde{\varphi}_2 + \varphi_0\wedge *v_1 + \varphi_2\wedge *v_3),\\
&\forall\ V = v_1+v_3 \in C^1_1(\mathbb{R}^3)\oplus C^1_3(\mathbb{R}^3)\Big\}.
\end{aligned}$$

Lemma 1. *If $U \in \mathcal{U}$ then $U \in C^\infty(\Omega)$ and $dU = \delta U$ in Ω.*

Proof. Note that we have in particular

$$\int_\Omega (dv_1\wedge *u_2 - \delta v_1\wedge *u_0 - \delta v_3\wedge *u_2) = 0 \qquad (14)$$

for any $V \in \mathring{C}^\infty_1(\Omega)\oplus \mathring{C}^\infty_3(\Omega)$. Taking $v_1 = -\delta v_2, v_3 = dv_2$, where v_2 belongs to $\mathring{C}^\infty_2(\Omega)$, we get

$$\int_\Omega \Delta v_2\wedge *u_2 = 0$$

for any $v_2 \in \mathring{C}^\infty_2(\Omega)$. From the classic Caccioppoli–Weyl Lemma we deduce that $u_2 \in C^\infty_2(\Omega)$ and $\Delta u_2 = 0$ in Ω. Analogously, taking in (14) $v_1 = dv_0, v_3 = 0$, v_0 being in $\mathring{C}^\infty_0(\Omega)$, we find

$$\int_\Omega \Delta v_0\wedge *u_0 = 0$$

for any $v_0 \in \mathring{C}^\infty_0(\Omega)$ and then $u_0 \in C^\infty_0(\Omega)$ and $\Delta u_0 = 0$ in Ω.

In view of (7) and (10) we get

$$\begin{aligned}
\delta v_1\wedge *u_0 &= u_0\wedge *\delta v_1 = -u_0\wedge d*v_1,\\
\delta v_3\wedge *u_2 &= u_2\wedge *\delta v_3 = -u_2\wedge d*v_3
\end{aligned}$$

and we can write (14) as

$$\int_\Omega (dv_1\wedge *u_2 + u_0\wedge d*v_1 + u_2\wedge d*v_3) = 0$$

for any $V \in \mathring{C}_1^\infty(\Omega) \oplus \mathring{C}_3^\infty(\Omega)$.

Since $u_0 + u_2$ is C^∞ in Ω, we can apply Stoke's theorem to the integral in (14) and find

$$\int_\Omega (v_1 \wedge d * u_2 - du_0 \wedge *v_1 - du_2 \wedge *v_3) = 0,$$

i.e.

$$\int_\Omega (v_1 \wedge (d * u_2 - *du_0) - du_2 \wedge *v_3) = 0\,.$$

Because of the arbitrariness of V in $\mathring{C}_1^\infty(\Omega) \oplus \mathring{C}_3^\infty(\Omega)$, we get

$$d * u_2 - *du_0 = 0, \quad du_2 = 0 \quad \text{in } \Omega.$$

This shows that U is self–conjugate. □

Let us introduce now the space $\mathcal{V}$ defined as

$$\Big\{ u_0 + u_2 \in L_0^1(\Omega) \oplus L_2^1(\Omega) : \exists\, \Phi = \varphi_0 + \varphi_2,\ \widetilde{\Phi} = \widetilde{\varphi}_2 \in L_0^1(\Sigma) \oplus L_1^1(\Sigma) \oplus L_2^1(\Sigma)$$

such that

$$\int_{+\Sigma} (\varphi_0(y) \wedge \underset{y}{*} d_y s_0(x,y) + d_y s_0(x,y) \wedge \widetilde{\varphi}_2(y)) = \begin{cases} u_0(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \overline{\Omega}; \end{cases},$$

$$\int_{+\Sigma} (\varphi_2(y) \wedge \underset{y}{*} d_y s_2(x,y) - \delta_y s_2(x,y) \wedge \widetilde{\varphi}_2(y) - \varphi_0(y) \wedge \underset{y}{*} \delta_y s_2(x,y)) \qquad (15)$$
$$= \begin{cases} u_2(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \overline{\Omega}. \end{cases} \Big\}$$

Roughly speaking, we can say that $\mathcal{U}$ is the space self-conjugate differential forms of the type $u_0 + u_2$ which have traces on Σ in some weak sense, given by formula (13), while in the space $\mathcal{V}$ there are self-conjugate differential forms in $L^1(\Omega)$ for which the "Cauchy formula" (15) holds. Actually these two spaces coincide.

Theorem 1. $\mathcal{U} = \mathcal{V}$.

Proof. For the proof we refer to [7, Th. III], where the result is proved for general self-conjugate differential forms in any number of variables. We remark that, looking at the proof of this theorem, we see that Φ and $\widetilde{\Phi}$ in (13) and (15) coincide. □

Thanks to Theorems 3.2 and 3.3 in [14], we have that, if $U = u_0 + u_2 \in \mathcal{U} = \mathcal{V}$, then there exist the interior non–tangential traces of u_0, u_2 and $*u_2$

a.e. on Σ, and they are given by φ_0, φ_2 and $\widetilde{\varphi}_2$, respectively. This shows that U can be considered solution of the Dirichlet problem

$$\begin{cases} U \in \mathcal{U}, \\ dU - \delta U = 0 & \text{in } \Omega, \\ U = \Phi, \; *U = \widetilde{\Phi} & \text{on } \Sigma. \end{cases} \tag{16}$$

Let us denote by $\omega_h^{i_1\cdots i_k}$ the k-form $\omega_h dx^{i_1}\cdots dx^{i_k}$, where $\{\omega_h\}$ is a complete system of homogeneous harmonic polynomials. Such a system can be obtained by ordering in one sequence the polynomials:

$$|x|^k Y_s^k\left(\frac{x}{|x|}\right), \; s = 1, \ldots, 2k+1, \; k = 0, 1, 2, \ldots;$$

where $Y_1^k, \ldots, Y_{2k+1}^k$ is a complete system of (surface) spherical harmonics of degree k.

Theorem 2. *Let Ω be a bounded domain such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. Let $\Phi = \varphi_0 + \varphi_2$, $\widetilde{\Phi} = \widetilde{\varphi}_2 \in L_0^1(\Sigma) \oplus L_1^1(\Sigma) \oplus L_2^1(\Sigma)$ be given forms on Σ. There exists a non homogeneous form $U = u_0 + u_2$ solution of the boundary value problem* (16) *if and only if the following infinite compatibility conditions are satisfied:*

$$\begin{gathered} \int_{+\Sigma} (\varphi_0 \wedge *d\omega_h + d\omega_h \wedge \widetilde{\varphi}_2) = 0, \\ \int_{+\Sigma} (\varphi_2 \wedge *d\omega_h^{i_1 i_2} - \delta\omega_h^{i_1 i_2} \wedge \widetilde{\varphi}_2 - \varphi_0 \wedge *\delta\omega_h^{i_1 i_2}) = 0, \end{gathered} \tag{17}$$

for any $h = 1, 2, \ldots$ and for any choice of $1 \le i_1 < i_2 \le 3$. The solution U is unique.

Proof. Uniqueness. Suppose there are two self-conjugate differential forms U_1, U_2 in $\mathcal{U}$ such that their traces on Σ are equal to φ and the traces of $*U_1$ and $*U_2$ are equal to $\widetilde{\varphi}$. Let $U = U_1 - U_2$. Conditions (13) lead to

$$\int_\Omega (dv_1 \wedge *u_2 - \delta v_1 \wedge *u_0 - \delta v_3 \wedge *u_2) = 0, \quad \forall\, V = v_1 + v_3 \in C_1^1(\mathbb{R}^3) \oplus C_3^1(\mathbb{R}^3).$$

Reasoning as before, we find

$$\int_\Omega (\Delta v_2 \wedge *u_2 + \Delta v_0 \wedge *u_0) = 0, \forall\, v_0 + v_2 \in C_0^\infty(\mathbb{R}^3) \oplus C_2^\infty(\mathbb{R}^3).$$

It is easy to deduce that this implies $U = u_0 + u_2 = 0$.

Existence. If a solution does exist, (17) follows from (13). Indeed, the first relations are obtained taking $v_1 = d\omega_h$, $v_3 = 0$ and observing that

$\delta v_1 = \delta d\omega_h = -\Delta\omega_h = 0$. For the other ones, take $v_1 = -\delta\omega_h^{i_1 i_2}$, $v_3 = d\omega_h^{i_1 i_2}$ in (13). We get

$$\int_{+\Sigma} (\varphi_2 \wedge *d\omega_h^{i_1 i_2} - \delta\omega_h^{i_1 i_2} \wedge \widetilde{\varphi}_2 - \varphi_0 \wedge *\delta\omega_h^{i_1 i_2})$$
$$= \int_\Omega (-d\delta\omega_h^{i_1 i_2} - \delta d\omega_h^{i_1 i_2}) \wedge *u_2 = \int_\Omega \Delta\omega_h^{i_1 i_2} \wedge *u_2 = 0.$$

Conversely, let us suppose that (17) are satisfied. It is known that there exists $\delta > 0$ such that if $0 < r < R$, with $r \leq \delta R$, the development

$$s_0(x-y) = \sum_{k=0}^{\infty} \sum_{h=1}^{2k+1} \frac{Q_{hk}(x) P_{hk}(y)}{|x|^{2k+1}} \tag{18}$$

holds uniformly for $|y| \leq r$ and $|x| \geq R$, where P_{hk} and Q_{hk} are homogeneous harmonic polynomials of degree k (see [5, Lemma 3.2]). Therefore, taking $r = \max_{x\in\overline{\Omega}} |x|$, we can say that we can find R such that (18) holds uniformly for $y \in \Sigma$ and $|x| \geq R$. Thanks to (17) we deduce that the potentials

$$u_0(x) = \int_{+\Sigma} (\varphi_0(y) \wedge \underset{y}{*} d_y s_0(x,y) + d_y s_0(x,y) \wedge \widetilde{\varphi}_2(y)),$$
$$u_2(x) = \int_{+\Sigma} (\varphi_2(y) \wedge \underset{y}{*} d_y s_2(x,y) - \delta_y s_2(x,y) \wedge \widetilde{\varphi}_2(y) - \varphi_0(y) \wedge \underset{y}{*} \delta_y s_2(x,y))$$

vanish for $|x| \geq R$. Since they are analytic in $\mathbb{R}^3 \setminus \Sigma$, we have

$$u_0(x) = 0, \quad u_2(x) = 0, \quad \forall\, x \notin \overline{\Omega}.$$

This shows that $U = u_0 + u_2$ is in $\mathcal{V}$ and then, in view of Theorem 1, $U \in \mathcal{U}$ and the boundary value problem (16) is satisfied. □

Actually the two conditions in (17) are not independent. This is a consequence of the following Lemma. Saying that ω is a polynomial differential form, we mean that all its coefficients are polynomials.

Lemma 2. *Let ω be a scalar harmonic polynomial. There exists a polynomial 2-form w_2 such that*

$$\delta w_2 = d\omega, \quad dw_2 = 0. \tag{19}$$

Proof. We recall that, by the classical homotopy formula, given a closed C^1 2-form $\alpha = \alpha_1 dx^2 dx^3 + \alpha_2 dx^3 dx^1 + \alpha_3 dx^1 dx^2$, the 1-form

$$I(\alpha) = \left(\int_0^1 \alpha_1(tx)\, t\, dt\right)(x_2 dx^3 - x_3 dx^2) \tag{20}$$
$$+ \left(\int_0^1 \alpha_2(tx)\, t\, dt\right)(x_3 dx^1 - x_1 dx^3) + \left(\int_0^1 \alpha_3(tx)\, t\, dt\right)(x_1 dx^2 - x_2 dx^1)$$

is such that $dI(\alpha)=\alpha$ (see, e.g., [20, pp.29–30]).

Let now ω be a scalar harmonic polynomial. Since $d*d\omega=0$, the 2-form $w_2^0=*I(*d\omega)$ is such that $d*w_2^0=dI(*d\omega)=*d\omega$, i.e. $\delta w_2^0=d\omega$. We remark that, in view of the expression (20), the coefficients of w_2^0 are polynomials.

Let now θ be a 2-form with polynomial coefficients such that $\Delta\theta=w_2^0$. Thanks to (11), the existence of θ follows immediately from the analogous result for scalar functions (see [30, Th.1]). We have

$$-d\delta d\theta=-d(\delta d+d\delta)\theta=d\Delta\theta=dw_2^0.$$

This implies $d(w_2^0+\delta d\theta)=0$ and, since $\delta(w_2^0+\delta d\theta)=\delta w_2^0=d\omega$, the form $w_2=w_2^0+\delta d\theta$ satisfies (19) and its coefficients are polynomials. □

Lemma 3. *If* $\Phi=\varphi_0+\varphi_2$, $\widetilde{\Phi}=\widetilde{\varphi}_2\in L_0^1(\Sigma)\oplus L_1^1(\Sigma)\oplus L_2^1(\Sigma)$ *are such that*

$$\int_{+\Sigma}(\varphi_2\wedge *d\omega_h^{i_1i_2}-\delta\omega_h^{i_1i_2}\wedge\widetilde{\varphi}_2-\varphi_0\wedge *\delta\omega_h^{i_1i_2})=0 \tag{21}$$

for any $h=1,2,\dots$ *and for any choice of* $1\le i_1<i_2\le 3$*, then we have also*

$$\int_{+\Sigma}(\varphi_0\wedge *d\omega_h+d\omega_h\wedge\widetilde{\varphi}_2)=0 \tag{22}$$

for any $h=1,2,\dots$.

Proof. It is clear that condition (21) implies that

$$\int_{+\Sigma}(\varphi_2\wedge *dw_2-\delta w_2\wedge\widetilde{\varphi}_2-\varphi_0\wedge *\delta w_2)=0 \tag{23}$$

for any harmonic polynomial 2-form w_2. Now, let us consider the harmonic polynomial ω_h. By Lemma 2, there exists a polynomial 2-form w_2 such that $\delta w_2=d\omega_h$, $dw_2=0$. Note that this implies that $\Delta w_2=0$, and then (23) holds. This means that (22) is true, proving the Lemma. □

Theorem 3. *Let* Ω *be a bounded domain such that* $\mathbb{R}^3\setminus\overline{\Omega}$ *is connected. Let* $\Phi=\varphi_0+\varphi_2$, $\widetilde{\Phi}=\widetilde{\varphi}_2\in L_0^1(\Sigma)\oplus L_1^1(\Sigma)\oplus L_2^1(\Sigma)$ *be given forms on* Σ*. There exists a non homogeneous form* $U=u_0+u_2$ *solution of the boundary value problem* (16) *if and only if the following infinite compatibility conditions are satisfied:*

$$\int_{+\Sigma}(\varphi_2\wedge *d\omega_h^{i_1i_2}+\widetilde{\varphi}_2\wedge\delta\omega_h^{i_1i_2}-\varphi_0\wedge *\delta\omega_h^{i_1i_2})=0, \tag{24}$$

for any $h=1,2,\dots$ *and for any choice of* $1\le i_1<i_2\le 3$*. The solution* U *is unique.*

Proof. This follows immediately from Theorem 2 and Lemma 3. □

4 Completeness theorems

Theorem 4. *Let $1 \leq p < \infty$. Let Ω be a bounded domain such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. The system*

$$\{(\delta\omega_j^i, -*d\omega_j^i, -d\omega_j^i),\ j = 1, 2, \ldots, i = 1, 2, 3\} \tag{25}$$

is complete in the subspace of $L_0^p(\Sigma) \times L_1^p(\Sigma) \times L_2^p(\Sigma)$ defined as

$$\Lambda^p = \Big\{(\varphi_0, \varphi_1, \varphi_2) \in L_0^p(\Sigma) \times L_1^p(\Sigma) \times L_2^p(\Sigma) \text{ such that}$$

$$\int_{+\Sigma} (\varphi_0 \wedge *\delta\omega_h^{i_1 i_2} - \varphi_1 \wedge \delta\omega_h^{i_1 i_2} - \varphi_2 \wedge *d\omega_h^{i_1 i_2}) = 0, \tag{26}$$

$$\text{for any } h = 1, 2, \ldots \text{ and for any choice of } 1 \leq i_1 < i_2 \leq 3 \Big\}.$$

Proof. We first observe that system (25) is contained in Λ^p. Indeed, if we set $u_0 = \delta\omega_j^i$, $u_2 = -d\omega_j^i$, we have $du_0 = d\delta\omega_j^i = -\delta d\omega_j^i = \delta u_2$, $du_2 = -d^2\omega_j^i = 0$. This shows that $\delta\omega_j^i$, $-*d\omega_j^i$ and $-d\omega_j^i$ are the traces on Σ of u_0, $*u_2$ and u_2, respectively, with $u_0 + u_2$ self–conjugate. Thus conditions (24), and then (26), are satisfied.

Let $1 < p < \infty$. In order to prove the completeness of system (25) in Λ^p, we have to show that, if (ψ_2, ψ_1, ψ_0) in $(L_0^p(\Sigma) \times L_1^p(\Sigma) \times L_2^p(\Sigma))^* = L_2^q(\Sigma) \times L_1^q(\Sigma) \times L_0^q(\Sigma)$ is such that

$$\int_{+\Sigma} (\psi_2 \wedge \delta\omega_j^i - \psi_1 \wedge *d\omega_j^i - \psi_0 \wedge d\omega_j^i) = 0 \tag{27}$$

for any $j = 1, 2, \ldots$, $i = 1, 2, 3$, then

$$\int_{+\Sigma} (\psi_2 \wedge \varphi_0 + \psi_1 \wedge \varphi_1 + \psi_0 \wedge \varphi_2) = 0 \tag{28}$$

for any $(\varphi_0, \varphi_1, \varphi_2) \in \Lambda^p$. It is clear that (27) is equivalent to say that

$$\int_{+\Sigma} (\psi_2 \wedge \delta\omega_1 - \psi_1 \wedge *d\omega_1 - \psi_0 \wedge d\omega_1) = 0$$

for any polynomial harmonic 1-form ω_1. Setting $\omega_2 = *\omega_1$ (which implies $\omega_1 = *\omega_2$), we have $\delta\omega_1 = -*d*\omega_1 = -*d\omega_2$, $*d\omega_1 = *d*\omega_2 = \delta\omega_2$ and $d\omega_1 = d*\omega_2$. Therefore we may write

$$\int_{+\Sigma} (\psi_2 \wedge *d\omega_2 + \psi_1 \wedge \delta\omega_2 + \psi_0 \wedge d*\omega_2) = 0 \tag{29}$$

for any polynomial harmonic 2-form ω_2. By Theorem 3 there exists a self–conjugate differential form $V \in \mathcal{U}$ such that $V = -\psi_0 + \psi_2$, $*V = \psi_1$ on Σ.

Thanks to Theorem 1, we have also $V = v_0 + v_2$, where

$$\int_{+\Sigma} (-\psi_0(y) \wedge \underset{y}{*} d_y s_0(x,y) + d_y s_0(x,y) \wedge \psi_1(y)) = \begin{cases} v_0(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \overline{\Omega}; \end{cases},$$

$$\int_{+\Sigma} (\psi_2(y) \wedge \underset{y}{*} d_y s_2(x,y) - \delta_y s_2(x,y) \wedge \psi_1(y) + \psi_0(y) \wedge \underset{y}{*} \delta_y s_2(x,y))$$
$$= \begin{cases} v_2(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \overline{\Omega}. \end{cases}$$

Analogously, (26) leads to

$$\int_{+\Sigma} (\varphi_0(y) \wedge \underset{y}{*} d_y s_0(x,y) + d_y s_0(x,y) \wedge \varphi_1(y)) = \begin{cases} u_0(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \overline{\Omega}; \end{cases},$$

$$\int_{+\Sigma} (\varphi_2(y) \wedge \underset{y}{*} d_y s_2(x,y) - \delta_y s_2(x,y) \wedge \varphi_1(y) - \varphi_0(y) \wedge \underset{y}{*} \delta_y s_2(x,y)) \quad (30)$$
$$= \begin{cases} u_2(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \overline{\Omega}, \end{cases}$$

$U = u_0 + u_2$ being in $\mathcal{U}$.

Let us consider now a harmonic simple layer potential

$$g(x) = \int_{\Sigma} f(y)\, s_0(x,y)\, d\sigma_y$$

with density f belonging to $L^p(\Sigma)$.

Given $0 < \alpha < 1$, consider the non tangential maximal function

$$(\nabla g)^*(x) = \sup \Big\{ |\nabla g(\xi)| : \xi \in \Omega,\ |\xi - x| < \delta,\ \langle \xi - x, \nu(x) \rangle > \alpha |\xi - x| \Big\}. \quad (31)$$

By a fundamental result by Fabes, Jodeit and Rivière [16, Th.1.10], there exists a constant $\delta = \delta_{\Omega,\alpha}$ such that the function (31) belongs to $L^p(\Sigma)$ and

$$\|(\nabla g)^*\|_{L^p(\Sigma)} \leq C \|f\|_{L^p(\Sigma)} \quad (32)$$

with C independent of f.

Let us introduce a family of surfaces which are "parallel" to $\Sigma = \partial\Omega$ in some sense. Let $\lambda(x)$ be a unit vector defined and continuously differentiable on Σ such that $-\lambda(x) \cdot \nu(x) \geqslant \beta_0 > 0$. Let Σ_ϱ be the surface $x_\varrho = x + \varrho\lambda(x), x \in \Sigma$, where $|\varrho| \leqslant \varrho_0$. We can choose $\varrho_0 > 0$ such that the surface Σ_ϱ is the boundary of a domain contained in Ω (containing Ω) if $0 < \varrho \leq \varrho_0$ (if $-\varrho_0 \leq \varrho < 0$). One can prove that if $\Sigma \in C^1$, then such a vector does exist (see [19, pp.273–275], [22, p.40–41]). It is clear that we can choose the constants β_0 and ϱ_0 in such a way the point x_ϱ belongs to the cone considered in the definition (31). Then we have the inequality

$$|\nabla g(x_\varrho)| \leq |(\nabla g)^*(x)|\,, \quad (33)$$

for any $x \in \Sigma$ and for any $0 < \varrho \leq \varrho_0$, with $(\nabla g)^*(x)$ satisfying the estimate (32).

We claim that

$$\lim_{\varrho \to 0^+} \int_{+\Sigma_\varrho} (v_2 \wedge u_0 + *v_2 \wedge *u_2 - v_0 \wedge u_2) = \int_{+\Sigma} (\psi_2 \wedge \varphi_0 + \psi_1 \wedge \varphi_1 + \psi_0 \wedge \varphi_2). \quad (34)$$

Indeed, from (30) we see that the components of $u_0 + u_2$ can be written as linear combinations of first derivatives of simple layer potentials having L^p densities. Thanks to inequalities (32) and (33), we have that there exists a function $H \in L^p(\Sigma)$ such that

$$|U(x_\varrho)| \leq H(x), \qquad \text{a.e. } x \in \Sigma, 0 < \varrho \leq \varrho_0,$$

where by $|U(x_\varrho)|$ we denote the sum of the modulus of all the coefficients of u_0 and u_2. Analogously, there exists a function $K \in L^q(\Sigma)$ such that

$$|V(x_\varrho)| \leq K(x), \qquad \text{a.e. } x \in \Sigma, 0 < \varrho \leq \varrho_0.$$

Therefore

$$|v_2 \wedge u_0 + *v_2 \wedge *u_2 - v_0 \wedge u_2| \leq H(x)\, K(x)$$

a.e. on Σ_ϱ. Since $H\,K$ is in $L^1(\Sigma)$, and

$$\lim_{\varrho \to 0^+} u_0 = \varphi_0, \ \lim_{\varrho \to 0^+} u_2 = \varphi_2, \ \lim_{\varrho \to 0^+} *u_2 = \varphi_1,$$
$$\lim_{\varrho \to 0^+} v_0 = \psi_0, \ \lim_{\varrho \to 0^+} v_2 = \psi_2, \ \lim_{\varrho \to 0^+} *v_2 = \psi_1$$

a.e. on Σ, Lebesgue's dominated convergence theorem shows that (34) holds.

By the Stokes Theorem (see also (9)), we get

$$\int_{+\Sigma_\varrho} (v_2 \wedge u_0 + *v_2 \wedge *u_2 - v_0 \wedge u_2) = \int_{\Omega_\varrho} (dv_2 \wedge u_0 + v_2 \wedge du_0$$
$$+ d(*v_2) \wedge *u_2 - *v_2 \wedge d(*u_2) - dv_0 \wedge u_2 - v_0 \wedge du_2)\,.$$

Recalling (6), (8) and (10) we find

$$d(*v_2) \wedge *u_2 = u_2 \wedge *d(*v_2) = u_2 \wedge \delta v_2 = \delta v_2 \wedge u_2 = dv_0 \wedge u_2,$$
$$*v_2 \wedge d(*u_2) = *d(*u_2) \wedge v_2 = \delta u_2 \wedge v_2 = v_2 \wedge \delta u_2 = v_2 \wedge du_0,$$

which, together with $du_2 = dv_2 = 0$, lead to

$$\int_{+\Sigma_\varrho} (v_2 \wedge u_0 + *v_2 \wedge *u_2 - v_0 \wedge u_2) = 0.$$

In view of (34), we have that (28) holds for any $(\varphi_0, \varphi_1, \varphi_2) \in \Lambda^p$ and the proof is complete when $1 < p < \infty$. The result for $p = 1$ easily follows from the completeness for $p > 1$. $\square$

5 The Brothers Riesz Theorem and the completeness in the uniform norm

The classical theorem of F. and M. Riesz [25] states that if μ is a complex measure defined on the circle $T = \{z \in \mathbb{C} \mid |z| = 1\}$ which satisfies

$$\int_T e^{in\vartheta} d\mu_\vartheta = 0 \quad n = 1, 2, 3, \dots$$

then μ is absolutely continuous, i.e. there exists a function $f \in L^1(T)$ such that

$$\mu(B) = \int_B f(\vartheta) d\vartheta$$

for any Borel set $B \subset T$. This theorem has inspired many generalizations and for a survey of the more "concrete" ones we refer to [8]. In particular, a Brothers Riesz theorem for conjugate differential forms in any dimension was proved in [6].

In this section we suppose that Σ is a Lyapunov boundary. This is because we are going to use the results proven in [6], in which this assumption was made.

Theorem 5. *Let Ω be a bounded domain such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. If $(\mu_0, \mu_1, \mu_2) \in M_0(\Sigma) \times M_1(\Sigma) \times M_2(\Sigma)$ is such that*

$$\int_{+\Sigma} (\mu_0 \wedge *\delta\omega_h^{i_1 i_2} - \mu_1 \wedge \delta\omega_h^{i_1 i_2} - \mu_2 \wedge *d\omega_h^{i_1 i_2}) = 0, \tag{35}$$

for any $h = 1, 2, \dots$ and for any choice of $1 \le i_1 < i_2 \le 3$, then μ_0, μ_1 and μ_2 are absolutely continuous k-measures.

Proof. This theorem was essentially proved in [6, Th. XI], where it is shown that if $(\mu_0, \mu_1, \mu_2) \in M_0(\Sigma) \times M_1(\Sigma) \times M_2(\Sigma)$ is such that

$$\int_{+\Sigma} (\mu_0 \wedge *d\omega_h + d\omega_h \wedge \mu_1) = 0, \quad h = 1, 2, \dots \tag{36}$$

and (35) holds, then μ_0, μ_1 and μ_2 are absolutely continuous. On the other hand, as in Lemma 3, conditions (35) imply conditions (36) and this concludes the proof. □

Remark 1. Conditions (35) are equivalent to

$$\int_{+\Sigma} (\mu_0 \wedge *\delta\omega - \mu_1 \wedge \delta\omega - \mu_2 \wedge *d\omega) = 0, \tag{37}$$

for any polynomial harmonic 2-form ω.

We are now in a position to prove the completeness of system (25) in the uniform norm.

Theorem 6. *Let Ω be a bounded domain such that $\mathbb{R}^3\setminus\overline{\Omega}$ is connected. The system* (25) *is complete in the subspace Λ^0 of $C^0_0(\Sigma)\times C^0_1(\Sigma)\times C^0_2(\Sigma)$ defined as the class of $(\varphi_0,\varphi_1,\varphi_2)\in C^0_0(\Sigma)\times C^0_1(\Sigma)\times C^0_2(\Sigma)$ such that conditions* (26) *hold for any $h=1,2,\dots$ and for any choice of $1\le i_1<i_2\le 3$.*

Proof. As in Theorem 4, one can see that the system (25) is contained in Λ^0. We have to show that if $(\mu_2,\mu_1,\mu_0)\in M_2(\Sigma)\times M_1(\Sigma)\times M_0(\Sigma)$ is such that

$$\int_{+\Sigma}(\mu_2\wedge\delta\omega^i_j-\mu_1\wedge *d\omega^i_j-\mu_0\wedge d\omega^i_j)=0 \tag{38}$$

for $j=1,2,\dots$ $i=1,2,3$, then

$$\int_{+\Sigma}(\mu_2\wedge\varphi_0+\mu_1\wedge\varphi_1+\mu_0\wedge\varphi_2)=0$$

for any $(\varphi_0,\varphi_1,\varphi_2)\in\Lambda^0$.

Conditions (38) mean that

$$\int_{+\Sigma}(\mu_2\wedge\delta\omega_1-\mu_1\wedge *d\omega_1-\mu_0\wedge d\omega_1)=0$$

for any polynomial harmonic 1-form ω_1. As in (29), setting $\omega_2=*\omega_1$, we may rewrite this conditions as

$$\int_{+\Sigma}(\mu_2\wedge *d\omega_2+\mu_1\wedge\delta\omega_2+\mu_0\wedge *\delta\omega_2)=0 \tag{39}$$

for any polynomial harmonic 2-form ω_2.

Thanks to Theorem 5, the k-measures μ_0, μ_1 and μ_2 are absolutely continuous. This means that $\mu_0=\psi_0$, $\mu_1=\psi_1$, $\mu_2=\psi_2$, where $(\psi_0,\psi_1,\psi_2)\in L^1_0(\Sigma)\times L^1_1(\Sigma)\times L^1_2(\Sigma)$. Conditions (39) show that $(\psi_0,-\psi_1,-\psi_2)$ belongs to Λ^1. Since the system (25) is complete in Λ^1 (see Theorem 4), there exists a sequence of polynomial harmonic 1-forms $\{\omega^1_n\}$ such that $(\delta\omega^1_n,-*d\omega^1_n,-d\omega^1_n)$ tends to $(\psi_0,-\psi_1,-\psi_2)$ in $L^1_0(\Sigma)\times L^1_1(\Sigma)\times L^1_2(\Sigma)$. This implies that

$$\int_{+\Sigma}(\mu_2\wedge\varphi_0+\mu_1\wedge\varphi_1+\mu_0\wedge\varphi_2)=\int_{+\Sigma}(\psi_2\wedge\varphi_0+\psi_1\wedge\varphi_1+\psi_0\wedge\varphi_2)$$
$$=\lim_{n\to\infty}\int_{+\Sigma}(d\omega^1_n\wedge\varphi_0+*d\omega^1_n\wedge\varphi_1+\delta\omega^1_n\wedge\varphi_2)\,.$$

As before, we may write

$$\int_{+\Sigma}(d\omega^1_n\wedge\varphi_0+*d\omega^1_n\wedge\varphi_1+\delta\omega^1_n\wedge\varphi_2)=\int_{+\Sigma}(d*\omega^2_n\wedge\varphi_0+\delta\omega^2_n\wedge\varphi_1-*d\omega^2_n\wedge\varphi_2),$$

where $\omega_n^2 = *\omega_n^1$. Therefore

$$\int_{+\Sigma} (\mu_2 \wedge \varphi_0 + \mu_1 \wedge \varphi_1 + \mu_0 \wedge \varphi_2)$$
$$= \lim_{n\to\infty} \int_{+\Sigma} (d * \omega_n^2 \wedge \varphi_0 + \delta\omega_n^2 \wedge \varphi_1 - *d\omega_n^2 \wedge \varphi_2)$$
$$= \lim_{n\to\infty} \int_{+\Sigma} (\varphi_0 \wedge *\delta\omega_n^2 - \varphi_1 \wedge \delta\omega_n^2 - \varphi_2 \wedge *d\omega_n^2) = 0,$$

because $(\varphi_0, \varphi_1, \varphi_2)$ belongs to Λ^0. This completes the proof. □

6 Reformulation of the main results

The aim of the present section is to reformulate the main results obtained in this paper in terms of functions rather than differential forms. If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are two vectors, $v \cdot w$ stands for their scalar product in $\mathbb{R}^3$: $\sum_{j=1}^3 v_j w_j$. Analogously, if $(\lambda_1, \lambda_2, \lambda_3)$ is a vector whose components are measures, we write $v \cdot d\lambda$ for the measure $\sum_{j=1}^3 v_j d\lambda_j$.

The cross product of v and w will be denoted by $v \times w$.

Let $\widetilde{\mathcal{U}}$ be the space defined as

$$\Big\{(u, v) \in L^1(\Omega) \times [L^1(\Omega)]^3 : \exists\, (f, g) \in L^1(\Sigma) \times [L^1(\Sigma)]^3$$

such that

$$\int_\Omega (u \,\operatorname{div} w + v \cdot \operatorname{curl} w + v \cdot \nabla w_0)\, dx$$
$$= \int_\Sigma (f\,(w \cdot \nu) + (g \cdot \nu)\, w_0 + (g \times \nu) \cdot w)\, d\sigma, \forall (w_0, w) \in C^1(\mathbb{R}^3) \times [C^1(\mathbb{R}^3)]^3 \Big\}.$$

It is not difficult to verify that if we identify (u, v) with the non homogeneous differential form $u_0 + u_2$, with $u_0 = u$, $u_2 = v_1 dx^2 dx^3 + v_2 dx^3 dx^1 + v_3 dx^1 dx^2$, then the space $\widetilde{\mathcal{U}}$ can be identified with the space $\mathcal{U}$.

Theorem 7. *Let Ω be a bounded domain such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. Let f, g_1, g_2, g_3 be L^1 scalar functions given on Σ. There exists a solution u, $v = (v_1, v_2, v_3)$ of the Dirichlet problem for the Moisil-Theodorescu system*

$$\begin{cases} (u, v) \in \widetilde{\mathcal{U}} \\ \nabla u = \operatorname{curl} v, \ \operatorname{div} v = 0, & \text{in } \Omega \\ u = f, \ v = g, & \text{on } \Sigma \end{cases} \tag{40}$$

(where $g = (g_1, g_2, g_3)$) if and only if the following infinite compatibility conditions are satisfied

$$\int_\Sigma ((\operatorname{div} \omega)\, g + g \times \operatorname{curl} \omega - f \operatorname{curl} \omega) \cdot \nu \, d\sigma = 0, \tag{41}$$

for any harmonic vector polynomial $\omega = (\omega_1, \omega_2, \omega_3)$. The solution $(u,\, v)$ is unique.

Proof. This is nothing but Theorem 3. Indeed we already know that $\nabla u = \operatorname{curl} v$, $\operatorname{div} v = 0$ is equivalent to $dU = \delta U$, where $U = u_0 + u_2 = u + (v_1 dx^2 dx^3 + v_2 dx^3 dx^1 + v_3 dx^1 dx^2)$. Let us consider a polynomial harmonic 2-form ω. We can write it as

$$\omega = \omega_1 dx^2 dx^3 + \omega_2 dx^3 dx^1 + \omega_3 dx^1 dx^2$$

where ω_1, ω_2 and ω_3 are harmonic polynomials. Keeping in mind the definition (6), we have

$$\delta\omega = *d*\omega = *d(\omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3) = *(\partial_h \omega_j dx^h dx^j)$$
$$= *((\operatorname{curl}\omega)_1\, dx^2 dx^3 + (\operatorname{curl}\omega)_2\, dx^3 dx^1 + (\operatorname{curl}\omega)_3\, dx^1 dx^2) = (\operatorname{curl}\omega)_j dx^j\,,$$

where $(\operatorname{curl}\omega)_j$ is the j- th component of the curl of the vector $(\omega_1, \omega_2, \omega_3)$, which we still denote by ω. We have also

$$*\delta\omega = (\operatorname{curl}\omega)_1\, dx^2 dx^3 + (\operatorname{curl}\omega)_2\, dx^3 dx^1 + (\operatorname{curl}\omega)_3\, dx^1 dx^2$$

and

$$*d\omega = *d(\omega_1 dx^2 dx^3 + \omega_2 dx^3 dx^1 + \omega_3 dx^1 dx^2) = *(\operatorname{div}\omega) dx^1 dx^2 dx^3 = \operatorname{div}\omega.$$

Let us suppose that (41) hold. Setting $\varphi_0 = f$, $\varphi_2 = g_1 dx^2 dx^3 + g_2 dx^3 dx^1 + g_3 dx^1 dx^2$, $\widetilde{\varphi}_2 = g_j\, dx^j$, and keeping in mind (4), we find on Σ

$$\varphi_2 \wedge *d\omega = (\operatorname{div}\omega)(g_1 dx^2 dx^3 + g_2 dx^3 dx^1 + g_3 dx^1 dx^2) = (\operatorname{div}\omega)\, g \cdot \nu\, d\sigma,$$
$$\widetilde{\varphi}_2 \wedge \delta\omega = g_h (\operatorname{curl}\omega)_j dx^h dx^j = \varepsilon_{p\,h\,j} g_h (\operatorname{curl}\omega)_j \nu_p\, d\sigma = (g \times \operatorname{curl}\omega) \cdot \nu,$$
$$\varphi_0 \wedge *\delta\omega = f\,((\operatorname{curl}\omega)_1\, dx^2 dx^3 + (\operatorname{curl}\omega)_2\, dx^3 dx^1 + (\operatorname{curl}\omega)_3\, dx^1 dx^2)$$
$$= f\,(\operatorname{curl}\omega) \cdot \nu\, d\sigma.$$

In view of (41), conditions (24) hold and then, in view of Theorem 3, there exists a self-conjugate non homogenous differential form $U = u_0 + u_2$ such that $U\big|_\Sigma = \varphi_0 + \varphi_2$, $*U\big|_\Sigma = \widetilde{\varphi}_2$, i.e.

$$u_0|_\Sigma = f \tag{42}$$
$$u_2|_\Sigma = (g_1 dx^2 dx^3 + g_2 dx^3 dx^1 + g_3 dx^1 dx^2)|_\Sigma = g \cdot \nu\, d\sigma \tag{43}$$
$$*u_2|_\Sigma = g_h dx^h|_\Sigma. \tag{44}$$

Let us show that (43)–(44) hold if and only if $v_h|_\Sigma = g_h$ $(h = 1, 2, 3)$. Let us consider a chart on Σ with local coordinates (s, t). Locally, equation (44) mean

$$v_h|_\Sigma(\partial_s x_h ds + \partial_t x_h dt) = g_h(\partial_s x_h ds + \partial_t x_h dt)$$

i.e.

$$v_h|_\Sigma\, \partial_s x_h = g_h\, \partial_s x_h, \quad v_h|_\Sigma\, \partial_t x_h = g_h\, \partial_t x_h,$$

while equation (43) means

$$v_h|_\Sigma \nu_h = g_h \nu_h.$$

Since the matrix

$$\begin{pmatrix} \partial_s x_1 & \partial_s x_2 & \partial_s x_3 \\ \partial_t x_1 & \partial_t x_2 & \partial_t x_3 \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} \tag{45}$$

is not singular, we have that (43)–(44) are equivalent to $v_h|_\Sigma = g_h$ $(h = 1,2,3)$. Therefore the system (42)–(44) holds if and only if the boundary conditions in BVP (40) are satisfied. We have proved that, if the orthogonality conditions (41) are satisfied, then the BVP (40) admits a solution. The viceversa can be proved in a similar way. We omit the details. The uniqueness follows immediately from Theorem 3. □

As far as the completeness results are concerned, let us denote by $\{\omega^{(n)}\}$ a sequence of polynomial harmonic vectors such that any polynomial harmonic vector can be written as a finite linear combination of $\omega^{(n)}$.

Theorem 8. *Let $1 \le p < \infty$. Let Ω be a bounded domain such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. The system*

$$\{(\operatorname{div}\omega^{(n)}, \operatorname{curl}\omega^{(n)})\} \tag{46}$$

is complete in the subspace of $L^p(\Sigma) \times [L^p(\Sigma)]^3$ defined as $\widetilde{\Lambda}^p = \{(f,g) \in L^p(\Sigma) \times [L^p(\Sigma)]^3$ such that conditions (41) hold for any harmonic vector polynomial $\omega = (\omega_1, \omega_2, \omega_3)\}$.

Proof. Let $(f,g) \in \widetilde{\Lambda}^p$. Arguing as in Theorem 7, defining $\varphi_0 = -f$, $\varphi_1 = -g_j\, dx^j$, $\varphi_2 = -(g_1 dx^2 dx^3 + g_2 dx^3 dx^1 + g_3 dx^1 dx^2)$, we see that conditions (26) are satisfied. In view of Theorem 4, we can find a sequence of polynomial harmonic 1-forms $\{\omega^{1,n}\}$ such that $(\delta\omega^{1,n}, -*d\omega^{1,n}, -d\omega^{1,n})$ tend to $(\varphi_0, \varphi_1, \varphi_2)$ in $L^p_0(\Sigma) \times L^p_1(\Sigma) \times L^p_2(\Sigma)$. Let us denote by the same symbol $\omega^{1,n}$ the vector whose components are the coefficients of the 1-form $\omega^{1,n}$. If we consider a chart with local coordinates (s,t), we have in local L^p norms

$$\begin{gathered} -\operatorname{div}\omega^{1,n} \to -f, \\ -\operatorname{curl}\omega^{1,n} \to -g_j\, dx^j, \\ -((\operatorname{curl}\omega^{1,n})_1 dx^2 dx^3 + (\operatorname{curl}\omega^{1,n})_2 dx^3 dx^1 + (\operatorname{curl}\omega^{1,n})_3 dx^1 dx^2) \\ \to -(g_1 dx^2 dx^3 + g_2 dx^3 dx^1 + g_3 dx^1 dx^2). \end{gathered}$$

The matrix (45) being non singular, the last two formulas imply that $(\operatorname{curl}\omega^{1,n})_j \to g_j$ locally.

This holding on any chart of Σ, we conclude that $(\operatorname{div}\omega^{1,n}, \operatorname{curl}\omega^{1,n}) \to (f,g)$ in $L^p(\Sigma) \times [L^p(\Sigma)]^3$. Since each $\omega^{1,n}$ can be written as a finite linear combination of vectors of the system $\{\omega^{(n)}\}$, we have proved the completeness of (46) in $\widetilde{\Lambda}^p$. □

Thanks to Theorem 6, we have a completeness result for system (46) in the uniform norm, too.

Theorem 9. *Let* Ω *be a bounded domain such that* $\mathbb{R}^3 \setminus \overline{\Omega}$ *is connected. The system* (46) *is complete in the subspace of* $C^0(\Sigma) \times [C^0(\Sigma)]^3$ *defined as* $\widetilde{\Lambda}^0 = \{(f, g) \in C^0(\Sigma) \times [C^0(\Sigma)]^3$ *such that conditions* (41) *hold for any harmonic vector polynomial* $\omega = (\omega_1, \omega_2, \omega_3)\}$.

The proof is similar to the previous one. We omit the details.

As far as the Brother Riesz Theorem is concerned, we have

Theorem 10. *Let* Ω *be a bounded domain with a Lyapunov boundary, such that* $\mathbb{R}^3 \setminus \overline{\Omega}$ *is connected. If* α_j *are measures in* $M(\Sigma)$ $(j = 0, 1, 2, 3)$ *such that*

$$\int_\Sigma (-\operatorname{curl} \omega \cdot \nu \, d\alpha_0 + \operatorname{div} \omega \, \nu \cdot d\alpha + (\operatorname{curl} \omega \times \nu) \cdot d\alpha) = 0, \tag{47}$$

for any harmonic vector polynomial $\omega = (\omega_1, \omega_2, \omega_3)$, *then* α_0, α_1, α_2, α_3 *are absolutely continuous with respect to the two-dimensional Lebesgue measure on* Σ.

Proof. Setting

$$\mu_0 = \alpha_0, \quad \mu_1 = \alpha_j \wedge dx^j, \quad \mu_2 = \alpha_1 \wedge dx^2 dx^3 + \alpha_2 \wedge dx^3 dx^1 + \alpha_3 \wedge dx^1 dx^2,$$

we can rewrite conditions (47) as (37). By Theorem 5 (see Remark 1) the k-measures μ_0, μ_1, μ_2 are absolutely continuous. This means that α_0, α_1, α_2, α_3 are absolutely continuous measures and the proof is complete. □

Remark 2. Conditions (47) are necessary and sufficient for the existence of a solution of the Dirichlet problem

$$\begin{cases} \nabla u = \operatorname{curl} v, \ \operatorname{div} v = 0, & \text{in } \Omega \\ u = \alpha_0, \ v = (\alpha_1, \alpha_2, \alpha_3), & \text{on } \Sigma \end{cases}$$

where the boundary conditions are assumed in the sense of measures. Then Theorem 10 says that, if the boundary values of a solution of the Moisil-Theodorescu system are measures, then all these measures have to be absolutely continuous.

It is interesting to see that Theorem 10 can be written in another way. Before stating this result, let us see a lemma.

Lemma 4. *Let* λ_j *be given measures in* $M(\Sigma)$ $(j = 0, 1, 2, 3)$. *There exist measures* $\beta_j \in M(\Sigma)$ $(j = 1, 2, 3)$ *such that*

$$\begin{cases} \nu_j \, d\beta_j = d\lambda_0 \,, \\ \varepsilon_{hsj} \, \nu_s \, d\beta_j = (\delta_{hj} - \nu_h \nu_j) \, d\lambda_j \,, \quad (h = 1, 2, 3). \end{cases} \tag{48}$$

Proof. Equation (48) means that

$$\int_\Sigma (w\,\nu_j + v_h\,\varepsilon_{hsj}\,\nu_s)\,d\beta_j = \int_\Sigma (w\,d\lambda_0 + v_h\,(\delta_{hj} - \nu_h\nu_j)\,d\lambda_j) \tag{49}$$

for any $(w,v) \in C^0(\Sigma)\times[C^0(\Sigma)]^3$ $(v = (v_1,v_2,v_3))$. In order to prove that there exists a solution of this problem, we are going to use a general existence principle due to Fichera (see [18, Lecture 2])). Setting $V = B_1 = C^0(\Sigma)\times[C^0(\Sigma)]^3$, $B_2 = V = [C^0(\Sigma)]^3$, $M_1 v = (w, v - (v\cdot\nu)\nu)$, $M_2 v = w\,\nu + v\times\nu$ in this principle (see [18] for the notations), we have that, given $(\lambda_0,\lambda_1,\lambda_2,\lambda_3) \in [M(\Sigma)]^4$, there exist $(\beta_1,\beta_2,\beta_3) \in [M(\Sigma)]^3$ such that (49) holds for any $(w,v) \in C^0(\Sigma)\times[C^0(\Sigma)]^3$ if and only if there exists a constant K such that the following inequality holds

$$\|w\|_\infty + \|v - (v\cdot\nu)\nu\|_\infty \le K\,\|w\,\nu + v\times\nu\|_\infty, \tag{50}$$

$$\forall\ (w,v) \in C^0(\Sigma)\times[C^0(\Sigma)]^3.$$

But this inequality is trivial, since, for any (w,v), we have

$$\begin{aligned}(|w| + |v - (v\cdot\nu)\nu|)^2 &\le 2(|w|^2 + |v - (v\cdot\nu)\nu|^2)\\ &= 2(|w|^2 + |v\times\nu|^2) = 2|w\,\nu + v\times\nu|^2\end{aligned}$$

pointwise on Σ, and then

$$|w| + |v - (v\cdot\nu)\nu| \le \sqrt{2}\,|w\,\nu + v\times\nu| \le \sqrt{2}\,\|w\,\nu + v\times\nu\|_\infty\,.$$

This implies inequality (50) and the proof is complete. □

Theorem 11. *Let Ω be a bounded domain with a Lyapunov boundary, such that $\mathbb{R}^3\setminus\overline{\Omega}$ is connected. If λ_j are measures in $M(\Sigma)$ $(j = 0,1,2,3)$ such that*

$$\int_\Sigma (\operatorname{div}\omega\,d\lambda_0 + \operatorname{curl}\omega\cdot d\lambda) = 0 \tag{51}$$

for any harmonic vector polynomial $\omega = (\omega_1,\omega_2,\omega_3)$, then λ_0, λ_1, λ_2, λ_3 are absolutely continuous with respect to the two-dimensional Lebesgue measure on Σ.

Proof. Our proof starts with the observation that we can write

$$\begin{aligned}&\int_\Sigma \operatorname{curl}\omega\cdot d\lambda\\ &= \int_\Sigma (\operatorname{curl}\omega\cdot\nu)\,\nu\cdot d\lambda + \int_\Sigma [\operatorname{curl}\omega - ((\operatorname{curl}\omega)\cdot\nu)\,\nu]\cdot d\lambda\,.\end{aligned}$$

In the second integral we can write

$$[\operatorname{curl}\omega - ((\operatorname{curl}\omega)\cdot\nu)\,\nu]\cdot d\lambda = (\operatorname{curl}\omega)_h\,(\delta_{hj} - \nu_h\nu_j)\,d\lambda_j\,.$$

By Lemma 4, there exist $\beta_j \in M(\Sigma)$ ($j = 1, 2, 3$) such that equations (48) hold. Therefore

$$[\operatorname{curl}\omega - ((\operatorname{curl}\omega)\cdot\nu)\,\nu]\cdot d\lambda = \varepsilon_{hsj}\,(\operatorname{curl}\omega)_h\,\nu_s\,d\beta_j = (\operatorname{curl}\omega\times\nu)\cdot d\beta\,.$$

and

$$\operatorname{div}\omega\,d\lambda_0 = \operatorname{div}\omega\,(\nu\cdot d\beta)\,.$$

Setting $d\beta_0 = -\nu\cdot d\lambda$, we rewrite (51) as

$$\int_\Sigma (-\operatorname{curl}\omega\cdot\nu\,d\beta_0 + \operatorname{div}\omega\,(\nu\cdot d\beta) + (\operatorname{curl}\omega\times\nu)\cdot d\beta) = 0.$$

By Theorem 10, the measures β_0, β_1, β_2, β_3 are absolutely continuous. Since $d\lambda_0 = \nu\cdot d\beta$, we get the absolute continuity of λ_0. Moreover we have

$$(\delta_{hj} - \nu_h\nu_j)\,d\lambda_j = d\lambda_h - \nu_h\,(\nu\cdot d\lambda) = d\lambda_h + \nu_h\,d\beta_0$$

from which it follows

$$d\lambda_h = -\nu_h\,d\beta_0 + \varepsilon_{hsj}\,\nu_s\,d\beta_j\,, \quad (h = 1, 2, 3).$$

This shows that also λ_1, λ_2, λ_3 are absolutely continuous. □

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