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A complement to potential theory in the Cosserat elasticity

Alberto Cialdea*,†, Emanuela Dolce and Angelica Malaspina

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In this paper, we investigate the internal second, third, and fourth boundary value problems of the three-dimensional Cosserat elasticity by means of potential theory. The obtained integral representations differ from the classical ones. These results complete the ones related to the first BVP, which have recently been obtained by the authors. Copyright © 2014 John Wiley & Sons, Ltd.

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1. Introduction

The main boundary value problems of the Cosserat theory are four. They consist in finding the elastic oscillation state when on the boundary is given the displacements and rotations in the first problem, the force and couple-stress in the second problem, the displacements and couple-stresses in the third problem, and the rotations and force-stresses in the fourth problem.

These problems have been studied by means of potential theory (see, e.g., [1, 2] and [3–5] and the reference therein for the first and second problems in plane, anti-plane deformations and in the bending of plates).

Recently, in [6], we have considered the first problem, and by using an indirect boundary integral method, we have obtained the solution in the form of a simple layer potential instead of the usual double layer potential. The method we used was introduced for the first time in [7] for the Dirichlet problem for Laplacian and, later, applied also to other PDEs ([8–13]). It hinges on the theory of reducible operators (see, e.g., [14, 15]) and on the theory of differential forms (see, e.g, [16, 17]). We remark that our method uses neither the theory of pseudodifferential operators nor the concept of hypersingular integral. For a sketch of the method in the simple but significant case of Laplacian, we refer to [11, Section 2, p. 2].

In the present paper, we consider the other three main problems in the Cosserat theory, and extending the results of Cialdea *et al*. [6], we obtain integral representations different from the ones obtained in [2].

We remark that these boundary value problems could be posed in the fractional Sobolev space $W^{1+\frac{1}{p},p}(\Omega)$ (see, e.g., [18, Sec. 14.3, pp. 483-489]). By means of mapping properties of integral operators and uniqueness theorems (in suitable quotient spaces) for the boundary value problems considered here and in [6], one could deduce that the potentials we find are the solutions in $W^{1+\frac{1}{p},p}(\Omega)$. However, we have preferred to avoid the related technicalities and focus on the analysis of the integral equations arising from our ansatz.

2. Preliminary

Throughout this paper, we consider Ω as a bounded domain of $\mathbb{R}^3$ such that its boundary $\partial\Omega$ is a Lyapunov surface Σ (i.e., Σ has a uniformly Hölder continuous normal field of some exponent $l \in (0,1]$) and such that $\mathbb{R}^3 - \overline{\Omega}$ is connected; $\nu(y) = (\nu_1(y), \nu_2(y), \nu_3(y))$ denotes the outward unit normal vector at the point $y = (y_1, y_2, y_3) \in \Sigma$.

Given the set of constants $\lambda, \mu, \alpha, \varepsilon, v, \beta$ satisfying the conditions

$$\alpha, \beta, \mu, v > 0; \quad 3\lambda + 2\mu > 0; \quad 3\varepsilon + 2v > 0,$$

[1] *Department of Mathematics, Computer Science and Economics, University of Basilicata, V.le dell'Ateneo Lucano, 10, Campus of Macchia Romana, Potenza, Italy*
* *Correspondence to: Alberto Cialdea, Department of Mathematics, Computer Science and Economics, University of Basilicata, V.le dell'Ateneo Lucano, 10, Campus of Macchia Romana, Potenza, Italy.*
† *E-mail: cialdea@email.it*

the homogeneous equation of statics of a Cosserat continuum has the form [2, p. 50]

$$\begin{cases} (\mu+\alpha)\Delta u + (\lambda+\mu-\alpha)\text{grad div} u + 2\alpha \,\text{rot}\, \omega = 0 & \text{in } \Omega \\ (v+\beta)\Delta\omega + (\varepsilon + v - \beta)\text{grad div}\omega + 2\alpha \,\text{rot}\, u - 4\alpha\omega = 0 & \text{in } \Omega \end{cases} \tag{1}$$

where $u = (u_1, u_2, u_3)$ is the displacement vector and $\omega = (\omega_1, \omega_2, \omega_3)$ is the rotation vector. It is convenient to write the basic equations (1) in a matrix form. To this end, we introduce the block matrix

$$M = \begin{pmatrix} M^1 & M^2 \\ M^3 & M^4 \end{pmatrix}$$

whose entries are (3×3)-matrices of differential operators given by

$$M^1_{ij} = (\mu+\alpha)\delta_{ij}\Delta + (\lambda+\mu-\alpha)\frac{\partial^2}{\partial x_i \partial x_j},$$

$$M^2_{ij} = M^3_{ij} = -2\alpha \sum_{k=1}^{3} \delta_{ijk}\frac{\partial}{\partial x_k},$$

$$M^4_{ij} = \delta_{ij}[(v+\beta)\Delta - 4\alpha] + (\varepsilon + v - \beta)\frac{\partial^2}{\partial x_i \partial x_j}$$

for $i, j = 1, 2, 3$, where δ_{kj} denotes Kronecker's symbol and δ_{jkp} is the Levi-Civita's symbol [‡]. Thus, Equation (1) becomes

$$M\mathscr{U} = 0, \qquad \text{in } \Omega \tag{2}$$

where $\mathscr{U} = (u, \omega)'$ is a six-component column vector.
We denote by T the stress operator [2, p. 59]

$$T = \begin{pmatrix} T^{(1)} & T^{(2)} \\ 0 & T^{(4)} \end{pmatrix} \qquad T^{(i)} = \left(T^{(i)}_{kj}\right) \qquad k, j = 1, 2, 3, \quad i = 1, 2, 4, \tag{3}$$

where

$$T^{(1)}u = \lambda(\text{div} u)\nu + (2\mu)\frac{\partial u}{\partial \nu} + (\mu-\alpha)(\nu \wedge \text{rot}\, u),$$

$$T^{(2)}u = 2\alpha(\nu \wedge u),$$

$$T^{(4)}u = \varepsilon(\text{div}\, u)\nu + (2v)\frac{\partial u}{\partial \nu} + (v-\beta)(\nu \wedge \text{rot}\, u).$$

The block matrix of the fundamental solution of the homogeneous system (2) is given by

$$\Psi(x) = \begin{pmatrix} \Psi^{(1)}(x) & \Psi^{(2)}(x) \\ \Psi^{(3)}(x) & \Psi^{(4)}(x) \end{pmatrix} \qquad x \in \mathbb{R}^3 \setminus \{(0,0,0)\}$$

where $\Psi^{(i)}(x) = \left(\Psi^{(i)}_{kj}(x)\right)$ $k, j = 1, 2, 3$, $i = 1, \ldots, 4$ are the following (3×3)-matrices [2, p. 93]:

$$\Psi^{(1)}_{kj}(x) = \frac{\delta_{kj}}{2\pi}\left[\frac{1}{\mu|x|} - \frac{\alpha}{\mu(\alpha+\mu)}\frac{e^{-\sigma|x|}}{|x|}\right] + \frac{1}{2\pi\mu}\frac{\partial^2}{\partial x_k \partial x_j}\left[-\frac{(\lambda+\mu)}{2(\lambda+2\mu)}|x| + \frac{\beta+v}{4\mu}\frac{e^{-\sigma|x|}-1}{|x|}\right];$$

$$\Psi^{(2)}_{kj}(x) = \Psi^{(3)}_{kj}(x) = \frac{1}{4\pi\mu}\sum_{p=1}^{3}\delta_{jkp}\frac{\partial}{\partial x_p}\frac{1-e^{-\sigma|x|}}{|x|};$$

$$\Psi^{(4)}_{kj}(x) = \frac{\delta_{kj}}{2\pi(\beta+v)}\frac{e^{-\sigma|x|}}{|x|} + \frac{1}{8\pi}\frac{\partial^2}{\partial x_k \partial x_j}\left[\frac{e^{-\rho|x|}-e^{-\sigma|x|}}{\alpha|x|} - \frac{e^{-\sigma|x|}-1}{\mu|x|}\right],$$

$\sigma = \sqrt{\frac{4\alpha\mu}{(\mu+\alpha)(v+\beta)}}$ and $\rho = \sqrt{\frac{4\alpha}{\varepsilon+2v}}$.

[‡] $\delta_{jkp} = 1$ *or* $\delta_{jkp} = -1$ *depending on whether* j, k, p *have an even or odd number of transpositions of the numbers* $1, 2, 3$ *and* $\delta_{jkp} = 0$ *if at least two of three indices* j, k, p *are equal.*

Lemma 1
The matrix $\Psi(x)$ defined by (2) can be written as

$$\begin{aligned}
\Psi^1_{kj}(x) &= \frac{1}{4\pi}\left[\frac{\lambda+3\mu+\alpha}{(\mu+\alpha)(\lambda+2\mu)}\frac{\delta_{kj}}{|x|}+\frac{\lambda+\mu-\alpha}{(\mu+\alpha)(\lambda+2\mu)}\frac{x_k x_j}{|x|^3}\right]+C_{kj}(x),\\
\Psi^2_{kj}(x) &= \Psi^3_{kj}(x) = \mathcal{O}(1),\\
\Psi^4_{kj}(x) &= \frac{1}{4\pi}\left[\frac{\varepsilon+3\nu+\beta}{(\nu+\beta)(\varepsilon+2\nu)}\frac{\delta_{kj}}{|x|}+\frac{\varepsilon+\nu-\beta}{(\nu+\beta)(\varepsilon+2\nu)}\frac{x_k x_j}{|x|^3}\right]+D_{kj}(x),
\end{aligned}$$

where

$$C_{kj}(x) = \frac{e^{-\sigma|x|}-1}{|x|}\left[-\frac{\delta_{kj}}{2\pi\mu}\frac{\alpha}{\mu+\alpha}+\frac{\alpha}{2\pi\mu(\mu+\alpha)}\frac{x_k x_j}{|x|^2}\right]+\frac{1}{2\pi\mu}\frac{\beta+\nu}{4\mu}\left(\frac{3x_k x_j}{|x|^2}-\delta_{kj}\right)\left[\frac{(1+\sigma|x|)e^{-\sigma|x|}-1+\frac{1}{2}\sigma^2|x|^2}{|x|^3}\right], \tag{4}$$

$$\begin{aligned}
D_{kj}(x) = {}&\left[\frac{\delta_{kj}}{2\pi(\beta+\nu)}-\frac{1}{8\pi}\left(\frac{1}{\alpha}+\frac{1}{\mu}\right)\frac{x_k x_j}{|x|^2}\sigma^2\right]\left[\frac{e^{-\sigma|x|}-1}{|x|}\right]+\frac{1}{8\pi\alpha}\frac{x_k x_j}{|x|^2}\rho^2\left[\frac{e^{-\rho|x|}-1}{|x|}\right]+\\
&+\left(\frac{3}{8\pi\alpha}\frac{x_k x_j}{|x|^2}-\frac{1}{8\pi\alpha}\delta_{kj}\right)\left[\frac{(1+\rho|x|)e^{-\rho|x|}-1+\frac{1}{2}\rho^2|x|^2}{|x|^3}\right]+\\
&+\left[\frac{1}{8\pi}\left(\frac{1}{\alpha}+\frac{1}{\mu}\right)\delta_{kj}-\frac{3}{8\pi}\left(\frac{1}{\alpha}+\frac{1}{\mu}\right)\frac{x_k x_j}{|x|^2}\right]\left[\frac{(1+\sigma|x|)e^{-\sigma|x|}-1+\frac{1}{2}\sigma^2|x|^2}{|x|^3}\right].
\end{aligned} \tag{5}$$

The functions $C_{kj}(x)$ and $D_{kj}(x)$ are bounded.

2.1. *Basic problems*

The basic problems of statics consist in finding a six-component vector $\mathscr{U}$ solution of (2) and satisfying one of the following boundary conditions, where f is an assigned vector function:

- the first internal basic problem or Problem $(I)^+$:

$$\mathscr{U}^+(y) = f(y), \qquad \forall y \in \Sigma;$$

- the second internal basic problem or Problem $(II)^+$:

$$[T\mathscr{U}]^+(y) = f(y), \qquad \forall y \in \Sigma;$$

  where T is given by (3);
- the third internal basic problem or Problem $(III)^+$:

$$[H\mathscr{U}]^+(y) = f(y), \qquad \forall y \in \Sigma,$$

  where

$$H = \begin{pmatrix} I & 0 \\ 0 & -T^{(4)} \end{pmatrix};$$

- the fourth internal basic problem or Problem $(IV)^+$:

$$[R\mathscr{U}]^+(y) = f(y), \qquad \forall y \in \Sigma,$$

  where

$$R = \begin{pmatrix} T^{(1)} & T^{(2)} \\ 0 & I \end{pmatrix}.$$

We observe the following identity

$$[R_x[H_y\Psi(x-y)]']' = H_y[R_x\Psi(y-x)]'. \tag{6}$$

Let us consider some potential-type integrals:

$$\mathscr{W}[\Phi](x) = \int_\Sigma [T_y\Psi(y-x)]'\Phi(y)d\sigma_y; \tag{7}$$

$$\mathscr{U}[\Phi](x) = \int_\Sigma \Psi(y-x)\Phi(y)d\sigma_y; \tag{8}$$

$$\mathscr{R}[\Phi](x) = \int_\Sigma [R_y\Psi(y-x)]'\Phi(y)d\sigma_y; \tag{9}$$

$$\mathscr{H}[\Phi](x) = \int_\Sigma [H_y\Psi(y-x)]'\Phi(y)d\sigma_y. \tag{10}$$

Integrals (7) and (8) are the double and simple layer potential, respectively. Usually (see, e.g., [2]), the solutions of the problems $(I)^+ - (IV)^+$ are sought in the form of the potentials (7)–(10), respectively.

In this paper, p indicates a real number such that $p \in]1, +\infty[$. We denote by $[L^p(\Sigma)]^j$ ($j \in \mathbb{N} \setminus \{0\}$) the space of all measurable vector-valued functions $u = (u_1, \dots, u_j)$ such that $|u_i|^p$ ($i = 1, \dots, j$) is integrable over Σ. $\left[W^{1,p}(\Sigma)\right]^j$ ($j \in \mathbb{N} \setminus \{0\}$) is the space of all measurable vector-valued functions $u = (u_1, \dots, u_j)$ such that u_i ($i = 1, \dots, j$) belongs to the Sobolev space $W^{1,p}(\Sigma)$. The symbols $\mathfrak{D}^p$, $\mathfrak{R}^p$, and $\mathfrak{H}^p$ stand for the class of double layer potentials (7) with density in $\left[W^{1,p}(\Sigma)\right]^6$, the class of potentials (9) with density in $\left[W^{1,p}(\Sigma)\right]^3 \times [L^p(\Sigma)]^3$ and the class of potentials (10) with density in $[L^p(\Sigma)]^3 \times \left[W^{1,p}(\Sigma)\right]^3$, respectively.

Let us consider the following singular integral systems:

$$\mp \Phi(z) + \int_\Sigma [T_y \Psi(y-z)]' \Phi(y) d\sigma_y = f(z); \tag{11}$$

$$\mp \Phi(z) + \int_\Sigma H_z [R_y \Psi(y-z)]' \Phi(y) d\sigma_y = f(z); \tag{12}$$

$$\pm \Phi(z) + \int_\Sigma R_z [H_y \Psi(y-z)]' \Phi(y) d\sigma_y = f(z). \tag{13}$$

The corresponding homogeneous integral equations ($f = 0$) will be denoted by the same symbols with the subscript 0.

We recall that a complete system of linearly independent solutions of $(11)_0^-$ $\left\{\varphi^{(h)}(x)\right\}, h = 1, \dots, 6$ is given by [2, Theorem 2.7, p. 502]:

$$\begin{array}{ll} \varphi^{(1)}(x) = (1,0,0,0,0,0); & \varphi^{(4)}(x) = (0,-x_3,x_2,1,0,0); \\ \varphi^{(2)}(x) = (0,1,0,0,0,0); & \varphi^{(5)}(x) = (x_3,0,-x_1,0,1,0); \\ \varphi^{(3)}(x) = (0,0,1,0,0,0); & \varphi^{(6)}(x) = (-x_2,x_1,0,0,0,1). \end{array} \tag{14}$$

The system $\left\{\varphi^{(h)}(x)\right\}, h = 1,2,3$ forms a complete system of linearly independent solutions of $(12)_0^-$ [2, Theorem 2.12, p. 507].

The following result is proved in [6, Theorem 3.1, p. 10] (note that in [6], we have considered the generalized stress operator S, which includes T as a particular case).

Theorem 1

Let $\mathscr{W}$ be the double layer potential (7) with density $\mathscr{U} = (u, \omega)' \in \left[W^{1,p}(\Sigma)\right]^6$. We have for any $x \in \Omega$

$$\frac{\partial}{\partial x_s} \mathscr{W}_j(x) = \mathscr{K}_{js}(du)(x) + \frac{\partial}{\partial x_s} \int_\Sigma (T_y\Psi)^3_{kj}(y-x)\omega_k(y) d\sigma_y,$$

$$\frac{\partial}{\partial x_s} \mathscr{W}_{j+3}(x) = \mathscr{F}_{js}(d\omega)(x) + \frac{\partial}{\partial x_s} \int_\Sigma \left[\widetilde{D}_{kj}(y-x)\omega_k(y) + (T_y\Psi)^2_{kj}(y-x)u_k(y)\right] d\sigma_y,$$

where $du = (du_1, du_2, du_3)$, $d\omega = (d\omega_1, d\omega_2, d\omega_3)$,

$$\mathscr{K}_{js}(\psi)(x) = 2\Theta_s(\psi_j)(x) - \delta^{123}_{pkq} \int_\Sigma \frac{\partial}{\partial x_s}\left[H^1_{jp}(y-x)\right] \wedge \psi_k(y) \wedge dy^q, \qquad \psi \in \left[L^p_1(\S)\right]^3,$$

$$\mathscr{F}_{js}(\varphi)(x) = 2\Theta_s(\varphi_j)(x) - \delta^{123}_{pkq} \int_\Sigma \frac{\partial}{\partial x_s}\left[H^2_{jp}(y-x)\right] \wedge \varphi_k(y) \wedge dy^q, \qquad \varphi \in \left[L^p_1(\S)\right]^3,$$

$$\begin{aligned} H^1_{jp}(y-x) = {} & \frac{1}{4\pi}\left[\frac{(\mu+\xi)(\lambda+3\mu+\alpha)}{(\mu+\alpha)(\lambda+2\mu)} - 2\right]\frac{\delta_{jp}}{|y-x|} + \\ & + \frac{1}{4\pi}\frac{(\mu+\xi)(\lambda+\mu-\alpha)}{(\mu+\alpha)(\lambda+2\mu)}\frac{1}{|y-x|}\frac{\partial}{\partial y_j}|y-x|\frac{\partial}{\partial y_p}|y-x| + (\mu+\xi)C_{jp}(y-x), \end{aligned}$$

$$\begin{aligned} H^2_{jp}(y-x) = {} & \frac{1}{4\pi}\left[\frac{(\chi+v)(\varepsilon+3v+\beta)}{(v+\beta)(\varepsilon+2v)} - 2\right]\frac{\delta_{jp}}{|y-x|} + \\ & + \frac{1}{4\pi}\frac{(\chi+v)(\varepsilon+v-\beta)}{(v+\beta)(\varepsilon+2v)}\frac{1}{|y-x|}\frac{\partial}{\partial y_j}|y-x|\frac{\partial}{\partial y_p}|y-x| + (\chi+v)D_{jp}(y-x) \end{aligned}$$

and

$$\widetilde{D}_{kj}(y-x) = \frac{1}{2\pi}\nu_k(y)\frac{\partial}{\partial y_j}\frac{e^{-\rho|y-x|}-1}{|y-x|} - \frac{1}{2\pi}\nu_j(y)\frac{\partial}{\partial y_k}\frac{e^{-\sigma|y-x|}-1}{|y-x|} + \frac{1}{2\pi}\delta_{kj}\frac{\partial}{\partial \nu}\frac{e^{-\sigma|y-x|}-1}{|y-x|}. \tag{15}$$

Here, S, C_{jp}, and D_{jp} are given by (3), (4), and (5), respectively.

Lemma 2

Let $\mathscr{W} \in \mathscr{D}^2$ be a double layer potential with density $\Phi = (\varphi, \vartheta) \in \left[W^{1,2}(\Sigma)\right]^6$. Then

$$\int_\Omega E(\mathscr{W}, \mathscr{W})\, dx = \int_\Sigma [\mathscr{W} T \mathscr{W}]^+ \, d\sigma, \tag{16}$$

where

$$
\begin{aligned}
E(\mathscr{U},\mathscr{U}') = {} & \frac{3\lambda+2\mu}{3}\sum_{i,j}\frac{\partial u_i}{\partial x_i}\frac{\partial u_j'}{\partial x_j} + \frac{\mu}{2}\sum_{i,j}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\sum_k\frac{\partial u_k}{\partial x_k}\right] \\
& \times\left[\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}-\frac{2}{3}\delta_{ij}\sum_k\frac{\partial u_k'}{\partial x_k}\right] + \frac{\alpha}{2}\sum_{ij}\left[\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}+2\sum_k\varepsilon_{kji}\omega_k\right] \\
& \times\left[\frac{\partial u_j'}{\partial x_i}-\frac{\partial u_i'}{\partial x_j}+2\sum_k\varepsilon_{kji}\omega_k'\right] + \frac{3\varepsilon+2\upsilon}{3}\sum_{i,j}\frac{\partial \omega_i}{\partial x_i}\frac{\partial \omega_j'}{\partial x_j} \\
& + \frac{\upsilon}{2}\sum_{i,j}\left[\frac{\partial \omega_i}{\partial x_j}+\frac{\partial \omega_j}{\partial x_i}-\frac{2}{3}\delta_{ij}\sum_k\frac{\partial \omega_k}{\partial x_k}\right]\left[\frac{\partial \omega_i'}{\partial x_j}+\frac{\partial \omega_j'}{\partial x_i}-\frac{2}{3}\delta_{ij}\sum_k\frac{\partial \omega_k'}{\partial x_k}\right] \\
& + \frac{\beta}{2}\sum_{ij}\left[\frac{\partial \omega_j}{\partial x_i}-\frac{\partial \omega_i}{\partial x_j}\right]\left[\frac{\partial \omega_j'}{\partial x_i}-\frac{\partial \omega_i'}{\partial x_j}\right],
\end{aligned}
$$

with $\mathscr{U}=(u,\omega)$, $\mathscr{U}'=(u',\omega')$ six-component vectors and $u=(u_1,u_2,u_3)$, $\omega=(\omega_1,\omega_2,\omega_3)$, $u'=(u_1',u_2',u_3')$, $\omega'=(\omega_1',\omega_2',\omega_3')$.

Proof
Let $(\Phi_k)_{k\geq 1}$ be a sequence of functions in $\left[C^{1,h}(\Sigma)\right]^6$ ($0<h<l$, l being the Lyapunov exponent of Σ) such that $\Phi_k\to\Phi$ in $\left[W^{1,2}(\Sigma)\right]^6$. Setting

$$\mathscr{W}_k[\Phi_k](x)=\int_\Sigma\left[T_y\Psi(x-y)\right]'\,\Phi_k(y)\,d\sigma_y,$$

we have that $\mathscr{W}_k[\Phi_k]\in\left[C^{1,h}(\overline{\Omega})\right]^6$, $M\mathscr{W}_k[\Phi_k]=0$ and then

$$\int_\Omega E(\mathscr{W}_k,\mathscr{W}_k)\,dx=\int_\Sigma[\mathscr{W}_k T\mathscr{W}_k]^+\,d\sigma. \tag{17}$$

From $\Phi_k\to\Phi$ in $\left[L^2(\Sigma)\right]^6$, it follows that $\mathscr{W}_k\to\mathscr{W}$ in $\left[L^2(\Sigma)\right]^6$ because of [2, Theorem 3.36, p. 211].
One can check that $\frac{\partial}{\partial x_s}\left(T_y\Psi\right)^2_{kj}(x-y)=\mathcal{O}\left(\frac{1}{|x-y|^2}\right)$ and $\frac{\partial}{\partial x_s}\left(T_y\Psi\right)^3_{kj}(x-y)=\mathcal{O}\left(\frac{1}{|x-y|^2}\right)$, while $\frac{\partial}{\partial x_s}\widetilde{D}(x-y)=\mathcal{O}\left(\frac{1}{|x-y|}\right)$. This implies

$$\mathscr{K}_{sj}(d\varphi_k)\to\mathscr{K}_{sj}(d\varphi),\quad \mathscr{F}_{sj}(d\vartheta_k)\to\mathscr{F}_{sj}(d\vartheta),$$

$$\frac{\partial}{\partial x_s}\int_\Sigma\left[(T_y\Psi)^3\right]'(y-x)\vartheta_k(y)d\sigma_y\to\frac{\partial}{\partial x_s}\int_\Sigma\left[(T_y\Psi)^3\right]'(y-x)\vartheta(y)d\sigma_y,$$

$$\frac{\partial}{\partial x_s}\int_\Sigma\left[\widetilde{D}'(y-x)\vartheta_k(y)+\left[(T_y\Psi)^2\right]'(y-x)\varphi_k(y)\right]d\sigma_y\to\frac{\partial}{\partial x_s}\int_\Sigma\left[\widetilde{D}'(y-x)\vartheta(y)+\left[(T_y\Psi)^2\right]'(y-x)\varphi(y)\right]d\sigma_y$$

in $L^2(\Omega)$. By applying Theorem 1, we see that $\nabla\mathscr{W}_k\to\nabla\mathscr{W}$ in $\left[L^2(\Omega)\right]^6$. Moreover, because $\mathscr{K}_{sj}(d\varphi_k)\to\mathscr{K}_{sj}(d\varphi)$ $\left(\mathscr{F}_{sj}(d\vartheta_k)\to\mathscr{F}_{sj}(d\vartheta)\right)$ also in $L^2(\Sigma)$ [9, Lemma 3.2 and Lemma 3.3], it follows from [6, Lemma 4.3] that $T\mathscr{W}_k\to T\mathscr{W}$ in $\left[L^2(\Sigma)\right]^6$. We obtain the claim by letting $k\to+\infty$ in (17). □

3. Problem $(II)^+$

In this section, we look for the solution of the second BVP (18) in the form of a double layer potential instead of the simple layer potential.

Theorem 2
Given $f\in[L^p(\Sigma)]^6$, the second BVP

$$\begin{cases}\mathscr{U}\in\mathfrak{D}^p \\ M\mathscr{U}=0 & \text{in }\Omega \\ [T\mathscr{U}]^+=f & \text{on }\Sigma\end{cases} \tag{18}$$

admits a solution if, and only if,

$$
\begin{aligned}
&\int_\Sigma f_k(y)d\sigma_y=0, \qquad k=1,2,3; \\
&\int_\Sigma\left[f_{3+k}(y)+\sum_{i,j=1}^3\delta_{kij}y_i f_j(y)\right]d\sigma_y=0, \qquad k=1,2,3.
\end{aligned} \tag{19}
$$

The density of double layer potential (7) is given by a simple layer potential (8) $\mathscr{U}[\Phi]$, $\Phi \in [L^p(\Sigma)]^6$ being a solution of the singular integral system

$$-\Phi + J^2\Phi = f, \tag{20}$$

where

$$J\Phi(x) = \int_\Sigma T_x[\Psi(y-x)]\Phi(y)d\sigma_y, \quad x \in \Sigma. \tag{21}$$

Moreover, the solution is determined up to an additive rigid displacement (u, ω), where $u = a \wedge x + b, \omega = a, \left(a, b \in \mathbb{R}^3\right)$.

Proof

Let $\mathscr{W}$ be a double layer potential with density $\mathscr{U} \in \left[W^{1,p}(\Sigma)\right]^6$. It is proved in [6, p. 17] that

$$T\mathscr{W}[\mathscr{U}[\Phi]](x) = -\Phi(x) + J^2\Phi(x), \qquad x \in \Sigma,$$

where J is defined in (21). The boundary condition $[T\mathscr{W}]^+ = f$ can be written as in (20). There exists a solution of (18) in the form of a double layer potential if, and only if, the singular system (20) is solvable. To this end, assume that conditions (19) hold and consider the following integral system

$$-\gamma + J\gamma = f. \tag{22}$$

Because the homogeneous adjoint of (22) (which is $(11)_0^+$, see [2, Theorem 2.1, p. 496]) has only the trivial solution [2, Theorem 2.5, p. 500], there exists a solution $\gamma \in [L^p(\Sigma)]^6$ of (22) for any $f \in [L^p(\Sigma)]^6$. We now consider

$$\Phi + J\Phi = \gamma, \tag{23}$$

where γ is a solution of (22). In this case, there exists $\Phi \in [L^p(\Sigma)]^6$ solution of (23) if, and only if,

$$\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x = 0, \tag{24}$$

where $\{\varphi^{(h)}(x)\}$ is a complete system of linearly independent solutions of $(11)_0^-$ given by (14). We have

$$\begin{aligned}
\int_\Sigma f(x)\varphi^{(h)}(x)d\sigma_x &= \int_\Sigma [-I + J]\gamma(x)\varphi^{(h)}(x)d\sigma_x = \\
&= -\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x + \int_\Sigma \varphi^{(h)}(x)d\sigma_x \int_\Sigma [T_x\Psi(y-x)]\gamma(y)d\sigma_y = \\
&= -\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x + \int_\Sigma \gamma(y)d\sigma_y \int_\Sigma [T_x\Psi(y-x)]'\varphi^{(h)}(x)d\sigma_x = \\
&= -\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x - \int_\Sigma \gamma(y)\varphi^{(h)}(y)d\sigma_y = \\
&= -2\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x.
\end{aligned}$$

Then compatibility conditions (24) hold by virtue of (19).

Conversely, if $\mathscr{U}$ is a solution of (18), from [6, Lemma 4.3] and Lemma 2, we have

$$\int_\Sigma f(y)\varphi^{(h)}(y)d\sigma_y = \int_\Sigma T\mathscr{U}(y)\varphi^{(h)}(y)d\sigma_y = \int_\Omega E(\mathscr{U}(y), \varphi^{(h)}(y))dy = 0,$$

because $E(\mathscr{U}, \varphi^{(h)}) = 0$ (as one can be verified by a straightforward calculation).

Finally, we pass to discuss the uniqueness. Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be solutions of (18) with datum f. Then, $\mathscr{U} = \mathscr{V}_1 - \mathscr{V}_2$ is the solution of the corresponding homogeneous problem. We observe that

$$\mathscr{U}T\mathscr{U} = u(T^{(1)}u + T^{(2)}\omega) + \omega T^{(4)}\omega, \qquad \text{on } \Sigma, \tag{25}$$

where $\mathscr{U} = (u, \omega)$. Because $[T\mathscr{U}]^+ = 0$, it follows from (16) that $E(\mathscr{U}, \mathscr{U}) = 0$. Then, $u = a \wedge x + b, \omega = a, (a, b \in \mathbb{R}^3)$. □

4. Problem $(III)^+$

Here, we solve the third BVP (29) by means of the potential $\mathscr{H}$ (10). In this case, we need an additional term to $\mathscr{H}$ (34). We start by a lemma.

Lemma 3

We have that

$$[H\mathscr{H}[R\mathscr{R}[\Phi]]]^+ = -\Phi + L^2\Phi, \quad \Phi \in \left[W^{1,p}(\Sigma)\right]^3 \times [L^p(\Sigma)]^3$$

where

$$L\Phi(x) = \int_\Sigma H_x[R_y\Psi(y-x)]'\Phi(y)d\sigma_y, \quad x \in \Sigma \tag{26}$$

$$\mathscr{H}[R\mathscr{R}[\Phi]](x) = \int_\Sigma [H_y\Psi(y-x)]' R\mathscr{R}[\Phi](y)d\sigma_y, \quad x \in \Omega \tag{27}$$

and $\mathscr{R}[\Phi]$ is given by (9).

Proof

Let $\Phi \in \left[W^{1,p}(\Sigma)\right]^3 \times [L^p(\Sigma)]^3$. ~~Because~~ $\mathscr{R}[\Phi] \in \left[W^{1,p}(\Sigma)\right]^6$, we have $R\mathscr{R}[\Phi] \in [L^p(\Sigma)]^3 \times \left[W^{1,p}(\Sigma)\right]^3$. Then $\mathscr{H}[R\mathscr{R}[\Phi]] \in \left[W^{1,p}(\Sigma)\right]^6$.

From [2, Theorem 1.2, p. 492], we have

$$\mathscr{H}[R\mathscr{R}](x) = 2\mathscr{R}(x) + \int_\Sigma [R_y\Psi(y-x)]' H\mathscr{R}(y)d\sigma_y, \qquad x \in \Omega.$$

It is also known that [2, (1.24), p. 493]

$$\left[H\left(\int_\Sigma [R_y\Psi(y-x)]'\Phi(y)d\sigma_y\right)\right]^+ = -\Phi(x) + H_x\int_\Sigma [R_y\Psi(y-x)]'\Phi(y)d\sigma_y, \qquad x \in \Sigma. \tag{28}$$

So we have

$$\begin{aligned}
[H\mathscr{H}[R\mathscr{R}](x)]^+ &= \left[H\left(2\mathscr{R}(x) + \int_\Sigma [R_y\Psi(y-x)]'H\mathscr{R}(y)d\sigma_y\right)\right]^+ = \\
&= \left[2H\mathscr{R}(x) + H_x\int_\Sigma [R_y\Psi(y-x)]'H\mathscr{R}(y)d\sigma_y\right]^+ = \\
&= 2H\mathscr{R}(x) - H\mathscr{R}(x) + H_x\int_\Sigma [R_y\Psi(y-x)]'H\mathscr{R}(y)d\sigma_y = \\
&= H\mathscr{R}(x) + H_x\int_\Sigma [R_y\Psi(y-x)]'H\mathscr{R}(y)d\sigma_y.
\end{aligned}$$

Keeping in mind (9), (28), and (26)

$$\begin{aligned}
&[H\mathscr{H}[R\mathscr{R}[\Phi]](x)]^+ = H_x\left[\int_\Sigma [R_y\Psi(y-x)]'\Phi(y)d\sigma_y\right] + \\
&H_x\int_\Sigma [R_y\Psi(y-x)]'H_y\left[\int_\Sigma [R_z\Psi(z-y)]'\Phi(z)d\sigma_z\right]d\sigma_y = \\
&= -\Phi(x) + H_x\int_\Sigma [R_y\Psi(y-x)]'\Phi(y)d\sigma_y - H_x\int_\Sigma [R_y\Psi(y-x)]'\Phi(y)d\sigma_y + \\
&H_x\int_\Sigma [R_y\Psi(y-x)]'H_y\int_\Sigma [R_z\Psi(z-y)]'\Phi(z)d\sigma_z d\sigma_y = \\
&= -\Phi(x) + \int_\Sigma H_x[R_y\Psi(y-x)]'\int_\Sigma H_y[R_z\Psi(z-y)]'\Phi(z)d\sigma_z d\sigma_y = \\
&= -\Phi(x) + L^2\Phi(x).
\end{aligned}$$

□

Proposition 1

Given $f \in [W^{1,p}(\Sigma)]^3 \times [L^p(\Sigma)]^3$, the third BVP

$$\begin{cases}\mathscr{U} \in \mathfrak{H}^p \\ M\mathscr{U} = 0 \quad \text{in } \Omega \\ [H\mathscr{U}]^+ = f \ \text{ on } \Sigma\end{cases} \tag{29}$$

admits a unique solution represented by the potential (27) if the following conditions

$$\int_\Sigma f(x)\psi^{(h)}(x)d\sigma_x = 0, \quad h = 1,2,3 \tag{30}$$

are satisfied, $\{\psi^{(h)}\}$ being a complete system of linearly independent solutions of $(13)_0^+$. Moreover, the density of (27) is given by the potential $R\mathscr{R}[\Phi]$, where $\Phi \in \left[W^{1,p}(\Sigma)\right]^3 \times [L^p(\Sigma)]^3$ satisfies the following singular integral system

$$-\Phi + L^2\Phi = f \tag{31}$$

L being given by (26).

Proof

Assume that (30) hold. The system (31) can be rewritten as $(-I + L)(\Phi + L\Phi) = f$. Let $\gamma \in \left[W^{1,p}(\Sigma)\right]^3 \times [L^p(\Sigma)]^3$ be a solution of

$$-\gamma + L\gamma = f.$$

Such a solution does exist, because its homogeneous adjoint system $(13)_0^-$ has no eigensolutions [2, Theorem 2.10, p. 505]. The following system

$$\Phi + L\Phi = \gamma$$

admits solution if and only if

$$\int_\Sigma \gamma(x)\psi^{(h)}(x)d\sigma_x = 0, \quad h = 1,2,3, \tag{32}$$

where $\{\psi^{(h)}\}$ forms a complete system of linearly independent solutions of $(13)_0^+$. Keeping in mind (6) and that $\psi^{(h)}$ satisfies $(13)_0^+$, we have

$$\begin{aligned}
&\int_\Sigma f(x)\psi^{(h)}(x)d\sigma_x = \int_\Sigma [-I+L]\gamma(x)\psi^{(h)}(x)d\sigma_x = \\
&= -\int_\Sigma \gamma(x)\psi^{(h)}(x)d\sigma_x + \int_\Sigma \psi^{(h)}(x)d\sigma_x \int_\Sigma H_x[R_y\Psi(y-x)]'\gamma(y)d\sigma_y = \\
&= -\int_\Sigma \gamma(x)\psi^{(h)}(x)d\sigma_x + \int_\Sigma \gamma(y)d\sigma_y \int_\Sigma R_y[H_x\Psi(y-x)]'\psi^{(h)}(x)d\sigma_x = \\
&= -\int_\Sigma \gamma(x)\psi^{(h)}(x)d\sigma_x - \int_\Sigma \gamma(y)\psi^{(h)}(y)d\sigma_y = \\
&= -2\int_\Sigma \gamma(x)\psi^{(h)}(x)d\sigma_x.
\end{aligned}$$

Then, (32) is satisfied by virtue of (30).

Let now $\mathscr{V}_1$ and $\mathscr{V}_2$ be the solutions of (29) with datum f. Then, $\mathscr{U} = \mathscr{V}_1 - \mathscr{V}_2$ is the solution of the corresponding homogeneous problem. Because $[H\mathscr{U}]^+ = 0$, from (25), we have that $[\mathscr{U}T\mathscr{U}]^+ = 0$ on Σ. Then, $\mathscr{U}$ is a double layer potential whose first three components are zero. It follows from (16) that $E(\mathscr{U},\mathscr{U}) = 0$. Then, $u = a \wedge x + b, \omega = a, (a,b \in \mathbb{R}^3)$. Because $u = 0$, we have $a = b = 0$. □

By the symbol $\mathfrak{C}^p$, we denote the class of all linear combinations of potentials (10) and (9).

Theorem 3

Given $f \in [W^{1,p}(\Sigma)]^3 \times [L^p(\Sigma)]^3$, the following BVP

$$\begin{cases} \mathscr{U} \in \mathfrak{C}^p \\ M\mathscr{U} = 0 \quad \text{in } \Omega \\ [H\mathscr{U}]^+ = f \quad \text{on } \Sigma \end{cases} \tag{33}$$

admits a unique solution represented by

$$\mathscr{U} = \mathscr{H} + \mathscr{R}\left[-\frac{f}{2}\right], \tag{34}$$

where $\mathscr{H}$ is the potential (27) and $\mathscr{R}\left[-\frac{f}{2}\right]$ is the potential (9).

Proof

If $\mathscr{U}$ is a solution of (33) in the form (34), the boundary condition $[H\mathscr{U}]^+ = f$ is equivalent to

$$H\mathscr{H} = f - H\mathscr{R}\left[-\frac{f}{2}\right].$$

From the Proposition (1), the third BVP (29) admits a solution if

$$\int_\Sigma \left(f - H\mathscr{R}\left[-\frac{f}{2}\right]\right)\psi^{(h)}d\sigma = 0, \quad h = 1,2,3,$$

where $\{\psi^{(h)}\}$ is a complete system of linearly independent solutions of $(13)_0^+$. In fact,

$$\begin{aligned}
&\int_\Sigma H\mathscr{R}\left[-\frac{f}{2}\right](x)\psi^{(h)}(x)d\sigma_x = \\
&= -\int_\Sigma \left(-\frac{f}{2}\right)(x)\psi^{(h)}(x)d\sigma_x + \int_\Sigma \psi^{(h)}(x)d\sigma_x \int_\Sigma H_x[R_y\Psi(y-x)]'\left(-\frac{f}{2}(y)\right)d\sigma_y = \\
&= \int_\Sigma \frac{f}{2}(x)\psi^{(h)}(x)d\sigma_x - \int_\Sigma \frac{f}{2}(y)d\sigma_y \int_\Sigma R_y[H_x\Psi(y-x)]'\psi^{(h)}(x)d\sigma_x = \\
&= \int_\Sigma \frac{f}{2}(x)\psi^{(h)}(x)d\sigma_x + \int_\Sigma \frac{f}{2}(y)\psi^{(h)}(y)d\sigma_y = \int_\Sigma f(x)\psi^{(h)}(x)d\sigma_x.
\end{aligned}$$

Finally, the uniqueness of the solution follows from the uniqueness of the problem (29). □

5. Problem $(IV)^+$

In the last section, we represent the solution of the fourth BVP (40) as a potential (9).

Lemma 4
The following singular integral system

$$-\Phi + K^2\Phi = f \tag{35}$$

where $f \in [L^p(\Sigma)]^3 \times \left[W^{1,p}(\Sigma)\right]^3$ and

$$K\Phi(x) = \int_\Sigma R_x[H_y\Psi(y-x)]'\Phi(y)d\sigma_y$$

admits a solution $\Phi \in [L^p(\Sigma)]^3 \times \left[W^{1,p}(\Sigma)\right]^3$ if, and only if, the following conditions

$$\int_\Sigma f_k(y)d\sigma_y = 0, \quad k = 1,2,3 \tag{36}$$

are satisfied.

Proof
Let $\Phi \in [L^p(\Sigma)]^3 \times \left[W^{1,p}(\Sigma)\right]^3$. ~~Because~~ $\mathscr{H}[\Phi] \in \left[W^{1,p}(\Sigma)\right]^6$, we have $K[\Phi] \in [L^p(\Sigma)]^3 \times \left[W^{1,p}(\Sigma)\right]^3$.
The system (35) can be rewritten as $(-I+K)(\Phi + K\Phi) = f$. Consider

$$-\gamma + K\gamma = f. \tag{37}$$

Its homogeneous adjoint system is $(12)_0^+$, which has only trivial solutions [2, Theorem 2.10, p. 505]. Then there exists $\gamma \in [L^p(\Sigma)]^3 \times [W^{1,p}(\Sigma)]^3$ solution of (37) for any f. Consider now

$$\Phi + K\Phi = \gamma. \tag{38}$$

It admits a solution $\Phi \in [L^p(\Sigma)]^3 \times [W^{1,p}(\Sigma)]^3$ if, and only if, γ is orthogonal to every element of $(12)_0^-$. A complete system of linearly independent solutions of $(12)_0^-$ $\{\varphi^{(h)}\}$ is given by (14), for $h = 1,2,3$. Then there exists a solution γ of (38) if, and only if,

$$\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x = 0, \quad h = 1,2,3. \tag{39}$$

Keeping in mind (6) and that $\{\varphi^{(h)}\}$, $h = 1,2,3$, satisfies $(12)_0^-$, we have

$$\begin{aligned}
&\int_\Sigma f(x)\varphi^{(h)}(x)d\sigma_x = \int_\Sigma [-I+K]\gamma(x)\varphi^{(h)}(x)d\sigma_x = \\
&= -\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x + \int_\Sigma \varphi^{(h)}(x)d\sigma_x \int_\Sigma R_x[H_y\Psi(y-x)]'\gamma(y)d\sigma_y = \\
&= -\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x + \int_\Sigma \gamma(y)d\sigma_y \int_\Sigma H_y[R_x\Psi(y-x)]'\varphi^{(h)}(x)d\sigma_x = \\
&= -\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x - \int_\Sigma \gamma(y)\varphi^{(h)}(y)d\sigma_y = \\
&= -2\int_\Sigma \gamma(x)\varphi^{(h)}(x)d\sigma_x.
\end{aligned}$$

Thus, (39) are equivalent to (36). □

Theorem 4
Given $f \in [L^p(\Sigma)]^3 \times [W^{1,p}(\Sigma)]^3$, the fourth BVP

$$\begin{cases} \mathscr{U} \in \mathfrak{R}^p \\ M\mathscr{U} = 0 \quad \text{in } \Omega \\ [R\mathscr{U}]^+ = f \;\text{ on } \Sigma \end{cases} \tag{40}$$

admits a solution if, and only if, (36) is satisfied. The solution is determined up to an additive rigid translation, that is, an expression of the type (u,ω), where $u = b$ and $\omega = 0$, b being an arbitrary constant vector.
Moreover, the solution of (40) is represented by a potential (9) $\mathscr{R}[\Lambda]$ where its density is given by

$$\Lambda(x) = H_x \int_\Sigma [H_y\Psi(y-x)]'\Phi(y)d\sigma_y, \qquad x \in \Sigma, \tag{41}$$

$\Phi \in [L^p(\Sigma)]^3 \times [W^{1,p}(\Sigma)]^3$ being a solution of the singular integral system (35).

Proof

Let $\Phi \in [L^p(\Sigma)]^3 \times \left[W^{1,p}(\Sigma)\right]^3$. Because $\mathscr{H}[\Phi] \in \left[W^{1,p}(\Sigma)\right]^6$, we have $H\mathscr{H}[\Phi] \in \left[W^{1,p}(\Sigma)\right]^3 \times [L^p(\Sigma)]^3$. Then $\mathscr{R}[H\mathscr{H}[\Phi]] \in \left[W^{1,p}(\Sigma)\right]^6$. We consider a potential $\mathscr{U} \in \mathfrak{R}^p$ with density (41). The boundary condition $[R\mathscr{R}[\Lambda]]^+ = f$ turns into the system (35). In fact, by [2, Theorem 1.2, p. 492], we have

$$\mathscr{R}[H\mathscr{H}](x) = -2\mathscr{H}(x) + \int_\Sigma [H_y\Psi(y-x)]' R\mathscr{H}(y)d\sigma_y, \qquad x \in \Omega.$$

It is also known that [2, (1.23), p. 493]

$$\left[R\left(\int_\Sigma [H_y\Psi(y-x)]'\Phi(y)d\sigma_y\right)\right]^+ = \Phi(x) + R_x\int_\Sigma [H_y\Psi(y-x)]'\Phi(y)d\sigma_y, \qquad x \in \Sigma.$$

So we have

$$\begin{aligned}
[R\mathscr{R}[H\mathscr{H}](x)]^+ &= \left[R\left(-2\mathscr{H}(x) + \int_\Sigma [H_y\Psi(y-x)]'R\mathscr{H}(y)d\sigma_y\right)\right]^+ = \\
&= \left[-2R\mathscr{H}(x) + R_x\int_\Sigma [H_y\Psi(y-x)]'R\mathscr{H}(y)d\sigma_y\right]^+ = \\
&= -2R\mathscr{H}(x) + R\mathscr{H}(x) + R_x\int_\Sigma [H_y\Psi(y-x)]'R\mathscr{H}(y)d\sigma_y = \\
&= -R\mathscr{H}(x) + R_x\int_\Sigma [H_y\Psi(y-x)]'R\mathscr{H}(y)d\sigma_y.
\end{aligned}$$

Keeping in mind (10),

$$\begin{aligned}
&[R\mathscr{R}[H\mathscr{H}[\Phi]](x)]^+ = -R_x\left[\int_\Sigma [H_y\Psi(y-x)]'\Phi(y)d\sigma_y\right] + \\
&R_x\int_\Sigma [H_y\Psi(y-x)]'R_y\left[\int_\Sigma [H_z\Psi(z-y)]'\Phi(z)d\sigma_z\right]d\sigma_y = \\
&= -\Phi(x) - R_x\int_\Sigma [H_y\Psi(y-x)]'\Phi(y)d\sigma_y + R_x\int_\Sigma [H_y\Psi(y-x)]'\Phi(y)d\sigma_y + \\
&R_x\int_\Sigma [H_y\Psi(y-x)]'R_y\int_\Sigma [H_z\Psi(z-y)]'\Phi(z)d\sigma_z d\sigma_y = \\
&= -\Phi(x) + \int_\Sigma R_x[H_y\Psi(y-x)]'\int_\Sigma R_y[H_z\Psi(z-y)]'\Phi(z)d\sigma_z d\sigma_y = \\
&= -\Phi(x) + K^2\Phi(x)
\end{aligned}$$

and thus, because of $[R\mathscr{R}]^+ = f$, we obtain (35). Then there exists a solution of the fourth BVP if, and only if, (35) is solvable. If (36) is satisfied from Lemma 4, we have the assert.

Conversely, if $\mathscr{U} = (u, \omega)$ is a solution of (40), from [6, Lemma 4.3] and Lemma 2, we have

$$\int_\Sigma f(y)\varphi^{(h)}(y)d\sigma_y = \int_\Sigma R\mathscr{U}(y)\varphi^{(h)}(y)d\sigma_y = \int_\Omega E(\mathscr{U}(y), \varphi^{(h)}(y))dy = 0,$$

$(h = 1, 2, 3)$ because $E(\mathscr{U}, \varphi^{(h)}) = 0$.

Finally, we discuss the uniqueness. Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be solutions of (40) with datum f. Then, $\mathscr{U} = \mathscr{V}_1 - \mathscr{V}_2$ is the solution of the corresponding homogeneous problem. Because $[R\mathscr{U}]^+ = 0$, from (25), we have that $[\mathscr{U}T\mathscr{U}]^+ = 0$ on Σ. Then, $\mathscr{U}$ is a double layer potential whose last three components are zero. It follows from (16) that $E(\mathscr{U}, \mathscr{U}) = 0$. Then, $u = a \wedge x + b$, $\omega = a$, $(a, b \in \mathbb{R}^3)$. Because $\omega = 0$, we have $a = 0$. $\square$

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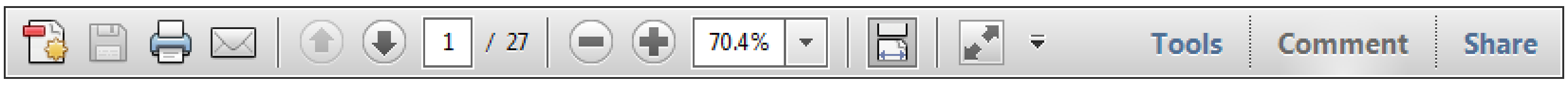

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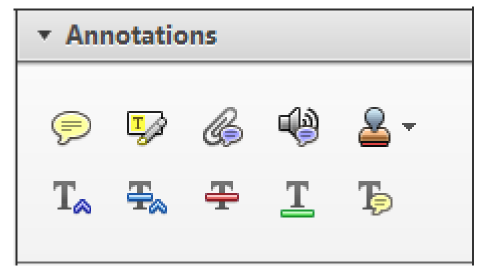

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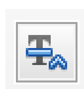

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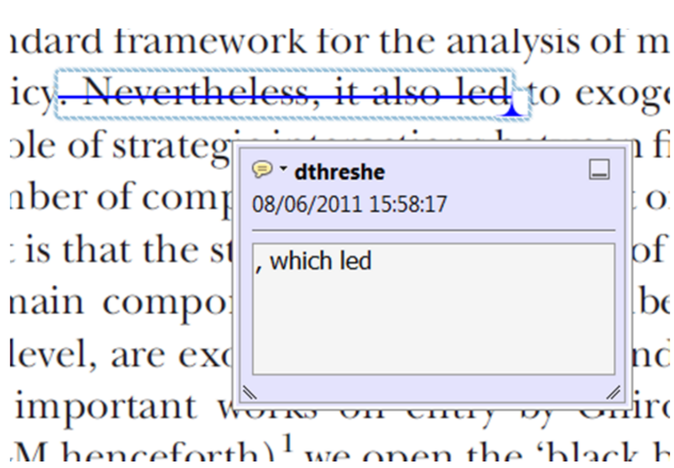

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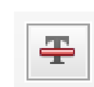

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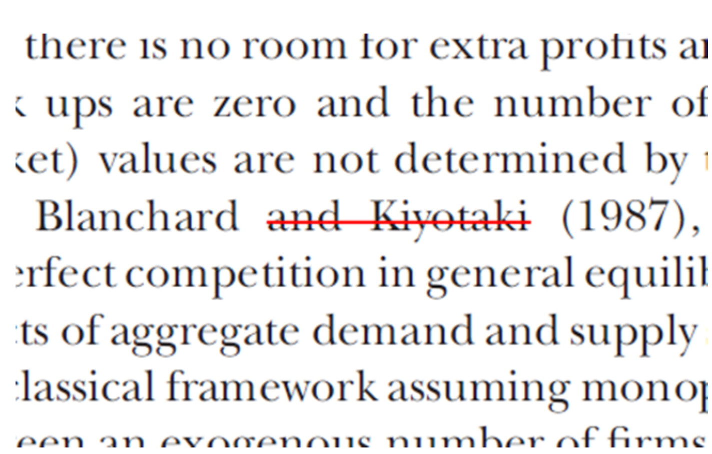
- Highlight a word or sentence.
- Click on the Strikethrough (Del) icon in the Annotations section.

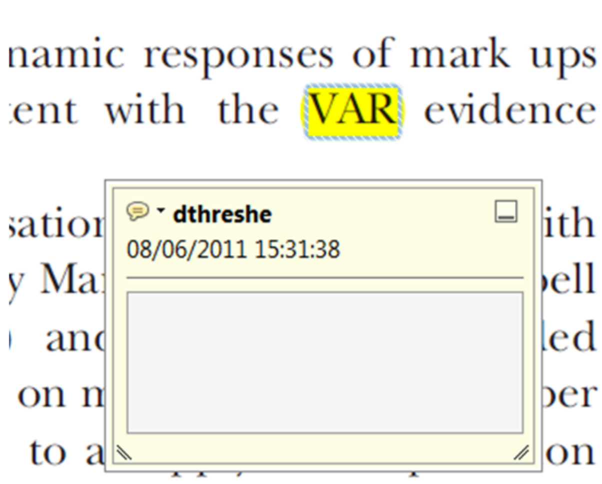


3. Add note to text Tool – for highlighting a section to be changed to bold or italic.

Highlights text in yellow and opens up a text box where comments can be entered.

How to use it

- Highlight the relevant section of text.
- Click on the Add note to text icon in the Annotations section.
- Type instruction on what should be changed regarding the text into the yellow box that appears.

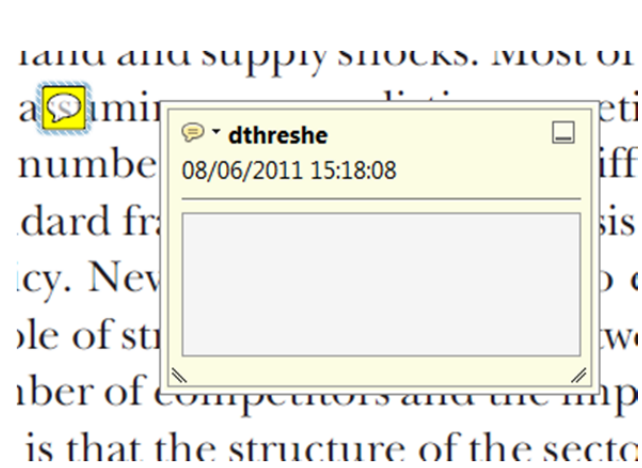


4. Add sticky note Tool – for making notes at specific points in the text.

Marks a point in the proof where a comment needs to be highlighted.

How to use it

- Click on the Add sticky note icon in the Annotations section.
- Click at the point in the proof where the comment should be inserted.
- Type the comment into the yellow box that appears.



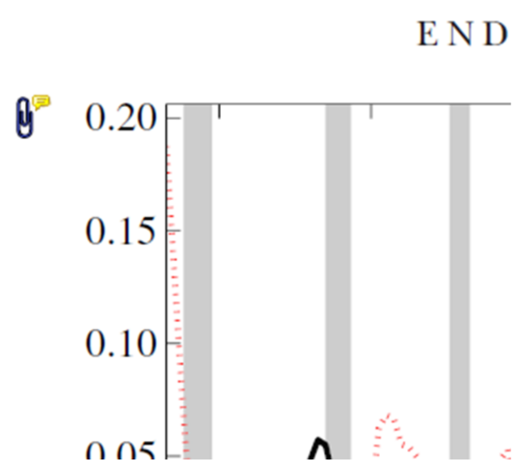
USING e-ANNOTATION TOOLS FOR ELECTRONIC PROOF CORRECTION

5. Attach File Tool – for inserting large amounts of text or replacement figures.

Inserts an icon linking to the attached file in the appropriate pace in the text.

How to use it

- Click on the Attach File icon in the Annotations section.
- Click on the proof to where you'd like the attached file to be linked.
- Select the file to be attached from your computer or network.
- Select the colour and type of icon that will appear in the proof. Click OK.

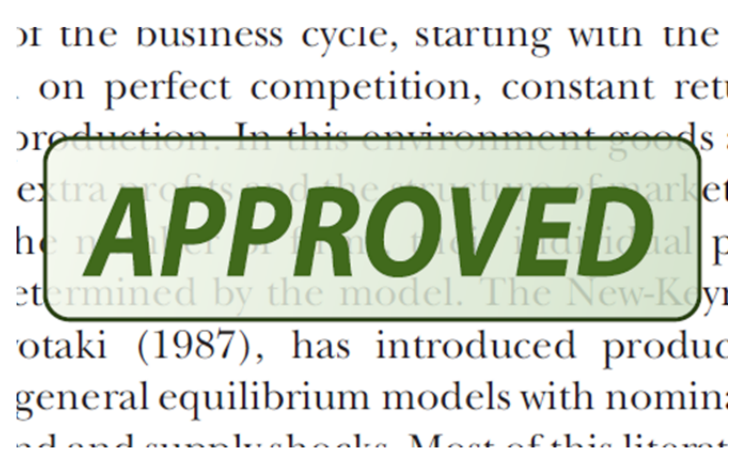


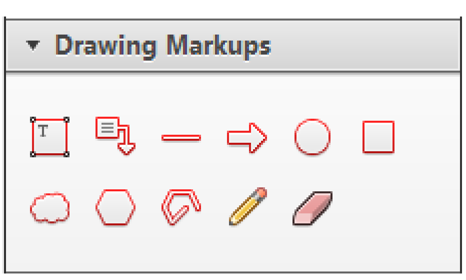
6. Add stamp Tool – for approving a proof if no corrections are required.

Inserts a selected stamp onto an appropriate place in the proof.

How to use it

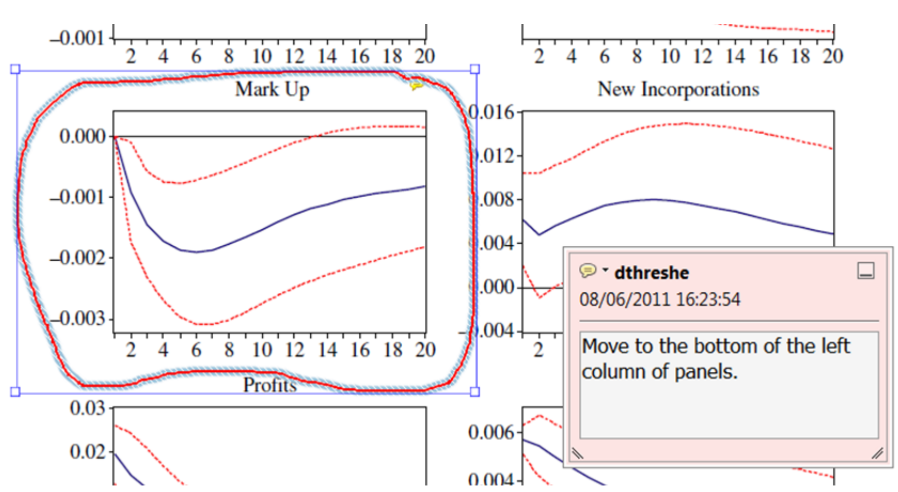
- Click on the Add stamp icon in the Annotations section.
- Select the stamp you want to use. (The Approved stamp is usually available directly in the menu that appears).
- Click on the proof where you'd like the stamp to appear. (Where a proof is to be approved as it is, this would normally be on the first page).





7. Drawing Markups Tools – for drawing shapes, lines and freeform annotations on proofs and commenting on these marks.

Allows shapes, lines and freeform annotations to be drawn on proofs and for comment to be made on these marks..



How to use it

- Click on one of the shapes in the Drawing Markups section.
- Click on the proof at the relevant point and draw the selected shape with the cursor.
- To add a comment to the drawn shape, move the cursor over the shape until an arrowhead appears.
- Double click on the shape and type any text in the red box that appears.

For further information on how to annotate proofs, click on the Help menu to reveal a list of further options: