An application of the theory of self-conjugate differential forms to the Dirichlet problem for Cimmino system

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Abstract. In the present paper we find necessary and sufficient conditions for the solvability of the Dirichlet problem for Cimmino system in simply and multiply connected domains. Our results hinge on the theory of self-conjugate differential forms, which are non homogeneous differential forms U such that $dU = \delta U$.

Introduction

In paper [6] Dragomir and Lanconelli studied the system (introduced by Cimmino [4])

$$\begin{cases} f_{0_{x_1}} - f_{1_{x_2}} + f_{2_{x_3}} - f_{3_{x_4}} = 0 \\ f_{0_{x_2}} + f_{1_{x_1}} - f_{2_{x_4}} - f_{3_{x_3}} = 0 \\ f_{0_{x_3}} - f_{1_{x_4}} - f_{2_{x_1}} + f_{3_{x_2}} = 0 \\ f_{0_{x_4}} + f_{1_{x_3}} + f_{2_{x_2}} + f_{3_{x_1}} = 0, \end{cases} \tag{1}$$

where $f_i : \Omega \subset \mathbb{R}^4 \to \mathbb{R}(i = 0, \dots, 3)$. Setting $u = f_0 + if_1$ and $v = f_2 + if_3$, Cimmino system (1) can be written in the equivalent form

$$\begin{cases} u_{\overline{z}} + \overline{v}_w = 0 \\ u_{\overline{w}} - \overline{v}_z = 0. \end{cases}$$

In [6] Dragomir and Lanconelli obtained, among many other results, a necessary condition for the resolubility of the Dirichlet problem for Cimmino system:

Let $\Omega \subset \mathbb{C}^2$ be a bounded domain on which Green's formula holds and Σ its boundary; let $f,\ g \in L^2(\Omega)$, $\Phi,\ \Psi \in L^2(\Sigma)$. If there is a solution $u, v \in C^1(\Omega) \cap C^0(\overline{\Omega})$ to the boundary value problem

$$\begin{cases} u_{\overline{z}} + \overline{v}_w = f, \quad u_{\overline{w}} - \overline{v}_z = g & \textit{in } \Omega \\ \qquad u = \Phi \qquad v = \Psi & \textit{on } \Sigma, \end{cases} \tag{2}$$

then (f, g, Φ, Ψ) *satisfies the compatibility relations*

$$Re\left\{2\int_\Omega (f\overline{h} + g\overline{k})dV - \int_\Sigma \left\{\Phi\left[(n_1 + in_2)\overline{h} + (n_3 + in_4)\overline{k}\,\right]\right.\right.$$
$$\left.\left. +\Psi\left[(n_3 + in_4)h - (n_1 + in_2)k\right]\right\}d\sigma\right\} = 0 \quad (3)$$

for any solution $h, k \in C^1(\Omega) \cap C^0(\overline{\Omega})$ *to*

$$h_z + k_w = 0, \qquad h_{\overline{w}} - k_{\overline{z}} = 0 \qquad in\ \Omega,$$

where (n_1, n_2, n_3, n_4) *is the outward unit normal on* Σ.

In the same paper the Authors address the problem whether conditions (3) are also sufficient for the resolubility of the Dirichlet problem (2).

More recently, Abreu Blaya et al. [1] studied (1) by means of quaternionic analysis. In particular, they found some different necessary and sufficient conditions involving some particular integral operators. From this they deduce that (3) are also sufficient when $f = g = 0$ and Ω is a simply connected domain.

In this paper we study the Dirichlet problem (2) with a different approach. We obtain necessary and sufficient conditions for the resolubility of this problem. In the case of a simply connected domain, we have proved that there exists a solution of the Dirichlet problem (2) if, and only if, a certain denumerable set of orthogonality conditions are satisfied. These conditions involve the data f, g, Φ and Ψ and a particular system of differential forms whose coefficients are harmonic polynomials.

We also extend this result to multiply connected domains.

Our approach is based on the theory of self-conjugate differential forms. Following [3], we say that a non homogeneous differential form $U = \sum_{k=0}^{n} u_k$, u_k being a k-form, is self-conjugate if it is solution of the equation $dU = \delta U$, where d and δ are the differential and the codifferential operators respectively.

The paper is organized as follows.

In section 1, we collect some properties of self-conjugate differential forms.

In section 2, we study the Dirichlet problem for the equation $dU - \delta U = F$ in simply and multiply connected domains. In particular we give necessary and sufficient conditions for its solvability.

Hinging on these results, in the last section we find necessary and sufficient conditions for the resolubility of the Dirichlet problem for the Cimmino system. Finally, we remark that our results show that Dragomir-Lanconelli conditions (3) are not only necessary but also sufficient for the solvability of (2).

1 Definitions and preliminary results

In this section, we introduce some notations and recall some of the basic facts about differential forms. For more details see, e.g., [7], [8].

A *differential form of degree* k, or briefly a k*-form*, on a domain $T \subset \mathbb{R}^n$ is a function defined on T whose values are in the k-covectors space of $\mathbb{R}^n$. In an admissible coordinate system $(x_1, \ldots, x_n)$, a k-form u is expressed as

$$u = \frac{1}{k!} u_{s_1,\ldots,s_k} dx_{s_1} \ldots dx_{s_k},$$

where $u_{s_1,\ldots,s_k}$ are the components of a k-covector, i.e. the components of a skew-symmetric covariant tensor.

By $C^q_k(T)$ we denote the space of the k-forms whose coefficients are continuously up to the order q in a coordinate system of class C^{q+1} (and then in every coordinate system of class C^{q+1}). Moreover the symbol L^p_k stands for the space of all k-forms whose coefficients are L^p real valued functions.

If $u \in C^0_k(T)$, the *adjoint* of u is the following $(n-k)$-form

$$*u = \frac{1}{(n-k)!} \frac{1}{k!} \delta^{1,\ldots,n}_{s_1,\ldots,s_k,i_1,\ldots,i_{n-k}} u_{s_1,\ldots,s_k} dx_{i_1} \ldots dx_{i_{n-k}}.$$

We remark that $** u = (-1)^{k(n-k)} u$.

If $u \in C^1_k(T)$, the *differential* of u is the following $(k+1)$-form

$$du = \frac{1}{k!} \frac{\partial}{\partial x_j} u_{s_1,\ldots,s_k} dx_j dx_{s_1} \ldots dx_{s_k},$$

while the *codifferential* of u is the $(k-1)$-form defined as follows

$$\delta u = (-1)^{n(k+1)+1} * d * u.$$

These operators are strictly related to the Laplacian; indeed if $u \in C^2_k$

$$-(d\delta + \delta d) u = \Delta u = \frac{1}{k!} \Delta u_{s_1,\ldots,s_k} dx_{s_1} \ldots dx_{s_k},$$

where $\Delta u_{s_1,\ldots,s_k} = \sum_{h=1}^{n} \frac{\partial^2}{\partial x_h^2} u_{s_1,\ldots,s_k}$.

These definitions can be immediately extended to non homogeneous differential forms; if $U = \sum_{k=0}^{n} u_k$, where u_k is a k-form, we set

$$dU = \sum_{k=0}^{n-1} du_k, \quad \delta U = \sum_{k=1}^{n} \delta u_k, \quad \Delta U = \sum_{k=0}^{n} \Delta u_k.$$

Since $d^2 = 0$ and $\delta^2 = 0$, we can write

$$\Delta = (d - \delta)^2. \tag{4}$$

We denote by $C^k(\Omega)$ the space $C^k_0(\Omega) \oplus \ldots \oplus C^k_n(\Omega)$; similarly $L^p(\Omega) = L^p_0(\Omega) \oplus \ldots \oplus L^p_n(\Omega)$ is the space composed by k-forms whose coefficients are L^p real valued functions defined in Ω.

Definition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $U \in C^1(\Omega)$; we say that U is self-conjugate if*

$$dU = \delta U \qquad in\ \Omega$$

i.e.

$$\delta u_1 = 0, \quad du_{k-1} = \delta u_{k+1}\ (k = 1, \ldots, n-1), \quad du_{n-1} = 0.$$

From (4) it follows that if U is self-conjugate then U is harmonic, i.e. all the coefficients of u_k are harmonic functions.
Such forms generalize the concept of holomorphic functions of one complex variable. Moreover solutions of the Moisil-Theodorescu system and Fueter system can be considered as very particular case of self-conjugate differential forms in $\mathbb{R}^3$ and in $\mathbb{R}^4$ respectively. Other examples in $\mathbb{R}^n$ are given by harmonic vectors, i.e. vectors $(w_1, \ldots, w_n)$ such that $div(w_1, \ldots, w_n) = 0$, $curl(w_1, \ldots, w_n) = 0$, and harmonic forms, i.e. k-form u_k which are solutions of $du_k = \delta u_k = 0$ (see [3]).

In a similar way we have a relation between self-conjugate differential forms and solutions of the Cimmino system. Indeed a straightforward calculation shows that

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^4$ be an open set and*

$$U = u_0 + u_2 + u_4,$$

where

$$\begin{aligned} u_0 = f_0, \quad u_2 = {} & f_1(dx_1dx_2 + dx_3dx_4) - f_2(dx_1dx_3 + dx_4dx_2) \\ & + f_3(dx_1dx_4 + dx_2dx_3), \quad u_4 = -f_0dx_1dx_2dx_3dx_4. \end{aligned}$$

U is self-conjugate in Ω if and only if (f_0, f_1, f_2, f_3) is solution of the Cimmino system

$$\begin{cases} {f_0}_{x_1} - {f_1}_{x_2} + {f_2}_{x_3} - {f_3}_{x_4} = 0 \\ {f_0}_{x_2} + {f_1}_{x_1} - {f_2}_{x_4} - {f_3}_{x_3} = 0 \\ {f_0}_{x_3} - {f_1}_{x_4} - {f_2}_{x_1} + {f_3}_{x_2} = 0 \\ {f_0}_{x_4} + {f_1}_{x_3} + {f_2}_{x_2} + {f_3}_{x_1} = 0. \end{cases}$$

Let us consider now the double k-form introduced by Hodge

$$s_k(x,y) = \sum_{j_1<\ldots<j_k} s(x-y)dx_{j_1}\ldots dx_{j_k}dy_{j_1}\ldots dy_{j_k},$$

where

$$s(x-y) = \begin{cases} \dfrac{1}{2\pi}\log|x-y| & \text{if } n=2 \\ -\dfrac{1}{(n-2)\omega_n}|x-y|^{2-n} & \text{if } n>2 \end{cases}$$

(ω_n being the hypersurface measure of the unit sphere of $\mathbb{R}^n$) is the fundamental solution of Laplace equation. It satisfies the following identities for $x \neq y$:

$$d_y s_k(x,y) = \delta_x s_{k+1}(x,y), \qquad k=0,\ldots,n-1, \tag{5}$$

(see [5]) from which one can prove that

$$\begin{cases} \delta_x \underset{y}{*} d_y s_k(x,y) = 0, & \delta_x d_y s_k(x,y) = 0, \\ d_x \delta_y s_k(x,y) = 0, & d_x \underset{y}{*} \delta_y s_k(x,y) = 0, \end{cases} \tag{6}$$

$$\begin{cases} d_x \underset{y}{*} d_y s_k(x,y) = -\delta_x \underset{y}{*} \delta_y s_{k+2}(x,y), \\ d_x d_y s_k(x,y) = -\delta_x \delta_y s_{k+2}(x,y). \end{cases} \tag{7}$$

Moreover

$$\underset{x}{*}\, d_x s_k(x,y) = (-1)^{nk+1} \underset{y}{*}\, d_y s_{n-1-k}(x,y), \tag{8}$$

$$\underset{x}{*}\, s_k(x,y) = (-1)^{(n-k)k} \underset{y}{*}\, s_{n-k}(x,y). \tag{9}$$

Let now Ω be a *regular domain;* this means that Ω is a bounded domain, its boundary Σ is an orientable $(n-1)$-dimensional C^1 differentiable manifold and for any $u \in C^0_{n-1}(\overline{\Omega}) \cap C^1_{n-1}(\Omega)$ such that $du \in C^0_n(\overline{\Omega})$ the Stokes formula holds

$$\int_\Omega du = \int_{+\Sigma} u.$$

This implies

$$\int_\Omega du \wedge *v = \int_{+\Sigma} u \wedge *v + \int_\Omega \delta v \wedge *u \quad \forall u \in C^1_k(\overline{\Omega}),\ v \in C^1_{k+1}(\overline{\Omega}).$$

If $U = \sum_{k=0}^{n} u_k \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ is self-conjugate, we may write

$$\begin{cases} \displaystyle\int_\Omega dv \wedge *u_1 = \int_{+\Sigma} v \wedge *u_1\,, & \forall v \in C^1_0(\overline{\Omega}) \\ \displaystyle\int_\Omega [dv \wedge *u_{k+2} - \delta v \wedge *u_k] = \int_{+\Sigma} [u_k \wedge *v + v \wedge *u_{k+2}]\,, & \\ & \forall v \in C^1_{k+1}(\overline{\Omega}), \quad k = 0, \dots, n-2 \\ \displaystyle -\int_\Omega \delta v \wedge *u_{n-1} = \int_{+\Sigma} u_{n-1} \wedge *v\,, & \forall v \in C^1_n(\overline{\Omega}). \end{cases}$$

Theorem 1.1. *If Ω is a regular domain and $U \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ is such that $dU - \delta U = F \in C^0(\overline{\Omega})$, then the following Cauchy integral formula holds*

$$\begin{aligned} -\int_\Omega [d_y s_k(x,y) \wedge *F_{k+1}(y) - \delta_y s_k(x,y) \wedge *F_{k-1}(y)] + \\ + \int_{+\Sigma} \Big[u_k(y) \wedge \underset{y}{*} d_y s_k(x,y) - \delta_y s_k(x,y) \wedge *u_k(y) + \\ + d_y s_k(x,y) \wedge *u_{k+2}(y) - u_{k-2}(y) \wedge \underset{y}{*} \delta_y s_k(x,y) \Big] = \\ = \begin{cases} u_k(x) & x \in \Omega \\ 0 & x \notin \overline{\Omega} \end{cases} \end{aligned} \tag{10}$$

$(k = 0, \dots, n)$, where $U = \sum_{k=0}^{n} u_k$, $F = \sum_{k=0}^{n} F_k$, $u_k \equiv 0, k = -2, -1, n+1, n+2$; $F_k \equiv 0, k = -1, n+1$.

This theorem is proved in [3, Th. II] under the hypothesis $F = 0$. The same arguments apply to prove the slightly more general Theorem 1.1.
We remark that in the case $n = 2$ formula (10) gives

$$\begin{aligned} -\frac{1}{2\pi} \int_\Omega d_\zeta \log|z-\zeta| \wedge *F_1(\zeta) - \frac{1}{2\pi} \int_\Sigma \Big[u(\zeta) \frac{\partial}{\partial n_\zeta} \log|z-\zeta| \\ - v(\zeta) \frac{\partial}{\partial s_\zeta} \log|z-\zeta| \Big] ds_\zeta = \begin{cases} u(z) & z \in \Omega \\ 0 & z \notin \overline{\Omega}, \end{cases} \end{aligned}$$

$$\begin{aligned} \frac{1}{2\pi} \int_\Omega - * d_\zeta \log|z-\zeta| \wedge *F_1(\zeta) - \frac{1}{2\pi} \int_\Sigma \Big[v(\zeta) \frac{\partial}{\partial n_\zeta} \log|z-\zeta| \\ + u(\zeta) \frac{\partial}{\partial s_\zeta} \log|z-\zeta| \Big] ds_\zeta = \begin{cases} v(z) & z \in \Omega \\ 0 & z \notin \overline{\Omega}. \end{cases} \end{aligned}$$

Putting $f(z) = u(z) + iv(z)$ we reobtain the classical formula

$$\frac{1}{2\pi i} \int_{+\Sigma} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \int_\Omega \frac{f_{\overline{\zeta}}(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z) & z \in \Omega \\ 0 & z \notin \overline{\Omega}. \end{cases}$$

2 Main Result

In this section we want to study the boundary behaviour of forms such that $dU - \delta U = F$. We say that $U \in L^1_{\mathrm{loc}}(\Omega)$ is a weak solution of $dU - \delta U = F \in L^1_{\mathrm{loc}}(\Omega)$ if

$$\int_\Omega (d\phi - \delta\phi) \wedge *U = -\int_\Omega F \wedge *\phi \qquad \forall \phi \in \mathring{C}^\infty(\Omega). \tag{11}$$

We denote $L^1_0(\Sigma) \oplus \ldots \oplus L^1_{n-1}(\Sigma)$ by $L^1(\Sigma)$. Let us introduce the following spaces:

$$\begin{aligned}\mathcal{U} = \Bigg\{ & U \in L^1(\Omega) : \exists\, \phi = \sum_{k=0}^{n-1} \phi_k,\ \widetilde{\phi} = \sum_{k=0}^{n-1} \widetilde{\phi}_{n-k} \in L^1(\Sigma),\ F = \sum_{k=0}^{n} F_k \in L^1(\Omega) \\ & \text{such that } \sum_{k=0}^{n-1} \int_\Omega dv_k \wedge *u_{k+1} - \sum_{k=1}^{n} \int_\Omega \delta v_k \wedge *u_{k-1} - \sum_{k=0}^{n} \int_\Omega v_k \wedge *F_k = \\ = & \sum_{k=0}^{n-1} \int_{+\Sigma} v_k \wedge \widetilde{\phi}_{k+1} + \sum_{k=1}^{n} \int_{+\Sigma} \phi_{k-1} \wedge * v_k \qquad \text{for any } V = \sum_{k=0}^{n} v_k \in C^1(\mathbb{R}^n) \Bigg\},\end{aligned} \tag{12}$$

$$\begin{aligned}\mathcal{V} = \Bigg\{ & U \in L^1(\Omega) : \exists\, \phi = \sum_{k=0}^{n-1} \phi_k,\ \widetilde{\phi} = \sum_{k=0}^{n-1} \widetilde{\phi}_{n-k} \in L^1(\Sigma),\ F = \sum_{k=0}^{n} F_k \in L^1(\Omega) \\ & \text{such that } -\int_\Omega [d_y s_k(x,y) \wedge *F_{k+1}(y) - \delta_y s_k(x,y) \wedge *F_{k-1}(y)] + \\ + & \int_{+\Sigma} \Big[\phi_k(y) \wedge \underset{y}{*} d_y s_k(x,y) - \delta_y s_k(x,y) \wedge *\phi_k(y) + d_y s_k(x,y) \wedge *\phi_{k+2}(y) \\ & -\phi_{k-2}(y) \wedge \underset{y}{*} \delta_y s_k(x,y) \Big] = \begin{cases} u_k(x) & x \in \Omega \\ 0 & x \notin \overline{\Omega} \end{cases} \quad k = 0, \ldots, n \\ & (\phi_k \equiv 0, k = -2, -1; \widetilde{\phi}_k \equiv 0, k = n+1, n+2; F_k \equiv 0, k = -1, n+1) \Bigg\}.\end{aligned}$$

Roughly speaking the space $\mathcal{U}$ is given by the L^1 differential forms solutions of $dU - \delta U = F$ in Ω having L^1 traces in a weak sense (see (11)), while $\mathcal{V}$ is the space of the L^1 forms in Ω such that there exist L^1 forms on Σ for which Cauchy integral formula holds.

Actually these two spaces are equal.

Theorem 2.1. $\mathcal{U} = \mathcal{V}$

This theorem is proved in [3, Th. III] in the case $F = 0$. By a similar argument, it is possible to prove the slightly more general Theorem 2.1.

2.1 The Dirichlet problem for the equation $dU - \delta U = F$ in a simply connected domain

Lemma 2.1. *There exists $\delta > 0$ such that if $0 < r < R$, with $r \leq \delta R$, the development*

$$s(x-y) = \sum_{k=0}^{\infty} \sum_{h=1}^{p_{nk}} \frac{Q_{hk}(x) P_{hk}(y)}{|x|^{2k-2+n}}$$

holds uniformly for $|y| \leq r$ and $|x| \geq R$, where P_{hk} and Q_{hk} are homogeneous harmonic polynomials of degree k.

Proof. Fix ξ such that $|\xi| = 1$. Since s is analytic, there exists $r_\xi > 0$ such that

$$s(t-\xi) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!} \left[D^\alpha s(\nu)\right]_{\nu=-\xi} t^\alpha \tag{13}$$

uniformly for $|t| \leq r_\xi$.
From the compactness of the unit sphere it is possible to find a finite cover $B_{r_{\xi_j}}(\xi_j)$, $j = 1, \dots, m$, $|\xi_j| = 1$. If we put $\delta = \min_j r_{\xi_j}$ we have that (13) holds uniformly for $|t| \leq \delta$, for any ξ such that $|\xi| = 1$.
Fix $0 < r < R$ with $r \leq \delta R$. Observing that

$$\frac{1}{|x-y|^{n-2}} = \frac{1}{|x|^{n-2} \left| \dfrac{x}{|x|} - \dfrac{y}{|x|} \right|^{n-2}}$$

we have

$$s(x-y) = |x|^{2-n} \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!} \left[D^\alpha s(\nu)\right]_{\nu=-\frac{x}{|x|}} \left(\frac{y}{|x|} \right)^\alpha,$$

uniformly for $|y| \leq r,\ |x| \geq R$.
Since $D^\alpha s$ is homogeneous of degree $2 - n - |\alpha|$, we have

$$\left[D^\alpha s(\nu)\right]_{\nu=-\frac{x}{|x|}} = |x|^{n-2+|\alpha|} \left[D^\alpha s(\nu)\right]_{\nu=-x},$$

and then

$$s(x-y) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{\alpha!} \left[D^\alpha s(\nu)\right]_{\nu=-x} y^\alpha \tag{14}$$

uniformly for $|y| \leq r,\ |x| \geq R$.
We have thus obtained, uniformly for $|y| \leq r,\ |x| \geq R$.

$$s(x-y) = \sum_{k=0}^{\infty} \sum_{h=1}^{p_{nk}} R_{hk}(x) P_{hk}(y),$$

where $R_{1k}(x), \ldots, R_{p_{nk}k}(x)$ form a basis for the functions $[D^\alpha s(\nu)]_{\nu=-x}$ ($|\alpha| = k$) and $P_{hk}(y)$ are homogeneous polynomials of degree k.
Series (14) being a power series in y, we can derive it term by term with respect to y, obtaining

$$0 = \Delta_y s(x-y) = \sum_{k=0}^{\infty} \sum_{h=1}^{p_{nk}} R_{hk}(x) \Delta P_{hk}(y).$$

Since

$$\sum_{h=1}^{p_{nk}} R_{hk}(x) \Delta P_{hk}(y)$$

are homogeneous polynomials of degree $k-2$ with respect to y, it follows that (see, e.g. [2], p.23)

$$\sum_{h=1}^{p_{nk}} R_{hk}(x) \Delta P_{hk}(y) = 0.$$

Linearly independence of $R_{1k}(x), \ldots, R_{p_{nk}k}(x)$ implies $\Delta P_{hk}(y) = 0$. Therefore $P_{hk}(y)$ are homogeneous harmonic polynomials of degree k.
Moreover, by induction, one can prove that

$$[D^\alpha s(\nu)]_{\nu=-x} = \frac{Q_\alpha(x)}{|x|^{n-2+2|\alpha|}},$$

where $Q_\alpha(x)$ are homogeneous polynomials of degree $|\alpha|$. The left-hand side being harmonic, we find

$$\Delta \left(\frac{Q_\alpha(x)}{|x|^{n-2+2|\alpha|}} \right) = 0.$$

It is well known that this implies $\Delta Q_\alpha = 0$. It follows that $R_{hk}(x)$ can be written as $\dfrac{Q_{hk}(x)}{|x|^{n-2+2k}}$, where Q_{hk} are homogeneous harmonic polynomials of degree k. This completes the proof. $\square$

Let us denote by $w_h^{i_1,\ldots,i_k}$ the $k-$form $w_h dx_{i_1} \ldots dx_{i_k}$, where $\{w_h\}$ is a complete system of homogeneous harmonic polynomials. Such a system can be obtained by ordering in one sequence the polynomials:

$$|x|^k Y_s^k \left(\frac{x}{|x|} \right), \qquad k = 0, 1, 2, \ldots;$$

$$s = 1, \ldots, p_{nk}; \qquad p_{nk} = (2k+n-2)\frac{(k+n-3)!}{(n-2)!k!},$$

where $Y_1^k(\omega), \ldots, Y_{p_{nk}}^k(\omega)$ is a complete system of (surface) spherical harmonics of degree k.

Theorem 2.2. *Let Ω be a regular domain such that $\mathbb{R}^n \setminus \Omega$ is connected. Let $\phi = \sum_{k=0}^{n-1} \phi_k$, $\widetilde{\phi} = \sum_{k=0}^{n-1} \widetilde{\phi}_{n-k} \in L^1(\Sigma)$ and $F = \sum_{k=0}^{n} F_k \in L^1(\Omega)$ be given forms. There exists a non homogeneous differential form $U \in L^1(\Omega)$ solution*[1] *of*

$$\begin{cases} dU - \delta U = F & \text{in } \Omega \\ U = \phi \quad * U = \widetilde{\phi} & \text{on } \Sigma \end{cases} \tag{15}$$

if and only if

$$\begin{aligned} (-1)^{(n-1)(k-1)+1} & \int_\Omega \left[*F_{k+1} \wedge dw_h^{i_1,\dots,i_k} - *F_{k-1} \wedge \delta w_h^{i_1,\dots,i_k} \right] \\ & - \frac{1}{2} \left\{ \int_{+\Sigma} \left[\phi_k \wedge * dw_h^{i_1,\dots,i_k} - \delta w_h^{i_1,\dots,i_k} \wedge \widetilde{\phi}_k \right. \right. \\ & \left. \left. + dw_h^{i_1,\dots,i_k} \wedge \widetilde{\phi}_{k+2} - \phi_{k-2} \wedge * \delta w_h^{i_1,\dots,i_k} \right] \right\} = 0 \end{aligned} \tag{16}$$

for any $1 \le i_1 < \ldots < i_k \le n, h = 1, 2, \ldots,\ k = 0, 1, \ldots, n$ ($\phi_k \equiv 0, k = -2, -1; \widetilde{\phi}_k \equiv 0, k = n+1, n+2; F_k \equiv 0, k = -1, n+1$).

Proof. If there exists U solution of (15), then $U \in \mathcal{U}$. In particular, taking $V = -\delta w_h^{i_1,\dots,i_k} + dw_h^{i_1,\dots,i_k}$ in (12), we obtain (16). Conversely, let us suppose that (16) are satisfied. Taking $r > \max_{y \in \Sigma} |y|$ in Lemma 2.1, the following development

$$s(x-y) = \sum_{k=0}^{\infty} \sum_{h=1}^{p_{nk}} \frac{Q_{hk}(x) P_{hk}(y)}{|x|^{2k-2+n}} \tag{17}$$

holds uniformly for any $y \in \Sigma$ and for any $x \in \mathbb{R}^n \setminus B$, B being a ball of radius $R \ge \frac{r}{\delta}$ centered at 0. It follows from (16) that

$$\begin{aligned} (-1)^{(n-1)(k-1)+1} & \int_\Omega [*F_{k+1}(y) \wedge d_y [s(x-y) dy_{i_1} \ldots dy_{i_k}] - *F_{k-1}(y) \wedge \delta_y [s(x-y) dy_{i_1} \ldots dy_{i_k}]] \\ -\frac{1}{2} & \left\{ \int_{+\Sigma} \left[\phi_k(y) \wedge \underset{y}{*} d_y [s(x-y) dy_{i_1} \ldots dy_{i_k}] - \delta_y [s(x-y) dy_{i_1} \ldots dy_{i_k}] \wedge \widetilde{\phi}_k(y) \right. \right. \\ & \left. \left. + d_y [s(x-y) dy_{i_1} \ldots dy_{i_k}] \wedge \widetilde{\phi}_{k+2}(y) - \phi_{k-2}(y) \wedge \underset{y}{*} \delta_y [s(x-y) dy_{i_1} \ldots dy_{i_k}] \right] \right\} = 0, \end{aligned}$$

for any $x \in \mathbb{R}^n \setminus B$. Since $\mathbb{R}^n \setminus \overline{\Omega}$ is connected this is still true for any

[1] In the problem (15) the equation $dU - \delta U = F$ is considered in the weak sense (11).

$x \in \mathbb{R}^n \setminus \overline{\Omega}$. Then

$$
\begin{aligned}
(-1)^{(n-1)(k-1)+1} \int_{\Omega} & [*F_{k+1}(y) \wedge d_y s_k(x,y) - *F_{k-1}(y) \wedge \delta_y s_k(x,y)] \\
& - \frac{1}{2} \left\{ \int_{+\Sigma} \left[\phi_k(y) \wedge \underset{y}{*} d_y s_k(x,y) - \delta_y s_k(x,y) \wedge \widetilde{\phi}_k(y) \right.\right. \\
& \left.\left. + d_y s_k(x,y) \wedge \widetilde{\phi}_{k+2}(y) - \phi_{k-2}(y) \wedge \underset{y}{*} \delta_y s_k(x,y) \right] \right\} = 0 \qquad \forall x \notin \overline{\Omega}. \quad (18)
\end{aligned}
$$

Let us denote by u_k the left-hand side of (18), when $x \in \Omega$. We have

$$
\begin{aligned}
du_k(x) = & (-1)^{(n-1)(k-1)+1} d_x \left\{ \int_{\Omega} [*F_{k+1}(y) \wedge d_y s_k(x,y) - *F_{k-1}(y) \wedge \delta_y s_k(x,y)] \right\} \\
& - \frac{1}{2} d_x \left\{ \int_{+\Sigma} \left[\phi_k(y) \wedge \underset{y}{*} d_y s_k(x,y) - \delta_y s_k(x,y) \wedge \widetilde{\phi}_k(y) \right.\right. \\
& \left.\left. + d_y s_k(x,y) \wedge \widetilde{\phi}_{k+2}(y) - \phi_{k-2}(y) \wedge \underset{y}{*} \delta_y s_k(x,y) \right] \right\}, \quad x \in \Omega.
\end{aligned}
$$

Since $d^2 = \delta^2 = 0$ and using (5)

$$
\begin{aligned}
du_k(x) = & (-1)^{(n-1)(k-1)+1} d_x \delta_x \left[\int_{\Omega} *F_{k+1}(y) \wedge s_{k+1}(x,y) \right] \\
& - \frac{1}{2} \left\{ \int_{+\Sigma} \left[\phi_k(y) \wedge d_x \underset{y}{*} d_y s_k(x,y) - d_x \delta_y s_k(x,y) \wedge \widetilde{\phi}_k(y) \right.\right. \\
& \left.\left. + d_x d_y s_k(x,y) \wedge \widetilde{\phi}_{k+2}(y) - \phi_{k-2}(y) \wedge d_x \underset{y}{*} \delta_y s_k(x,y) \right] \right\}, \quad x \in \Omega. \quad (19)
\end{aligned}
$$

In a similar way

$$
\begin{aligned}
\delta u_{k+2}(x) = & (-1)^{(n-1)(k-1)} \delta_x d_x \left[\int_{\Omega} *F_{k+1}(y) \wedge s_{k+1}(x,y) \right] \\
& - \frac{1}{2} \left\{ \int_{+\Sigma} \left[\phi_{k+2}(y) \wedge \delta_x \underset{y}{*} d_y s_{k+2}(x,y) - \delta_x \delta_y s_{k+2}(x,y) \wedge \widetilde{\phi}_{k+2}(y) \right.\right. \\
& \left.\left. + \delta_x d_y s_{k+2}(x,y) \wedge \widetilde{\phi}_{k+4}(y) - \phi_k(y) \wedge \delta_x \underset{y}{*} \delta_y s_{k+2}(x,y) \right] \right\}, \quad x \in \Omega. \quad (20)
\end{aligned}
$$

From (19) and (20) it follows

$$du_k(x)-\delta u_{k+2}(x) = (-1)^{(n-1)(k-1)+1}(d\delta+\delta d)\left[\int_\Omega *F_{k+1}(y)\wedge s_{k+1}(x,y)\right] =$$

$$= (-1)^{(n-1)(k-1)}\Delta\int_\Omega [*F_{k+1}(y)\wedge \delta_x s_{k+1}(x,y)] =$$

$$= (-1)^{(n-1)(k-1)}\Delta\int_\Omega \Bigg[\sum_{j_1<\ldots<j_{k+1}} \delta^{1,\ldots,n}_{j_1,\ldots,j_{k+1},r_1,\ldots,r_{n-k-1}} F_{j_1,\ldots,j_{k+1}}(y)dy_{r_1}\ldots dy_{r_{n-k-1}}$$

$$s(x-y)dy_{i_1}\ldots dy_{i_{k+1}}\big]\, dx_{i_1}\ldots dx_{i_{k+1}} =$$

$$= (-1)^{(n-1)(k-1)} \sum_{j_1<\ldots<j_{k+1}} \delta^{1,\ldots,n}_{j_1,\ldots,j_{k+1},r_1,\ldots,r_{n-k-1}}\delta^{1,\ldots,n}_{r_1,\ldots,r_{n-k-1},i_1,\ldots,i_{k+1}}$$

$$\Delta\int_\Omega \left[F_{j_1,\ldots,j_{k+1}}(y)s(x-y)dy\right] dx_{i_1}\ldots dx_{i_{k+1}} =$$

$$= (-1)^{(n-1)(k-1)} \sum_{j_1<\ldots<j_{k+1}} \delta^{j_1,\ldots,j_{k+1}}_{i_1,\ldots,i_{k+1}}(-1)^{(n-1)(k-1)}$$

$$\Delta\int_\Omega \left[F_{j_1,\ldots,j_{k+1}}(y)s(x-y)dy\right] dx_{i_1}\ldots dx_{i_{k+1}} =$$

$$= \sum_{j_1<\ldots<j_{k+1}} \Delta\int_\Omega \left[F_{j_1,\ldots,j_{k+1}}(y)s(x-y)dy\right] dx_{j_1}\ldots dx_{j_{k+1}}, \quad x\in\Omega.$$

Poisson's formula leads to

$$du_k(x)-\delta u_{k+2}(x) = \sum_{j_1<\ldots<j_{k+1}} F_{j_1,\ldots,j_{k+1}}(x)dx_{j_1}\ldots dx_{j_{k+1}} = F_{k+1}(x).$$

Analogously, we obtain

$$du_{n-1}(x) = F_n(x), \qquad -\delta u_1(x) = F_0(x), \qquad x\in\Omega.$$

Therefore $U = \sum_{k=0}^{n} u_k$, u_k being defined by (18), satisfies $dU-\delta U = F$ in Ω. Moreover $U\in\mathcal{V}$, and by Theorem 2.1 the traces of U and $*U$ are ϕ and $\widetilde{\phi}$ respectively, which completes the proof. □

2.2 The Dirichlet problem for the equation $dU-\delta U = F$ in a multiply connected domain

We consider now a domain Ω of the form

$$\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m}\overline{\Omega}_j,$$

where Ω_j $(j = 0, \dots, m)$ are bounded connected domains of $\mathbb{R}^n$, whose boundaries Σ_j are connected Lyapunov surfaces, such that

$$\overline{\Omega}_j \subset \Omega_0 \ \text{ and } \ \overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset \qquad j,k = 1,\dots,m, \quad j \neq k.$$

For brevity, we shall call such a domain an $(m+1)$-*connected domain.*

Theorem 2.3. *Let* $\Omega = \Omega_0 \setminus \bigcup_{j=1}^m \overline{\Omega}_j$ *be an* $(m+1)$*-connected domain. Let* $\phi = \sum_{k=0}^{n-1} \phi_k$, $\widetilde{\phi} = \sum_{k=0}^{n-1} \widetilde{\phi}_{n-k} \in L^1(\Sigma)$ *and* $F = \sum_{k=0}^{n} F_k \in L^1(\Omega)$ *be given forms. There exists a non homogeneous differential form* $U \in L^1(\Omega)$ *solution of*

$$\begin{cases} dU - \delta U = F & \text{in } \Omega \\ U = \phi \quad *U = \widetilde{\phi} & \text{on } \Sigma \end{cases} \tag{21}$$

if and only if

$$\begin{aligned}
(-1)^{(n-1)(k-1)+1} &\int_\Omega \left[*F_{k+1} \wedge dw_h^{i_1,\dots,i_k} - *F_{k-1} \wedge \delta w_h^{i_1,\dots,i_k} \right] \\
&- \frac{1}{2} \left\{ \int_{+\Sigma} \left[\phi_k \wedge *dw_h^{i_1,\dots,i_k} - \delta w_h^{i_1,\dots,i_k} \wedge \widetilde{\phi}_k \right.\right. \\
&\qquad \left.\left. + dw_h^{i_1,\dots,i_k} \wedge \widetilde{\phi}_{k+2} - \phi_{k-2} \wedge *\delta w_h^{i_1,\dots,i_k} \right] \right\} = 0
\end{aligned} \tag{22}$$

$$\begin{aligned}
&(-1)^{(n-1)(k-1)+1} \int_\Omega \Big[*F_{k+1}(y) \wedge d_y[|y - x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y - x^j)] \\
&- * F_{k-1}(y) \wedge \delta_y[|y - x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y - x^j)] \Big] \\
&- \frac{1}{2} \left\{ \int_{+\Sigma} \Big[\phi_k(y) \wedge \underset{y}{*} d_y[|y - x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y - x^j)] \right. \\
&- \delta_y[|y - x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y - x^j)] \wedge \widetilde{\phi}_k(y) \\
&+ d_y[|y - x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y - x^j)] \wedge \widetilde{\phi}_{k+2}(y) \\
&\left. - \phi_{k-2}(y) \wedge \underset{y}{*} \delta_y[|y - x^j|^{2-n-2k} w_h^{i_1,\dots,i_k}(y - x^j)] \Big] \right\} = 0 \quad j = 1,\dots,m.
\end{aligned} \tag{23}$$

for any $1 \le i_1 < \ldots < i_k \le n, h = 1,2,\dots,\ k = 0,1,\dots,n$ $(\phi_k \equiv 0, k = -2,-1; \widetilde{\phi}_k \equiv 0, k = n+1, n+2; F_k \equiv 0, k = -1, n+1)$. *Here* x^j *is a fixed point in* Ω_j $(j = 1,\dots,m)$.

Proof. The necessity follows as in Theorem 2.2. Conversely, from (17) and

(22) we obtain

$$(-1)^{(n-1)(k-1)+1}\int_{\Omega}[*F_{k+1}(y)\wedge d_y s_k(x,y)-*F_{k-1}(y)\wedge\delta_y s_k(x,y)]$$
$$-\frac{1}{2}\left\{\int_{+\Sigma}\left[\phi_k(y)\wedge\underset{y}{*}d_y s_k(x,y)-\delta_y s_k(x,y)\wedge\widetilde{\phi}_k(y)\right.\right.$$
$$\left.\left.+d_y s_k(x,y)\wedge\widetilde{\phi}_{k+2}(y)-\phi_{k-2}(y)\wedge\underset{y}{*}\delta_y s_k(x,y)\right]\right\}=0\qquad\forall x\notin\overline{\Omega}_0. \quad (24)$$

Applying Lemma 2.1 with $0<R<\min_{y\in\Sigma}|y-x_j|$, the following development

$$s(x-y)=s(x-x^j-(y-x^j))=\sum_{k=0}^{\infty}\sum_{h=1}^{p_{nk}}\frac{P_{hk}(x-x^j)Q_{hk}(y-x^j)}{|y-x^j|^{2k-2+n}} \quad (25)$$

holds uniformly for any $y\in\Sigma$ and for any $x\in B_j$, B_j being a ball of radius $r\le\delta R$ centered at x^j.
By (23) and (25) we have

$$(-1)^{(n-1)(k-1)+1}\int_{\Omega}[*F_{k+1}(y)\wedge d_y s_k(x,y)-*F_{k-1}(y)\wedge\delta_y s_k(x,y)]$$
$$-\frac{1}{2}\left\{\int_{+\Sigma}\left[\phi_k(y)\wedge\underset{y}{*}d_y s_k(x,y)-\delta_y s_k(x,y)\wedge\widetilde{\phi}_k(y)\right.\right.$$
$$\left.\left.+d_y s_k(x,y)\wedge\widetilde{\phi}_{k+2}(y)-\phi_{k-2}(y)\wedge\underset{y}{*}\delta_y s_k(x,y)\right]\right\}=0\qquad\forall x\in B_j \quad (26)$$

and since Ω_j is connected this is still true for any $x\in\Omega_j$, $j=1,\dots,m$.
From (24) and (26) it follows

$$(-1)^{(n-1)(k-1)+1}\int_{\Omega}[*F_{k+1}(y)\wedge d_y s_k(x,y)-*F_{k-1}(y)\wedge\delta_y s_k(x,y)]$$
$$-\frac{1}{2}\left\{\int_{+\Sigma}\left[\phi_k(y)\wedge\underset{y}{*}d_y s_k(x,y)-\delta_y s_k(x,y)\wedge\widetilde{\phi}_k(y)\right.\right.$$
$$\left.\left.+d_y s_k(x,y)\wedge\widetilde{\phi}_{k+2}(y)-\phi_{k-2}(y)\wedge\underset{y}{*}\delta_y s_k(x,y)\right]\right\}=0\qquad\forall x\notin\overline{\Omega}. \quad (27)$$

If we denote by u_k the left-hand side of (27) when $x\in\Omega$ and we proceed as in Theorem 2.2, we obtain the result. $\square$

3 The Dirichlet problem for the Cimmino system

Theorem 2.3 provides necessary and sufficient conditions for the solvability of the Dirichlet problem (21). After a lemma characterizing the self-conjugate 2-forms in $\mathbb{R}^4$, in Theorem 3.1 we consider (21) with the data chosen in a certain way. For such data we obtain the relevant necessary and sufficient conditions for the resolubility of the corresponding Dirichlet problem. Moreover we prove that the solution U has a particular structure (see (32) below). Theorem 3.2 will show that the problem considered in Theorem 3.1 is equivalent to the Dirichlet problem for Cimmino system.

Lemma 3.1. *Let us consider a 2-form defined in an open set $\Omega \subset \mathbb{R}^4$. Then $u_2 = *u_2$ if and only if there exist f_1, f_2 and f_3 such that*

$$u_2 = f_1(dx_1dx_2+dx_3dx_4)-f_2(dx_1dx_3+dx_4dx_2)+f_3(dx_1dx_4+dx_2dx_3). \quad (28)$$

Proof. Let $u_2 = \frac{1}{2}u_{jk}dx_jdx_k$ be a 2-form. Therefore

$$*u_2 = \frac{1}{4}u_{jk}\delta^{1234}_{jkpq}dx_pdx_q = \frac{1}{4}u_{pq}\delta^{1234}_{pqjk}dx_jdx_k.$$

If $u_2 = *u_2$ we have $u_{jk} = \frac{1}{2}u_{pq}\delta^{1234}_{pqjk}$ and then $u_{12} = u_{34},\ u_{13} = -u_{24},\ u_{14} = u_{23}$. Putting $u_{12} = f_1,\ u_{13} = -f_2$ and $u_{14} = f_3$ we obtain (28). Conversely, if u_2 is defined as in (28), a straightforward computation shows that $u_2 = *u_2$. □

Theorem 3.1. *Let $\Omega = \Omega_0 \setminus \bigcup_{j=1}^m \overline{\Omega}_j \subset \mathbb{R}^4$ be an $(m+1)$-connected domain. Let $\phi = (\phi_0, 0, \phi_2, 0)$, $\widetilde{\phi} = (-\phi_0, 0, \phi_2, 0) \in L^1(\Sigma)$ and $F = F_1 - *F_1 \in L^1(\Omega)$ be given forms, where $F_1 = \gamma_k dx_k$. There exists a non homogeneous differential form $U \in L^1(\Omega)$ solution of*

$$\begin{cases} dU - \delta U = F & \text{in } \Omega \\ U = \phi \quad *U = \widetilde{\phi} & \text{on } \Sigma \end{cases} \quad (29)$$

if and only if

$$\int_\Omega *F_1(y) \wedge dw_h(y) - \frac{1}{2}\int_{+\Sigma} [\phi_0(y) \wedge *dw_h(y) + dw_h(y) \wedge \phi_2(y)] = 0;$$

$$\int_\Omega *F_1(y) \wedge d_y[|y-x^j|^{-2}w_h(y-x^j)] - \frac{1}{2}\left\{\int_{+\Sigma}\Big[\phi_0(y) \wedge \underset{y}{*}d_y[|y-x^j|^{-2}w_h(y-x_j)]\right.$$
$$\left. + d_y[|y-x^j|^{-2}w_h(y-x^j)] \wedge \phi_2(y)\Big]\right\} = 0, \qquad j = 1, \ldots, m; \quad (30)$$

$$
\int_{\Omega}\Big[F_1(y)\wedge dw_h^{i_1,i_2}(y)-*F_1(y)\wedge\delta w_h^{i_1,i_2}(y)\Big]-\frac{1}{2}\Big\{\int_{+\Sigma}\Big[\phi_2(y)\wedge *dw_h^{i_1,i_2}(y)
$$
$$
-\delta w_h^{i_1,i_2}(y)\wedge\phi_2(y)+dw_h^{i_1,i_2}(y)\wedge-\phi_0(y)-\phi_0(y)\wedge*\delta w_h^{i_1,i_2}(y)\Big]\Big\}=0;
$$
$$
\int_{\Omega}\Big[F_1(y)\wedge d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]-*F_1(y)\wedge\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\Big]
$$
$$
-\frac{1}{2}\Big\{\int_{+\Sigma}\Big[\phi_2(y)\wedge\underset{y}{*}d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]-\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\wedge\phi_2(y)
$$
$$
+d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\wedge-\phi_0(y)-\phi_0(y)\wedge\underset{y}{*}\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\Big]\Big\}=0,
$$
$$
j=1,\dots,m, \quad (31)
$$

for any $1\le i_1<i_2\le 4$, $h=1,2,\dots$. *Here* x^j *is a fixed point in* Ω_j $(j=1,\dots,m)$.
Moreover the solution U *can be written as*

$$
\begin{aligned}
&U=u_0+u_2+u_4,\\
&\qquad u_0=f_0,\quad u_2=f_1(dx_1dx_2+dx_3dx_4)-f_2(dx_1dx_3+dx_4dx_2)\\
&\qquad\qquad +f_3(dx_1dx_4+dx_2dx_3),\quad u_4=-f_0dx_1dx_2dx_3dx_4. \quad (32)
\end{aligned}
$$

Proof. In the present case, formulas (22) and (23) become

$$
\int_{\Omega}*F_1(y)\wedge dw_h(y)-\frac{1}{2}\int_{+\Sigma}\Big[\phi_0(y)\wedge*dw_h(y)+dw_h(y)\wedge\widetilde{\phi}_2(y)\Big]=0;
$$
$$
\int_{\Omega}*F_1(y)\wedge d_y[|y-x^j|^{-2}w_h(y-x^j)]-\frac{1}{2}\Big\{\int_{+\Sigma}\Big[\phi_0(y)\wedge\underset{y}{*}d_y[|y-x^j|^{-2}w_h(y-x_j)]
$$
$$
+d_y[|y-x^j|^{-2}w_h(y-x^j)]\wedge\widetilde{\phi}_2(y)\Big]\Big\}=0,\qquad j=1,\dots,m;
$$

$$
\int_{\Omega}\Big[F_1(y)\wedge dw_h^{i_1,i_2}(y)-*F_1(y)\wedge\delta w_h^{i_1,i_2}(y)\Big]-\frac{1}{2}\Big\{\int_{+\Sigma}\Big[\phi_2(y)\wedge *dw_h^{i_1,i_2}(y)
$$
$$
-\delta w_h^{i_1,i_2}(y)\wedge\widetilde{\phi}_2(y)+dw_h^{i_1,i_2}(y)\wedge\widetilde{\phi}_4(y)-\phi_0(y)\wedge*\delta w_h^{i_1,i_2}(y)\Big]\Big\}=0;
$$
$$
\int_{\Omega}\Big[F_1(y)\wedge d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]-*F_1(y)\wedge\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\Big]
$$
$$
-\frac{1}{2}\Big\{\int_{+\Sigma}\Big[\phi_2(y)\wedge\underset{y}{*}d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]-\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\wedge\widetilde{\phi}_2(y)
$$
$$
+d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\wedge\widetilde{\phi}_4(y)-\phi_0(y)\wedge\underset{y}{*}\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\Big]\Big\}=0,
$$
$$
j=1,\dots,m,\ \ 1\le i_1<i_2\le 4;
$$

$$
\begin{aligned}
&-\int_{\Omega} F_1(y)\wedge\delta w_h^{1234}(y)+\frac{1}{2}\int_{+\Sigma}\Big[\delta w_h^{1234}(y)\wedge\widetilde{\phi}_4(y)+\phi_2(y)\wedge *\delta w_h^{1234}(y)\Big]=0;\\
&-\int_{\Omega} F_1(y)\wedge\delta_y[|y-x^j|^{-2}w_h^{1234}(y-x^j)]+\frac{1}{2}\Bigg\{\int_{+\Sigma}\Big[\delta_y[|y-x^j|^{-2}w_h^{1234}(y-x_j)]\wedge\widetilde{\phi}_4\\
&\qquad +\phi_2\wedge \underset{y}{*}\delta_y[|y-x^j|^{-2}w_h^{1234}(y-x^j)]\Big]\Bigg\}=0,\qquad j=1,\dots,m.
\end{aligned}
$$

By means of the particular expression of ϕ and $\widetilde{\phi}$, the previous conditions can be written as

$$
\begin{aligned}
&\int_{\Omega} *F_1(y)\wedge dw_h(y)-\frac{1}{2}\int_{+\Sigma}[\phi_0(y)\wedge *dw_h(y)+dw_h(y)\wedge\phi_2(y)]=0;\\
&\int_{\Omega} *F_1(y)\wedge d_y[|y-x^j|^{-2}w_h(y-x^j)]-\frac{1}{2}\Bigg\{\int_{+\Sigma}\Big[\phi_0(y)\wedge \underset{y}{*}d_y[|y-x^j|^{-2}w_h(y-x_j)]\\
&\qquad +d_y[|y-x^j|^{-2}w_h(y-x^j)]\wedge\phi_2(y)\Big]\Bigg\}=0,\qquad j=1,\dots,m; \quad (33)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\Omega}\Big[F_1(y)\wedge dw_h^{i_1,i_2}(y)-*F_1(y)\wedge\delta w_h^{i_1,i_2}(y)\Big]-\frac{1}{2}\Bigg\{\int_{+\Sigma}\Big[\phi_2(y)\wedge *dw_h^{i_1,i_2}(y)\\
&\qquad -\delta w_h^{i_1,i_2}(y)\wedge\phi_2(y)+dw_h^{i_1,i_2}(y)\wedge -\phi_0(y)-\phi_0(y)\wedge *\delta w_h^{i_1,i_2}(y)\Big]\Bigg\}=0;\\
&\int_{\Omega}\Big[F_1(y)\wedge d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]-*F_1(y)\wedge\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\Big]\\
&-\frac{1}{2}\Bigg\{\int_{+\Sigma}\Big[\phi_2(y)\wedge \underset{y}{*}d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]-\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\wedge\phi_2(y)\\
&+d_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\wedge -\phi_0(y)-\phi_0(y)\wedge \underset{y}{*}\delta_y[|y-x^j|^{-6}w_h^{i_1,i_2}(y-x^j)]\Big]\Bigg\}=0,\\
&\qquad\qquad j=1,\dots,m,\ \ 1\le i_1<i_2\le 4;
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{\Omega} F_1(y)\wedge\delta w_h^{1234}(y)+\frac{1}{2}\int_{+\Sigma}\big[\delta w_h^{1234}(y)\wedge -\phi_0(y)+\phi_2(y)\wedge *\delta w_h^{1234}(y)\big]=0;\\
&-\int_{\Omega} F_1(y)\wedge\delta_y[|y-x^j|^{-2}w_h^{1234}(y-x^j)]+\frac{1}{2}\Bigg\{\int_{+\Sigma}\Big[\delta_y[|y-x^j|^{-2}w_h^{1234}(y-x_j)]\wedge -\phi_0\\
&\qquad +\phi_2\wedge \underset{y}{*}\delta_y[|y-x^j|^{-2}w_h^{1234}(y-x^j)]\Big]\Bigg\}=0,\qquad j=1,\dots,m. \quad (34)
\end{aligned}
$$

In view of the equality $\delta w_h^{1234}(y) = - * dw_h(y)$, we find

$$
\begin{aligned}
-\int_\Omega F_1(y)\wedge\delta w_h^{1234}(y) + \frac{1}{2}\int_{+\Sigma}\left[\delta w_h^{1234}(y)\wedge -\phi_0(y) + \phi_2(y)\wedge *\delta w_h^{1234}(y)\right] = \\
= \int_\Omega F_1(y)\wedge *dw_h(y) + \frac{1}{2}\int_{+\Sigma}\left[*dw_h(y)\wedge\phi_0(y) + \phi_2(y)\wedge dw_h(y)\right] = \\
= \int_\Omega *F_1(y)\wedge *dw_h(y) + \frac{1}{2}\int_{+\Sigma}\left[\phi_0(y)\wedge *dw_h(y) + dw_h(y)\wedge\phi_2(y)\right].
\end{aligned}
$$

By a similar argument we deduce that (33) are equivalent to (34).
Keeping in mind Theorem 2.3, the first part of the theorem is proved.
By the Cauchy integral formula we obtain

$$
\begin{aligned}
u_0(x) =& \int_\Omega *F_1(y)\wedge d_y s_0(x,y) - \frac{1}{2}\int_{+\Sigma}\left[\phi_0(y)\wedge \underset{y}{*} d_y s_0(x,y) + d_y s_0(x,y)\wedge\phi_2(y)\right],\\
u_1(x) =& 0,\\
u_2(x) =& \int_\Omega \left[F_1(y)\wedge d_y s_2(x,y) - *F_1(y)\wedge\delta_y s_2(x,y)\right] - \frac{1}{2}\int_{+\Sigma}\Big[\phi_2(y)\wedge \underset{y}{*} d_y s_2(x,y)\\
&- \delta_y s_2(x,y)\wedge\phi_2(y) - d_y s_2(x,y)\wedge\phi_0(y) - \phi_0(y)\wedge \underset{y}{*}\delta_y s_2(x,y)\Big],\\
u_3(x) =& 0,\\
u_4(x) =& \int_\Omega -F_1(y)\wedge\delta_y s_4(x,y) - \frac{1}{2}\int_{+\Sigma}\left[\delta_y s_4(x,y)\wedge\phi_0(y) - \phi_2\wedge \underset{y}{*}\delta_y s_4(x,y)\right].
\end{aligned}
$$

Then the solution of (29) is $U = u_0 + u_2 + u_4$. On the other hand,

$$
\begin{aligned}
*u_0(x) =& \int_\Omega *F_1(y)\wedge \underset{x}{*} d_y s_0(x,y) - \frac{1}{2}\int_{+\Sigma}\left[\phi_0(y)\wedge \underset{x}{*}\,\underset{y}{*}\, d_y s_0(x,y) + \underset{x}{*} d_y s_0(x,y)\wedge\phi_2(y)\right],\\
*u_1(x) =& 0,\\
u_2(x) =& \int_\Omega \left[F_1(y)\wedge \underset{x}{} d_y s_2(x,y) - *F_1(y)\wedge \underset{x}{*}\delta_y s_2(x,y)\right] - \frac{1}{2}\int_{+\Sigma}\Big[\phi_2(y)\wedge \underset{x}{*}\,\underset{y}{*}\, d_y s_2(x,y)\\
&- \underset{x}{*}\,\delta_y s_2(x,y)\wedge\phi_2(y) - \underset{x}{*} d_y s_2(x,y)\wedge\phi_0(y) - \phi_0(y)\wedge \underset{x}{*}\,\underset{y}{*}\,\delta_y s_2(x,y)\Big],\\
*u_3(x) =& 0,\\
u_4(x) =& \int_\Omega -F_1(y)\wedge \underset{x}{}\delta_y s_4(x,y) - \frac{1}{2}\int_{+\Sigma}\left[\underset{x}{*}\delta_y s_4(x,y)\wedge\phi_0(y) - \phi_2\wedge \underset{x}{*}\,\underset{y}{*}\,\delta_y s_4(x,y)\right].
\end{aligned}
$$

By (5)-(9), we have

$$
\begin{aligned}
&\underset{x}{*}\underset{y}{*}\delta_y s_4(x,y) = d_y s_0(x,y),\\
&\underset{x}{*}\,\delta_y s_4(x,y) = -\underset{y}{*}\,d_y s_0(x,y),\\
&\underset{x}{*}\underset{y}{*}d_y s_2(x,y) = -\underset{x}{*}\underset{x}{*}d_x s_1(x,y) = -d_x s_1(x,y) = -\delta_y s_2(x,y),\\
&-\underset{x}{*}\delta_y s_2(x,y) = \underset{x}{*}\underset{y}{*}\,d_y\underset{y}{*}\,s_2(x,y) = \underset{y}{*}d_y\underset{y}{*}\underset{x}{*}s_2(x,y) = \underset{y}{*}d_y s_2(x,y),\\
&\underset{y}{*}\,\delta_y s_2(x,y) = -\underset{y}{*}\underset{y}{*}d_y\underset{x}{*}\,s_2(x,y) = \underset{x}{*}d_y s_2(x,y),\\
&\underset{x}{*}\underset{y}{*}\delta_y s_2(x,y) = -\underset{y}{*}\underset{y}{*}d_y\underset{y}{*}\underset{y}{*}s_2(x,y) = d_y s_2(x,y).
\end{aligned}
$$

In conclusion the coefficients satisfy $u_0 = - * u_4$ and $u_2 = *u_2$. Lemma 3.1 completes the proof. $\square$

Theorem 3.2. *Let $\Omega \subset \mathbb{C}^2$ be a regular domain and Σ its boundary. Let $f, g \in L^1(\Omega)$ and $F, G \in L^1(\Sigma)$. Conditions (30) and (31) are necessary and sufficient for the resolubility of the boundary value problem for Cimmino system*

$$
\begin{cases} u_{\overline{z}} + \overline{v}_w = f, \quad u_{\overline{w}} - \overline{v}_z = g & \text{in } \Omega \\ \qquad u = F \qquad\quad v = G & \text{on } \Sigma. \end{cases} \tag{35}
$$

Proof. If we put $u = f_0 + if_1$, $v = f_2 + if_3$, $f = \frac{1}{2}(\gamma_1 + i\gamma_2)$, $g = \frac{1}{2}(\gamma_3 + i\gamma_4)$, $F = F_0 + iF_1$ and $G = G_0 + iG_1$, (35) is equivalent to

$$
\begin{cases}
f_{0_{x_1}} - f_{1_{x_2}} + f_{2_{x_3}} - f_{3_{x_4}} = \gamma_1 & \\
f_{0_{x_2}} + f_{1_{x_1}} - f_{2_{x_4}} - f_{3_{x_3}} = \gamma_2 & \text{in } \Omega\\
f_{0_{x_3}} - f_{1_{x_4}} - f_{2_{x_1}} + f_{3_{x_2}} = \gamma_3 & \\
f_{0_{x_4}} + f_{1_{x_3}} + f_{2_{x_2}} + f_{3_{x_1}} = \gamma_4 & \\
f_0 = F_0 \quad f_1 = F_1 \quad f_2 = G_0 \quad f_3 = G_1 & \text{on } \Sigma
\end{cases} \tag{36}
$$

Let us consider the 2-form defined on Σ

$$
\phi_2 = F_1(dx_1dx_2 + dx_3dx_4) - G_0(dx_1dx_3 - dx_2dx_4) - G_1(dx_1dx_4 + dx_2dx_3)
$$

and set

$$
\phi = (F_0, 0, \phi_2, 0), \quad \widetilde{\phi} = (-F_0, 0, \phi_2, 0), \quad F = F_1 - *F_1, \tag{37}
$$

with $F_1 = \gamma_k dx_k$. Let U be the solution of the Dirichlet problem (29) with these data. As in Proposition 1.1, and keeping in mind (32), one can show that (f_0, f_1, f_2, f_3) satisfies the problem (36). Conversely, if (f_0, f_1, f_2, f_3) is solution of (36), then the non homogeneous differential form U given by (32) is solution of (29).
By Theorem 3.1, conditions (30) and (31) are necessary and sufficient for the solvability of (29). The problems (29) (with data (37)), (35) and (36) being equivalent, we get the result.

$\square$

Remark 3.1. *Finally we remark that our results show that Dragomir and Lanconelli conditions are not only necessary, but also sufficient. Indeed, in [6], it is proved that*

$$Re\left\{2\int_\Omega (f\overline{h}+g\overline{k})dV - \int_{\partial\Omega}\left\{F\left[(n_1+in_2)\overline{h}+(n_3+in_4)\overline{k}\right]\right.\right.$$
$$\left.\left. +G\left[(n_3+in_4)h-(n_1+in_2)k\right]\right\}d\sigma\right\}=0 \quad (38)$$

for any $h, k \in C^1(\Omega)\cap C^0(\overline{\Omega})$ *such that*

$$h_z + k_w = 0, \qquad h_{\overline{w}} - k_{\overline{z}} = 0 \qquad in\ \Omega$$

are necessary conditions for solving (35)*. If we show that* (38) *implies* (30) *and* (31)*, we can conclude that* (38) *is not only necessary but also sufficient for the solvability of* (35)*. Let us consider the first of* (30)*; it can be written as*

$$\begin{aligned}
&\int_\Omega\left[\gamma_1\frac{\partial w_h}{\partial y_1}+\gamma_2\frac{\partial w_h}{\partial y_2}+\gamma_3\frac{\partial w_h}{\partial y_3}+\gamma_4\frac{\partial w_h}{\partial y_4}\right]dy\\
&-\frac{1}{2}\int_{+\Sigma}\left[\left(F_0\frac{\partial w_h}{\partial y_1}+F_1\frac{\partial w_h}{\partial y_2}-G_0\frac{\partial w_h}{\partial y_3}+G_1\frac{\partial w_h}{\partial y_4}\right)n_1\right.\\
&+\left(F_0\frac{\partial w_h}{\partial y_2}-F_1\frac{\partial w_h}{\partial y_1}+G_0\frac{\partial w_h}{\partial y_4}+G_1\frac{\partial w_h}{\partial y_3}\right)n_2\\
&+\left(F_0\frac{\partial w_h}{\partial y_3}+F_1\frac{\partial w_h}{\partial y_4}+G_0\frac{\partial w_h}{\partial y_1}-G_1\frac{\partial w_h}{\partial y_2}\right)n_3\\
&\left.+\left(F_0\frac{\partial w_h}{\partial y_4}-F_1\frac{\partial w_h}{\partial y_3}-G_0\frac{\partial w_h}{\partial y_2}-G_1\frac{\partial w_h}{\partial y_1}\right)n_4\right]d\sigma=0.
\end{aligned} \quad (39)$$

If we put $h_0 = \dfrac{\partial w_h}{\partial y_1}, h_1 = \dfrac{\partial w_h}{\partial y_2}, k_0 = \dfrac{\partial w_h}{\partial y_3}, k_1 = \dfrac{\partial w_h}{\partial y_4}$, *we have that* (38) *implies* (39)*. In a similar way, it is possible to prove that* (38) *implies* (30) *and* (31)*.*

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