

An alternate description of a $(q + 1, 8)$ -cage

Marièn Abreu *

*Dipartimento di Matematica Informatica ed Economia,
Università degli Studi della Basilicata,
Viale dell'Ateneo Lucano 10, I-85100 Potenza, Italy*

Gabriela Araujo-Pardo †

*Instituto de Matemáticas, Universidad Nacional Autónoma de México,
México D. F., México*

Camino Balbuena

*Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya,
Campus Nord, Edifici C2, C/ Jordi Girona 1 i 3 E-08034, Barcelona, Spain*

Domenico Labbate ‡

*Dipartimento di Matematica Informatica ed Economia,
Università degli Studi della Basilicata,
Viale dell'Ateneo Lucano 10, I-85100 Potenza, Italy*

Received 15 November 2017, accepted 4 June 2018, published online 23 July 2019

Abstract

Let $q \geq 2$ be a prime power. In this note we present an alternate description of the known $(q + 1, 8)$ -cages which has allowed us to construct small (k, g) -graphs for $k = q - 1, q$ and $g = 7, 8$ in other papers on this same topic.

Keywords: Cages, girth, Moore graphs, perfect dominating sets.

Math. Subj. Class.: 05C35, 05C69, 05B25

*This research was carried out within the activity of INdAM-GNSAGA and supported by the Italian Ministry MIUR. Research supported by PAPIIT-México under Project IN107218.

†Research supported by CONACyT-México under Project 282280 and PAPIIT-México under Projects IN106318 and IN107218.

‡Corresponding author. This research was carried out within the activity of INdAM-GNSAGA and supported by the Italian Ministry MIUR. Research supported by PAPIIT-México under Project IN107218.

E-mail addresses: marien.abreu@unibas.it (Marièn Abreu), garaujo@matem.unam.mx (Gabriela Araujo-Pardo), m.camino.balbuena@upc.edu (Camino Balbuena), domenico.labbate@unibas.it (Domenico Labbate)

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Bondy and Murty [14] for terminology and notation.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of G is the number $g = g(G)$ of edges in a shortest cycle. For every $v \in V$, $N_G(v)$ denotes the *neighbourhood* of v , i.e. the set of all vertices adjacent to v , and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighbourhood* of v . The *degree* of a vertex $v \in V$ is the cardinality of $N_G(v)$. Let $S \subset V(G)$, then we denote by $N_G(S) = \cup_{s \in S} N_G(s) - S$ and by $N_G[S] = S \cup N_G(S)$.

A graph is called *regular* if all its vertices have the same degree. A (k, g) -*graph* is a k -regular graph with girth g . Erdős and Sachs [15] proved the existence of (k, g) -graphs for all values of k and g provided that $k \geq 2$. Since then most work carried out has focused on constructing a smallest (k, g) -graph (cf. e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 20, 21, 22]). A (k, g) -*cage* is a k -regular graph with girth g having the smallest possible number of vertices. Cages have been intensely studied since they were introduced by Tutte [25] in 1947. More details about constructions of cages can be found in the recent survey by Exoo and Jajcay [17].

In this note we are interested in $(k, 8)$ -cages. Counting the number of vertices in the distance partition with respect to an edge yields the following lower bound on the order of a $(k, 8)$ -cage:

$$n_0(k, 8) = 2(1 + (k - 1) + (k - 1)^2 + (k - 1)^3). \quad (1.1)$$

A $(k, 8)$ -cage with $n_0(k, 8)$ vertices is called a Moore $(k, 8)$ -*graph* (cf. [14]). These graphs have been constructed as the incidence graphs of generalized quadrangles $Q(4, q)$ and $W(q)$ [12, 17, 24], which are known to exist for q a prime power and $k = q + 1$ and no example is known when $k - 1$ is not a prime power (cf. [11, 13, 19, 27]). Since they are incidence graphs, these cages are bipartite and have diameter 4. Recall also that if q is even, $Q(4, q)$ is isomorphic to the dual of $W(q)$ and viceversa. Hence, the corresponding $(q + 1, 8)$ -cages are isomorphic.

In this note we present an alternate description of the known $(q + 1, 8)$ -cages with $q \geq 2$ a prime power as follows:

Definition 1.1. Let \mathbb{F}_q be a finite field with $q \geq 2$ a prime power and ϱ be a symbol not belonging to \mathbb{F}_q . Let $\Gamma_q = \Gamma_q[W_0, W_1]$ denote a bipartite graph with vertex sets $W_i = \mathbb{F}_q^3 \cup \{(\varrho, b, c)_i, (\varrho, \varrho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_i\}$, $i = 0, 1$, and edge set defined as follows:

For all $a, b, c \in \mathbb{F}_q$

$$N_{\Gamma_q}((a, b, c)_1) = \{(w, aw + b, a^2w + 2ab + c)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, a, c)_0\};$$

$$N_{\Gamma_q}((\varrho, b, c)_1) = \{(c, b, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, c)_0\};$$

$$N_{\Gamma_q}((\varrho, \varrho, c)_1) = \{(\varrho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\};$$

$$N_{\Gamma_q}((\varrho, \varrho, \varrho)_1) = \{(\varrho, \varrho, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\}.$$

Or equivalently,

For all $i, j, k \in \mathbb{F}_q$

$$N_{\Gamma_q}((i, j, k)_0) = \{(w, j - wi, w^2i - 2wj + k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, j, i)_1\}$$

$$N_{\Gamma_q}((\varrho, j, k)_0) = \{(j, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, j)_1\}$$

$$N_{\Gamma_q}((\varrho, \varrho, k)_0) = \{(\varrho, w, k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\};$$

$$N_{\Gamma_q}((\varrho, \varrho, \varrho)_0) = \{(\varrho, \varrho, w)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_1\}.$$

Note that ϱ is just a symbol not belonging to \mathbb{F}_q and no arithmetical operation will be performed with it.

Theorem 1.2. *The graph Γ_q given in Definition 1.1 is a Moore $(q + 1, 8)$ -graph for each prime power $q \geq 2$.*

The proof of the above theorem shows that the graph Γ_q described in Definition 1.1 is in fact a labelling for a $(q + 1, 8)$ -cage, for each prime power $q \geq 2$. We need to settle this alternate description because it is used in [2, 3, 4] to construct small (k, g) -graphs for $k = q - 1, q$ and $g = 7, 8$.

2 Proof of Theorem 1.2

2.1 Preliminaries: the graphs H_q and B_q

In order to prove Theorem 1.2 we will first define two q -regular bipartite graphs H_q and B_q (cf. Definitions 2.1 and 2.4). The graph H_q was also introduced by Lazebnik, Ustimenko and Woldar [20] with a different formulation.

Definition 2.1. Let \mathbb{F}_q be a finite field with $q \geq 2$. Let $H_q = H_q[U_0, U_1]$ be a bipartite graph with vertex set $U_r = \mathbb{F}_q^3$, $r = 0, 1$, and edge set $E(H_q)$ defined as follows:

For all $a, b, c \in \mathbb{F}_q$

$$N_{H_q}((a, b, c)_1) = \{(w, aw + b, a^2w + c)_0 : w \in \mathbb{F}_q\}.$$

Note that throughout the proofs equalities and operations are intended in \mathbb{F}_q .

Lemma 2.2. *Let H_q be the graph from Definition 2.1. For any given $a \in \mathbb{F}_q$, the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four. And, for any given $i \in \mathbb{F}_q$, the vertices in the set $\{(i, j, k)_0 : j, k \in \mathbb{F}_q\}$ are mutually at distance at least four.*

Proof. Suppose that there exists a path of length two between distinct vertices of the form $(a, b, c)_1 (w, j, k)_0 (a', b', c')_1$ in H_q . By Definition 2.1, $j = aw + b = aw + b'$ and $k = a^2w + c = a^2w + c'$. Combining the equations we get $b = b'$ and $c = c'$ which implies that $(a, b, c)_1 = (a, b', c')_1$ contradicting the assumption that the path has length two. Similarly suppose that there exists a path of length two $(i, j, k)_0 (a, b, c)_1 (i', j', k')_0$. Reasoning as before, we obtain $j = ai + b = j'$, and $k = a^2i + c = k'$ yielding $(i, j, k)_0 = (i, j', k')_0$ which is a contradiction. \square

Proposition 2.3. *The graph H_q from Definition 2.1 is q -regular, bipartite, of girth 8 and order $2q^3$.*

Proof. For $q = 2$ it can be checked that H_2 consists of two disjoint cycles of length 8. Thus we assume that $q \geq 3$. Clearly H_q has order $2q^3$ and every vertex of U_1 has degree q . Let $(x, y, z)_0 \in U_0$. By definition of H_q ,

$$N_{H_q}((x, y, z)_0) = \{(a, y - ax, z - a^2x)_1 : a \in \mathbb{F}_q\}. \tag{2.1}$$

Hence every vertex of U_0 has also degree q and H_q is q -regular. Next, let us prove that H_q has no cycles of length smaller than 8. Otherwise suppose that there exists in H_q a cycle

$$C_{2t+2} = (a_0, b_0, c_0)_1 (x_0, y_0, z_0)_0 (a_1, b_1, c_1)_1 \cdots (x_t, y_t, z_t)_0 (a_0, b_0, c_0)_1$$

of length $2t + 2$ with $t \in \{1, 2\}$. By Lemma 2.2, $a_k \neq a_{k+1}$ and $x_k \neq x_{k+1}$ (subscripts being taken modulo $t + 1$). Then

$$\begin{aligned} y_k &= a_k x_k + b_k = a_{k+1} x_k + b_{k+1}, & k = 0, \dots, t, \\ z_k &= a_k^2 x_k + c_k = a_{k+1}^2 x_k + c_{k+1}, & k = 0, \dots, t, \end{aligned}$$

subscripts k being taken modulo $t + 1$. Summing all these equalities we get

$$\begin{aligned} \sum_{k=0}^{t-1} (a_k - a_{k+1})x_k &= (a_0 - a_t)x_t, & t = 1, 2; \\ \sum_{k=0}^{t-1} (a_k^2 - a_{k+1}^2)x_k &= (a_0^2 - a_t^2)x_t, & t = 1, 2. \end{aligned} \tag{2.2}$$

If $t = 1$, then (2.2) leads to $(a_0 - a_1)(x_1 - x_0) = 0$. System (2.2) gives $x_0 = x_1 = x_2$ which is a contradiction to Lemma 2.2. This means that H_q has no squares so that we may assume that $t = 2$. The coefficient matrix of (2.2) has a Vandermonde determinant, i.e.

$$\begin{vmatrix} a_1 - a_0 & a_0 - a_2 \\ a_1^2 - a_0^2 & a_0^2 - a_2^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_0 & a_2 \\ a_1^2 & a_0^2 & a_2^2 \end{vmatrix} = \prod_{0 \leq k < j \leq 2} (a_j - a_k).$$

This determinant is different from zero because by Lemma 2.2, $a_{k+1} \neq a_k$ (the subscripts being taken modulo 3). Using Cramer’s rule to solve it we obtain $x_1 = x_0 = x_2$ which is a contradiction to Lemma 2.2.

Hence, H_q has girth at least 8. Furthermore, when $q \geq 3$ the minimum number of vertices of a q -regular bipartite graph of girth greater than 8 must be greater than $2q^3$. Thus we conclude that the girth of H_q is exactly 8. □

Next, we will make use of the following induced subgraph B_q of Γ_q .

Definition 2.4. Let $B_q = B_q[V_0, V_1]$ be a bipartite graph with vertex set $V_i = \mathbb{F}_q^3, i = 0, 1$, and edge set $E(B_q)$ defined as follows:

$$\begin{aligned} \text{For all } a, b, c \in \mathbb{F}_q \\ N_{B_q}((a, b, c)_1) &= \{(j, aj + b, a^2j + 2ab + c)_0 : j \in \mathbb{F}_q\}. \end{aligned}$$

Lemma 2.5. *The graph B_q is isomorphic to the graph H_q .*

Proof. Let H_q be the bipartite graph from Definition 2.1. Since the map $\sigma: B_q \rightarrow H_q$ defined by $\sigma((a, b, c)_1) = (a, b, 2ab+c)_1$ and $\sigma((x, y, z)_0) = (x, y, z)_0$ is an isomorphism, the result holds. \square

Hence, the graph B_q is also q -regular, bipartite, of girth 8 and order $2q^3$.

In what follows, we will obtain the graph Γ_q from the graph B_q by adding some new vertices and edges. We need a preliminary lemma.

Lemma 2.6. *Let B_q be the graph from Definition 2.4. Then the following hold:*

- (i) *The vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four for all $a \in \mathbb{F}_q$.*
- (ii) *The vertices in the set $\{(i, j, k)_0 : j, k \in \mathbb{F}_q\}$ are mutually at distance at least four for all $i \in \mathbb{F}_q$.*
- (iii) *The q vertices of the set $\{(x, y, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least six for all $x, y \in \mathbb{F}_q$.*

Proof. The proof of items (i) and (ii) is almost identical to that of Lemma 2.2.

(iii): By (ii), the vertices in $\{(x, y, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least four. Suppose by contradiction that B_q contains the following path of length four:

$$(x, y, j)_0 (a, b, c)_1 (x', y', j')_0 (a', b', c')_1 (x, y, j'')_0, \text{ for some } j'' \neq j.$$

Then $y = ax+b = a'x+b'$ and $y' = ax'+b = a'x'+b'$. It follows that $(a-a')(x-x') = 0$, which is a contradiction since, by the previous statements, $a \neq a'$ and $x \neq x'$. \square

2.2 The conclusion

Figure 1 shows a spanning tree of Γ_q with the vertices labelled according to Definition 1.1. Note that the lower level of such a tree corresponds to the set of vertices of B_q .

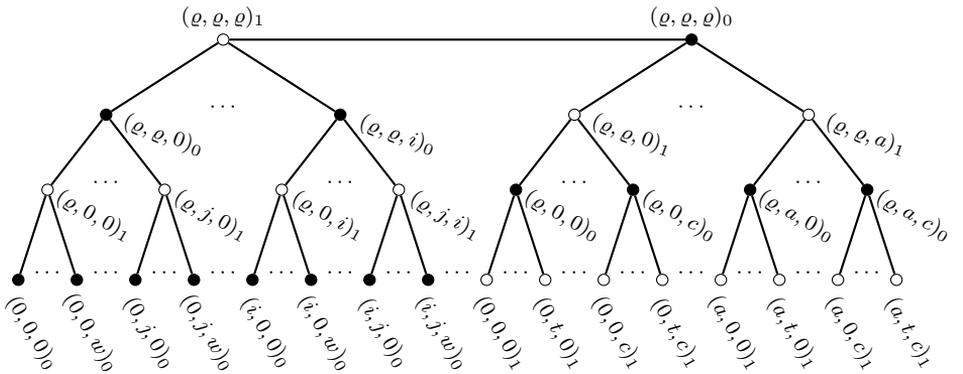


Figure 1: Spanning tree of Γ_q .

We are now ready to prove Theorem 1.2:

Proof of Theorem 1.2. Let $B'_q = B'_q[V_0, V_1]$ be the bipartite graph obtained from $B_q = B_q[V_0, V_1]$ by adding q^2 new vertices to V_1 labeled $(\varrho, b, c)_1, b, c \in \mathbb{F}_q$ (i.e., $V'_1 = V_1 \cup \{(\varrho, b, c)_1 : b, c \in \mathbb{F}_q\}$), and new edges $N_{B'_q}((\varrho, b, c)_1) = \{(c, b, j)_0 : j \in \mathbb{F}_q\}$ (see Figure 1). Then B'_q has $|V'_1| + |V_0| = 2q^3 + q^2$ vertices, every vertex of V_0 has degree $q + 1$, and every vertex of V'_1 has still degree q . Note that the girth of B'_q is 8 by Lemma 2.6(iii). The statements from Lemma 2.6 still partially hold in B'_q , as stated in the following claim.

Claim 1. *For any given $a \in \mathbb{F}_q \cup \{\varrho\}$, the vertices of the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four in B'_q .*

Proof. For $a = \varrho$, it is clear from Lemma 2.6(i), since the new vertices do not change the distance among the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$. For $a \neq \varrho$, the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four since each vertex of the form $(i, j, k)_0$ has exactly one neighbour in this set, so the result follows from the bipartition of B'_q . □

Claim 2. *For all $a \in \mathbb{F}_q \cup \{\varrho\}$ and for all $c \in \mathbb{F}_q$, the q vertices of the set $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B'_q .*

Proof. By Claim 1, for all $a \in \mathbb{F}_q \cup \{\varrho\}$ the q vertices of $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 4 in B'_q . Suppose that there exists in B'_q the following path of length four:

$$(a, t, c)_1 (x, y, z)_0 (a', t', c')_1 (x', y', z')_0 (a, t'', c)_1, \text{ for some } t'' \neq t.$$

If $a = \varrho$, then $x = x' = c, y = t, y' = t''$ and $a' \neq \varrho$ by Claim 1. Then $y = a'x + t' = a'x' + t' = y'$ yielding that $t = t''$ which is a contradiction. Therefore $a \neq \varrho$. If $a' = \varrho$, then $x = x' = c'$ and $y = y' = t'$. Thus $y = ax + t = ax' + t'' = y'$ yielding that $t = t''$ which is a contradiction. Hence we may assume that $a' \neq \varrho$ and $a \neq a'$ by Claim 1. In this case we have:

$$\begin{aligned} y &= ax + t = a'x + t'; \\ y' &= ax' + t'' = a'x' + t'; \\ z &= a^2x + 2at + c = a'^2x + 2a't' + c'; \\ z' &= a^2x' + 2at'' + c = a'^2x' + 2a't' + c'. \end{aligned}$$

Thus,

$$\begin{aligned} (a - a')(x - x') &= t'' - t; & (2.3) \\ (a^2 - a'^2)(x - x') &= 2a(t'' - t). & (2.4) \end{aligned}$$

If q is even, (2.4) leads to $x = x'$ and (2.3) leads to $t'' = t$ which is a contradiction with our assumption. Thus assume q is odd. If $a + a' = 0$, then (2.4) gives $2a(t'' - t) = 0$, so that $a = 0$ yielding that $a' = 0$ (because $a + a' = 0$) which is again a contradiction. If $a + a' \neq 0$, multiplying equation (2.3) by $a + a'$ and subtracting both equations we obtain $(2a - (a + a'))(t'' - t) = 0$. Then $a = a'$ because $t'' \neq t$, which is a contradiction to Claim 1. Therefore, Claim 2 holds. □

Let $B''_q = B''_q[V'_0, V'_1]$ be the graph obtained from $B'_q = B'_q[V_0, V_1]$ by adding $q^2 + q$ new vertices to V_0 labeled $(\varrho, a, c)_0$, $a \in \mathbb{F}_q \cup \{\varrho\}$, $c \in \mathbb{F}_q$, and new edges $N_{B''_q}((\varrho, a, c)_0) = \{(a, t, c)_1 : t \in \mathbb{F}_q\}$ (see Figure 1). Then B''_q has $|V'_1| + |V'_0| = 2q^3 + 2q^2 + q$ vertices such that every vertex has degree $q + 1$ except the new added vertices which have degree q . Moreover the girth of B''_q is 8 by Claim 2.

Claim 3. For all $a \in \mathbb{F}_q \cup \{\varrho\}$, the q vertices of the set $\{(\varrho, a, j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B''_q .

Proof. Clearly these q vertices are mutually at distance at least 4 in B''_q . Suppose that there exists in B''_q the following path of length four:

$$(\varrho, a, j)_0 (a, b, j)_1 (x, y, z)_0 (a, b', j')_1 (\varrho, a, j')_0, \text{ for some } j' \neq j.$$

If $a = \varrho$ then $x = j = j'$ which is a contradiction. Therefore $a \neq \varrho$. In this case $y = ax + b = ax + b'$ which implies that $b = b'$. Hence $z = a^2x + 2ab + j = a^2x + 2ab' + j'$ yielding that $j = j'$ which is again a contradiction. \square

Let $B'''_q = B'''_q[V'_0, V'_1]$ be the graph obtained from B''_q by adding $q + 1$ new vertices to V'_1 labeled $(\varrho, \varrho, a)_1$, $a \in \mathbb{F}_q \cup \{\varrho\}$, and new edges $N_{B'''_q}(\varrho, \varrho, a)_1 = \{(\varrho, a, c)_0 : c \in \mathbb{F}_q\}$, see Figure 1. Then B'''_q has $|V'_1| + |V'_0| = 2q^3 + 2q^2 + 2q + 1$ vertices such that every vertex has degree $q + 1$ except the new added vertices which have degree q . Moreover the girth of B'''_q is 8 by Claim 3 and clearly these $q + 1$ new vertices are mutually at distance 6. Finally, the graph Γ_q is obtained by adding to B'''_q another new vertex labeled $(\varrho, \varrho, \varrho)_0$ and edges $N_{\Gamma_q}((\varrho, \varrho, \varrho)_0) = \{(\varrho, \varrho, i)_1 : i \in \mathbb{F}_q \cup \{\varrho\}\}$. The graph Γ_q has $2(q^3 + q^2 + q + 1)$ vertices, it is $(q + 1)$ -regular and has girth 8, so by the uniqueness of a $(q + 1, 8)$ -cage (see e.g. [29]), Γ_q is indeed a $(q + 1, 8)$ Moore graph. \square

Remark 2.7. Coordinatizations of classical generalized quadrangles $Q(4, q)$ and $W(q)$ in four dimensions are discussed in [23, 26, 28]. The alternate description of a Moore $(q + 1, 8)$ -graph given in Theorem 1.2 in three dimensions is equivalent to this coordinatization.

References

- [1] M. Abreu, G. Araujo-Pardo, C. Balbuena and D. Labbate, Families of small regular graphs of girth 5, *Discrete Math.* **312** (2012), 2832–2842, doi:10.1016/j.disc.2012.05.020.
- [2] M. Abreu, G. Araujo-Pardo, C. Balbuena and D. Labbate, A construction of small $(q - 1)$ -regular graphs of girth 8, *Electron. J. Combin.* **22** (2015), #P2.10 (8 pages), <https://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i2p10>.
- [3] M. Abreu, G. Araujo-Pardo, C. Balbuena, D. Labbate and J. Salas, Families of small regular graphs of girth 7, *Electron. Notes Discrete Math.* **40** (2013), 341–345, doi:10.1016/j.endm.2013.05.060.
- [4] M. Abreu, G. Araujo-Pardo, C. Balbuena, D. Labbate and J. Salas, Small regular graphs of girth 7, *Electron. J. Combin.* **22** (2015), #P3.5 (16 pages), <https://www.combinatorics.org/ojs/index.php/eljc/article/view/v22i3p5>.
- [5] M. Abreu, M. Funk, D. Labbate and V. Napolitano, On (minimal) regular graphs of girth 6, *Australas. J. Combin.* **35** (2006), 119–132, https://ajc.maths.uq.edu.au/pdf/35/ajc_v35_p119.pdf.

- [6] M. Abreu, M. Funk, D. Labbate and V. Napolitano, A family of regular graphs of girth 5, *Discrete Math.* **308** (2008), 1810–1815, doi:10.1016/j.disc.2007.04.031.
- [7] G. Araujo-Pardo, C. Balbuena and T. Héger, Finding small regular graphs of girths 6, 8 and 12 as subgraphs of cages, *Discrete Math.* **310** (2010), 1301–1306, doi:10.1016/j.disc.2009.12.014.
- [8] C. Balbuena, Incidence matrices of projective planes and of some regular bipartite graphs of girth 6 with few vertices, *SIAM J. Discrete Math.* **22** (2008), 1351–1363, doi:10.1137/070688225.
- [9] C. Balbuena, A construction of small regular bipartite graphs of girth 8, *Discrete Math. Theor. Comput. Sci.* **11** (2009), 33–46, <https://dmtcs.episciences.org/461>.
- [10] E. Bannai and T. Ito, On finite Moore graphs, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **20** (1973), 191–208, <http://hdl.handle.net/2261/6123>.
- [11] L. M. Batten, *Combinatorics of Finite Geometries*, Cambridge University Press, Cambridge, 2nd edition, 1997, doi:10.1017/cbo9780511665608.
- [12] C. T. Benson, Minimal regular graphs of girths eight and twelve, *Canadian J. Math.* **18** (1966), 1091–1094, doi:10.4153/cjm-1966-109-8.
- [13] N. Biggs, *Algebraic Graph Theory*, volume 67 of *Cambridge Mathematical Library*, Cambridge University Press, Cambridge, 2nd edition, 1993, doi:10.1017/cbo9780511608704.
- [14] J. A. Bondy and U. S. R. Murty, *Graph Theory*, volume 244 of *Graduate Texts in Mathematics*, Springer, New York, 2008, doi:10.1007/978-1-84628-970-5.
- [15] P. Erdős and H. Sachs, Reguläre Graphen gegebener Tailenweite mit minimaler Knotenzahl, *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe* **12** (1963), 251–257, https://users.renyi.hu/~p_erdos/1963-16.pdf.
- [16] G. Exoo, A simple method for constructing small cubic graphs of girths 14, 15, and 16, *Electron. J. Combin.* **3** (1996), #R30 (3 pages), <https://www.combinatorics.org/ojs/index.php/eljc/article/view/v3i1r30>.
- [17] G. Exoo and R. Jajcay, Dynamic cage survey, *Electron. J. Combin.* (2013), #DS16 (55 pages), <https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS16>.
- [18] A. Gács and T. Héger, On geometric constructions of (k, g) -graphs, *Contrib. Discrete Math.* **3** (2008), 63–80, doi:10.11575/cdm.v3i1.61998.
- [19] C. Godsil and G. Royle, *Algebraic Graph Theory*, volume 207 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2001, doi:10.1007/978-1-4613-0163-9.
- [20] F. Lazebnik, V. A. Ustimenko and A. J. Woldar, New upper bounds on the order of cages, *Electron. J. Combin.* **4** (1997), #R13 (11 pages), <https://www.combinatorics.org/ojs/index.php/eljc/article/view/v4i2r13>.
- [21] M. Meringer, Fast generation of regular graphs and construction of cages, *J. Graph Theory* **30** (1999), 137–146, doi:10.1002/(sici)1097-0118(199902)30:2<137::aid-jgt7>3.0.co;2-g.
- [22] M. O’Keefe and P. K. Wong, The smallest graph of girth 6 and valency 7, *J. Graph Theory* **5** (1981), 79–85, doi:10.1002/jgt.3190050105.
- [23] S. E. Payne, Affine representations of generalized quadrangles, *J. Algebra* **16** (1970), 473–485, doi:10.1016/0021-8693(70)90001-3.
- [24] S. E. Payne and J. A. Thas, *Finite Generalized Quadrangles*, volume 110 of *Research Notes in Mathematics*, Pitman Advanced Publishing Program, Boston, MA, 1984.
- [25] W. T. Tutte, A family of cubical graphs, *Proc. Cambridge Philos. Soc.* **43** (1947), 459–474, doi:10.1017/s0305004100023720.

- [26] V. A. Ustimenko, A linear interpretation of the flag geometries of Chevalley groups, *Ukr. Math. J.* **42** (1990), 341–344, doi:10.1007/bf01057020.
- [27] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, Cambridge University Press, Cambridge, 1992.
- [28] H. van Maldeghem, *Generalized Polygons*, volume 93 of *Monographs in Mathematics*, Birkhäuser Verlag, Basel, 1998, doi:10.1007/978-3-0348-0271-0.
- [29] P. K. Wong, Cages—a survey, *J. Graph Theory* **6** (1982), 1–22, doi:10.1002/jgt.3190060103.