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Quadrature methods for integro-differential equations of Prandtl's type in weighted spaces of continuous functions

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ABSTRACT

The paper deals with the approximate solution of integro-differential equations of Prandtl's type. Quadrature methods involving "optimal" Lagrange interpolation processes are proposed and conditions under which they are stable and convergent in suitable weighted spaces of continuous functions are proved.

The efficiency of the method has been tested by some numerical experiments, some of them including comparisons with other numerical procedures. In particular, as an application, we have implemented the method for solving Prandtl's equation governing the circulation air flow along the contour of a plane wing profile, in the case of elliptic or rectangular wing-shape.

1. Introduction

Hypersingular Integro-Differential Equations (IDE) find application in the treatment of many physics and engineering problems (for instance, see [1–6] and the references therein). In particular, the IDE of Prandtl's type

$$\sigma(y)\zeta(y) + a\zeta'(y) + \frac{b}{\pi}\int_{-1}^{1}\frac{\zeta'(x)}{x-y}dx + \frac{1}{\pi}\int_{-1}^{1}\bar{k}(x,y)\zeta(x)dx = g(y), \quad y \in (-1,1), \tag{1}$$

with $\sigma(y), \bar{k}(x,y)$ and $g(y)$ given functions, the constants $a, b \in \mathbb{R}$ s.t. $a^2+b^2=1$, and the unknown solution ζ a differentiable function, satisfying the zero boundary condition

$$\zeta(-1) = \zeta(1) = 0, \tag{2}$$

is well-known in aerodynamics. In fact, the solution ζ can represent the circulation distribution of air flow along the contour of a wing profile (see, for instance, [3,7–10] and the references therein). (Some experiments concerned with this application will be proposed in Section 4).

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Taking into account the zero boundary condition (2), the solution ζ is conveniently represented as the product of a smooth function f for a Jacobi weight, i.e.

$$\zeta(x) = f(x)v^{\alpha,\beta}(x), \quad v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \ \alpha,\beta > 0. \tag{3}$$

Several authors have studied this kind of IDEs and introduced numerical methods for approximating their solutions (see [11–16] and the references therein), mainly in the case $\alpha = \beta = \frac{1}{2}$. In [14] and [16], when $\sigma \equiv 0$, the equation has been also considered in the more general case $0 < \alpha < 1, \ \beta = 1-\alpha$. In particular in [14] the authors introduced collocation and quadrature methods based on Jacobi zeros studying stability and convergence in weighted L^2 spaces and in [16] a regularized version of (1) has been investigated in a scale of pairs of weighted Besov spaces.

Here, we consider the Eq. (1) both for $\sigma \not\equiv 0, \alpha = \beta = \frac{1}{2}$ and for $\sigma \equiv 0$, $0 < \alpha < 1, \ \beta = 1-\alpha$. In both cases we seek the solution in a couple of weighted Zygmund-type spaces equipped with uniform norm. Two quadrature methods which make use of optimal Lagrange interpolation processes are proposed and for them conditions assuring stability and convergence are determined. The error estimates in weighted uniform norm and the conditioning of the final linear systems are studied. Finally, some numerical tests, which confirm the agreement among the theoretical estimates and the numerical results, are provided.

The plan of the paper is the following. Next section contains some basic results and notation used throughout the paper. In Section 3 the numerical procedures are described and the results about their stability and convergence are stated. Section 4 contains some numerical tests to show the efficiency of the proposed procedure, some of them in comparison with other ones. In Section 5 the proofs of the main results are given, while Section 6 contains conclusions and a brief discussion on the numerical experiments.

2. Preliminaries

From now on the following setting will be used along all the paper:

$$u = v^{\gamma,\delta}, \gamma,\delta \geq 0, \quad w = v^{1-\alpha,\alpha}, \quad \rho = v^{\alpha,1-\alpha}, \quad 0 < \alpha < 1. \tag{4}$$

Moreover the constant $\mathcal{C}$ will be used several times, having different meaning in different formulas. We will write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ to say that $\mathcal{C}$ is a positive constant independent of the parameters $a, b, \ldots,$ and $\mathcal{C} = \mathcal{C}(a,b,\ldots)$ to say that $\mathcal{C}$ depends on $a, b, \ldots$. If $A, B > 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a constant $0 < \mathcal{C} \neq \mathcal{C}(A,B)$ such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B.$$

$\mathbb{P}_m$ will denote the space of the algebraic polynomials of degree at most m. For a bivariate function $k(x, y)$ we use k_x (or k_y) to regard k as function of the only variable y (or x).

Many properties holding for finite part integrals in the Hadamard sense can be found in [17,18] (see also [19] and the references therein). Here we recall [20, Lemma 6.1, Cap II]

$$\frac{d}{dy}\int_{-1}^{1} \frac{g(x)}{x-y}dx = \int_{-1}^{1} \frac{g'(x)}{x-y}dx - \frac{g(-1)}{1+y} - \frac{g(1)}{1-y}, \quad -1 < y < 1,$$

holding if g has a generalized derivative $g' \in L_p(-1,1)$, for some $p > 1$. Then, under the zero endpoints conditions (2)-(3) with $\beta = 1-\alpha$, Eq. (1) can be rewritten as

$$(M_{\sigma\rho} + DA^{\rho} + K + H)f(y) = g(y),$$

where

$$(M_{\sigma\rho}f)(y) = (\sigma\rho f)(y), \quad (Dq)(y) = \frac{d}{dy}q(y),$$

$$(A^{\rho}f)(y) = a(f\rho)(y) + \frac{b}{\pi}\int_{-1}^{1} \frac{(f\rho)(x)}{x-y}dx,$$

$$(Kf)(y) = \frac{1}{\pi}\int_{-1}^{1} k(x,y)(f\rho)(x)dx, \quad (Hf)(y) = \frac{1}{\pi}\int_{-1}^{1} h(x,y)(f\rho)(x)dx,$$

with $\sigma(y)$ a given function, $k(x, y)$ and $h(x, y)$ smooth and weakly singular kernels, respectively, such that $\bar{k}$ in (1) satisfies $\bar{k}(x,y) = k(x,y) + h(x,y)$. The Fredholm index of the Cauchy singular integral operator $A^{\rho} : L^2_{\rho} \to L^2_{\rho}$ is equal to -1 if $a = \cos(\pi\alpha), \ b = -\sin(\pi\alpha)$ (see, for instance, [21]). Here, L^2_{ρ} is the Hilbert space defined by the inner product

$$< f, g >_{\rho} = \int_{-1}^{1} f(x)\overline{g(x)}\rho(x)dx. \tag{5}$$

2.1. Function spaces

We consider the space of functions

$$C_u = \begin{cases} \left\{f \in C^0((-1,1)) \ : \ \lim_{x\to\pm1^{\mp}}(fu)(x)=0\right\}, & \gamma>0, \delta>0 \\ \left\{f \in C^0((-1,1]) \ : \ \lim_{x\to-1^{+}}(fu)(x)=0\right\}, & \gamma=0, \delta>0 \\ \left\{f \in C^0([-1,1)) \ : \ \lim_{x\to1^{-}}(fu)(x)=0\right\}, & \gamma>0, \delta=0 \\ C^0([-1,1]), & \gamma=\delta=0 \end{cases},$$

equipped with the norm

$$\|f\|_{C_u} := \|fu\|_\infty = \max_{|x|\le1}|(fu)(x)|.$$

Somewhere, for brevity, we will set $\|f\|_A := \max_{x\in A}|f(x)|$.

Note that the limit conditions are necessary for the validity of the Weierstrass theorem in C_u. Then, denoting by

$$E_m(f)_u = \inf_{P_m\in\mathbb{P}_m}\|(f-P_m)u\|_\infty$$

the error of best polynomial approximation of $f\in C_u$ by means of polynomials of degree at most m, we have [22, p. 172 (2.5.23)]

$$\lim_m E_m(f)_u = 0. \tag{6}$$

Let $\varphi(x)=\sqrt{1-x^2}$. For a given integer $k\ge1$, let $I_{k\tau}=[-1+(2k\tau)^2, 1-(2k\tau)^2]$, with $|\tau|<\frac{1}{2k}$. For any $f\in C_u$ the main part of the *φ-modulus of smoothness* is defined as [23, p. 90]

$$\Omega_\varphi^k(f,t)_u = \sup_{0<\tau\le t}\|u\Delta_{\tau\varphi}^k f\|_{I_{k\tau}}, \quad 0<t\le t_0, \tag{7}$$

for some $t_0<1$, being

$$\Delta_{\tau\varphi}^k f(x) = \sum_{i=0}^{k}(-1)^i\binom{k}{i} f\left(x+\frac{\tau\varphi(x)}{2}(k-2i)\right).$$

It is understood whenever one of the points $x\pm\frac{\tau\varphi(x)}{2}k$ doesn't belong to $(-1,1)$ the corresponding addendum in the summation is null.

By means of $\Omega_\varphi^k(f,t)_u$, we define the Zygmund space of order $r\in\mathbb{R}, r>0$,

$$Z_{r,k}(u) = \left\{f\in C_u \ : \ sup_{t>0}\frac{\Omega_\varphi^k(f,t)_u}{t^r}<+\infty\right\}, \quad k\ge r,$$

endowed with the norm

$$\|f\|_{Z_{r,k}(u)} = \|fu\|_\infty + \sup_{t>0}\frac{\Omega_\varphi^k(f,t)_u}{t^r}. \tag{8}$$

The following equivalence holds true (see, for instance, [22, p. 172])

$$\sup_{t>0}\frac{\Omega_\varphi^k(f,t)_u}{t^r} \sim \sup_{i\ge0}(1+i)^r E_i(f)_u, \tag{9}$$

where the constants in " ∼ " depend on r. Such norms equivalence ensure that the definition of the Zygmund space doesn't depend on $k\ge r$ and therefore we will set $Z_r(u) := Z_{r,k}(u)$.

When r is a positive integer, we define the Sobolev space

$$W_r(u) = \left\{f\in C_u : f^{(r-1)}\in AC(-1,1), \quad \|f^{(r)}\varphi^r u\|_\infty<\infty\right\},$$

where $AC(-1,1)$ denotes the set of the functions which are absolutely continuous on every closed subinterval of $(-1,1)$, equipped with the norm

$$\|f\|_{W_r(u)} = \|fu\|_\infty + \|f^{(r)}\varphi^r u\|_\infty.$$

In order to estimate $E_m(f)_u$ we recall the Favard inequality (see, for instance, [22, p. 172])

$$E_m(f)_u \le \frac{\mathcal{C}}{m^r}\|f\|_{Z_r(u)}, \quad \forall f\in Z_r(u), \tag{10}$$

where the constant $\mathcal{C}$ does not depend on m and f but depends on r. Moreover, letting $E_m(f)_{Z_r(u)} = \inf_{P_m\in\mathbb{P}_m}\|f-P_m\|_{Z_r(u)}$, we recall [24, p. 33]

$$E_m(f)_{Z_r(u)} \le \mathcal{C}\sup_{k\ge1}k^r E_k(f)_u, \quad \mathcal{C}\ne\mathcal{C}(m,f). \tag{11}$$

In the sequel we will write $Z_r := Z_r(v^{0,0})$ and $E_m(f)_{v^{0,0}} := E_m(f)$. Moreover, we refer to the space C_u when the notations $Z_0(u)$ and $W_0(u)$ arise.

2.2. Lagrange interpolation

For a given Jacobi weight $\theta = v^{\alpha,\beta}$, $\alpha, \beta > -1$, let $\{p_m^\theta\}_{m=0}^\infty$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients and let $\{\lambda_{m,k}^\theta\}_{k=1}^m$ be the Christoffel numbers w.r.t. θ. Let ρ and w be defined in (4). Let $L_m^w(G,x)$ be the Lagrange polynomial interpolating a given function $G \in C_{u\varphi}$ at the zeros $\{x_i\}_{i=1}^m$ of p_m^w and let $L_m^\rho(G,x)$ be the Lagrange polynomial interpolating $G \in C_{u\rho}$ at the zeros $\{t_i\}_{i=1}^m$ of p_m^ρ. Following an idea in [25,26], we represent $L_m^w(G,x)$ in the basis

$$\psi_i^w(x) = \frac{\lambda_{m,i}^w \sum_{j=0}^{m-1} p_j^w(x_i) p_j^w(x)}{(u\varphi)(x_i)}, \quad i = 1, 2, \cdots, m,$$

of $\mathbb{P}_{m-1}$ and $L_m^\rho(G,x)$ in the basis

$$\psi_i^\rho(x) = \frac{\lambda_{m,i}^\rho \sum_{j=0}^{m-1} p_j^\rho(t_i) p_j^\rho(x)}{(u\rho)(t_i)}, \quad i = 1, 2, \cdots, m,$$

of $\mathbb{P}_{m-1}$. More precisely, we write

$$L_m^w(G,x) = \sum_{i=1}^m \psi_i^w(x)(u\varphi G)(x_i) \tag{12}$$

and

$$L_m^\rho(G,x) = \sum_{i=1}^m \psi_i^\rho(x)(u\rho G)(t_i). \tag{13}$$

The choice of these bases is crucial in the study of the conditioning of the linear systems involved in our numerical methods (see Theorems 3.3 and 3.6).

Next lemma, a consequence of [27, Theorem 2.2], states the conditions under which the above introduced Lagrange processes are optimal:

Lemma 2.1. *Let* $0 < \alpha < 1$. *If* γ, δ *satisfy*

$$-\frac{\alpha}{2} + \frac{1}{4} \le \gamma < -\frac{\alpha}{2} + \frac{5}{4}, \qquad \frac{\alpha}{2} - \frac{1}{4} \le \delta < \frac{\alpha}{2} + \frac{3}{4}, \tag{14}$$

then

$$\|L_m^w(f)u\varphi\|_\infty \le \mathcal{C}\log m \|fu\varphi\|_\infty, \quad \forall f \in C_{u\varphi}, \tag{15}$$

$$\|L_m^\rho(f)u\rho\|_\infty \le \mathcal{C}\log m \|fu\rho\|_\infty, \quad \forall f \in C_{u\rho}, \tag{16}$$

where $\mathcal{C} \neq \mathcal{C}(m,f)$.

The following lemma, a special case of [28, Th.1, p. 680], will be also useful in the sequel.

Lemma 2.2. *Let* $0 < \alpha < 1$. *If* γ, δ *satisfy* $0 \le \gamma < -\frac{\alpha}{2} + \frac{3}{4}$, $0 \le \delta < \frac{\alpha}{2} + \frac{1}{4}$, *then, for any* $f \in C^0([-1,1])$,

$$\int_{-1}^1 |L_m^\rho(f,x)|u^{-1}(x)dx \le \mathcal{C}\|f\|_\infty, \quad \mathcal{C} \neq \mathcal{C}(m,f).$$

3. Main results

We present now our main results, concerned with the equations

$$(DA^\rho + K + H)f = g, \quad 0 < \alpha < 1, \tag{17}$$

$$(M_{\sigma\varphi} + DA^\varphi + K + H)f = g. \tag{18}$$

We start investigating (17) in the pair of Zygmund spaces $(Z_r(u\rho), Z_{r-1}(u\varphi))$.

Theorem 3.1. *Let* $0 < \alpha < 1$. *Assume that with* γ, δ *satisfying*

$$\max\left\{0, -\frac{\alpha}{2} + \frac{1}{4}\right\} \le \gamma < -\frac{\alpha}{2} + \frac{1}{2}, \qquad \max\left\{0, \frac{\alpha}{2} - \frac{1}{4}\right\} \le \delta < \frac{\alpha}{2}, \tag{19}$$

and for some $s > 0$ *it is*

$$k_x \in Z_s(u\varphi) \quad \text{uniformly with respect to}\, x, \tag{20}$$

$$\sup_{|y|\le 1}(u\varphi)(y)\int_{-1}^{1}|h(x,y)|u^{-1}(x)dx<+\infty, \tag{21}$$

and

$$A(\tau):=\sup_{y\in I_\tau}(u\varphi)(y)\int_{-1}^{1}|\Delta_{\tau\varphi(y)}h(x,y)|u^{-1}(x)dx<\mathcal{C}\tau^s, \tag{22}$$

with $\mathcal{C}\neq\mathcal{C}(\tau)$, $I_\tau=[-1+(2\tau)^2,1-(2\tau)^2]$. *If* $Ker(DA^\rho+K+H)=\{0\}$ *in* $Z_r(u\rho)$ *with* $1<r<s+1$, *then Eq. (17) admits a unique solution* f^* *in* $Z_r(u\rho)$ *for any* $g\in Z_{r-1}(u\varphi)$.

Provided the conditions assuring existence and uniqueness of the solution of Eq. (17), we go to describe the numerical method proposed to approximate its solution. Letting

$$(K_m f)(y)=\frac{1}{\pi}\int_{-1}^{1}L_m^\rho(k_y,x)(f\rho)(x)dx, \tag{23}$$

we proceed to solve the finite dimensional equation

$$(DA^\rho+L_m^wK_m+L_m^wH)f_m=L_m^wg,\quad m\ge 1, \tag{24}$$

in the unknown f_m, where

$$f_m(y)=\sum_{k=1}^{m}\psi_k^\rho(y)a_k. \tag{25}$$

Since by [21, Theorems 9.9 and 9.14, Remark 9.15] and [29, (4.21.7)] we get

$$DA^\rho p_m^\rho=(m+1)p_m^w,\quad m=0,1,\dots, \tag{26}$$

Eq. (24) can be written as

$$L_m^w(DA^\rho f_m+K_mf_m+Hf_m)=L_m^w(g)$$

and collocating it at the zeros $\{x_i:=x_i^w\}_{i=1}^m$ of p_m^w, we get, for $i=1,\dots,m$,

$$(u\varphi DA^\rho f_m)(x_i)+(u\varphi K_mf_m)(x_i)+(u\varphi Hf_m)(x_i)=(u\varphi g)(x_i). \tag{27}$$

In view of (13) and (26)

$$(DA^\rho f_m)(x_i)=\sum_{k=1}^{m}\frac{a_k}{(u\rho)(t_k)}\lambda_{m,k}^\rho\sum_{j=0}^{m-1}p_j^\rho(t_k)(j+1)p_j^w(x_i) \tag{28}$$

and, by (23),

$$(K_mf_m)(x_i)=\frac{1}{\pi}\sum_{k=1}^{m}\frac{a_k}{(u\rho)(t_k)}\lambda_{m,k}^\rho k(t_k,x_i). \tag{29}$$

Moreover, we have

$$(Hf_m)(x_i)=\frac{1}{\pi}\sum_{k=1}^{m}\frac{a_k\lambda_{m,k}^\rho}{(u\rho)(t_k)}\sum_{j=0}^{m-1}p_j^\rho(t_k)c_j(x_i),\quad c_j(y)=\int_{-1}^{1}h(x,y)p_j^\rho(x)\rho(x)dx, \tag{30}$$

where the Fourier coefficients $c_j(y)$, also known as *Modified Moments*, can be computed by suitable recurrence relations (see e.g. [30, p. 333]). Thus, combining (28), (29) and (30) with (27), setting $\mathbf{a}_m=[a_1,\dots,a_m]^T$, we get the linear system

$$\mathbf{A}_m\mathbf{a}_m=\mathbf{b}_m, \tag{31}$$

$$\mathbf{A}_m=\mathbf{U}_m(\mathbf{V}_m[\mathbf{D}_m\mathbf{Z}_m+\mathbf{W}_m]+\mathbf{K}_m)\mathbf{\Lambda}_m,\qquad \mathbf{b}_m=\mathbf{U}_m\mathbf{g}_m, \tag{32}$$

with $\mathbf{U}_m=\mathrm{diag}((u\varphi)(x_1),\dots,(u\varphi)(x_m))$, $\mathbf{g}_m=[g(x_1),\dots,g(x_m)]^T$,

$$\left\{\mathbf{W}_m(i,j)=\frac{1}{\pi}c_i(x_j)\right\}_{\substack{i=0,1,\dots,m-1\\ j=1,\dots,m}},\quad \left\{\mathbf{K}_m(i,k)=\frac{1}{\pi}k(t_k,x_i)\right\}_{\substack{i=1,\dots,m\\ k=1,\dots,m}},$$

$$\mathbf{D}_m=\mathrm{diag}(1,\dots,m),\quad \mathbf{\Lambda}_m=\mathrm{diag}\left(\frac{\lambda_{m,1}^\rho}{(u\rho)(t_1)},\dots,\frac{\lambda_{m,m}^\rho}{(u\rho)(t_m)}\right),$$

$$\{\mathbf{V}_m(i,j)=p_j^\rho(t_i)\}_{\substack{i=1,\dots,m\\ j=0,1,\dots,m-1}},\quad \{\mathbf{Z}_m(i,j)=p_i^w(x_j)\}_{\substack{i=0,1,\dots,m-1\\ j=1,\dots,m}}.$$

Therefore, if $\mathbf{a}_m^* = [a_1^*, \dots, a_m^*]^T$ is the unique solution of the linear system (31), we construct the unique solution of the Eq. (24) as follows

$$f_m^*(y) = \sum_{k=1}^m \psi_k^\rho(y)\ a_k^*.$$

About the stability and the convergence of the method, we prove the following

Theorem 3.2. *Let $0 < \alpha < 1$. Let us assume that (19) holds and that, for some $s > 0$, the kernel k satisfies the assumptions (20) and*

$$(u\varphi)(y)k_y \in Z_s \quad \text{uniformly with respect to}\, y, \tag{33}$$

h satisfy (21)-(22), $g \in Z_s(u\varphi)$, and $Ker(DA^\rho + K + H) = \{0\}$ in $Z_r(u\rho)$ with $1 < r < s+1$.

Then, for m sufficiently large (say $m > m_0$), the operators $DA^\rho + L_m^w K_m + L_m^w H : (\mathbb{P}_{m-1}, \|\cdot\|_{Z_r(u\rho)}) \to (\mathbb{P}_{m-1}, \|\cdot\|_{Z_{r-1}(u\varphi)})$ are invertible and their inverses are uniformly bounded. Moreover, the unique solution f^* of (17) belongs to $Z_{s+1}(u\rho)$ and if f_m^* denotes the unique solution of (24), for all $1 < r < s+1$, the following error estimate holds true

$$\|f^* - f_m^*\|_{Z_r(u\rho)} \le \mathcal{C}\left(\frac{\log m}{m^{s-r+1}}\right)\|f^*\|_{Z_{s+1}(u\rho)}, \tag{34}$$

where the constant $\mathcal{C}$ is independent of m and f^*.

We conclude with the study of the linear system conditioning.

Theorem 3.3. *Under the assumptions of Theorem 3.2, denoting by $\operatorname{cond}(\mathbf{A}_m)$ the condition number of $\mathbf{A}_m$ in infinity norm, we have*

$$\operatorname{cond}(\mathbf{A}_m) \le \mathcal{C}\ \left\|\left((DA^\rho + L_m^w K_m + L_m^w H)\ _{|\mathbb{P}_{m-1}}\right)^{-1}\right\|_{C_{u\varphi}\to C_{u\rho}} m\log^3 m, \tag{35}$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

Now we treat the case of the Eq. (18). Next theorem assigns sufficient conditions under which it is unisolvent.

Theorem 3.4. *Let us assume that for some $s > 0$, $0 \le \gamma < \frac{1}{4}$ and $0 \le \delta < \frac{1}{4}$, the kernels k and h satisfy the assumptions (20)–(22) and (33) with $\alpha = \frac{1}{2}$ and $\sigma\varphi \in Z_s$. If $Ker(M_{\sigma\varphi} + DA^\varphi + K + H) = \{0\}$ in $Z_r(u\varphi)$ with $1 < r < s+1$, then Eq. (18) admits a unique solution f^* in $Z_r(u\varphi)$ for any $g \in Z_{r-1}(u\varphi)$.*

Now, to approximate the solution of Eq. (18) we solve the following finite dimensional equation

$$(L_m^\varphi M_{\sigma\varphi} + DA^\varphi + L_m^\varphi K_m + L_m^\varphi H)f_m = L_m^\varphi(g), \quad m \ge 1, \tag{36}$$

in the unknown

$$f_m(y) = \sum_{k=1}^m \psi_k^\varphi(y)\bar{a}_k, \quad \psi_k^\varphi(y) = \frac{\ell_{m,k}^\varphi(y)}{(u\varphi)(x_k^\varphi)}, \quad x_k^\varphi \text{ zeros of } p_m^\varphi. \tag{37}$$

By (28), (29), (30) with $\alpha = \frac{1}{2}$ and

$$(u\varphi M_{\sigma\varphi} f_m)(x_i^\varphi) = (\sigma\varphi)(x_i^\varphi)a_i, \quad 1 \le i \le m,$$

the finite dimensional Eq. (36) is equivalent to the linear system

$$\bar{\mathbf{A}}_m\bar{\mathbf{a}}_m = \mathbf{b}_m,$$

where

$$\bar{\mathbf{a}}_m = [\bar{a}_1, \dots, \bar{a}_m]^T, \quad \mathbf{b}_m = \mathbf{U}_m\mathbf{g}_m, \quad \bar{\mathbf{A}}_m = \mathbf{\Gamma}_m + \mathbf{A}_m,$$

with $\mathbf{A}_m$ and $\mathbf{b}_m$ defined in (32) and $\mathbf{\Gamma}_m = \operatorname{diag}((\sigma\varphi)(x_1^\varphi), \dots, (\sigma\varphi)(x_m^\varphi))$.

About the stability and the convergence of the method and the conditioning of the linear systems, next theorems hold true.

Theorem 3.5. *Under the same assumptions of Theorem 3.2 with $\alpha = \frac{1}{2}$, if for some $s > 0$, $\sigma\varphi \in Z_s$ and $Ker(M_{\sigma\varphi} + DA^\varphi + K + H) = \{0\}$ in $Z_r(u\varphi)$, with $1 < r < s+1$, then, for m sufficiently large, the operators $L_m^\varphi M_{\sigma\varphi} + DA^\varphi + L_m^\varphi K_m + L_m^\varphi H : (\mathbb{P}_{m-1}, \|\cdot\|_{Z_r(u\varphi)}) \to (\mathbb{P}_{m-1}, \|\cdot\|_{Z_{r-1}(u\varphi)})$ are invertible and their inverses are uniformly bounded.*

Moreover, the unique solution f_m^* *of* (36) *converges to the unique solution* $f^* \in Z_{s+1}(u\varphi)$ *of* (18) *and, for all* $1 < r < s+1$, *the following error estimate holds*

$$\|f^* - f_m^*\|_{Z_r(u\varphi)} \leq \mathcal{C}\left(\frac{\log m}{m^{s-r+1}}\right)\|f^*\|_{Z_{s+1}(u\varphi)}, \tag{38}$$

where the constant $\mathcal{C}$ *is independent of* m *and* f^*.

Theorem 3.6. *Under the assumptions of Theorem* 3.5, *denoting by* $\mathrm{cond}(\bar{\mathbf{A}}_m)$ *the condition number of the matrix* $\bar{\mathbf{A}}_m$ *in infinity norm, we get*

$$\mathrm{cond}(\bar{\mathbf{A}}_m) \leq \mathcal{C}\left\| \left((L_m^\varphi M_{\sigma\varphi} + DA^\varphi + L_m^\varphi K_m + L_m^\varphi H)_{\ |\mathbb{P}_{m-1}} \right)^{-1} \right\|_{C_{u\varphi}\to C_{u\varphi}} m \log^3 m, \tag{39}$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

Remark 3.1. Firstly we recall that the following subspace of L^2_ρ

$$L_\rho^{2,r+1} = \left\{ f \in L^2_\rho \ : \ \|f\|_{L_\rho^{2,r+1}} := \left(\sum_{n=0}^{\infty} (1+n)^{2(r+1)} c_n^2 \right)^{\frac{1}{2}} < +\infty \right\},$$

where $\{c_n\}_{n=0}^\infty$ are the Fourier coefficients of f in the orthonormal system $\{p_n^\rho\}_{n=0}^\infty$ w.r.t. the inner product (5), is embedded in the Zygmund space $Z_r(u\rho)$ (see [31,32]), i.e.,

$$\|f\|_{Z_r(u\rho)} \leq \mathcal{C}\|f\|_{L_\rho^{2,r+1}}, \quad \mathcal{C} \neq \mathcal{C}(f).$$

This observation could allow to deduce error estimates in $\|\cdot\|_{Z_r(u\rho)}$ starting from that obtained in $\|\cdot\|_{L_\rho^{2,r+1}}$. In fact, by using the estimate in [14, Theorem 3.1], one can prove

$$\|f^* - f_m^*\|_{Z_r(u\rho)} \leq \mathcal{C}\|f^* - f_m^*\|_{L_\rho^{2,r+1}} \leq \frac{\mathcal{C}}{m^{s-r}}\|f^*\|_{L_\rho^{2,s+1}}, \quad \mathcal{C} \neq \mathcal{C}(m, f^*).$$

However, comparing the latter bound with the one in (34), it is clear that a direct estimate in Zygmund norm, let us get a better rate of convergence.

Now, if f^* is the solution of the Eq. (17) (or (18)) and f_m^* is the solution of (24) (or (36)), we denote by $\zeta^* = f^*\rho$ the exact solution of the initial Prandtl's equation (1) and by $\zeta_m^* := f_m^*\rho$ its $m-$th approximation. By Theorems 3.2 and 3.5 we can easily deduce the following

Corollary 3.1. *Under the assumptions of Theorems* 3.2 *or* 3.5, *for any* $\varepsilon > 0$, *one has*

$$\|(\zeta^* - \zeta_m^*)u\|_\infty = \|(f^* - f_m^*)u\rho\|_\infty \leq \mathcal{C}\|f^* - f_m^*\|_{Z_{1+\varepsilon}(u\rho)} \leq \mathcal{C}\ \frac{\log m}{m^{s-\varepsilon}}, \tag{40}$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

Remark 3.2. Estimate (40) will be useful in the practical evaluation of the error in the numerical tests, since the discrete absolute error on the left hand side is what we want in order to deduce the number of the exact digits we can reach. From (40) we can deduce that the convergence order of the proposed method is at least $s - \varepsilon$. We recall that for functions belonging to $Z_{s+1}(u\rho)$ the convergence order of the polynomial of best approximation is $s+1$ (see (11)).

Remark 3.3. As you can see, the estimates of $\mathrm{cond}(\mathbf{A}_m)$ and $\mathrm{cond}(\bar{\mathbf{A}}_m)$ given in Theorems 3.3 and 3.6 are not complete, since we are not able to state the uniformly boundedness of the norms in (35) and (39). Nevertheless, the numerical evidences provided by the numerical tests (see Section 4) encourage us to believe that such norms do not increase w.r.t. m.

We conclude by showing some weakly singular kernels satisfying (21)–(22).

Proposition 3.1. *Under the assumptions* $0 \leq \gamma,\ \delta < 1$ *and* $-1 < \mu < 0$, *the kernels*

$$h(x,y) = \begin{cases} |x-y|^\mu, \\ |x-y|^\mu \mathrm{sgn}(x-y), \\ \log|x-y|, \\ |x-y|^\mu \log|x-y|, \end{cases}$$

satisfy (21) *and, with* $A(t)$ *as in* (22), *next estimates hold*

$$A(\tau) \leq \mathcal{C}\begin{cases} \tau^{\mu+1} & h(x,y) = |x-y|^\mu \\ \tau^{\mu+1} & h(x,y) = |x-y|^\mu \mathrm{sgn}(x-y) \\ \tau \log \tau^{-1} & h(x,y) = \log|x-y| \\ \tau^{\mu+1}\log\tau^{-1} & h(x,y) = |x-y|^\mu \log|x-y|. \end{cases} \tag{41}$$

Table 1
Example 4.1.

m	$\mathrm{cond}(\bar{\mathbf{A}}_m)$	Err_m
8	$4.9982e+00$	$1.5099e-03$
16	$9.0130e+00$	$7.0718e-05$
32	$1.6870e+01$	$1.6872e-06$
64	$3.2465e+01$	$4.5720e-08$
128	$6.3581e+01$	$8.8290e-10$
256	$1.2576e+02$	$2.5805e-11$
512	$2.5011e+02$	$6.4149e-13$

4. Numerical tests

Now we show the performance of our methods by some numerical examples, where the exact solution ζ of (1) will be approximated by $\zeta_m := f_m\rho$, with f_m given in (25) or (37). When ζ is unknown we will retain the approximation ζ_{1024} as exact.

In the tables we will report, for each m, the maximum absolute error attained by ζ_m at a sufficiently large uniform mesh $\mathrm{X} \subset [-1, 1]$, i.e.

$$Err_m = \max_{y_i\in \mathrm{X}} u(y_i)|\zeta(y_i) - \zeta_m(y_i)|, \quad err_m = \max_{y_i\in \mathrm{X}} u(y_i)|\zeta_{1024}(y_i) - \zeta_m(y_i)|. \tag{42}$$

In order to make comparisons with other methods existing in the literature, in Example 1 we show the numerical results obtained approximating an IDE considered in [15] and in Example 2 we compare our results with those achieved with the method in [16].

Moreover, to verify the effectiveness of our theoretical estimates, in Examples 2 and 3 we consider suitable test IDEs and we will report the Estimated Order of Convergence (EOC) for increasing values of m, i.e.

$$EOC_m = \frac{\log(err_m/err_{2m})}{\log 2}.$$

According to Theorems 3.3 and 3.6, the condition numbers of the linear systems increase with m at least as $m\log^3 m$. Presuming a more general increasing behaviour of the condition numbers of order m^ν, with $\nu > 0$, for the Examples 2 and 3, we will report for each m the following estimators of ν

$$\nu(m) = \frac{\log\left(\frac{\mathrm{cond}(\mathbf{A}_{2m})}{\mathrm{cond}(\mathbf{A}_m)}\right)}{\log 2} \quad \text{and} \quad \overline{\nu}(m) = \frac{\log\left(\frac{\mathrm{cond}(\bar{\mathbf{A}}_{2m})}{\mathrm{cond}(\bar{\mathbf{A}}_m)}\right)}{\log 2},$$

respectively.

The values $\mathrm{cond}(\mathbf{A}_m)$ and $\mathrm{cond}(\bar{\mathbf{A}}_m)$ are computed using the `MatLab` function `cond.m` with parameter $p = inf$.

Finally, in Section 4.1 we show how our numerical method can be used to approximate the solutions of some special IDEs of Prandtl's type coming from some problems in aerodynamics.

All the computations were performed in 16−digits arithmetic.

Example 4.1. Let us consider the IDE of Prandtl's type (18) with

$$\sigma(y)=2, \quad k(x,y)\equiv 0, \;\; h(x,y) = \log|x-y|,$$

and $g(x)$ such that the exact solution is $\zeta(x) = \sqrt{1-x^2}f(x)$ with $f(x) = \sqrt{(1-x^2)^3}$. This equation has been considered in [15]. The authors show (see [15, Table 2]) only the approximations of the solution obtained for $n = 15$: they get at most 2 exact decimal digits. As one can see inspecting Table 1, our results are more satisfactory. In fact with $m = 16$ we get 3 exact decimal digits and with $m = 512$ we attain 11 exact decimal digits.

The same integral equation has been considered in [15] also with $g(x)$ such that the exact solution is $\zeta(x) = \sqrt{1-x^2}f(x)$ with $f(x) = x$. Applying their method with $n = 35$ the authors get approximations of the solution with at most 3 exact decimal digits. On the contrary, our method allows us to attain approximations of the solution with the machine precision by solving a linear system of order $m = 2$.

Example 4.2. Now we consider the Eq. (17) with $\alpha = \frac{1}{4}$,

$$k(x,y) = \left|\cos\left(y - \frac{\pi}{4}\right)\right|^{\frac{9}{2}} + |\sin(x)|^{\frac{7}{2}}, \;\; h(x,y) = |x-y|^{-\frac{1}{3}}, \;\; g(y) = |y|^{\frac{11}{2}}.$$

The solution is $\zeta(x) = (1-x)^{\frac{1}{4}}(1+x)^{\frac{3}{4}}f(x)$, f unknown. Here, choosing $\gamma = \frac{1}{8}$ and $\delta = 0$ (according to (19)), k satisfies (20) with $s = \frac{9}{2}$ and (33) with $s = \frac{7}{2}$, h satisfies (22) with $s = \frac{2}{3}$ (see Proposition 3.1) and $g \in Z_{\frac{11}{2}}(v^{\frac{5}{8},\frac{1}{2}})$. Thus, by Theorem 3.2, $f \in Z_{\frac{5}{3}}(v^{\frac{3}{8},\frac{3}{4}})$ and, by Remark 3.2, the error behaves at least as $\frac{\log m}{m^{\frac{2}{3}-\varepsilon}}$. This slow convergence is confirmed

Table 2
Example 4.2.

m	cond($\mathbf{A}_m$)	$\nu(m)$	err_m	EOC_m
8	$5.5777e+00$		$3.5841e-02$	
16	$1.1021e+01$	$9.82621e-01$	$2.1644e-02$	0.7276
32	$2.1911e+01$	$9.91306e-01$	$9.3647e-03$	1.2086
64	$4.3681e+01$	$9.95335e-01$	$4.7068e-03$	0.9924
128	$8.7390e+01$	$1.00044e+00$	$2.2208e-03$	1.0836
256	$1.7485e+02$	$1.00058e+00$	$9.5749e-04$	1.2137
512	$3.4982e+02$	$1.00051e+00$	$3.2044e-04$	1.5791

Table 3
Example 4.2: Numerical results obtained using the method in [16].

m	cond($\mathbf{A}_m$)	err_m
8	$2.0128e+01$	$6.1062e+01$
16	$9.3696e+01$	$6.0582e+01$
32	$4.8291e+02$	$5.9618e+01$
64	$2.6155e+03$	$5.7693e+01$
128	$1.4510e+04$	$5.3846e+01$
256	$8.1311e+04$	$4.6153e+01$
512	$4.5778e+05$	$3.0769e+01$

Table 4
Example 4.3.

m	cond$\left(\bar{\mathbf{A}}_m\right)$	$\overline{\nu}(m)$	err_m	EOC_m
8	$4.9498e+00$		$9.7163e-05$	
16	$9.1339e+00$	$8.8384e-01$	$5.3368e-06$	4.18634
32	$1.7478e+01$	$9.3631e-01$	$3.1042e-07$	4.10367
64	$3.4150e+01$	$9.6627e-01$	$1.5510e-08$	4.32298
128	$6.7481e+01$	$9.8259e-01$	$7.4500e-10$	4.37980
256	$1.3413e+02$	$9.9114e-01$	$5.1794e-11$	3.84638
512	$2.6744e+02$	$9.9552e-01$	$7.6090e-12$	2.76699

inspecting Table 2. In fact, the arithmetic mean of the estimated orders of convergence EOC_m is almost 1.1342. In this case the estimator $\nu(m)$ shows that $\nu \sim 1.00051$.

Note that, the integrals c_j in (30) have been computed using the recurrence relation showed in [30, p. 333].

Applying the numerical method proposed in [16, p. 160] for the numerical resolution of the above integral equation you get the results presented in Table 3. As you can see, since the obtained linear systems have higher condition numbers, no correct digits are achieved for the approximations of the solution.

Example 4.3. Consider the integral Eq. (18) with

$$(\sigma\varphi)(y) = y^2+1, \quad k(x,y) = \frac{\cos(x+y)}{(x^2+y^2+20)^2}, \ \ h \equiv 0, \ \ g(y) = \left|y+\frac{3}{10}\right|^{\frac{7}{2}} + y\sin(y).$$

In this case the solution has the form $\zeta(x) = \sqrt{1-x^2}f(x)$, f unknown. According to (19) we take $\gamma = \delta = 0$. Since $\sigma\varphi \in Z_s(\varphi)$ for any $s > 0$, k satisfies (20) and (33) for any $s > 0$ and $g \in Z_{\frac{7}{2}}(\varphi)$, by Theorem 3.5, $s = \frac{7}{2}$ and therefore $f \in Z_{\frac{9}{2}}(\varphi)$. So, according to Remark 3.2, the errors behave at least as $\frac{\log m}{m^{\frac{7}{2}-\varepsilon}}$. By Table 4, we can conclude that the theoretical expectations are verified, the arithmetic mean of the EOC_m being ~ 3.9343. In this case, we have cond$(\bar{\mathbf{A}}_m) \sim m$ at most.

4.1. An application

The Prandtl's equation (see [8],[9])

$$\beta C(z) = \frac{\alpha(z)}{2}\int_{-b}^{b}\frac{C(\eta)}{(\eta-z)^2}d\eta + j(z), \tag{43}$$

with the zero boundary conditions $C(\pm b) = 0$, governs the (unknown) circulation air flow $C(y)$ along the contour of a plane wing profile. The constant $\beta = \sqrt{1-M^2}$, where M the Mach number in undisturbed motion, j, α are given functions depending on the geometry of the wing and the solution $C(z) = \sqrt{b^2-z^2}c(z)$.

Table 5
Rectangular wing.

m	$\text{cond}(\bar{\mathbf{A}}_m)$	err_m
8	$2.9237e+00$	$1.0186e-03$
16	$4.8340e+00$	$4.5344e-05$
32	$8.3045e+00$	$1.6127e-06$
64	$1.5198e+01$	$5.7771e-08$
128	$2.8963e+01$	$2.1943e-09$
256	$5.6503e+01$	$8.7550e-11$

4.1.1. Elliptic wing

In this case, being $x^2+z^2/b^2=1$ the ellipsis equation, $2b$ is the wingspan,

$$\alpha(z)=b^{-1}\sqrt{b^2-z^2}, \quad j(z)=2\pi\alpha(y)\epsilon,$$

with ϵ acute angle between the direction of the relative wind and the chord of the wing (the angle of attack).

Introducing the changes of variable $z=by,\ \eta=bx,$ we have the equivalent equation

$$\beta\zeta(y)=\frac{\tilde{\alpha}(y)}{2b}\int_{-1}^{1}\frac{\zeta(x)}{(x-y)^2}dx+\tilde{j}(y), \tag{44}$$

with $\zeta(y)=\sqrt{1-y^2}d_1(y)$ and therefore

$$\frac{2b\beta}{\pi}d_1(y)-\frac{1}{\pi}\int_{-1}^{1}\frac{d_1(x)}{(x-y)^2}\varphi(x)dx=4b\epsilon.$$

Setting

$$\sigma_1(y)=\frac{2b\beta}{\pi\varphi(y)},\ \ \rho(y)=\varphi(y),\ \ g_1(y)=4b\epsilon$$

we have to solve

$$(M_{\sigma_1\varphi}+DA^{\varphi})d_1(y)=g_1(y). \tag{45}$$

The exact solution is known in this case

$$\tilde{C}(y)=\sqrt{1-y^2}d_1(y)=\sqrt{1-y^2}\frac{4\epsilon b}{1+\frac{2b\beta}{\pi}}.$$

We have tested our method selecting $\beta=1, b=10$ and choosing two different values for the angle of attack, $\epsilon=0.1$ and $\epsilon=0.0872$. In both the cases, since the solution belongs to $Z_s(\varphi)$ for any $s>1$, the machine precision is attained by solving a linear system of order $m=2$. We point out that in [9], for the same tests, by using a discretization based on a $N-th$ Gauss rule, only two exact digits are achieved with $N=50$.

4.1.2. Rectangular wing

In this case $2b$ is the length of the rectangular's largest dimension and

$$\alpha(z)=1, \quad j(z)=2\pi\epsilon.$$

By (44) with $\zeta(y)=\sqrt{1-y^2}d_2(y)$

$$\frac{2b\beta}{\pi}d_2(y)\varphi(y)-\frac{1}{\pi}\int_{-1}^{1}\frac{d_2(x)}{(x-y)^2}\varphi(x)dx=4b\epsilon,$$

and setting

$$\sigma_2(y)=\frac{2b\beta}{\pi},\ \ \rho(y)=\varphi(y),\ \ g_2(y)=4b\epsilon$$

the equation can be rewritten

$$(M_{\sigma_2\varphi}+DA^{\varphi})d_2(y)=g_2(y). \tag{46}$$

For this case the exact solution is unknown.

Since $\sigma_2\varphi\in Z_1$, according to Remark 3.2, the error behaves as $\mathcal{O}\left(\frac{\log m}{m^{1-\varepsilon}}\right)$. Inspecting Table 5, one can see that the numerical results are better than the expected ones, as order.

5. The proofs

5.1. Proof of Theorem 3.1

In order to prove the theorem we need to study the mapping properties of the operators D, A^ρ, K and H. To this end, for $0 < \alpha < 1$, we consider the following subspace of $Z_r(u)$

$$Z_{r,0}(u) := \left\{ f \in Z_r(u) \ : \ \int_{-1}^{1} f(x)\rho^{-1}(x)dx = 0 \right\}, \ \|f\|_{Z_{r,0}(u)} := \|f\|_{Z_r(u)}.$$

The following lemma states the boundedness of the operator $D : Z_r(u) \to Z_{r-1}(u\varphi), r > 1$.

Lemma 5.1. *For $r > 1$*

$$f' \in Z_{r-1}(u\varphi) \quad \Leftrightarrow \quad f \in Z_r(u), \tag{47}$$

and

$$\|f'\|_{Z_{r-1}(u\varphi)} \le \mathcal{C}\|f\|_{Z_r(u)}, \quad \mathcal{C} \neq \mathcal{C}(f). \tag{48}$$

In particular, (48) is not true for $r = 1$.

Proof. (47) follows by arguments similar to those used in [33, p. 337–338].

Start from

$$\|f'\|_{Z_{r-1}(u\varphi)} = \|f'u\varphi\|_\infty + \sup_{t>0} \frac{\Omega_\varphi^k(f',t)_{u\varphi}}{t^{r-1}}.$$

First we prove

$$\sup_{t>0} \frac{\Omega_\varphi^k(f',t)_{u\varphi}}{t^{r-1}} \le \mathcal{C} \sup_{t>0} \frac{\Omega_\varphi^{k+1}(f,t)_u}{t^r}, \quad k > r-1. \tag{49}$$

Since $f \in C_u$, by (6) there exists a sequence $\{P_m\}_m$ of best approximation polynomials s.t. the series $P_m + \sum_{i=0}^{\infty}(P_{2^{i+1}m} - P_{2^i m})$ converges uniformly in $[-1,1]$ to f in C_u. If we prove that the series

$$\sum_{i=0}^{\infty}(P_{2^{i+1}m}(x) - P_{2^i m}(x))'(u\varphi)(x) \tag{50}$$

uniformly converges $\forall x \in [-1,1]$, then the equality

$$\left(\sum_{i=0}^{\infty}(P_{2^{i+1}m} - P_{2^i m})\right)' u\varphi = \sum_{i=0}^{\infty}(P_{2^{i+1}m} - P_{2^i m})'u\varphi$$

holds true and the series $P_m' + \sum_{i=0}^{\infty}(P_{2^{i+1}m} - P_{2^i m})'$ converges uniformly in $[-1,1]$ to f' in $C_{u\varphi}$.

By the Bernstein inequality [23, Th. 8.4.7] and the weak-Jackson inequality [23, Th. 8.2.1], we have

$$\begin{aligned}
&\|(P_{2^{i+1}m} - P_{2^i m})'u\varphi\|_\infty \le \mathcal{C}(2^{i+1}m)\|(P_{2^{i+1}m} - P_{2^i m})u\|_\infty \\
&\le \mathcal{C}(2^{i+1}m)E_{2^i m}(f)_u \le \mathcal{C}(2^{i+1}m)\int_0^{\frac{1}{2^i m}} \frac{\Omega_\varphi^{k+1}(f,t)_u}{t}dt \\
&\le \mathcal{C}\frac{1}{(2^i m)^{r-1}} \sup_{t>0} \frac{\Omega_\varphi^{k+1}(f,t)_u}{t^r}, \quad \mathcal{C} \neq \mathcal{C}(m).
\end{aligned}$$

Thus, by the assumption on f, we have

$$\sum_{i=0}^{\infty}\|(P_{2^{i+1}m} - P_{2^i m})'u\varphi\|_\infty \le \frac{\mathcal{C}}{m^{r-1}} \sup_{t>0} \frac{\Omega_\varphi^{k+1}(f,t)_u}{t^r}$$

and the series (50) uniformly converges $\forall x \in [-1,1]$. Then

$$\begin{aligned}
E_m(f')_{u\varphi} &\le \|(f - P_m)'u\varphi\|_\infty \le \sum_{i=0}^{\infty}\|(P_{2^{i+1}m} - P_{2^i m})'u\varphi\|_\infty \\
&\le \frac{\mathcal{C}}{m^{r-1}} \sup_{t>0} \frac{\Omega_\varphi^{k+1}(f,t)_u}{t^r},
\end{aligned} \tag{51}$$

where $\mathcal{C} \neq \mathcal{C}(m)$. Since, using (9)

$$\sup_{t>0} \frac{\Omega_\varphi^k(f',t)_{u\varphi}}{t^{r-1}} \le \mathcal{C} \sup_{m\ge1} m^{r-1}E_m(f')_{u\varphi},$$

taking into account (51), (49) follows.

Let Q_1 be the 1-degree polynomial of best approximation of $f' \in C_{u\varphi}$. We have

$$\|f'u\varphi\|_\infty \le E_1(f')_{u\varphi} + \|Q_1 u\varphi\|_\infty.$$

By (51) and $\|Q_1 u\varphi\|_\infty \le \mathcal{C}\|fu\|_\infty$,

$$\|f'u\varphi\|_\infty \le \mathcal{C}\left[\sup_{t>0}\frac{\Omega_\varphi^{k+1}(f,t)_u}{t^r} + \|fu\|_\infty\right] \le \mathcal{C}\|f\|_{Z_r(u)},\ \ k > r-1.$$

(48) follows by combining the last estimate with (49).

Finally, for $r = 1$ (48) becomes

$$\|f'u\varphi\|_\infty \le \|f\|_{Z_1(u)}$$

and the above inequality is not true, being $W_1(u) \subset Z_1(u)$. For example the function $f(x) = x\log|x|,\ |x| \le 1$, belongs to $Z_1(u)$ but does not belong to $W_1(u)$ (see [34, p. 54]). □

Lemma 5.2. *Let $\gamma,\ \delta \ge 0$ and $r > 1$. The operator $D : Z_{r,0}(u) \to Z_{r-1}(u\varphi)$ is continuous and invertible. Moreover its inverse is bounded.*

Proof. Since the continuity of D is a consequence of Lemma 5.1 it remains to prove only the invertibility. By (47) for any $g \in Z_{r-1}(u\varphi)$ there exists $f \in Z_r(u)$ s.t. $Df = g$, i.e. $D : Z_r(u) \to Z_{r-1}(u\varphi)$ is surjective. On the other hand, to any $f \in Z_r(u)$ we can associate the function $\bar f = f - \frac{\int_{-1}^1 f(x)\rho^{-1}(x)dx}{\int_{-1}^1 \rho^{-1}(x)dx}$ belonging to $Z_{r,0}(u)$, then $D : Z_{r,0}(u) \to Z_{r-1}(u\varphi)$ is surjective too. Since the injectivity can be easily proved, it follows that $D : Z_{r,0}(u) \to Z_{r-1}(u\varphi)$ is invertible for any $r > 1$. Moreover, by the open mapping theorem (see, for example, [35, p. 517]), the inverse of D is bounded.

Setting $$(A^{\rho^{-1}} f)(x) = (\cos\pi\alpha)\rho^{-1}(x)f(x) + \frac{\sin\pi\alpha}{\pi}\int_{-1}^1 f(y)\frac{\rho^{-1}(y)}{y-x}dy,$$

the following result can be found in [36, Corollary 2.2]. □

Lemma 5.3. *Let $0 < \alpha < 1$. Under the assumptions in (19) the linear maps*

$$A^{\rho^{-1}} : Z_{r,0}(u) \ \to\ Z_r(u\rho),\quad A^\rho : Z_r(u\rho) \to Z_{r,0}(u)$$

are both continuous for $r > 0$. Moreover, A^ρ is the inverse of $A^{\rho^{-1}}$ and the following equivalences hold true

$$\|A^{\rho^{-1}} f\|_{Z_r(u\rho)} \sim \|f\|_{Z_r(u)},\quad \|A^\rho f\|_{Z_r(u)} \sim \|f\|_{Z_r(u\rho)}, \tag{52}$$

where the constants in " $\sim$ " are independent of f.

As a consequence of Lemmas 5.2 and 5.3 we deduce the following result.

Corollary 5.1. *Let $0 < \alpha < 1$. Under the assumptions in (19) the operator $DA^\rho : Z_r(u\rho) \to Z_{r-1}(u\varphi)$ is continuous and invertible for each $r > 1$. Moreover its inverse is bounded.*

The following lemma will be useful in the sequel.

Lemma 5.4. *Let us assume that the kernel $k(x,\ y)$ satisfies (20). Then there exists a sequence $\{P_m\}_m$ of polynomials $P_m(x,y) = \sum_{i=0}^m p_{i,m}(x)y^i$, of degree not greater than m in y, such that $p_{i,m}(x)$ is piecewise constant for all $i = 0,\dots,m$ and*

$$\sup_{x,y\in[-1,1]}(u\varphi)(y)|P_m(x,y) - k(x,y)| \le \mathcal{C}\sup_{|x|\le 1}E_m(k_x)_{u\varphi}, \tag{53}$$

where $\mathcal{C} \ne \mathcal{C}(m)$.

Proof. The proof can be easily deduced following step by step the proof of Lemma 4.11 in [37]. □

Lemma 5.5. *Let $0 < \alpha < 1$ and let $\gamma,\ \delta < 1$. If for some $s > 0$ the kernel k satisfies (20), then $K : C_{u\rho} \to Z_{r-1}(u\varphi)$ is continuous for all $1 \le r \le s+1$ and compact for all $1 \le r < s+1$.*

Proof. Taking into account (20) we have

$$\begin{aligned}(u\varphi)(y)|(Kf)(y)| &\le \frac{1}{\pi}\|f u\rho\|_\infty (u\varphi)(y)\int_{-1}^1 |k(x,y)|u^{-1}(x)dx\\ &\le \mathcal{C}\|f\|_{C_{u\rho}}\sup_{|x|\le 1}\|k_x u\varphi\|_\infty \le \mathcal{C}\|f\|_{C_{u\rho}}.\end{aligned} \tag{54}$$

Let $\{P_m\}_m$ be the sequence of polynomials defined in Lemma 5.4. Then

$$E_m(Kf)_{u\varphi} \le \frac{1}{\pi}\sup_{|y|\le 1}(u\varphi)(y)\int_{-1}^1 |k(x,y) - P_m(x,y)||(f\rho)(x)|dx$$

$$\leq \mathcal{C}\|fu\rho\|_\infty \sup_{x,y\in[-1,1]} (u\varphi)(y)|k(x,y)-P_m(x,y)| \int_{-1}^{1} u^{-1}(x)dx$$
$$\leq \mathcal{C}\|fu\rho\|_\infty \sup_{|x|\leq 1} E_m(k_x)_{u\varphi}.$$

Since, under the assumption (20), $k_x \in Z_s(u\varphi)$, using (10), we get

$$E_m(Kf)_{u\varphi} \leq \frac{\mathcal{C}}{m^s}\|f\|_{C_{u\rho}}. \tag{55}$$

Combining (54) and (55) with (8) and (9), the continuity of $K : C_{u\rho} \to Z_{r-1}(u\varphi)$ with $1 \leq r \leq s+1$ follows.

Now, since for all $f \in Z_{r-1}(u\varphi)$ we have [27, p. 6]

$$\|f - L_m^w(f)\|_{Z_{r-1}(u\varphi)} \leq \frac{\mathcal{C}}{m^{s-r+1}}\|f\|_{Z_s(u\varphi)} \log m, \quad 0 \leq r-1 < s, \tag{56}$$

the imbedding operator $E : Z_s(u\varphi) \to Z_{r-1}(u\varphi)$ can be approximated by a sequence of finite dimensional operators and then $Z_s(u\varphi)$ is compactly imbedded in $Z_{r-1}(u\varphi)$ for $0 \leq r-1 < s$. Consequently, from the continuity of the operator K: $C_{u\rho} \to Z_s(u\varphi)$ we deduce the compactness of the operator $K : C_{u\rho} \to Z_{r-1}(u\varphi)$ for $1 \leq r < s+1$. □

Lemma 5.6. *Let $0 < \alpha < 1$. If the kernel $h(x, y)$ satisfies (21) and (22), then the operator $H : C_{u\rho} \to Z_{r-1}(u\varphi)$ is continuous for all $1 \leq r \leq s+1$ and compact for all $1 \leq r < s+1$.*

In particular, when $\alpha = \frac{1}{2}$ and $h(x,y) = \log|x-y|$, for all $r \geq 1$ the operator H is continuous as a map from $Z_r(\varphi)$ into $Z_r(\varphi)$ and compact as a map from $Z_r(\varphi)$ into $Z_{r-1}(\varphi)$.

Proof. We have

$$(u\varphi)(y)|(Hf)(y)| \leq \mathcal{C}\|fu\rho\|_\infty (u\varphi)(y) \int_{-1}^{1} |h(x,y)|u^{-1}(x)dx \tag{57}$$

and, by (21), we deduce the continuity of the operator $H : C_{u\rho} \to C_{u\varphi}$.

We observe that the compactness of $H : C_{u\rho} \to Z_{r-1}(u\varphi)$ can be proved if the following estimate holds

$$\frac{\Omega_\varphi(Hf,t)_{u\varphi}}{t^s} \leq \mathcal{C}\|f\|_{C_{u\rho}} \tag{58}$$

for some $s > r-1$. Indeed, by using the weak-Jackson inequality [23, Th. 8.2.1] together with (58), we get

$$E_m(Hf)_{u\varphi} \leq \frac{\mathcal{C}}{m^s}\|f\|_{C_{u\rho}}, \tag{59}$$

and, combining (57) and (59) with (8) and (9) we get the continuity of H: $C_{u\rho} \to Z_{r-1}(u\varphi)$, $1 \leq r \leq s+1$. Moreover, by (11) and (59), we obtain

$$E_m(Hf)_{Z_{r-1}(u\varphi)} \leq \mathcal{C}\sup_m m^{r-1}E_m(Hf)_{u\varphi} \leq \frac{\mathcal{C}}{m^{s-r+1}}\|f\|_{C_{u\rho}}, \quad s > r-1,$$

and, therefore, by [33, p. 44] it follows that $H : C_{u\rho} \to Z_{r-1}(u\varphi)$ is compact.

So, it remains to prove (58). By the assumption (22), we have

$$\|u\varphi\Delta_{\tau\varphi}Hf\|_{I_\tau} = \frac{1}{\pi}\sup_{y\in I_\tau}(u\varphi)(y)\left|\int_{-1}^{1}\Delta_{\tau\varphi(y)}h(x,y)f(x)\rho(x)dx\right|$$
$$\leq \mathcal{C}\|fu\rho\|_\infty \sup_{y\in I_\tau}(u\varphi)(y)\int_{-1}^{1}|\Delta_{\tau\varphi(y)}h(x,y)|u^{-1}(x)dx \leq \mathcal{C}\|f\|_{C_{u\rho}}\tau^s.$$

Thus, by definition of Ω_φ, (58) follows.

The case $\alpha = \frac{1}{2}$ and $h(x,y) = \log|x-y|$ is special since $\frac{d}{dy}(Hf)(y) = -(A^\varphi f)(y)$. The proof can be deduced following [38, Proof of Theorem 2.2]. □

Proof of Theorem 3.1.. The theorem follows by Corollary 5.1, Lemmas 5.5,5.6 and the Fredholm alternative Theorem (see, for instance, [39, Cor. 3.8]). □

5.2. Proofs of Theorems 3.2 and 3.3

In order to prove the theorems, we need the following lemmas.

Lemma 5.7. *Let $0 < \alpha < 1$. If, for some $s > 0$ and γ, δ satisfying (19), the kernel k satisfies (20) and (33), then, for every $1 \leq r < s+1$,*

$$\|(K - L_m^w K_m)f\|_{Z_{r-1}(u\varphi)} \leq \mathcal{C}\|f\|_{C_{u\rho}}\frac{\log m}{m^{s-r+1}}, \ \mathcal{C} \neq \mathcal{C}(m,f,k).$$

Proof. We have

$$\begin{aligned}(Kf)(y)-L_m^w(K_mf)(y) &= [(Kf)(y)-(K_mf)(y)]\\ &\quad+[(K_mf)(y)-L_m^w(K_mf)(y)]\\ &=: A(y)+B(y).\end{aligned} \tag{60}$$

Using Lemma 2.2, the assumption (33) and (10) we get

$$\begin{aligned}|(u\varphi)(y)A(y)| &\le \mathcal{C}\int_{-1}^{1}\left|(u\varphi)(y)k(x,y)-L_m^\rho((u\varphi)(y)k_y,x)\right||(fu\rho)(x)|\frac{dx}{u(x)}\\ &\le \mathcal{C}\|f\|_{C_{u\rho}}E_{m-1}((u\varphi)(y)k_y)\\ &\le \frac{\mathcal{C}}{m^s}\|f\|_{C_{u\rho}}.\end{aligned} \tag{61}$$

Concerning $|(u\varphi)(y)B(y)|$, if we prove that $K_m\colon C_{u\rho}\to Z_s(u\varphi)$ is a bounded operator then, by (15) and (10), we deduce

$$\begin{aligned}|(u\varphi)(y)B(y)| &\le \mathcal{C}\log m\; E_{m-1}(K_mf)_{u\varphi}\\ &\le \mathcal{C}\frac{\log m}{m^s}\|K_mf\|_{Z_s(u\varphi)}\\ &\le \mathcal{C}\frac{\log m}{m^s}\|f\|_{C_{u\rho}}.\end{aligned} \tag{62}$$

Consequently, combining (61) and (62) with (60), we deduce

$$\|u\varphi(K-L_m^wK_m)f\|_\infty \le \mathcal{C}\frac{\log m}{m^s}\|f\|_{C_{u\rho}},\quad \mathcal{C}=\mathcal{C}(s). \tag{63}$$

In order to prove that $K_mf\in Z_s(u\varphi)$ for any $f\in C_{u\rho}$, we estimate $\Omega_\varphi^k(K_mf,t)_{u\varphi}$. Using Lemma 2.2 we get

$$\begin{aligned}(u\varphi)(y)|\Delta_{h\varphi}^k(K_mf)(y)| &\le \mathcal{C}\|f\|_{C_{u\rho}}\int_{-1}^{1}|L_m^\rho((u\varphi)(y)\Delta_{h\varphi}^k k(\cdot,y),x)|\frac{dx}{u(x)}\\ &\le \mathcal{C}\|f\|_{C_{u\rho}}\sup_{|x|\le 1}|(u\varphi)(y)\Delta_{h\varphi}^k k_x(y)|.\end{aligned}$$

Taking first the supremum on $y\in I_{kh}$ and then the supremum on $0<h\le t$, we obtain

$$\frac{\Omega_\varphi^k(K_mf,t)_{u\varphi}}{t^s}\le \mathcal{C}\|f\|_{C_{u\rho}}\sup_{|x|\le 1}\frac{\Omega_\varphi^k(k_x,t)_{u\varphi}}{t^s}$$

and, under the assumption (20), we deduce the boundedness of the operator $K_m\colon C_{u\rho}\to Z_s(u\varphi)$.

Now, taking into account the equivalence (9), we get

$$\begin{aligned}\|u\varphi(K-L_m^wK_m)f\|_{Z_{r-1}(u\varphi)} &= \|u\varphi(K-L_m^wK_m)f\|_\infty\\ &\quad+\sup_k(1+k)^{r-1}E_k((K-L_m^wK_m)f)_{u\varphi}.\end{aligned}$$

Since for $k<m-1$, by (63), we have

$$\begin{aligned}\sup_k(1+k)^{r-1}E_k((K-L_m^wK_m)f)_{u\varphi} &\le \|u\varphi(K-L_m^wK_m)f\|_\infty\sup_k(1+k)^{r-1}\\ &\le \mathcal{C}\frac{\log m}{m^{s-r+1}}\|f\|_{C_{u\rho}}\end{aligned}$$

and, for $k\ge m-1$, using Lemma 5.5 and (10), we obtain

$$\begin{aligned}\sup_k(1+k)^{r-1}E_k((K-L_m^wK_m)f)_{u\varphi} &= \sup_k(1+k)^{r-1}E_k(Kf)_{u\varphi}\\ &\le \mathcal{C}\frac{\log m}{m^{s-r+1}}\|f\|_{C_{u\rho}},\end{aligned}$$

the thesis follows. □

Lemma 5.8. *Let* $0<\alpha<1$. *If, for some* $s>0$ *and* γ, δ *satisfying* (14), *the kernel* h *satisfies* (21)-(22), *then, for every* $1\le r<s+1$,

$$\|(H-L_m^wH)f\|_{Z_{r-1}(u\varphi)}\le \mathcal{C}\|f\|_{C_{u\rho}}\frac{\log m}{m^{s-r+1}},\quad \mathcal{C}\ne\mathcal{C}(m,f).$$

Proof. By Lemma 2.1 and using (59) we get

$$\|u\varphi(H-L_m^wH)f\|_\infty\le \mathcal{C}\log mE_{m-1}(Hf)_{u\varphi}\le \frac{\log m}{m^s}\|f\|_{C_{u\rho}}. \tag{64}$$

Using the equivalence (9) and (64) the theorem follows. □

Proof of Theorem 3.2.. We first note that, by Corollary 5.1, $g \in Z_s(u\varphi)$ implies $[DA^\rho]^{-1}g \in Z_{s+1}(u\rho)$ and, by Lemmas 5.5 and 5.6, $[DA^\rho]^{-1}Kf,\ [DA^\rho]^{-1}Hf \in Z_{s+1}(u\rho)$ for any $f \in C_{u\rho}$, then

$$f = -[DA^\rho]^{-1}Kf - [DA^\rho]^{-1}Hf + [DA^\rho]^{-1}g \in Z_{s+1}(u\rho)$$

too.

Since by Lemmas 5.7 and 5.8 we can choose m sufficiently large (say $m > m_0$) such that

$$\|(DA^\rho + K + H)^{-1}[(L_m^w K_m - K) + (L_m^w H - H)]\|_{Z_r(u\rho)\to Z_r(u\rho)} < 1,$$

then, using a well-known result (see, for example, [39, Theorem 10.1, p. 142]), the inverse operators $(DA^\rho + L_m^w K_m + L_m^w H)^{-1} : Z_{r-1}(u\varphi) \to Z_r(u\rho)$ exist and are uniformly bounded w.r.t. m, i.e.

$$\sup_{m\ge m_0} \|(DA^\rho + L_m^w K_m + L_m^w H)^{-1}\|_{Z_{r-1}(u\varphi)\to Z_r(u\rho)} < +\infty. \tag{65}$$

In order to prove (34), we use the following identity

$$(f - f_m) = (DA^\rho + L_m^w K_m + L_m^w H)^{-1}[(g - L_m^w g) + (L_m^w K_m - K)f + (L_m^w H - H)f]. \tag{66}$$

Since by (56) and the assumptions on g it is

$$\|g - L_m^w g\|_{Z_{r-1}(u\varphi)} \le \mathcal{C}\frac{\log m}{m^{s-r+1}}\|g\|_{Z_s(u\varphi)},$$

using (65) and Lemmas 5.7 and 5.8, we get

$$\|f - f_m\|_{Z_r(u\rho)} \le \mathcal{C}\frac{\log m}{m^{s-r+1}}\left[\|g\|_{Z_s(u\varphi)} + \|f\|_{Z_{s+1}(u\rho)}\right]. \tag{67}$$

On the other hand, since $g = (DA^\rho + K + H)^{-1}f$, by Theorem 3.1,

$$\|g\|_{Z_s(u\varphi)} \le \mathcal{C}\|f\|_{Z_{s+1}(u\rho)}. \tag{68}$$

Combining (68) and (67), (34) follows. □

Proof of Theorem 3.3.. Let

. Using [40, Th. 3.1] with $w_1 = u\rho$ and $w_2 = u$, we get

$$\begin{aligned}\|A^\rho P\|_{C_u} &\le \mathcal{C}\|P\|_{C_{u\rho}} + \mathcal{C}\int_0^1 \frac{\Omega_\varphi^k(P,t)_{u\rho}}{t}dt \\ &= \|P\|_{C_{u\rho}} + \mathcal{C}\left\{\int_0^{\frac{1}{m}} + \int_{\frac{1}{m}}^1\right\}\frac{\Omega_\varphi^k(P,t)_{u\rho}}{t}dt.\end{aligned}$$

Applying $\Omega_\varphi^k(P,t)_{u\rho} \le \mathcal{C}t\|P'\varphi u\rho\|_\infty$ and the Bernstein inequality [23, Th. 8.4.7] in the first integral and $\Omega_\varphi^k(P,t)_{u\rho} \le \mathcal{C}\|Pu\rho\|_\infty$ in the second one, it is easy to deduce that

$$\|A^\rho P\|_{C_u} \le \mathcal{C}\log m\|P\|_{C_{u\rho}}.$$

Using the above inequality together with the Bernstein inequality [23, Th. 8.4.7] we get

$$\|DA^\rho P\|_{C_{u\varphi}} \le \mathcal{C}m\log m\|P\|_{C_{u\rho}}. \tag{69}$$

Then, taking into account (26) and (see, for example, [16])

$$(DA^\rho)^{-1}p_m^w = \frac{1}{m+1}p_m^\rho, \tag{70}$$

the operator $DA^\rho : (\mathbb{P}_{m-1}, \|\cdot\|_{C_{u\rho}}) \to (\mathbb{P}_{m-1}, \|\cdot\|_{C_{u\varphi}})$

is continuous and invertible. Consequently, the operator $(DA^\rho + L_m^w K_m + L_m^w H) : (\mathbb{P}_{m-1}, \|\cdot\|_{C_{u\rho}}) \to (\mathbb{P}_{m-1}, \|\cdot\|_{C_{u\varphi}})$

is continuous and invertible too and its inverse is bounded (see, for example, [39, Theorem 3.4]).

Now, for every $\theta = (\theta_1, \dots, \theta_m)^T$ there exists $\eta = (\eta_1, \dots, \eta_m)^T$ such that $\mathbf{A}_m\theta = \eta$ iff $(DA^\rho + L_m^w K_m + L_m^w H)\tilde{\theta}(y) = \tilde{\eta}(y)$, where

$$\tilde{\theta}(y) = \sum_{i=1}^m \psi_i^\rho(y)\theta_i, \quad \tilde{\eta}(y) = \sum_{i=1}^m \psi_i^w(y)\eta_i,$$

with $\theta_i = (u\rho\tilde{\theta})(t_i)$ and $\eta_i = (u\varphi\tilde{\eta})(x_i)$.

Then, for every θ we have

$$\begin{aligned}\|\mathbf{A}_m\theta\|_\infty &= \|\eta\|_\infty = |\eta_\nu| = |(\tilde{\eta}v^{\gamma,\delta}\varphi)(x_\nu)| \le \|\tilde{\eta}\|_{C_{u\varphi}} \\ &= \|(DA^\rho + L_m^w K_m + L_m^w H)\tilde{\theta}\|_{C_{u\varphi}}\end{aligned}$$

$$\begin{aligned} &\le \ \|DA^\rho\tilde{\theta}\|_{C_{u\varphi}} + \|L_m^w K_m\tilde{\theta}\|_{C_{u\varphi}} + \|L_m^w H\tilde{\theta}\|_{C_{u\varphi}} \\ &=: N_1 + N_2 + N_3, \end{aligned}$$

where $|\eta_\nu| = \max_{1\le i\le n} |\eta_i|$. Using (69) and (16), we get

$$N_1 \le \mathcal{C} m \log^2 m \|\theta\|_\infty. \tag{71}$$

Moreover, by (15), the definition (23) of $K_m\tilde{\theta}$, Lemma 2.2, the assumption (20) and (16), we deduce

$$\begin{aligned} N_2 &\le \mathcal{C} \log m \|K_m\tilde{\theta}\|_{C_{u\varphi}} \\ &\le \mathcal{C} \log m \|\tilde{\theta}\|_{C_{u\varphi}} \sup_{|y|\le 1} (u\varphi)(y) \int_{-1}^{1} |L_m^\rho(k_y, x)| u^{-1}(x) dx \\ &\le \mathcal{C} \log m \|\tilde{\theta}\|_{C_{u\varphi}} \sup_{|y|\le 1} (u\varphi)(y) \|k_y\|_\infty \\ &= \mathcal{C} \log m \|\tilde{\theta}\|_{C_{u\varphi}} \sup_{|x|\le 1} \|k_x u\varphi\|_\infty \\ &\le \mathcal{C} \log^2 m \|\theta\|_{C_{u\varphi}}. \end{aligned}$$

Finally, by (15), the definition of $H\tilde{\theta}$, the assumption (21) and (16), we have

$$\begin{aligned} N_3 &\le \mathcal{C} \log m \|H\tilde{\theta}\|_{C_{u\varphi}} \\ &\le \mathcal{C} \log m \|\tilde{\theta}\|_{C_{u\varphi}} \sup_{|y|\le 1} (u\varphi)(y) \int_{-1}^{1} |h(x,y)| u^{-1}(x) dx \\ &\le \mathcal{C} \log^2 m \|\theta\|_\infty. \end{aligned}$$

Summing up, we get

$$\|\mathbf{A}_m\|_\infty \le \mathcal{C} m \log^2 m. \tag{72}$$

Similarly proceeding, for every η, using (15), we get

$$\begin{aligned} \|\mathbf{A}_m^{-1}\eta\|_\infty &\le \mathcal{C} \left\| \left((DA^\rho + L_m^w K_m + L_m^w H)_{\ |\mathbb{P}_{m-1}} \right)^{-1} \right\|_{C_{u\varphi}\to C_{u\rho}} \|\tilde{\eta}\|_{C_{u\varphi}} \\ &\le \mathcal{C} \log m \left\| \left((DA^\rho + L_m^w K_m + L_m^w H)_{\ |\mathbb{P}_{m-1}} \right)^{-1} \right\|_{C_{u\varphi}\to C_{u\rho}} \|\eta\|_\infty \end{aligned}$$

and, then,

$$\|\mathbf{A}_m^{-1}\|_\infty \le \mathcal{C} \log m \left\| \left((DA^\rho + L_m^w K_m + L_m^w H)_{\ |\mathbb{P}_{m-1}} \right)^{-1} \right\|_{C_{u\varphi}\to C_{u\rho}}. \tag{73}$$

Combining (72) and (73), the theorem follows. □

5.3. Proof of Theorem 3.4

Lemma 5.9. *Under the assumptions $0 \le \gamma < \frac{1}{2}$, $0 \le \delta < \frac{1}{2}$, and if $\sigma\varphi \in Z_r$ with $r \ge 0$, then the multiplying operator $M_{\sigma\varphi}$: $Z_r(u\varphi) \to Z_r(u\varphi)$ is continuous. Moreover, if $\sigma\varphi \in Z_r$ with $r \ge 1$, then $M_{\sigma\varphi} : Z_r(u\varphi) \to Z_{r-1}(u\varphi)$ is compact.*

Proof. Since

$$\|(M_{\sigma\varphi} f)u\varphi\|_\infty = \|\sigma\varphi f u\varphi\|_\infty \le \|\sigma\varphi\|_\infty \|f u\varphi\|_\infty \tag{74}$$

and, by standard computation and the Favard inequality (10), denoting by $\lfloor a \rfloor$ the greatest integer smaller or equal to $a > 0$, we have

$$E_m(M_{\sigma\varphi} f)_{u\varphi} \le \|\sigma\varphi\|_\infty E_{\lfloor \frac{m}{2} \rfloor}(f)_{u\varphi} + 2\|f u\varphi\|_\infty E_{\lfloor \frac{m}{2} \rfloor}(\sigma\varphi) \le \frac{\mathcal{C}}{m^r} \|f\|_{Z_r(u\varphi)} \|\sigma\varphi\|_{Z_r}, \tag{75}$$

it follows $\|M_{\sigma\varphi} f\|_{Z_r(u\varphi)} = \|(M_{\sigma\varphi} f)u\varphi\|_\infty + \sup_m m^r E_m(M_{\sigma\varphi} f)_{u\varphi} < \mathcal{C}\|f\|_{Z_r(u\varphi)}$, i.e., the operator $M_{\sigma\varphi}$: $Z_r(u\varphi) \to Z_r(u\varphi)$ is continuous. Then $M_{\sigma\varphi} : Z_r(u\varphi) \to Z_{r-1}(u\varphi)$ is compact, since $Z_r(u\varphi)$ is compactly imbedded into $Z_{r-1}(u\varphi)$ (see (56)). □

Proof of Theorem 3.4.. The theorem follows by Corollary 5.1, Lemmas 5.5, 5.6 and 5.9 and the Fredholm alternative Theorem. □

5.4. Proof of Theorems 3.5 and 3.6

Lemma 5.10. *If, for some $s>0$, $0\le\gamma<\frac{1}{4}$ and $0\le\delta<\frac{1}{4}$, we have $\sigma\varphi\in Z_s$, then, for every $1\le r<s+1$,*

$$\|(M_{\sigma\varphi}-L_m^{\varphi}M_{\sigma\varphi})f\|_{Z_{r-1}(u\varphi)}\le\mathcal{C}\|f\|_{Z_s(u\varphi)}\|\sigma\varphi\|_{Z_s}\frac{\log m}{m^{s-r+1}},\quad \mathcal{C}\ne\mathcal{C}(m,f,\sigma).$$

Proof. By Lemma 2.1 for $\alpha=\frac{1}{2}$, under the assumptions on γ, δ

$$\|u\varphi(M_{\sigma\varphi}-L_m^{\varphi}M_{\sigma\varphi})f\|_\infty\le\mathcal{C}\log mE_m(M_{\sigma\varphi}f)_{u\varphi}.$$

Since $\sigma\varphi\in Z_s$, by (75) with $r=s$, we obtain

$$\|u\varphi(M_{\sigma\varphi}-L_m^{\varphi}M_{\sigma\varphi})f\|_\infty\le\mathcal{C}\frac{\log m}{m^s}\|f\|_{Z_s(u\varphi)}\|\sigma\varphi\|_{Z_s}.$$

Finally, by equivalence (9), the lemma follows. □

Proof of Theorem 3.5.. Taking into account Lemma 5.10 the proof is similar to that of Theorem 3.2. □

Proof of Theorem 3.6.. The proof is similar to that of Theorem 3.3 taking into account that, by (15), (74) and (16), we get

$$\begin{aligned}\|L_m^{\varphi}M_{\sigma\varphi}\tilde{\theta}\|_{C_{u\varphi}}&\le\mathcal{C}\log m\|M_{\sigma\varphi}\tilde{\theta}\|_{C_{u\varphi}}\le\mathcal{C}\log m\|\tilde{\theta}\|_{C_{u\varphi}}\\&\le\mathcal{C}\log^2 m\|\theta\|_\infty.\end{aligned}$$

□

5.5. Proof of Proposition 3.1

We prove (41) for $h(x,y)=|x-y|^\mu$, since the other cases similarly follows.

Setting $\varphi_1(x)=\sqrt{1-|x|}$, by $\frac{\varphi(y)}{\sqrt{2}}\le\varphi_1(y)\le\varphi(y)$ it follows $\Omega_\varphi\sim\Omega_{\varphi_1}$ [23]. Assume at first $y\in[-1+4\tau^2,0]$ and consider the following decomposition

$$\begin{aligned}&(u\varphi)(y)\int_{-1}^{1}u^{-1}(x)|\Delta_{\tau\varphi_1(y)}k_x(y)|dx=(u\varphi)(y)\times\\&\times\left(\int_{-1}^{-1+\frac{1+y}{2}}+\int_{-1+\frac{1+y}{2}}^{y-\tau\varphi_1(y)}+\int_{y-\tau\varphi_1(y)}^{y}+\int_{y}^{y+\tau\varphi_1(y)}+\int_{y+\tau\varphi_1(y)}^{y+\frac{1+y}{2}}+\int_{y+\frac{1+y}{2}}^{1}\right)\\&u^{-1}(x)|\Delta_{\tau\varphi_1(y)}k_x(y)|dx=:\sum_{k=1}^{6}S_k(y).\end{aligned}$$

Since for $x<y-\frac{\tau}{2}\varphi_1(y)$ ([41, (13.5.3)])

$$|\Delta_{\tau\varphi_1(y)}k_x(y)|\le\tau\varphi_1(y)\left(y-\frac{\tau}{2}\varphi_1(y)-x\right)^{\mu-1},\tag{76}$$

and for $y\in[-1+4\tau^2,0]$, by $\mu-1<0$, $\left(y-\frac{\tau}{2}\varphi_1(y)-x\right)^{\mu-1}\le\left(\frac{1+y}{4}\right)^{\mu-1}$, we have

$$S_1(y)\le\mathcal{C}\tau(1+y)^{\delta+\mu}\int_{-1}^{-1+\frac{1+y}{2}}(1+x)^{-\delta}dx\le\mathcal{C}\tau(1+y)^{\mu+1}\le\mathcal{C}\tau,$$

being $\mu+1>0$. By (76) again

$$S_2(y)\le\mathcal{C}\tau(1+y)^{\delta+1}\int_{-1+\frac{1+y}{2}}^{y-\tau\varphi_1(y)}\left(y-\frac{\tau}{2}\varphi_1(y)-x\right)^{\mu-1}(1+x)^{-\delta}dx.$$

Then, by $(1+x)\ge\frac{1+y}{2}$ and setting $y-x=u\sqrt{1+y}$, it follows

$$S_2(y)\le\mathcal{C}\tau(1+y)^{1+\frac{\mu}{2}}\int_{\tau}^{\frac{\sqrt{1+y}}{2}}\left(u-\frac{\tau}{2}\right)^{\mu-1}dx=\mathcal{C}\tau\frac{(1+y)^{1+\frac{\mu}{2}}}{\mu}\left[\left(\frac{\sqrt{1+y}}{2}-\frac{\tau}{2}\right)^{\mu}-\tau^\mu\right]$$

and, using $\frac{\sqrt{1+y}}{2}\ge\tau$, we can conclude $S_2(y)\le\mathcal{C}\tau^{1+\mu}$. Similar estimates hold for S_5 and S_6. To estimate S_3 we use $|\Delta_{\tau\varphi_1(y)}k_x(y)|\le\left|y-\frac{\tau}{2}\varphi_1(y)-x\right|^{\mu}$ and $1+x\sim 1+y$. Then

$$S_3(y)\le\mathcal{C}(1+y)^{\frac{1}{2}}\int_{y-\tau\varphi_1(y)}^{y}\left|y-\frac{\tau}{2}\varphi_1(y)-x\right|^{\mu}dx$$

and by the change of variable $y-x-\frac{\tau}{2}\sqrt{1+y}=\theta$ it follows

$$S_3(y)\le \mathcal{C}(1+y)^{\frac{1}{2}}\int_{-\frac{\tau}{2}\sqrt{1+y}}^{\frac{\tau}{2}\sqrt{1+y}}|\theta|^{\mu}d\theta=\mathcal{C}\tau^{\mu+1}(1+y)^{\frac{3}{2}+\mu}\le \mathcal{C}\tau^{\mu+1}.$$

Similarly we estimate S_4 by using $|\Delta_{\tau\varphi_1(y)}k_x(y)|\le \left|y+\frac{\tau}{2}\varphi_1(y)-x\right|^{\mu}$, and the lemma is proved for $y\in[-1+4\tau^2,0]$. We omit the proof in the case $y\in[0,1-4\tau^2]$, since it follows by similar arguments.

6. Conclusions

In this paper we have proposed a numerical scheme based on Lagrange projection for solving integral equations of the kinds (17) and (18). The approximate solution has been obtained by solving a system of algebraic equations, whose conditioning has been studied. Stability and convergence have been proved, giving estimates of the errors in Zygmund norm. We have illustrated various aspects of the theory by means of some examples, evaluating the efficiency of the proposed scheme from different points of view. In the first test we have compared our results with those reached by the procedure proposed in [15], by showing that our method faster converges. Examples 2 and 3 have been devoted especially to test the agreement of the predicted orders of convergence with the numerical *EOCs*, choosing for this goal functions of different smoothness. In both examples numerical evidence shows also that the condition numbers of the linear systems diverge at most like $m\log^3 m$. This fact encourages us to believe that the norms in (35) and (39) do not increase w.r.t. m. Moreover, in Example 2 we have compared the condition numbers of the linear systems of our procedures with those of the procedure in [16], showing the substantial different behaviors between them (see Table 3). The better conditioning in our procedure is ascribable to the choice of the basis to represent the Lagrange polynomials (see [25,26]).

Finally we have considered the application of our method in solving the Prandtl's equation governing the circulation air flow along the contour of a plane wing profile, for two different wing-shapes. Also in these cases we have shown that our experimental results are more accurate than those obtained in [8,9], highlighting once again the efficiency of our approach.

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