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Some Estimates for the Error of Best Polynomial Approximation of Composite Functions

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Abstract. The purpose of the paper is to provide upper estimates of the error of best polynomial approximation of composite functions in weighted spaces. Such estimates are essential for the convergence analysis of numerical methods applied to non-linear problems or for numerical approaches making use of regularization techniques to cure the low smoothness of the solution. These results are obtained through a suitable estimate of the derivatives of composite functions in L^p and uniform weighted norms.

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1. Introduction

Consider a fixed Jacobi weight

$$u(x) = v^{\gamma,\delta}(x) := (1-x)^\gamma (1+x)^\delta, \qquad x \in (-1,1), \quad \gamma, \delta > -1.$$

In the case $\gamma, \delta \geq 0$, we denote by C_u the space of all functions f continuous on $(-1,1)$ and satisfying

$$\lim_{x\to+1} f(x)u(x) = 0 \quad \text{if } \gamma > 0, \qquad \text{and} \qquad \lim_{x\to-1} f(x)u(x) = 0 \quad \text{if } \delta > 0,$$

equipped with the norm

$$\|f\|_{C_u} := \|fu\|_\infty = \max_{x\in[-1,1]} |f(x)|u(x).$$

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If $1 \leq p < +\infty$, $\gamma, \delta > -\frac{1}{p}$, we denote by L^p_u the following space

$$L^p_u = \left\{ f : \int_{-1}^{1} |f(x)u(x)|^p \, dx < +\infty \right\}$$

equipped with the norm

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_{-1}^{1} |f(x)u(x)|^p \, dx \right)^{\frac{1}{p}}, \qquad 1 \leq p < +\infty.$$

In all the spaces L^p_u and C_u it is possible to define the error of best polynomial approximation

$$E_m(f)_{u,p} := \inf_{P \in \mathbb{P}_m} \|(f-P)u\|_p$$

where $\mathbb{P}_m$ denotes the set of the algebraic polynomials of degree at most m.

As it is well known, the rate of convergence of this error, as $m \to \infty$, depends on the smoothness of the function f. For instance, if some information about the derivatives of the function f are known, a renewed inequality used in estimating the error of best approximation is the following Favard type inequality (see for instance [6] and the references therein)

$$E_m(f)_{u,p} \leq \frac{\mathcal{C}}{m^r} \|f^{(r)} \varphi^r u\|_p, \quad 1 \leq p \leq +\infty, \tag{1}$$

where here and in the following $\mathcal{C}$ is a positive absolute constant, $f^{(r)}$ denotes the rth derivative of f and $\varphi(x) = \sqrt{1-x^2}$.

In various contexts in which the "global approximation" (i.e., in processes involving the polynomial approximation) is used, it becomes crucial to estimate the best polynomial approximation of composite functions in terms of the smoothness properties of their components.

This situation naturally occurs in non linear problems, but also in other contexts like for instance in methods using regularization techniques (see, for instance, [2,4,5,8,9], and [10]).

Just to give an idea, in the convergence analysis of a Nyström method, based on a Gaussian quadrature rule, applied for the numerical solution of Hammerstein integral equations

$$f(y) - \int_D k(x,y)h(x,f(x))dx = \phi(y), \qquad y \in D,$$

with D a closed interval in $\mathbb{R}$, a crucial step, in obtaining a convergence rate for the approximation error, is to estimate the best approximation error of the product $k(\cdot,y)h(\cdot,f)$ and, consequently, of the composite function $h(x,f(x))$. However, in order to obtain stability results, it is also essential that the norms of the involved functions appear in the estimate.

Another case in which the error estimation we are discussing could be useful is when numerical methods based on the polynomial approximation are employed to solve Fredholm linear integral equations of the second kind

$$f(y) - \int_D k(x,y)f(x)dx = \phi(y), \qquad y \in D,$$

having kernels and/or right-hand sides with a low smoothness. A strategy often adopted in order to accelerate the convergence of the method, which can be very slow due to the poor smoothness of the involved functions, is to regularize the equation by combining such functions with suitable smoothing transformations.

In this paper we propose upper estimates for the error of best polynomial approximation of composite functions in suitable weighted Sobolev type subspaces. Therefore for estimating $E_m(f \circ g)_{u,p}$, starting from the Favard type inequality (1), we will need an upper bound for the weighted norm of the derivatives of the composite functions.

As it will be clear in the following, a crucial step for our aim is to provide, in weighted norms, an estimate of the derivatives of a given function in terms of a fixed higher derivative of the same function. According to our knowledge, this is only known for functions belonging to $C([a,b])$ or to $L^p([a,b])$. In fact, in [3, Lemma 2.1] if $f, f^{(r)} \in L^p([a,b])$ or $f^{(r)} \in C([a,b])$ (with $f^{(r-1)}$ locally absolutely continuous), then the following inequality was stated

$$\|f^{(k)}\|_p \leq \mathcal{C}\left(\frac{\|f\|_p}{(b-a)^k} + (b-a)^{r-k}\|f^{(r)}\|_p\right), \quad 0 < k < r, \quad 1 \leq p \leq +\infty, \tag{2}$$

where $\mathcal{C}$ is a positive constant independent of the interval $[a,b]$ and f.

Therefore, we first prove the analogue of (2) in weighted L^p and uniform norms and then we give the estimate of the derivatives of a composite function in a very general case, i.e. the case in which f is a multivariate function and g is a vector of functions. As corollaries, as we already said, we will deduce the estimates for the best approximation errors from the Favard inequality.

We will present the results in $[-1,1]$, without loss of generality, since by linearity analogous results can be deduced in the generic interval $[a,b]$ of $\mathbb{R}$.

2. Sobolev-Type Spaces

For $1 \leq p \leq \infty$, let us introduce the weighted Sobolev–type space of order $1 \leq r \in \mathbb{N}$ [6]

$$W^r_{u,p} = \left\{ f \in L^p_u : f^{(r-1)} \in AC((-1,1)),\, \|f^{(r)}\varphi^r u\|_p < \infty \right\},$$

where $\varphi(x) = \sqrt{1-x^2}$ and $AC((-1,1))$ denotes the set of all absolutely continuous functions on on $(-1,1)$ and in the case $p = \infty$ we will consider C_u instead of L^∞_u. We equip $W^r_{u,p}$ with the norm

$$\|f\|_{W^r_{u,p}} := \|fu\|_p + \|f^{(r)}\varphi^r u\|_p.$$

For multivariate functions $f : \Omega \to \mathbb{R}$ with Ω denoting an open subset of $\mathbb{R}^n$, we define the Sobolev space $\boldsymbol{W}_p^r(\Omega)$ [1] as the set of all functions f in Ω such that for every n-tuples of nonnegative integers $\ell = (\ell_1, \ell_2, \dots, \ell_n)$, with $|\ell| = \sum_{i=1}^n \ell_i \le r$, the mixed partial derivatives

$$D^\ell f = \frac{\partial^{\ell_1+\ell_2+\dots+\ell_n} f}{\partial x_1^{\ell_1} \partial x_2^{\ell_2} \dots \partial x_n^{\ell_n}}$$

exist and $\|D^\ell f\|_p < \infty$. We endow this space with the norm

$$\|f\|_{\boldsymbol{W}_p^r(\Omega)} = \|f\|_p + \sum_{1 \le |\ell| \le r} \|D^\ell f\|_p. \tag{3}$$

The spaces $(W_{u,p}^r, \|\cdot\|_{W_{u,p}^r})$ and $(\boldsymbol{W}_p^r(\Omega), \|\cdot\|_{\boldsymbol{W}_p^r(\Omega)})$ are Banach spaces.

3. Faa di Bruno's Formula

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ and $g : A \subseteq \mathbb{R} \to \mathbb{R}^n$ with $g(x) = (g_1(x), g_2(x), \dots, g_n(x))$, be functions such that the range of g is contained in the domain of f and for which all the necessary derivatives are defined.
In the case $n = 1$ the well-known Fáa di Bruno formula states

$$(f \circ g)^{(r)}(x) = \sum \frac{r!}{k_1!k_2!\dots k_r!} f^{(k1+k2+\dots+k_r)}(g(x)) \prod_{i=1}^{r} \left(\frac{g^{(i)}(x)}{i!} \right)^{k_i} \tag{4}$$

where the sum is over all r-tuples of nonnegative integers $(k_1, k_2, \dots, k_r)$ such that $k_1 + 2k_2 + \dots + rk_r = r$.

Combining the terms with the same value of $k_1 + k_2 + \dots + k_r = k$ and noticing that k_j has to be zero for $j > r-k+1$ lead to the following equivalent but simpler formula

$$(f \circ g)^{(r)}(x) = \sum_{k=1}^{r} f^{(k)}(g(x))\, \mathbf{B}_{r,k}(g'(x), g''(x), \dots g^{(r-k+1)}(x))$$

where $\mathbf{B}_{r,k}(x_1, x_2, \dots x_{r-k+1})$ are the partial or incomplete exponential Bell polynomials defined as

$$\mathbf{B}_{r,k}(x_1, x_2, \dots x_{r-k+1}) = \sum \frac{r!}{k_1!k_2!\dots k_{r-k+1}!} \prod_{i=1}^{r-k+1} \left(\frac{x_i}{i!} \right)^{k_i},$$

with the sum taken over all sequences $(k_1, k_2, \dots, k_{r-k+1})$ of nonnegative integers such that these two conditions are satisfied

$$\sum_{i=1}^{r-k+1} k_i = k, \qquad \sum_{i=1}^{r-k+1} i k_i = r.$$

Formula (4) for the r-th derivative of a composite function can be generalized in the case $n > 1$ as (see [7])

$$\begin{aligned}(f \circ g)^{(r)}(x) \\ = \sum_0 \sum_1 \sum_2 \cdots \sum_r \frac{r!}{\prod_{i=1}^r \prod_{j=1}^n q_{ij}! \prod_{i=1}^r (i!)^{k_i}} \frac{\partial^k f(g(x))}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}} \\ \times \prod_{i=1}^r \left(g_1^{(i)}(x)\right)^{q_{i1}} \left(g_2^{(i)}(x)\right)^{q_{i2}} \cdots \left(g_n^{(i)}(x)\right)^{q_{in}} \end{aligned} \tag{5}$$

where the respective sums are over all nonnegative integer solutions of the Diophantine equations, as follows

$$\begin{aligned} \sum_0 \to k_1 + 2k_2 + \ldots + rk_r = r, \\ \sum_i \to q_{i1} + q_{i2} + \ldots + q_{in} = k_i, \quad i = 1, 2, \ldots, r, \end{aligned} \tag{6}$$

and

$$\begin{aligned} p_j = q_{1j} + q_{2j} + \ldots + q_{rj}, \quad j = 1, 2, \ldots, n, \\ k = p_1 + p_2 + \ldots + p_n = k_1 + k_2 + \ldots + k_r. \end{aligned} \tag{7}$$

4. Main Results

In this section, we will give an estimate for the error of best polynomial approximation of composite functions both in the space L^p_u, with $1 \le p < \infty$, and in the space C_u.

4.1. The C_u case

First, let us prove the following lemma, which gives an estimate for the norm of the derivatives of a function $f \in W^r_{u,\infty}$, generalizing (2).

Lemma 1. *Let $u(x) = (1-x)^\gamma (1+x)^\delta$ be a Jacobi weight with $0 \le \gamma, \delta < 1$ and $f \in W^r_{u,\infty}$. Then, for $0 < k < r$*

$$\|f^{(k)} \varphi^k u\|_\infty \le \mathcal{C} \left(\|fu\|_\infty + \|f^{(r)} \varphi^r u\|_\infty \right) \tag{8}$$

where $\mathcal{C} = \mathcal{C}(r, k, \gamma, \delta)$ is a positive constant independent of f.

Proof. First we note that it is enough to prove that, for any $0 < k < r$,

$$\|f^{(k)} \varphi^k u\|_\infty \le \mathcal{C} \left(\|fu\|_\infty + \|f^{(k+1)} \varphi^{k+1} u\|_\infty \right) \tag{9}$$

with $\mathcal{C} = \mathcal{C}(k)$, since (8) con be deduced from (9) by induction on k.
Fix $-1 \le x \le 0$ and $h = 1/k$. By using the Taylor formula with integral remainder, being $f^{(k)}$ locally absolutely continuous and consequently $f^{(k+1)}$ locally integrable, we can write

$$f(x+jh) = \sum_{i=0}^{k} \frac{(jh)^i}{i!} f^{(i)}(x) + \frac{1}{k!} \int_0^{jh} (jh-t)^k f^{(k+1)}(x+t)dt$$

from which it follows

$$\begin{aligned} h^k f^{(k)}(x) &= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x+jh) \\ &\quad - \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (-1)^{k-j} \int_0^{jh} (jh-t)^k f^{(k+1)}(x+t)dt \end{aligned}$$

and, then,

$$\begin{aligned} h^k |f^{(k)}(x)| \varphi^k(x) u(x) &\le \sum_{j=0}^{k} \binom{k}{j} |f(x+jh)| \varphi^k(x) u(x) \\ &+ \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} \int_0^{jh} (jh-t)^k \left| f^{(k+1)}(x+t) \right| \varphi^k(x) u(x) dt. \end{aligned} \tag{10}$$

Let us estimate the first term at the right-hand side. Since $\varphi^k(x) \le 1$ and

$$\frac{u(x)}{u(x+jh)} \le \frac{(1-x)^\gamma}{(1-x-jh)^\gamma} \le \frac{2^\gamma}{(1-jh)^\gamma}, \tag{11}$$

being $-1 \le x \le 0$ and $h > 0$, using $\binom{k}{j} \le 2^k$ for any $j \in \{0, 1, \dots, k\}$, we obtain

$$\begin{aligned} \sum_{j=0}^{k} \binom{k}{j} |f(x+jh)| \varphi^k(x) u(x) &\le \|fu\|_\infty \sum_{j=0}^{k} \binom{k}{j} \frac{u(x)}{u(x+jh)} \varphi^k(x) \\ &\le \|fu\|_\infty 2^{\gamma+k} \sum_{j=0}^{k} \frac{1}{(1-jh)^\gamma} \\ &\le \|fu\|_\infty 2^{\gamma+k} \int_0^k (1-zh)^{-\gamma} dz \\ &= \|fu\|_\infty \frac{2^{\gamma+k}}{(1-\gamma)h}. \end{aligned}$$

In order to estimate the second sum in (10), we observe that for $-1 \le x \le 0$, $1 \le j \le k$, $h = 1/k > 0$ and $0 \le t \le jh$ the following inequalities hold true

$$\frac{\varphi^k(x)}{\varphi^{k+1}(x+t)} \le \frac{2^{\frac{k}{2}}}{t^{\frac{1}{2}}(jh-t)^{\frac{k}{2}+\frac{1}{2}}},$$
$$\frac{u(x)}{u(x+t)} \le \frac{(1-x)^\gamma}{(1-x-t)^\gamma} \le \frac{2^\gamma}{(1-t)^\gamma} \le \frac{2^\gamma}{(jh-t)^\gamma}.$$

Consequently, we deduce

$$\begin{aligned}
&\frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}\int_0^{jh}(jh-t)^k\left|f^{(k+1)}(x+t)\right|\varphi^k(x)u(x)dt\\
&\le \frac{1}{k!}\left\|f^{(k+1)}\varphi^{k+1}u\right\|_\infty\sum_{j=1}^{k}\binom{k}{j}\int_0^{jh}(jh-t)^k\frac{\varphi^k(x)}{\varphi^{k+1}(x+t)}\frac{u(x)}{u(x+t)}dt\\
&\le \frac{2^{\frac{k}{2}+\gamma}}{k!}\left\|f^{(k+1)}\varphi^{k+1}u\right\|_\infty\sum_{j=1}^{k}\binom{k}{j}\int_0^{jh}(jh-t)^{\frac{k-1}{2}-\gamma}t^{-\frac{1}{2}}dt\\
&= \frac{1}{k!}2^{\frac{k}{2}+\gamma}\left\|f^{(k+1)}\varphi^{k+1}u\right\|_\infty B\left(1-\gamma,\frac{1}{2}\right)h^{\frac{k}{2}-\gamma}\sum_{j=1}^{k}\binom{k}{j}j^{\frac{k}{2}-\gamma}, \qquad (12)
\end{aligned}$$

where B denotes the beta function.
Now, we note that for any integer $k \ge 1$, it is

$$\sum_{j=1}^{k}\binom{k}{j}j^{\frac{k}{2}-\gamma} \le 2^k k^{\frac{k}{2}-\gamma},$$

from which, combining (10)-(12), we get

$$\begin{aligned}
&\max_{-1\le x\le 0}\left|f^{(k)}(x)\varphi^k(x)u(x)\right|\\
&\le \frac{1}{h^k}\left[\frac{2^{\gamma+k}}{(1-\gamma)h}\|fu\|_\infty + \frac{1}{k!}2^{\frac{k}{2}+\gamma}B\left(1-\gamma,\frac{1}{2}\right)h^{\frac{k}{2}-\gamma}2^k k^{\frac{k}{2}-\gamma}\left\|f^{(k+1)}\varphi^{k+1}u\right\|_\infty\right]\\
&= \frac{2^{\gamma+k}}{1-\gamma}k^{k+1}\|fu\|_\infty + \frac{k^k}{k!}2^{\frac{3}{2}k+\gamma}B\left(1-\gamma,\frac{1}{2}\right)\left\|f^{(k+1)}\varphi^{k+1}u\right\|_\infty. \qquad (13)
\end{aligned}$$

The case $0 \le x \le 1$ and $h = -1/k$ can be treated with similar arguments and leads to the following estimate

$$\begin{aligned}
\max_{0\le x\le 1}\left|f^{(k)}(x)\varphi^k(x)u(x)\right| &\le \frac{2^{\delta+k}}{1-\delta}k^{k+1}\|fu\|_\infty\\
&\quad + \frac{k^k}{k!}2^{\frac{3}{2}k+\delta}B\left(1-\delta,\frac{1}{2}\right)\left\|f^{(k+1)}\varphi^{k+1}u\right\|_\infty. \qquad (14)
\end{aligned}$$

Finally, combining (13) with (14), we can deduce (9) with the constant $\mathcal{C} = \mathcal{C}(k)$ given by

$$\mathcal{C}(k) = \max\left\{ \frac{2^{\gamma+k}}{1-\gamma} k^{k+1}, \frac{2^{\delta+k}}{1-\delta} k^{k+1}, \frac{k^k}{k!} 2^{\frac{3}{2}k+\gamma} B\left(1-\gamma, \frac{1}{2}\right), \right. \\ \left. \frac{k^k}{k!} 2^{\frac{3}{2}k+\delta} B\left(1-\delta, \frac{1}{2}\right) \right\},$$

where we emphasize that the expression obtained for the constant $\mathcal{C}$ is not the best one. What it is important is that it depends only on k, γ, δ but not on f. □

The previous lemma is crucial to prove our first main result.

Theorem 2. *Let $f : \Omega \to \mathbb{R}$ with Ω open subset of $\mathbb{R}^n$ and $g : (-1,1) \to \mathbb{R}^n$ such that $\operatorname{Im}(g) \subseteq \Omega$. Assume that $f \in \boldsymbol{W}^r_\infty(\Omega)$ and $g = (g_1, g_2, \ldots, g_n)$, $g_j \in W^r_{\sqrt[r]{u},\infty}$, $j = 1, 2, \ldots, n$, for a Jacobi weight $u(x) = (1-x)^\gamma(1+x)^\delta$ with $0 \le \gamma, \delta < r$. Then we have that $f \circ g \in W^r_{u,\infty}$ and*

$$\|(f \circ g)^{(r)} \varphi^r u\|_\infty \le \mathcal{C} n^r B_r \|f\|_{\boldsymbol{W}^r_\infty(\Omega)} \prod_{j=1}^n \|g_j\|^{s_j}_{W^r_{\sqrt[r]{u},\infty}} \tag{15}$$

where B_r is the r-th Bell number, $\mathcal{C} = \mathcal{C}(r, \gamma, \delta)$ is a positive constant independent of f and g, and

$$s_j = \begin{cases} 0, & \text{if } \|g_j\|_{W^r_{\sqrt[r]{u},\infty}} \le 1 \\ r, & \text{if } \|g_j\|_{W^r_{\sqrt[r]{u},\infty}} > 1 \end{cases}, \qquad j = 1, 2, \ldots, n. \tag{16}$$

Proof. Set $w = \sqrt[r]{u}$ (that is $u = w^r$). Faa di Bruno's formula (5), with (6)-(7), yields

$$\begin{aligned}
&\left|(f \circ g)^{(r)}(x)\varphi^r(x)u(x)\right| \\
&\le \sum_0 w^{r-k}(x) \sum_1 \sum_2 \cdots \sum_r \frac{r!}{\prod_{i=1}^r \prod_{j=1}^n q_{ij}!} \left|\frac{\partial^k f(g(x))}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}}\right| w^{p_1+\ldots+p_n}(x) \\
&\quad \times \prod_{i=1}^r \frac{\left|g_1^{(i)}(x)\varphi^i(x)\right|^{q_{i1}} \left|g_2^{(i)}(x)\varphi^i(x)\right|^{q_{i2}} \cdots \left|g_n^{(i)}(x)\varphi^i(x)\right|^{q_{in}}}{(i!)^{k_i}} \\
&\le \mathcal{C} \sum_0 \sum_1 \sum_2 \cdots \sum_r \frac{r!}{\prod_{i=1}^r \prod_{j=1}^n q_{ij}!} \left|\frac{\partial^k f(g(x))}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}}\right| \\
&\quad \times \prod_{i=1}^r \frac{\left|g_1^{(i)}(x)\varphi^i(x)w(x)\right|^{q_{i1}} \left|g_2^{(i)}(x)\varphi^i(x)w(x)\right|^{q_{i2}} \cdots \left|g_n^{(i)}(x)\varphi^i(x)w(x)\right|^{q_{in}}}{(i!)^{k_i}}.
\end{aligned}$$

Then, taking into account (8), we have

$$\left\|g_j^{(i)} \varphi^i w\right\|_\infty \le \mathcal{C}_i \|g_j\|_{W^r_{w,\infty}}, \qquad i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, n,$$

and, in virtue of (3), we deduce

$$\begin{aligned}\left|(f\circ g)^{(r)}(x)\varphi^r(x)u(x)\right| &\le \mathcal{C}\|f\|_{\boldsymbol{W}^r_\infty(\Omega)}\sum_0\sum_1\sum_2\cdots\sum_r \frac{r!}{\prod_{i=1}^r\prod_{j=1}^n q_{ij}!}\\ &\quad\times\prod_{i=1}^r\frac{\mathcal{C}_i^{q_{i1}+q_{i2}+\ldots+q_{in}}\prod_{j=1}^n\|g_j\|^{q_{ij}}_{W^r_{w,\infty}}}{(i!)^{k_i}}\\ &\le \mathcal{C}\|f\|_{\boldsymbol{W}^r_\infty(\Omega)}\sum_0\sum_1\sum_2\cdots\sum_r \frac{r!}{\prod_{i=1}^r\prod_{j=1}^n q_{ij}!}\\ &\quad\times\prod_{j=1}^n\|g_j\|^{p_j}_{W^r_{w,\infty}}\prod_{i=1}^r\frac{\mathcal{C}_i^{k_i}}{(i!)^{k_i}}.\end{aligned}$$

Since $p_j\le k\le r$, for $j=1,2,\ldots,n$, setting $\bar{\mathcal{C}}=\max\limits_{i=1,2,\ldots,r}\mathcal{C}_i$ and multiplying and dividing by $k_1!k_2!\cdots,k_r!$, we can write

$$\begin{aligned}\left|(f\circ g)^{(r)}(x)\varphi^r(x)u(x)\right| &\le \mathcal{C}\bar{\mathcal{C}}^r\|f\|_{\boldsymbol{W}^r_\infty(\Omega)}\prod_{j=1}^n\|g_j\|^{s_j}_{W^r_{w,\infty}}\sum_0\frac{r!}{k_1!k_2!\cdots,k_r!}\\ &\quad\times\left[\prod_{i=1}^r\frac{1}{(i!)^{k_i}}\right]\sum_1\sum_2\cdots\sum_r\prod_{i=1}^r\frac{k_i!}{\prod_{j=1}^n q_{ij}!}\end{aligned}$$

with the exponents s_j, $j=1,2,\ldots,n$, defined by (16). Now, we observe that

$$\begin{aligned}\sum_i\frac{k_i!}{\prod_{j=1}^n q_{ij}!} &= \sum_{q_{i1}+q_{i2}+\ldots+q_{in}=k_i}\binom{k_i}{q_{i1}\; q_{i2}\;\ldots\; q_{in}}\\ &=(1+1+\ldots+1)^{k_i}=n^{k_i}\qquad i=1,2,\ldots,r,\end{aligned}$$

from which it follows that

$$\begin{aligned}\sum_1\sum_2\cdots\sum_r\prod_{i=1}^r\frac{k_i!}{\prod_{j=1}^n q_{ij}!} &= \sum_1\frac{k_1!}{\prod_{j=1}^n q_{1j}!}\sum_2\frac{k_2!}{\prod_{j=1}^n q_{2j}!}\cdots\sum_r\frac{k_r!}{\prod_{j=1}^n q_{rj}!}\\ &= n^{k_1+k_2+\ldots+k_r}=n^k\le n^r\end{aligned}$$

and, consequently,

$$\begin{aligned}
&\left|(f \circ g)^{(r)}(x)\varphi^r(x)u(x)\right| \\
&\leq \mathcal{C} n^r \|f\|_{\boldsymbol{W}^r_\infty(\Omega)} \prod_{j=1}^n \|g_j\|^{s_j}_{W^r_{w,\infty}} \sum_0 \frac{r!}{k_1!k_2!\cdots,k_r!} \prod_{i=1}^r \frac{1}{(i!)^{k_i}} \\
&= \mathcal{C} n^r \|f\|_{\boldsymbol{W}^r_\infty(\Omega)} \prod_{j=1}^n \|g_j\|^{s_j}_{W^r_{w,\infty}} \sum_{k=1}^r B_{r,k}(1,1,\ldots,1) \\
&= \mathcal{C} n^r B_r \|f\|_{\boldsymbol{W}^r_\infty(\Omega)} \prod_{j=1}^n \|g_j\|^{s_j}_{W^r_{w,\infty}},
\end{aligned}$$

where $\mathcal{C}$ is a positive constant independent of f and g, i.e. the thesis (15). □

By combining (15) with (1) we can immediately deduce an estimate for $E_m(f \circ g)_u$.

Theorem 3. *Let $f : \Omega \to \mathbb{R}$ with Ω open subset of $\mathbb{R}^n$ and $g : (-1,1) \to \mathbb{R}^n$ such that $\mathrm{Im}(g) \subseteq \Omega$. Moreover, assume that $f \in \boldsymbol{W}^r_\infty(\Omega)$ and $g = (g_1, g_2, \ldots, g_n)$ with $g_j \in W^r_{\sqrt[r]{u},\infty}$, for $j = 1, 2, \ldots, n$, and $u(x) = (1-x)^\gamma(1+x)^\delta$, $0 \leq \gamma, \delta < r$. Then*

$$E_m(f \circ g)_{u,\infty} \leq \frac{\mathcal{C}}{m^r} n^r B_r \|f\|_{\boldsymbol{W}^r_\infty(\Omega)} \prod_{j=1}^n \|g_j\|^{s_j}_{W^r_{\sqrt[r]{u},\infty}}$$

where $\mathcal{C} = \mathcal{C}(r, \gamma, \delta)$ is a positive constant independent of f and g, B_r is the r-th Bell number, and the exponents s_j, $j = 1, 2, \ldots, n$, are given in (16).

Corollary 4. *In the special case $n = 1$ the previous estimate is simplified as follows*

$$E_m(f \circ g)_{u,\infty} \leq \frac{\mathcal{C}}{m^r} B_r \|f\|_{W^r_\infty(\Omega)} \|g\|^{s_1}_{W^r_{\sqrt[r]{u},\infty}}.$$

4.2. The L^p_u case

Also in this case we start by a generalization of the result (2) proved in the unweighted function space L^p in [3, Lemma 2.1]. The following lemma provides an estimate for the norm of the derivatives of a function $f \in W^r_{u,p}$, for $1 \leq p < \infty$.

Lemma 5. *Let $1 \leq p < \infty$. Assume that $u(x) = (1-x)^\gamma(1+x)^\delta$ is a Jacobi weight with $\max\left\{-\frac{1}{p}, -\frac{1}{2}\right\} < \gamma, \delta < 1$ and $f \in W^r_{u,p}$. Then, for $0 < k < r$*

$$\|f^{(k)}\varphi^k u\|_p \leq \mathcal{C}\left(\|fu\|_p + \|f^{(r)}\varphi^r u\|_p\right) \tag{17}$$

where $\mathcal{C} = \mathcal{C}(r, k, \gamma, \delta)$ is a positive constant independent of f.

Proof. As in the proof of Lemma 1, we limit ourselves to prove that

$$\|f^{(k)}\varphi^k u\|_p \le \mathcal{C}\left(\|fu\|_p + \|f^{(k+1)}\varphi^{k+1}u\|_p\right) \tag{18}$$

with $\mathcal{C} = \mathcal{C}(k)$.
First, we observe that

$$\|f^{(k)}\varphi^k u\|_p^p = \left\{\int_{-1}^{0} + \int_{0}^{1}\right\} \left|f^{(k)}(x)\varphi^k(x)u(x)\right|^p dx. \tag{19}$$

Now, setting $h = 1/k$ and using (10), we have

$$\begin{aligned} &h^k \left(\int_{-1}^{0} \left|f^{(k)}(x)\varphi^k(x)u(x)\right|^p dx\right)^{\frac{1}{p}} \\ &\le \sum_{j=0}^{k} \binom{k}{j} \left(\int_{-1}^{0} \left|f(x+jh)\varphi^k(x)u(x)\right|^p dx\right)^{\frac{1}{p}} \\ &\quad + \frac{1}{k!}\sum_{j=1}^{k} \binom{k}{j} \left(\int_{-1}^{0} \left|\int_{0}^{jh} (jh-t)^k f^{(k+1)}(x+t)dt\, \varphi^k(x)u(x)\right|^p dx\right)^{\frac{1}{p}} \\ &=: I_1 + I_2. \end{aligned} \tag{20}$$

Let us estimate I_1. First, we rewrite it as follows

$$I_1 = \sum_{j=0}^{k} \binom{k}{j} \left(\int_{-1}^{0} \left|f(x+jh)u(x+jh)\varphi^k(x)\frac{u(x)}{u(x+jh)}\right|^p dx\right)^{\frac{1}{p}},$$

and note that, being $0 \le 1+x \le 1$ and $1 \le 1-x \le 2$, under the assumptions on γ and δ, one has

$$\varphi^k(x)\frac{u(x)}{u(x+jh)} \le \begin{cases} 2^{k/2}(1+jh)^{-\delta}, & \gamma,\delta \le 0, \quad \text{or} \quad \delta \le 0,\, \gamma \ge 0, \\ 2^{k/2}, & \gamma \le 0,\, \delta \ge 0, \\ 2^{\gamma}(1-jh)^{-\gamma}, & \gamma,\delta \ge 0. \end{cases}$$

Therefore, if $\gamma, \delta \le 0$ or $\delta \le 0$ and $\gamma \ge 0$, one deduce

$$\begin{aligned} I_1 &\le 2^{\frac{3}{2}k}\|fu\|_p \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(1+jh)^{\delta}} \le 2^{\frac{3}{2}k}\|fu\|_p \int_0^k (1+zh)^{-\delta}\, dz \\ &= 2^{\frac{3}{2}k+1-\delta}\frac{k}{1-\delta}\,\|fu\|_p. \end{aligned} \tag{21}$$

In the remaining cases, following a similar process will yield

$$I_1 \le \|fu\|_p \begin{cases} 2^{\frac{3}{2}k}(k+1), & -\gamma \le 0,\, \delta \ge 0, \\ 2^{\gamma+k}\dfrac{k}{1-\gamma}, & \gamma, \delta \ge 0. \end{cases} \tag{22}$$

Let us now estimate I_2. By the Minkowski inequality we have

$$\begin{aligned} I_2 &\le \frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}\int_0^{jh}(jh-t)^k\left(\int_{-1}^{0}\left|f^{(k+1)}(x+t)\,\varphi^k(x)u(x)\right|^p dx\right)^{\frac{1}{p}} dt \\ &= \frac{1}{k!}\sum_{j=1}^{k}\binom{k}{j}\int_0^{jh}(jh-t)^k \\ &\left(\int_{-1}^{0}\left|(f^{(k+1)}\varphi^{k+1}u)(x+t)\right|^p\left(\frac{\varphi^k(x)}{\varphi^{k+1}(x+t)}\frac{u(x)}{u(x+t)}\right)^p dx\right)^{\frac{1}{p}} dt. \end{aligned}$$

Therefore, since by the assumtions on γ and δ, we deduce

$$\frac{\varphi^k(x)}{\varphi^{k+1}(x+t)}\frac{u(x)}{u(x+t)} \le \frac{1}{\sqrt{(jht)t}}\frac{2^{\gamma+\frac{k}{2}}}{(1-t)^{\gamma+\frac{k}{2}}} \le \frac{1}{\sqrt{(jh-t)t}}\frac{2^{\gamma+\frac{k}{2}}}{(jht)^{\gamma+\frac{k}{2}}},$$

we can write

$$\begin{aligned} I_2 &\le \left\|f^{(k+1)}\varphi^{k+1}u\right\|_p \frac{2^{\gamma+\frac{k}{2}}}{k!}\sum_{j=1}^{k}\binom{k}{j}\int_0^{jh}(jh-t)^{\frac{k}{2}-\gamma-\frac{1}{2}}t^{-\frac{1}{2}}dt \\ &= \left\|f^{(k+1)}\varphi^{k+1}u\right\|_p \frac{2^{\gamma+\frac{k}{2}}}{k!}B\left(1-\gamma,\frac{1}{2}\right)h^{\frac{k}{2}-\gamma}\sum_{j=1}^{k}\binom{k}{j}j^{\frac{k}{2}-\gamma} \\ &\le \left\|f^{(k+1)}\ \varphi^{k+1}u\right\|_p \frac{2^{\gamma+\frac{k}{2}}}{k!}B\left(1-\gamma,\frac{1}{2}\right)\left(\frac{1}{k}\right)^{\frac{k}{2}-\gamma} 2^k k^{\frac{k}{2}-\gamma}. \end{aligned} \tag{23}$$

Hence, by combining (21), (22), and (23) in (20) and setting

$$\mathcal{C}_1(k) = 2^k k^k \max\left\{\frac{k\,2^{\frac{k}{2}+1-\delta}}{1-\delta}, 2^{\frac{k}{2}}(k+1), \frac{2^\gamma k}{1-\gamma}, \frac{2^{\gamma+\frac{k}{2}}kB\left(1-\gamma,\frac{1}{2}\right)}{k!}\right\},$$

we have

$$\left(\int_{-1}^{0}\left|f^{(k)}(x)\varphi^k(x)u(x)\right|^p dx\right)^{\frac{1}{p}} \le \mathcal{C}_1(k)\ \left\|f^{(k+1)}\ \varphi^{k+1}u\right\|_p. \tag{24}$$

If $0 \le x \le 1$ and $h = -1/k$, we can proceed in a similar way getting

$$\left(\int_{0}^{1}\left|f^{(k)}(x)\varphi^k(x)u(x)\right|^p dx\right)^{\frac{1}{p}} \le \mathcal{C}_2(k)\ \left\|f^{(k+1)}\ \varphi^{k+1}u\right\|_p, \tag{25}$$

where C_2 is a positive constant which depends on k and the parameters of the weight γ and δ. By replacing (24) and (25) in (19) we have the assertion. □

Theorem 6. *Let $f : \Omega \to \mathbb{R}$ with Ω open subset of $\mathbb{R}^n$ and $g : (-1,1) \to \mathbb{R}^n$ such that $\mathrm{Im}(g) \subseteq \Omega$. Assume that $f \in \boldsymbol{W}^r_{\bar{p}}(\Omega)$, for some $1 \leq \bar{p} < \infty$, and $g = (g_1, g_2, \ldots, g_n)$, $g_j \in W^r_{\sqrt[r]{u},\bar{q}_j}$, for some n-tuple $(\bar{q}_1, \bar{q}_2, \ldots, \bar{q}_n) \in \mathbb{R}^n$ such that $1 \leq \bar{q}_j < \infty$, $j = 1, 2, \ldots, n$, and*

$$(\forall\,(p_1, p_2, \ldots, p_n) \in \mathbb{N}_0^n) \quad \sum_{j=1}^n \frac{p_j}{\bar{q}_j} \leq 1, \quad \textit{such that} \quad 1 \leq \sum_{j=1}^n p_j \leq r,$$

and for a Jacobi weight $u(x) = (1-x)^\gamma (1+x)^\delta$ with $\max\left\{-\frac{1}{\bar{q}_j}, -\frac{1}{2}\right\} r < \gamma, \delta < r$, for $j = 1, 2, \ldots, n$. Then, if $1 \leq \bar{q} < \infty$ satisfies the condition

$$(\forall\,(p_1, p_2, \ldots, p_n) \in \mathbb{N}_0^n) \quad \frac{1}{\bar{q}} \geq \sum_{j=1}^n \frac{p_j}{\bar{q}_j}, \quad \textit{such that} \quad 1 \leq \sum_{j=1}^n p_j \leq r, \tag{26}$$

and we set

$$\frac{1}{p} = \frac{1}{\bar{p}} + \frac{1}{\bar{q}} \leq 1$$

we have that $f \circ g \in W^r_{u,p}$ and

$$\|(f \circ g)^{(r)} \varphi^r u\|_p \leq \mathcal{C} n^r B_r \|f\|_{\boldsymbol{W}^r_{\bar{p}}(\Omega)} \prod_{j=1}^n \|g_j\|^{s_j}_{W^r_{\sqrt[r]{u},\bar{q}_j}} \tag{27}$$

where B_r is the r-th Bell number, $\mathcal{C} = \mathcal{C}(r, \gamma, \delta)$ is a positive constant independent of f and g, and

$$s_j = \begin{cases} 0, & \text{if } \|g_j\|_{W^r_{\sqrt[r]{u},\bar{q}_j}} \leq 1 \\ r, & \text{if } \|g_j\|_{W^r_{\sqrt[r]{u},\bar{q}_j}} > 1 \end{cases}, \qquad j = 1, 2, \ldots, n. \tag{28}$$

Proof. We proceed as in the proof of Theorem 2, setting $w = \sqrt[r]{u}$. Taking into account formulas (5)-(7) and also using the generalized Minkowski and Hölder inequalities, we get

$$\begin{aligned}
\left\|(f \circ g)^{(r)} \varphi^r u\right\|_p &= \left(\int_{-1}^1 \left|(f \circ g)^{(r)}(x) \varphi^r(x) u(x)\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \mathcal{C} \sum_0 \sum_1 \sum_2 \cdots \sum_r \frac{r!}{\prod_{i=1}^r \prod_{j=1}^n q_{ij}!} \\
&\quad \times \left(\int_{-1}^1 \left|\left|\frac{\partial^k f(g(x))}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}}\right| \prod_{j=1}^n \prod_{i=1}^r \frac{\left|g_j^{(i)}(x) \varphi^i(x) w(x)\right|^{q_{ij}}}{(i!)^{k_i}}\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \mathcal{C} \sum_0 \sum_1 \sum_2 \cdots \sum_r \frac{r!}{\prod_{i=1}^r \prod_{j=1}^n q_{ij}!} \\
&\quad \times \left(\int_{-1}^1 \left|\frac{\partial^k f(g(x))}{\partial x_1^{p_1} \partial x_2^{p_2} \cdots \partial x_n^{p_n}}\right|^{\bar{p}} dx\right)^{\frac{1}{\bar{p}}} \left(\int_{-1}^1 \left|\prod_{j=1}^n \prod_{i=1}^r \frac{\left|g_j^{(i)}(x) \varphi^i(x) w(x)\right|^{q_{ij}}}{(i!)^{k_i}}\right|^{\bar{q}} dx\right)^{\frac{1}{\bar{q}}}.
\end{aligned}$$

Now, observing that, in virtue of (7) and assumption (26),

$$\sum_{j=1}^{n}\sum_{i=1}^{r}\sum_{h=1}^{q_{ij}}\frac{1}{\bar{q}_j}=\sum_{j=1}^{n}\frac{1}{\bar{q}_j}\sum_{i=1}^{r}q_{ij}=\sum_{j=1}^{n}\frac{p_j}{\bar{q}_j}\leq\frac{1}{\bar{q}},$$

by applying the generalized Hölder inequality again and estimate (17), we deduce

$$\begin{aligned}\left\|(f\circ g)^{(r)}\varphi^r u\right\|_p &\leq \mathcal{C}\|f\|_{\boldsymbol{W}^r_{\bar{p}}(\Omega)}\sum_0\sum_1\sum_2\cdots\sum_r\frac{r!}{\prod_{i=1}^r\prod_{j=1}^n q_{ij}!}\\ &\times\prod_{j=1}^{n}\prod_{i=1}^{r}\prod_{h=1}^{q_{ij}}\frac{1}{(i!)^{k_i}}\left(\int_{-1}^{1}\left|g_j^{(i)}(x)\varphi^i(x)w(x)\right|^{\bar{q}_j}\right)^{\frac{1}{\bar{q}_j}}\\ &\leq \mathcal{C}\|f\|_{\boldsymbol{W}^r_{\bar{p}}(\Omega)}\sum_0\sum_1\sum_2\cdots\sum_r\frac{r!}{\prod_{i=1}^r\prod_{j=1}^n q_{ij}!}\prod_{j=1}^{n}\prod_{i=1}^{r}\frac{1}{(i!)^{k_i}}\left\|g_j^{(i)}\varphi^i w\right\|_{\bar{q}_j}^{q_{ij}}\\ &\leq \mathcal{C}\|f\|_{\boldsymbol{W}^r_{\bar{p}}(\Omega)}\sum_0\sum_1\sum_2\cdots\sum_r\frac{r!}{\prod_{i=1}^r\prod_{j=1}^n q_{ij}!}\prod_{j=1}^{n}\prod_{i=1}^{r}\frac{1}{(i!)^{k_i}}\left\|g_j^{(r)}\varphi^r w\right\|_{\bar{q}_j}^{q_{ij}}\\ &\leq \mathcal{C}\|f\|_{\boldsymbol{W}^r_{\bar{p}}(\Omega)}\sum_0\sum_1\sum_2\cdots\sum_r\frac{r!}{\prod_{i=1}^r\prod_{j=1}^n q_{ij}!}\prod_{j=1}^{n}\|g_j\|_{W^r_{w,\bar{q}_j}}^{p_j}\prod_{i=1}^{r}\frac{1}{(i!)^{k_i}}.\end{aligned}$$

At this point, by multiplying and dividing by $k_1!k_2!\cdots,k_r!$ and, then, proceeding exactly as in the proof of Theorem 2, we have

$$\left\|(f\circ g)^{(r)}\varphi^r u\right\|_p\leq\mathcal{C}n^rB_r\|f\|_{\boldsymbol{W}^r_{\bar{p}}(\Omega)}\prod_{j=1}^{n}\|g_j\|_{W^r_{\sqrt[n]{u},\bar{q}_j}}^{s_j},$$

where the exponents s_j, $j=1,2,\ldots,n$, are given in (28). □

Starting, also in this case, from the Favard inequality (1) and using (27) we obtain the estimate for the best polynomial approximation of composite functions in weighted L^p spaces.

Theorem 7. *Assume that the functions f, g and the weight u satisfy the same assumptions of Theorem 6. Then*

$$E_m(f\circ g)_{u,p}\leq\frac{\mathcal{C}}{m^r}n^rB_r\|f\|_{\boldsymbol{W}^r_{\bar{p}}(\Omega)}\prod_{j=1}^{n}\|g_j\|_{W^r_{\sqrt[n]{u},\bar{q}_j}}^{s_j},\tag{29}$$

where $\mathcal{C}=\mathcal{C}(r,\gamma,\delta)$ is a positive constant independent of f and g, B_r is the r-th Bell number, and the exponents s_j, $j=1,2,\ldots,n$, are given in (28).

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References

[1] Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, Berlin (2011)

[2] Diogo, T., Lima, P.M., Pedas, A., Vainikko, G.: Smoothing transformation and spline collocation for weakly singular Volterra integro-differential equations. Appl. Numer. Math. **114**, 63–76 (2017)

[3] Ditzian, Z.: On interpolation of $L_p([a,b])$ and weighted Sobolev spaces. Pacific J. Math. **90**, 307–324 (1980)

[4] Fermo, L., Russo, M.G.: Numerical methods for Fredholm integral equations with singular right-hand sides. Adv. Comput. Math. **33**(3), 305–330 (2010)

[5] Galperin, E.A., Kansa, E.J., Makroglou, A., Nelson, S.A.: Variable transformations in the numerical solution of second kind Volterra integral equations with continuous and weakly singular kernels; extensions to Fredholm integral equations. J. Comput. Appl. Math. **115**(1–2), 193–211 (2000)

[6] Mastroianni, G., Milovanović, G.V.: Interpolation Processes. Basic Theory and Applications. Springer Monographs in Mathematics. Springer-Verlag, Berlin (2008)

[7] Mishkov, R.L.: Generalization of the formula of Faa di Bruno for a composite function with a vector argument. Internat. J. Math. & Math. Sci **24**(7), 481–491 (2000)

[8] Monegato, G., Scuderi, L.: High order methods for weakly singular integral equations with nonsmooth input functions. Math. Comput. **67**(224), 1493–1515 (1998)

[9] Pedas, A., Vainikko, G.: Smoothing transformation and piecewise polynomial projection methods for weakly singular Fredholm integral equations. Commun. Pure Appl. Anal. **5**(2), 395–413 (2006)

[10] Vainikko, E., Vainikko, G.: A spline product quasi-interpolation method for weakly singular Fredholm integral equations. SIAM J. Numer. Anal. **46**(4), 1799–1820 (2008)

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