On the Geometry of Coherent State Maps

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Abstract Given a mechanical system whose phase space $\mathfrak{M}^n$ is equipped with a complex structure J, and a Hermitian line bundle $(E, H) \to \mathfrak{M}$, a coherent state map is an anti-holomorphic embedding $\mathscr{K} : \mathfrak{M} \to \mathbb{CP}\big(\mathscr{M}\big)$ built in terms of (J, H), with $\mathscr{M} = H^0\Big(\mathfrak{M}, L^2\, \mathscr{O}\big(T^{*(n,0)}(\mathfrak{M}) \otimes E\big)\Big)$, such that for any pair of classical states $z, \zeta \in \mathfrak{M}$ the number $\langle \mathscr{K}(z), \mathscr{K}(\zeta)\rangle$ is the transition probability amplitude from the coherent state $\mathscr{K}(z)$ to $\mathscr{K}(\zeta)$. We examine three related questions, as follows: (i) We generalize Lichnerowicz's theorem (on $\pm$ holomorphic maps of finite-dimensional compact Kählerian manifolds) to describe anti-holomorphic maps $\mathscr{K} : \mathfrak{M} \to \mathbb{CP}(\mathscr{M})$ as harmonic maps that are absolute minima within their homotopy classes. (ii) If the phase space is a domain $\mathfrak{M} = \Omega \subset \mathbb{C}^n$ and $E \to \Omega$ is a trivial Hermitian line bundle such that $\gamma = H\big(\sigma_0\,,\, \sigma_0\big) \in AW(\Omega)$ (i.e., γ is an admissible weight), we discuss the use of $K_\gamma(z, \zeta)$ [the γ-weighted Bergman kernel of Ω] *vis-a-vis* to the calculation of the transition probability amplitudes, focusing on the case where $\Omega = \Omega_n$ is the Siegel domain and $\gamma(z) = \gamma_a(z) = \big(\mathrm{Im}(z_n) - |z'|^2\big)^a$, $a > -1$. (iii) We study the boundary behavior of a coherent state map $\mathscr{K} : \Omega \to \mathbb{CP}\big[L^2 H(\Omega_n\,,\, \gamma_a)\big]$.

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1 Coherent State Maps: Odzijewicz's Construction

Let $\mathfrak{M}^n$ be a complex n-dimensional manifold and $\pi : E \to \mathfrak{M}$ a complex line bundle equipped with i) a holomorphic structure $\big\{\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}\big\}_{\alpha \in I}$ and such that each U_α is the domain of a local complex coordinate system $\big(z^1_\alpha\,,\, \cdots\,,\, z^n_\alpha\big)$ on $\mathfrak{M}$ and with ii) a Hermitian bundle metric H. Let us set $\sigma_\alpha(z) = \phi_\alpha^{-1}(z, 1)$

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for any $z \in U_\alpha$. Let $\mathscr{M} = H^0\big(\mathfrak{M}, L^2\ \mathscr{O}\big(T^{*\,(n,0)}(\mathfrak{M}) \otimes E\big)\big)$ be the space of L^2 holomorphic sections in the holomorphic line bundle $T^{*\,(n,0)}(\mathfrak{M}) \otimes E \to \mathfrak{M}$. Every $s \in \mathscr{M}$ is locally represented as $s\big|_{U_\alpha} = \Psi_\alpha\, \sigma_\alpha \otimes dz^1_\alpha \wedge \cdots dz^n_\alpha$ for some $\Psi_\alpha \in \mathscr{O}\big(U_\alpha\big)$. A L^2 inner product on $\mathscr{M}$ (organizing $\mathscr{M}$ as a Hilbert space) is given by

$$\langle s^1\,,\ s^2\rangle = i^{n^2} \int_{\mathfrak{M}} H^*\big(s^1\,,\ s^2\big), \quad s^1,\ s^2 \in \mathscr{M}, \tag{1}$$

$$H^*\big(s^1\,,\ s^2\big)\big|_{U_\alpha} = \Psi^1_\alpha\, \overline{\Psi}^2_\alpha\, \gamma_\alpha\, dz^1_\alpha \wedge \cdots \wedge dz^n_\alpha \wedge d\overline{z}^1_\alpha \wedge \cdots \wedge d\overline{z}^n_\alpha\,,$$

$$\gamma_\alpha \equiv H\big(\sigma_\alpha\,,\ \sigma_\alpha\big), \quad \alpha \in I.$$

Let $z \in \mathfrak{M}$ and $\alpha \in I$ such that $z \in U_\alpha$, and let us set $\delta^\alpha_z(s) = \Psi_\alpha(z)$ for any $s \in \mathscr{M}$. By a result of K. Gawedzki (cf. [21])

$$\big|\Psi_\alpha(z)\big| \le C_\alpha \|s\| \tag{2}$$

hence, $\delta^\alpha_z : \mathscr{M} \to \mathbb{C}$ is continuous. By the Riesz representation theorem, there is $k_{z,\,\overline{\alpha}} \in \mathscr{M}$ such that

$$\delta^\alpha_z(s) = \big\langle s,\ k_{z,\,\overline{\alpha}}\big\rangle = i^{n^2} \int_{\mathfrak{M}} H^*\big(s,\ k_{z,\,\overline{\alpha}}\big)$$

where

$$k_{z,\,\overline{\alpha}}\big|_{U_\beta} = \overline{K_{\alpha\overline{\beta}}\,(z,\ \cdot\,)}\,\sigma_\beta \otimes\ dz^1_\beta \wedge \cdots \wedge dz^n_\beta\,,$$

$$H^*\big(s,\ k_{z,\,\overline{\alpha}}\big) = K_{\alpha\overline{\beta}}(z,\ \cdot\,)\ \Psi_\beta\, \gamma_\beta\, dz^1_\beta \wedge \cdots \wedge dz^n_\beta \wedge d\overline{z}^1_\beta \wedge \cdots \wedge d\overline{z}^n_\beta\,.$$

Let $\Omega \subset \mathbb{C}^n$ be an open set. A *weight* on Ω is a Lebesgue measurable function $\gamma : \Omega \to (0,\ +\infty)$. The set of all weights on Ω is denoted by $W(\Omega)$ (a Banach manifold modeled on $L^\infty(\Omega,\ \mathbb{R})$, cf. [32]). Let $L^2(\Omega,\ \gamma)$ consist of all measurable functions $\Psi : \Omega \to \mathbb{C}$ such that $\|\Psi\|_\gamma < \infty$ where

$$(\Psi,\ \Phi)_\gamma = \int_\Omega \Psi(z)\, \overline{\Phi(z)}\, \gamma(z)\, d\,\mu(z), \quad \|\Psi\|_\gamma = (\Psi,\ \Psi)^{1/2}_\gamma\,,$$

(μ is the Lebesgue measure on $\mathbb{R}^{2n}$) and let us set $L^2H(\Omega,\ \gamma) = \mathscr{O}(\Omega) \cap L^2(\Omega,\ \gamma)$. A weight $\gamma \in W(\Omega)$ is *admissible* if i) the evaluation functional $\delta_z : L^2H(\Omega,\ \gamma) \to \mathbb{C}$, $\delta_z(\Psi) = \Psi(z)$ and ii) $L^2H(\Omega,\ \gamma)$ is a closed subspace of $L^2(\Omega,\ \gamma)$. Let $AW(\Omega)$ be the set of all admissible weights (an open subset of $W(\Omega)$, cf. [32]). If $\mathfrak{M} = \Omega \subset \mathbb{C}^n$, $\phi : E \simeq \Omega \times \mathbb{C}$ (a vector bundle isomorphism), σ_0 is the (globally defined) holomorphic frame $\sigma_0(z) = \phi^{-1}(z,\ 1)$, and $\gamma = H(\sigma_0\,,\ \sigma_0) \in AW(\Omega)$, then

$$H^0\big(\Omega, L^2\,\mathscr{O}\big(T^{*\,(n,0)}(\Omega)\otimes E\big)\big) \simeq L^2 H(\Omega\,,\,\gamma)$$

(a Hilbert space isomorphism). When $\gamma = H\big(\sigma_0\,,\sigma_0\big)$ satisfies $\gamma^{-a} \in L^1(\Omega)$ for some $a > 0$, a simple proof to Gawedzki's lemma (2) was given by Z. Pasternak-Winiarski, [31]. As a remarkable feature of Pasternak-Winiarski's proof, it relies on the relationship between holomorphic and subharmonic functions, rather than a power series argument.[1] Indeed, for each $z \in \Omega$, let $r > 0$ such that $B_{2r}(z) \subset \Omega$. By Corollary 2.1.15 in [26], p. 75, if $\Psi \in \mathscr{O}(\Omega)$, then $|\Psi|^P$ is subharmonic for every $P > 0$. Let $p = (1+a)/a > 1$ and $P = 2/p$. Then, for every $\Psi \in L^2 H(\Omega,\,\gamma)$ and every $\zeta \in B_r(z)$

$$|\Psi(\zeta)|^{2/p} \leq \frac{1}{\text{Vol}\,\big[B_r(\zeta)\big]}\int_{B_r(\zeta)} |\Psi(w)|^{2/p}\,d\mu(w) \leq$$

(by Hölder's inequality with $1/p + 1/q = 1$, hence $q = 1+a$)

$$\leq \frac{1}{\text{Vol}\,\big[B_r(\zeta)\big]}\Big(\int_{B_r(\zeta)} |\Psi|^2\,\gamma\,d\mu\Big)^{1/p}\Big(\int_{B_r(\zeta)} \gamma^{-q/p}\,d\mu\Big)^{1/q}$$

yielding

$$|\Psi(\zeta)| \leq C\,\|\Psi\|_\gamma\,,\quad C = \frac{1}{\text{Vol}\,\big[B_r(\zeta)\big]^{(1+a)/(2a)}}\Big(\int_\Omega \gamma^{-a}\,d\mu\Big)^{1/(2a)}.$$

The *coherent state map* is

$$\mathscr{K} : \mathfrak{M} \to \mathbb{CP}(\mathscr{M}),\quad \mathscr{K}(z) = \big[k_{z,\,\overline{\alpha}}\big],\quad z \in \mathfrak{M},$$

where $\alpha \in I$ is picked up such that $z \in U_\alpha$ and $[s]$ denotes the projective ray through $s \in \mathscr{M}\setminus\{0\}$. See also [5, 6]. Let us consider the globally defined $(0, 2)$-tensor field g on $\mathfrak{M}$ such that

$$g\big|_{U_\alpha} = \sum_{j,k=1}^{n} \frac{\partial^2 K_{\alpha\overline{\alpha}}(z,\,z)}{\partial z^j_\alpha\,\partial\overline{z}^k_\alpha}\,dz^j_\alpha \odot d\overline{z}^k_\alpha\,,\quad \alpha \in I.$$

By a foundational result due to A. Odzijewicz [30], the coherent state map $\mathscr{K}$ is an anti-holomorphic embedding if and only if $\mathscr{K}$ is one to one and g is positive definite. If this is the case, g is a Kählerian metric (the *Bergman metric* of $\mathfrak{M}$) so that classical states of a mechanical system whose phase space is $\mathfrak{M}$ may be quantized (within the

[1] The typical proof of (2) in the case $\gamma \equiv 1$ (leading to the Bergman kernel of Ω, cf., e.g., [23]) is to represent $\Psi \in \mathscr{O}(\Omega)$ in power series in a polidisc $P(z,\,\epsilon) \subset \Omega$, and profit from the fact that monomials of the form $(\zeta - z)^\alpha$, $\alpha \in \mathbb{Z}^n_+$, are mutually orthogonal in $L^2\big(P(z,\,\epsilon)\big)$.

quantization scheme proposed by A. Odzijewicz, cf. *op. cit.*) only when $\mathfrak{M}$ meets the topological requirements needed to support globally defined Kählerian metrics. See also [29]. Given a set E, let $\mathscr{F}(E)$ denote the space of all functions $f : E \to \mathbb{C}$. Let

$$\mathscr{H} = H^0\big(\mathfrak{M},\ C^\infty\big(T^{*\,(n,0)}(\mathfrak{M}) \otimes E\big)\big)$$

[a Hilbert space with the inner product (1)]. For every $\alpha \in I$, we consider

$$T_\alpha : \mathscr{H} \to \mathscr{F}\big(U_\alpha\big), \quad \big(T_\alpha\, s\big)(z) = \langle s,\ k_{z,\overline{\alpha}}\rangle\,, \quad s \in \mathscr{H}, \quad z \in U_\alpha\,.$$

By a result of A. Odzijewicz (cf. [30]) $T_\alpha s \in \mathscr{O}\big(U_\alpha\big)$ for every $\alpha \in I$ if and only if $\mathscr{K} : \mathfrak{M} \to \mathbb{CP}(\mathscr{M})$ is anti-holomorphic. For every $s \in \mathscr{H}$, the holomorphic functions $T_\alpha s$ glue up to a unique holomorphic section Ts in $T^{*\,(n,0)}(\mathfrak{M}) \otimes E$ such that $(Ts)\big|_{U_\alpha} = T_\alpha s$ for any $\alpha \in I$, thus yielding a linear operator:

$$T : \mathscr{H} \to H^0\big(\mathfrak{M}, L^2\ \mathscr{O}\big(T^{*\,(n,0)}(\mathfrak{M}) \otimes E\big)\big)$$

(the *Bergman-Gawedzki-Odzijewicz projection*). Note that $Ts = s$, i.e., T reproduces the square integrable E-valued holomorphic n-forms. The kernel $\mathscr{N}(T)$ is a closed subspace of $\mathscr{H}$; hence, the range $\mathscr{R}(T)$ may be organized[2] as a Hilbert space with the inner product:

$$\langle s^1\,,\ s^2\rangle_{\mathscr{R}(T)} = \langle P\sigma^1\,,\ P\sigma^2\rangle\,, \quad \sigma^j \in T^{-1}\big(s^j\big), \quad j \in \{1,\ 2\},$$

where $P : \mathscr{H} \to \mathscr{H} \ominus \mathscr{N}(T)$ is the orthogonal projection. Then

$$\|s\|_{\mathscr{R}(T)} = \inf\Big\{\|\sigma\|_{\mathscr{H}}\ :\ \sigma \in T^{-1}(s)\Big\}.$$

Cf. also [2].

2 Kählerian Structure of $\mathbb{CP}(\mathscr{M})$

Let $\mathscr{M}$ be a separable Hilbert space. The present section reviews the complex structure J and the Fubini-Study metric g on the complex projective space $\mathscr{P} = \mathbb{CP}(\mathscr{M}) = \big(\mathscr{M} \setminus \{0\}\big)/\mathbb{C}^*$ (cf. [35]). For every point $p \in \mathscr{P}$, let Φ be a normalized representative of p, i.e., $p = [\Phi]$ and $\|\Phi\| = 1$. One sets

$$V_{[\Phi]}^{\perp} = \big\{\Psi \in \mathscr{M}\ :\ \big\langle\Phi,\ \Psi\big\rangle = 0\big\}, \quad \tilde{U}_{[\Phi]} = \mathscr{M} \setminus V_{[\Phi]}^{\perp}\,,$$

[2] By a result of S. Saitoh, [34].

$$\tilde{\varphi}_\Phi : \tilde{U}_{[\Phi]} \to V_{[\Phi]}^\perp, \quad \tilde{\varphi}_\Phi(\Psi) = \frac{1}{\langle \Psi,\ \Phi \rangle}\,\Psi - \Phi.$$

$V_{[\Phi]}^\perp$ is a closed subspace of $\mathscr{M}$ and hence a complex Hilbert space itself. If $\pi : \mathscr{M} \setminus \{0\} \to \mathscr{P}$ is the projection, let us set $U_{[\Phi]} = \pi\big(\tilde{U}_{[\Phi]}\big) \subset \mathscr{P}$. Then, $\tilde{\varphi}_\Phi$ descends to a bijection $\varphi_\Phi : U_{[\Phi]} \to V_{[\Phi]}^\perp$, so that $\big\{\big(U_{[\Phi]}\,,\ \varphi_\Phi\big) : \|\Phi\| = 1\}$ is a C^∞ atlas on $\mathscr{P}$. Given a point $p \in \mathscr{P}$, $p = [\Phi]$ with $\|\Phi\| = 1$, $(U_p\,,\ \varphi_\Phi)$ is a local chart about p, with the model space $V_p^\perp$. Let us consider the ordinary identification of the tangent space $T_p(\mathscr{P})$ with the model space of the local chart $(U_p\,,\ \varphi_\Phi)$ i.e.,

$$\varphi_p^{\mathscr{P}} : T_p(\mathscr{P}) \to V_p^\perp\,, \quad \varphi_p^{\mathscr{P}}\big(\dot{\gamma}\big) = d_0\big(\varphi_\Phi \circ \gamma\big)1.$$

The almost complex structure J on $\mathscr{P}$ is given by

$$J_p(w) = \big(\varphi_p^{\mathscr{P}}\big)^{-1}\Big[i\,\varphi_p^{\mathscr{P}}(w)\Big], \quad w \in T_p(\mathscr{P}), \quad i = \sqrt{-1}.$$

Let $J^{\mathbb{C}}$ be the $\mathbb{C}$-linear extension of J to $T(\mathscr{P}) \otimes \mathbb{C}$, so that Spec $\big(J^{\mathbb{C}}\big) = \{\pm i\}$. Let then

$$T^{1,0}(\mathscr{P})_p = \text{Eigen}\big(J^{\mathbb{C}}\,;\ +i\big) = \big\{X - i\,J_p\,X \ :\ X \in T_p(\mathscr{P})\big\}.$$

Let $\mathscr{F}_p(\mathscr{P})$ be the set of all C^∞ functions $f : U \to \mathbb{C}$, defined on some open neighborhood $U \subset \mathcal{P}$ of p. $\mathscr{F}_p(\mathscr{P})$ is a complex algebra, in a natural manner, except for the uniqueness of units. Let $\text{Der}_p(\mathscr{P})$ be the space of derivations $v : \mathscr{F}_p(\mathscr{P}) \to \mathbb{C}$ i.e. i) v is $\mathbb{C}$-linear, ii) $v(fg) = v(f)\,g(p) + f(p)\,g(v)$, and iii) for any open set $U \subset \mathscr{P}$ with $p \in U$, the function $v : C^\infty(U,\ \mathbb{C}) \to \mathbb{C}$ is continuous [with respect to the locally convex topology of $C^\infty(U,\ \mathbb{C})$ as a Fréchet space]. As well as in finite-dimensional manifold theory, the useful interpretation of $T_p(\mathscr{P})$ as the space of derivations on the algebra $\mathscr{F}_p(\mathscr{P})$ comes from the $\mathbb{C}$-linear isomorphism:

$$F_p : T_p(\mathscr{P}) \to \text{Der}_p(\mathscr{P}),$$
$$F_p(\dot{\gamma}) = v, \quad v(f) = (f \circ \gamma)'(0) \in \mathbb{C}, \quad f \in \mathscr{F}_p(\mathscr{P}).$$

Let $S = \{\Phi \in \mathscr{M} \ :\ \|\Phi\| = 1\}$ be the sphere in $\mathscr{M}$. Let $\Phi \in S$ and let $\{\Phi_\nu\}_{\nu \geq 0}$ be a complete orthonormal system in $\mathscr{M}$, adapted to the decomposition $\mathscr{M} = V_{[\Phi]}^\perp \oplus \mathbb{C}\,\Phi$ i.e. $\Phi_0 = \Phi$ and $\{\Phi_\nu\}_{\nu \geq 1}$ is a complete orthonormal system in $V_{[\Phi]}^\perp$. Every $\Psi \in \mathscr{M}$ decomposes as $\Psi = \sum_{\nu=0}^\infty a_\nu\,\Phi_\nu$, where $a_\nu = \langle \Psi\,,\ \Phi_\nu \rangle$, with convergence in the $\mathscr{M}$ norm. Let

$$w_\nu : \mathscr{M} \to \mathbb{C}, \quad w_\nu(\Psi) = \langle \Psi,\ \phi_\nu \rangle\,, \quad \Psi \in \mathscr{M}\,,$$

[the "complex coordinates" on $\mathscr{M}$ (associated to the complete orthonormal system $\{\Phi_\nu\}_{\nu\geq 0}$)]. For any function $f : A \to \mathbb{C}$ defined on an open subset $A \subset V_p^\perp$, any point $\Psi_0 \in A$, and any direction $\Psi \in V_p^\perp$, $\|\Psi\| = 1$, the directional derivative $(\partial f/\partial \Psi)(\Psi_0)$ is customarily:

$$\frac{\partial f}{\partial \Psi}(\Psi_0) = \lim_{t\to 0} \frac{1}{t}\Big\{ f\big(\Psi_0 + t\,\Psi\big) - f\big(\Psi_0\big)\Big\}.$$

Next, for every element $f : U \to \mathbb{C}$ of the algebra $\mathscr{F}_p(\mathscr{P})$, and for each point $q \in U \cap U_p \subset \mathscr{P}$ let

$$\left(\frac{\partial}{\partial \Phi_\nu}\right)_q \in \mathrm{Der}_q(\mathscr{P}), \quad \left(\frac{\partial}{\partial \Phi_\nu}\right)_q (f) = \frac{\partial (f \circ \varphi_\Phi^{-1})}{\partial \Phi_\nu}\big(\varphi_\Phi(q)\big). \tag{3}$$

Let $\Phi \in S$ and set $p = [\Phi] \in \mathscr{P}$. For every derivation $v \in \mathrm{Der}_p(\mathscr{P})$, the series

$$\sum_{\nu=1}^{\infty} v\big(w_\nu \circ \varphi_\Phi\big)\left(\frac{\partial}{\partial \Phi_\nu}\right)_p \tag{4}$$

converges in $\mathrm{Der}_p(\mathscr{P})$ and its sum is v. Let $U \subset \mathscr{M}$ be an open subset and let E be a complex Banach space. Let $f : U \to E$ be a $\mathbb{R}$-differentiable function. If $a \in U$, then let $Df(a) \in E$ be the differential of f at a. Next, we set

$$D'' f(a)(\Psi) := \frac{1}{2}\Big\{ Df(a)(\Psi) + i\, Df(a)(i\,\Psi)\Big\}, \quad \Psi \in \mathscr{M}.$$

As $Df : U \to \mathscr{L}(\mathscr{M},\, E)$

$$D'' f(a)(\Psi) = \sum_{\nu=0}^{\infty} \overline{w_\nu(\Psi)}\, \frac{\partial f}{\partial \overline{w}_\nu}(a)$$

where

$$\frac{\partial f}{\partial \overline{w}_\nu}(a) := \frac{1}{2}\left\{ \frac{\partial f}{\partial \Phi_\nu}(a) + i\, \frac{\partial f}{\partial (i\,\Phi_\nu)}(a)\right\} \in E.$$

We adopt the conventions and notations in [28]. $f : U \to E$ is *holomorphic* (cf. [28], p. 33) if for every $a \in U$, there is a ball $B_r(a) \subset U$ and a sequence of polynomials $P_m \in \mathscr{P}\big({}^m\mathscr{H},\, E\big)$ such that

$$f(\Psi) = \sum_{m=0}^{\infty} P_m(\Psi - a)$$

uniformly for $\Psi \in B_r(a)$. The common and useful characterization is (cf. Corollary 13.17 in [28], p. 108) $f : U \to E$ is holomorphic if and only if f is $\mathbb{R}$-differentiable and $D'' f = 0$. The theory in [28] is built for functions $f : U \to E$ defined on open subsets $U \subset \mathscr{M}$ where $\mathscr{M}$ is a complex Banach space. In the case at hand [i.e., $\mathscr{M}$ is a complex Hilbert space and a complete orthonormal system $\{\Phi_\nu\}_{\nu \geq 0}$ in $\mathscr{M}$ has been fixed], holomorphy is also equivalent to

$$\frac{\partial f}{\partial \overline{w}_\nu}(a) = 0, \quad a \in U, \quad \nu \geq 0.$$

Let us illustrate the calculus we introduced [an analog to the classical finite-dimensional Wirtinger calculus (cf. [36]) on the complex Hilbert space $\mathscr{M}$] by showing that the projection $\pi : \mathscr{M} \setminus \{0\} \to \mathscr{P}$ is a holomorphic mapping. The target space is an infinite-dimensional manifold [the infinite-dimensional complex projective space $\mathscr{P} = \mathbb{CP}(\mathscr{M})$] so for every $\Psi \in \mathscr{M} \setminus \{0\}$ one should prove that the local representation of π with respect to the local chart $\big(U_p\,,\,\varphi_\Phi\big)$ [with $\Phi = (1/\|\Psi\|)\,\Psi \in S$ and $p = \pi(\Phi)$] is a (vector-valued) holomorphic function. See also [13, 15]. Said local representation is $\tilde{\varphi}_\Phi : \tilde{U}_p \to V_p^\perp$, which is clearly $\mathbb{R}$-differentiable. Moreover, for every $\Psi_0 \in \tilde{U}_p$

$$\frac{\partial \tilde{\varphi}_\Phi}{\partial \Phi_\nu}(\Psi_0) = \lim_{s \to 0} \frac{1}{s}\Big\{\tilde{\varphi}_\Phi\big(\Psi_0 + s\,\Phi_\nu\big) - \tilde{\varphi}_\Phi\big(\Psi_0\big)\Big\}$$
$$= \begin{cases} \dfrac{1}{\langle \Psi_0\,,\,\Phi\rangle}\,\Phi_\nu & \text{if } \nu \geq 1, \\[2ex] \dfrac{1}{\langle \Psi_0\,,\,\Phi\rangle^2}\Big[\langle \Psi_0\,,\,\Phi\rangle\,\Phi - \Psi_0\Big] & \text{if } \nu = 0, \end{cases}$$

and a similar calculation shows that

$$\frac{\partial \tilde{\Phi}_\Phi}{\partial (i\,\Phi_\nu)}\big(\Psi_0\big) = i\,\frac{\partial \tilde{\varphi}_\Phi}{\partial \Phi_\nu}\big(\Psi_0\big)$$

i.e., $D''\tilde{\varphi}_\Phi\big(\Psi_0\big) = 0$. Q.e.d.

Let us recall the Fubini-Study metric $h = h_{\text{F-S}}$ on $\mathscr{P}$, i.e.,

$$h_p(X,\,Y) = \frac{1}{2}\,\mathrm{Re}\,\big\langle X - i\,J_p\,X,\,Y - i\,J_p\,Y\big\rangle_p\,, \quad X,\,Y \in T_p(\mathscr{P}), \tag{5}$$

$$\langle Z,\,W\rangle_p = \frac{1}{4}\big\langle \varphi_p^{\mathscr{P}}(Z),\,\varphi_p^{\mathscr{P}}(W)\big\rangle_{\mathscr{M}}\,, \quad Z,\,W \in T_p(\mathscr{P}) \otimes_{\mathbb{R}} \mathbb{C}\,. \tag{6}$$

Previous to Definition (6), one extends $\varphi_p^{\mathscr{P}} : T_p(\mathscr{P}) \to V_p^\perp$, by $\mathbb{C}$-linearity, to a map $\varphi_p^{\mathscr{P}} : T_p(\mathscr{P}) \otimes_{\mathbb{R}} \mathbb{C} \to V_p^\perp$. Note that

$$\langle Z,\ W\rangle_p = \left\langle \varphi_p^{\mathscr{P}}(X),\ \varphi_p^{\mathscr{P}}(Y)\right\rangle_{\mathscr{M}},\quad Z = X - i\,J_p\,X,\quad W = Y - i\,J_p\,Y,$$

and then

$$h_p(X,\ Y) = \frac{1}{2}\ \mathrm{Re}\left\langle \varphi_p^{\mathscr{P}}(X),\ \varphi_p^{\mathscr{P}}(Y)\right\rangle_{\mathscr{M}}.$$

The Hermitian form $\langle\cdot,\ \cdot\rangle_p$ may also be recovered as follows (cf., e.g., [35]). Given two tangent vectors $X,\ Y \in T_p(\mathscr{P})$, let $p_X,\ p_Y \in \mathscr{C}_p$ be two C^1 curves $(-\epsilon,\ \epsilon) \to \mathscr{P}$ such that $X = \dot{p}_X$ and $Y = \dot{p}_Y$. Let $\Phi \in S$ such that $p = [\Phi]$ and let $\Psi_X,\ \Psi_Y : (-\epsilon,\ \epsilon) \to S$ be, respectively, lifts of p_X and p_Y such that $\Psi_X(0) = \Phi$ and $\Psi_Y(0) = \Phi$. Also, let us set

$$V := \dot{\Psi}_X,\quad W := \dot{\Psi}_Y,\quad V,\ W \in T_\Phi(S).$$

Then

$$\langle\varphi_p^{\mathscr{P}}(X),\ \varphi_p^{\mathscr{P}}(Y)\rangle_{\mathscr{H}} = 4\Big\{\langle\varphi_\Phi^S(V),\ \varphi_\Phi^S(W)\rangle_{\mathscr{H}} - \left\langle\varphi_\Phi^S(V),\ \Phi\right\rangle_{\mathscr{H}}\left\langle\Phi,\ \varphi_\Phi^S(W)\right\rangle_{\mathscr{H}}\Big\}$$

where $\varphi_\Phi^S : T_\Phi(S) \to E_\Phi$ is the natural identification of the tangent space to the sphere with the model space of the local chart

$$\chi_{-\Phi} : S \setminus \{-\Phi\} \to E_\Phi,$$

$$\chi_{-\Phi}(\Psi) = \frac{1}{1 + \mathrm{Re}\,\langle\Psi\,,\ \phi\rangle_{\mathscr{H}}}\Big[\Psi - \mathrm{Re}\,\langle\Psi\,,\ \Phi\rangle_{\mathscr{H}}\,\Phi\Big],$$

$$E_\Phi = \big\{\psi \in \mathscr{H}\ :\ \mathrm{Re}\,\langle\Psi,\ \Phi\rangle_{\mathscr{H}} = 0\big\}.$$

The tangent space $T_p(\mathscr{P})$ at a point $p \in \mathscr{P}$ is organized in a natural manner as a complex Hilbert space [*via* the isomorphism $\varphi_p^{\mathscr{P}} : T_p(\mathscr{P}) \simeq V_p^{\perp}$]. If we set

$$V_\nu \equiv F_p^{-1}\left(\frac{\partial}{\partial\Phi_\nu}\right)_p,\quad \nu \geq 1,$$

then $\big\{V_\nu\ :\ \nu \geq 1\big\}$ is a complete orthonormal system in $T_p(\mathscr{P})$. Indeed, the scalar product on $T_p(\mathscr{P})$ is

$$\langle V,\ W\rangle = \left\langle\varphi_\Phi^{\mathscr{P}}V\,,\ \varphi_\Phi^{\mathscr{P}}W\right\rangle_{\mathscr{H}},\quad V,\ W \in T_p(\mathscr{P}). \tag{7}$$

For each fixed $\nu \geq 1$, we consider the curve $\gamma_\nu(s) = \varphi_\Phi^{-1}\big(s\,\Phi_\nu\big)$, $|s| < \epsilon$, so that $\gamma_\nu \in \mathscr{C}_p$. Then, $F_p\big(\dot\gamma_\nu\big) = \Big(\dfrac{\partial}{\partial \Phi_\nu}\Big)_p$ and $\big(\varphi_\Phi \circ \gamma_\nu\big)(s) = s\,\Phi_\nu$ yielding $\varphi_p^{\mathscr{P}}\big(\dot\gamma_\nu\big) = d_0\big(\varphi_\Phi \circ \gamma_\nu\big)\,1 = \Phi_\nu$, and hence, $\langle V_\nu\,,\ V_\mu\rangle = \delta_{\nu\mu}$. The completeness of $\{\Phi_\nu\ :\ \nu \geq 1\}$ in $V_p^\perp$ implies that of $\{V_\nu\ :\ \nu \geq 1\}$. Q.e.d.

3 A Lichnerowicz-Type Homotopy Formula

Let $\mathfrak{M}$ be a complex n-dimensional Hermitian manifold, with perhaps nonempty boundary $\partial\mathfrak{M}$, equipped with the Hermitian metric g. Let $(J,\ h)$ be the complex structure and the Fubini-Study metric on $\mathscr{P} = \mathbb{CP}(\mathscr{M})$. Let $\mathscr{K}\ :\ \mathfrak{M} \to \mathbb{CP}(\mathscr{M})$ be a C^1 map, and let $d\mathscr{K}\ :\ T(\mathfrak{M}) \to T\big(\mathscr{P}\big)$ be the tangent map. Let us consider the maps:

$$\partial\mathscr{K}\ :T^{1,0}(\mathfrak{M}) \to T^{1,0}(\mathscr{P}),\ \ \partial\overline{\mathscr{K}}\ :T^{1,0}(\mathfrak{M}) \to T^{0,1}(\mathscr{P}),$$
$$\overline{\partial}\mathscr{K}\ :T^{0,1}(\mathfrak{M}) \to T^{1,0}(\mathscr{P}),\ \ \overline{\partial}\,\overline{\mathscr{K}}\ :T^{0,1}(\mathfrak{M}) \to T^{0,1}(\mathscr{P}),$$

defined by

$$\begin{array}{ccccc}
T^{0,1}(\mathscr{P}) & \stackrel{\partial\overline{\mathscr{K}}}{\longleftarrow} & T^{1,0}(\mathfrak{M}) & \stackrel{\partial\mathscr{K}}{\longrightarrow} & T^{1,0}(\mathscr{P}) \\
\Pi^{0,1}\uparrow & & j^{1,0}\downarrow & & \uparrow\Pi^{1,0} \\
T(\mathscr{P})\otimes\mathbb{C} & \stackrel{(d\mathscr{K})^{\mathbb{C}}}{\longleftarrow} & T(\mathfrak{M})\otimes\mathbb{C} & \stackrel{(d\mathscr{K})^{\mathbb{C}}}{\longrightarrow} & T(\mathscr{P})\otimes\mathbb{C} \\
\Pi^{0,1}\downarrow & & j^{0,1}\uparrow & & \downarrow\Pi^{1,0} \\
T^{0,1}(\mathscr{P}) & \stackrel{\overline{\partial}\,\overline{\mathscr{K}}}{\longleftarrow} & T^{0,1}(\mathfrak{M}) & \stackrel{\overline{\partial}\mathscr{K}}{\longrightarrow} & T^{1,0}(\mathscr{P})
\end{array}$$

A C^1 map $\mathscr{K}\ :\mathfrak{M} \to \mathscr{P}$ is holomorphic (respectively, anti-holomorphic) if $\overline{\partial}\mathscr{K} = 0$ (respectively, if $\partial\mathscr{K} = 0$). Next, let

$$\|\partial\mathscr{K}\| : \mathfrak{M} \to [0,\ +\infty),\quad \|\overline{\partial}\mathscr{K}\| : \mathfrak{M} \to [0,\ +\infty),$$

be defined as follows. Let $z \in \mathfrak{M}$ and let $\{V_j\ :\ 1 \leq j \leq n\}$ be a local frame in the holomorphic tangent bundle $T^{1,0}(\mathfrak{M})$, defined on an open neighborhood $U \subset \mathfrak{M}$ of z, such that

$$g\big(V_j\,,\ V_{\overline{k}}\big) = \delta_{jk}\,,\quad V_{\overline{k}} = \overline{V_k}\,,\quad 1 \leq j,\ k \leq n.$$

Then

$$\|\partial\mathscr{K}\|_z^2 = \sum_{j=1}^{n} h^{\mathscr{K}}\big((\partial\mathscr{K})V_j\,,\,\overline{(\partial\mathscr{K})V_j}\big)_z\,,$$

$$\|\overline{\partial}\mathscr{K}\|_z^2 = \sum_{j=1}^{n} h^{\mathscr{K}}\big((\overline{\partial}\mathscr{K})V_j\,,\,\overline{(\overline{\partial}\mathscr{K})V_j}\big)_z\,.$$

Here, $h^{\mathscr{K}} = \mathscr{K}^{-1}h$ is the bundle metric induced by h on the pullback bundle $\mathscr{K}^{-1}T(\mathscr{P}) \to \mathfrak{M}$. For every C^1 map $\mathscr{K} : \mathfrak{M} \to \mathscr{P}$

$$\mathrm{Tr}_g\left(\mathscr{K}^*h\right) = 2\Big(\big\|\partial\mathscr{K}\big\|^2 + \big\|\overline{\partial}\mathscr{K}\big\|^2\Big).$$

Let $D \subset\subset \mathfrak{M}$ be a relatively compact domain and let us consider the functionals:

$$E_D\,,\ E'_D\,,\ E''_D : C^1\big(\mathfrak{M},\ \mathscr{P}\big) \to \mathbb{R}\,,$$

$$E_D(\mathscr{K}) = \frac{1}{2}\int_D \mathrm{Tr}_g\left(\mathscr{K}^*h\right)\, d\ \mathrm{v}_g\,,$$

$$E'_D(\mathscr{K}) = \int_D \big\|\partial\,\mathscr{K}\big\|^2\, d\ \mathrm{v}_g\,, \quad E''_D(\mathscr{K}) = \int_D \big\|\overline{\partial}\mathscr{K}\big\|^2\, d\ \mathrm{v}_g\ .$$

Then

$$\mathscr{K} \ \text{ is holomorphic} \Longleftrightarrow E''_D(\mathscr{K}) = 0 \ \text{ for any } \ D \subset\subset \mathfrak{M},$$

$$\mathscr{K} \ \text{ is anti-holomorphic} \Longleftrightarrow E'_D(\mathscr{K}) = 0 \ \text{ for any } \ D \subset\subset \mathfrak{M}.$$

A C^1 map $\mathscr{K} : \mathfrak{M} \to \mathscr{P} = \mathbb{CP}(\mathscr{M})$ is *harmonic* if

$$\frac{d}{dt}\Big\{E_D\big(\mathscr{K}_t\big)\Big\}_{t=0} = 0$$

for any relatively compact domain $D \subset\subset \mathfrak{M}$, and any smooth 1-parameter variation $\big\{\mathscr{K}_t\big\}_{|t|<\epsilon} \subset C^1(\mathfrak{M},\ \mathscr{P})$ of $\mathscr{K}_0 = \mathscr{K}$ such that $\mathrm{Supp}(V) \subset D$, where $V \in C\big(\mathscr{K}^{-1}T\,\mathscr{P}\big)$ is the infinitesimal variation induced by $\big\{\mathscr{K}_t\big\}_{|t|<\epsilon}$, i.e.,

$$V(z) = \big(d_{(z,0)}\mathbf{K}\big)\left(\frac{\partial}{\partial t}\right)_{(z,0)}, \quad \mathbf{K}(z,t) = \mathscr{K}_t(z), \quad z \in \mathfrak{M}, \quad |t| < \epsilon\ .$$

Cf. [14] for the basic notions and results in harmonic maps theory, in the finite-dimensional setting. Let h be an arbitrary Kählerian metric on $\mathscr{P}$ and let Ω be its Kählerian 2-form. Let $\mathscr{K} : \mathfrak{M} \to \mathscr{P}$ be a C^2 map, and let us set

$$\ell(\mathscr{K}) = \|\partial\mathscr{K}\|^2 - \|\overline{\partial}\mathscr{K}\|^2\,, \quad L_D(\mathscr{K}) = \int_D \ell(\mathscr{K})\, d\, \mathrm{v}_g\ .$$

Theorem 1 *Let $\mathfrak{M}$ be a compact complex n-dimensional Hermitian manifold-with-boundary $\partial\mathscr{M}$, equipped with the Hermitian metric g, and let $\mathscr{K}_t : \mathfrak{M} \to \mathscr{P}$, $0 \leq t \leq 1$, be a smooth 1-parameter family of C^2 maps, and let us consider the 1-parameter family of 1-forms on $\mathfrak{M}$:*

$$\alpha_{\mathbf{K}}(\tau) = \int_0^\tau \mathscr{K}_t^*\Big(\frac{\partial\mathscr{K}_t}{\partial t} \rfloor \Omega\Big)\, d\,t \in \Omega^1(\mathfrak{M}), \quad 0 \leq \tau \leq 1. \tag{8}$$

(i) If g is a locally conformal Kähler metric, with the fundamental 2-form $\Omega^{\mathfrak{M}}$ and the Lee form ω, then

$$L\big(\mathscr{K}_\tau\big) - L\big(\mathscr{K}_0\big) = \int_{\partial\mathfrak{M}} \beta_{\mathbf{K}}(\tau) + (n-1)\int_{\mathfrak{M}} \omega \wedge \beta_{\mathbf{K}}(\tau), \tag{9}$$

for any $0 \leq \tau \leq 1$, where

$$\beta_{\mathbf{K}}(\tau) = \frac{1}{(n-1)!}\,\alpha_{\mathbf{K}}(\tau) \wedge \big(\Omega^{\mathfrak{M}}\big)^{n-1}\,.$$

(ii) If g is Kählerian, then

$$L\big(\mathscr{K}_\tau\big) - L\big(\mathscr{K}_0\big) = \int_{\partial\mathscr{M}} \beta_{\mathbf{K}}(\tau).$$

In particular, if $\mathfrak{M}$ is closed, then the map

$$t \in [0,\ 1] \longmapsto L\big(\mathscr{K}_t\big) \in \mathbb{R}$$

is constant.

(iii) Every (anti) holomorphic map $\mathscr{K} : \mathfrak{M} \to \mathscr{P}$ from a closed Kählerian manifold $\mathfrak{M}$ is an absolute minimum point of the energy functional $E : C^2(\mathfrak{M}, \mathscr{P}) \to [0, +\infty)$ within its homotopy class $[\mathscr{K}]$ and in particular $\mathscr{K}$ is a harmonic map.

Here, $L \equiv L_{\mathfrak{M}}$ and $X \rfloor \Omega$ denotes the interior product with the vector field X. To prove Theorem 1, one starts by observing that[3]

$$\langle\Omega^{\mathfrak{M}},\ \mathscr{K}^*\Omega\rangle = 2\,\ell(\mathscr{K}), \tag{10}$$

[3] The proof of (10) is straightforward. The proof of (11) is a rather involved calculation not reported here.

$$\frac{\partial}{\partial t}\left(\mathscr{K}_t^{\,*}\Omega\right) = 2\,d\left[\mathscr{K}_t^{\,*}\left(\frac{\partial \mathscr{K}_t}{\partial t}\,\rfloor\,\Omega\right)\right] \tag{11}$$

for any smooth 1-parameter family of C^2 maps $\mathscr{K}_t : \mathfrak{M} \to \mathscr{P}$, $t \in \mathbb{R}$. By (10)

$$2\,\ell(\mathscr{K}_t)\;d\;\mathrm{v}_g = \langle \Omega^{\mathfrak{M}}\,,\;\mathscr{K}_t^{\,*}\Omega\rangle\;d\;\mathrm{v}_g =$$

[by (2.7.11) in [22], p. 70]

$$= \Omega^{\mathfrak{M}} \wedge *\left(\mathscr{K}_t^{\,*}\Omega\right) =$$

[by the property (iii) in [22], p. 70, of the Hodge operator $*$ on $(\mathfrak{M},\; g)$]

$$= \left(\mathscr{K}_t^{\,*}\Omega\right) \wedge *\Omega^{\mathfrak{M}}$$

so that

$$2\left[L\left(\mathscr{K}_t\right) - L\left(\mathscr{K}_0\right)\right] = \int_{\mathfrak{M}} \left[\left(\mathscr{K}_t^{\,*}\Omega - \mathscr{K}_0^{\,*}\Omega\right) \wedge *\Omega^{\mathfrak{M}}\right]. \tag{12}$$

Let us consider the 1-parameter family of 1-forms $\left\{\alpha_{\mathbf{K}}(t)\right\}_{0\le t\le 1} \subset \Omega^1(\mathfrak{M})$ given by (8). Then [by (11)]

$$\mathscr{K}_t^{\,*}\Omega - \mathscr{K}_0^{\,*}\Omega = \int_0^t \frac{\partial}{\partial \tau}\left\{\mathscr{K}_\tau^{\,*}\Omega\right\}\,d\,\tau = d\,\alpha_{\mathbf{K}}(t).$$

Consequently [by (12)]

$$L\left(\mathscr{K}_t\right) - L\left(\mathscr{K}_0\right) = \int_{\mathfrak{M}} \left(d\;\alpha_{\mathbf{K}}(t)\right) \wedge *\,\Omega^{\mathfrak{M}}. \tag{13}$$

From now on, let g be a locally conformal Kähler (l.c.K.) metric with the Lee form $\omega \in \Omega^1(\mathfrak{M})$, so that $d\Omega^{\mathfrak{M}} = \omega \wedge \Omega^{\mathfrak{M}}$ (cf. [11]). Then

$$d\left(*\;\Omega^{\mathfrak{M}}\right) = d\left[\frac{1}{(n-1)!}\left(\Omega^{\mathfrak{M}}\right)^{n-1}\right] = \frac{1}{(n-2)!}\,d\Omega^{\mathfrak{M}} \wedge \left(\Omega^{\mathfrak{M}}\right)^{n-2} = \frac{1}{(n-2)!}\,\omega \wedge \left(\Omega^{\mathfrak{M}}\right)^{n-1},$$

$$\left(d\,\alpha_{\mathbf{K}}(t)\right) \wedge *\Omega^{\mathfrak{M}} = d\left(\alpha_{\mathbf{K}}(t) \wedge *\Omega^{\mathfrak{M}}\right) - \alpha_{\mathbf{K}}(t) \wedge d\left(*\;\Omega^{\mathfrak{M}}\right) = d\,\beta_{\mathbf{K}}(t) + (n-1)\,\omega \wedge \beta_{\mathbf{K}}(t).$$

Q.e.d.

Finally, we prove statement (iii) in Theorem 1. Let $\mathscr{K} : \mathfrak{M} \to \mathscr{P}$ be an (anti) holomorphic map, and let $\mathscr{L} : \mathfrak{M} \to \mathscr{P}$ be an arbitrary C^2 map homotopic to $\mathscr{K}$. Let $\mathbf{K} : \mathfrak{M} \times [0,\, 1] \to \mathscr{P}$ be a smooth homotopy $\mathbf{K} : \mathscr{K} \simeq \mathscr{L}$ and let us set $\mathscr{K}_t(z) = \mathbf{K}(z,\, t)$. As $E'(\mathscr{L}) \geq 0$ and $E''(\mathscr{L}) \geq 0$

$$E(\mathscr{L}) = E'(\mathscr{L}) + E''(\mathscr{L}) \geq \left| E'(\mathscr{L}) - E''(\mathscr{L}) \right| = |L(\mathscr{L})| = |L(\mathscr{K}_1)| =$$

[by (ii) in Theorem 1]

$$= |L(\mathscr{K}_0)| = |L(\mathscr{K})| = \left| E'(\mathscr{K}) - E''(\mathscr{K}) \right|$$

$$= \begin{cases} E'((\mathscr{K}) & \text{if } \mathscr{K} \text{ is holomorphic} \\ \\ E''(\mathscr{K}) & \text{if } \mathscr{K} \text{ is anti-holomorphic} \end{cases}$$

$$= E'(\mathscr{K}) + E''(\mathscr{K}) = E(\mathscr{K}).$$

In particular, $\mathscr{K}_0 = \mathscr{K}$ is a critical point of the function $t \longmapsto E(\mathscr{K}_t)$. Q.e.d.

4 Transition Probability Amplitudes

Let $z \in \mathfrak{M}$ be a classical state and let $\mathscr{K}(z) = \left[k_{z,\,\overline{\alpha}}\right] \in \mathbb{CP}(\mathscr{M})$ be the corresponding coherent state. The *transition probability amplitude* from z to ζ is

$$a_{\beta\overline{\alpha}}(\zeta,\, z) = \left\langle \frac{k_{z,\,\overline{\alpha}}}{\left\| k_{z,\,\overline{\alpha}} \right\|} \, , \, \frac{k_{\zeta,\,\overline{\beta}}}{\left\| k_{\zeta,\,\overline{\beta}} \right\|} \right\rangle$$

and $\left| a_{\beta\overline{\alpha}}(\zeta,\, z) \right|^2$ is the *transition probability density*. Therefore, the novelty brought forth by A. Odzijewicz (cf. [30]) is reducing the calculation of the transition probability amplitudes $a_{\beta\overline{\alpha}}(\zeta,\, z)$ to the calculation of the kernels $K_{\beta\,\overline{\alpha}}(\zeta,\, z)$. When $\mathfrak{M} = \Omega \subset \mathbb{C}^n$ and $E \simeq \Omega \times \mathbb{C}$, these are weighted Bergman kernels that are at least as difficult to compute as the ordinary (unweighted) Bergman kernels. Yet the explicit expression of reproducing kernels of sorts is available only for a handful of particular domains (cf., e.g., S. Krantz, [26], pp. 47-51, J.P. D'Angelo, [9], G. Francsics & N. Hanges, [20], for the unit ball $\mathbb{B}^n = \{z \in \mathbb{C}^n \,:\, |z| < 1\}$ and certain complex ovals in the unweighted case, and Z. Pasternak-Winiarski, [31], F. Forellli & W. Rudin, [19], for the unit ball $\mathbb{B}^1$ in the weighted case). See also [7, 8]. We emphasize that the construction of $\mathscr{K}$ and its use in the calculation of the transition probability amplitudes relies on a number of structural assumptions, such as the following:

(I) $\mathfrak{M}$ is sufficiently ample (cf. [30]).
(II) The pullback by $\mathscr{K}$ of the Fubini-Study metric on $\mathbb{CP}(\mathscr{M})$ is a globally defined Kählerian metric on $\mathfrak{M}$.
(III) (E, H) is a quantum bundle [i.e., the curvature form of the canonical Hermitian connection on (E, H) is a symplectic structure on $\mathfrak{M}$].
(IV) The measure on the phase space got from the data (E, H)

$$K_{\alpha\overline{\alpha}}(z, z)\,\gamma_\alpha(z)\,dz_\alpha^1 \wedge \cdots \wedge dz_\alpha^n \wedge d\overline{z}_\alpha^1 \wedge \cdots \wedge d\overline{z}_\alpha^n$$

coincides with the Liouville measure (up to a multiplicative constant).

When applied (as in [3]) to the Siegel domain $\mathfrak{M} = \Omega_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \operatorname{Im}(z_n) > |z'|^2\}$, the trivial line bundle $E = \Omega_n \times \mathbb{C}$ and the family of Hermitian metrics

$$H_a\big(\sigma_0\,,\,\sigma_0\big) = \rho^a\,, \quad a > -1, \quad \rho(z) = \operatorname{Im}(z_n) - |z'|^2\,, \quad z \in \Omega_n\,,$$

the coherent state map $\mathscr{K}_a : \Omega_n \to \mathbb{CP}\Big[L^2 H\big(\Omega_n\,,\,\gamma_a\big)\Big]$ is determined by the weighted Bergman kernel $K_{\gamma_a}(\zeta, z)$ associated to the (admissible) weight $\gamma_a \in AW(\Omega_n)$, $\gamma_a = \rho^a$, that was explicitly computed by E. Barletta and S. Dragomir (cf. [1]) by combining a very general method discovered by S. Saitoh (cf. *op. cit.*) with the use of an integral transform on the Siegel domain due to M.M. Djrbashian and A.H. Karapetyan (cf. [10]), thus allowing for the calculation of the transition probability amplitudes:

$$a_{0\overline{0}}(\zeta, z) = \left[\frac{2\,\rho(z)^{1/2}\,\rho(\zeta)^{1/2}}{i\,\big(\overline{z}_1 - \zeta_1\big) - 2\,\langle\zeta'\,,\,z'\rangle}\right]^{n+a+1}.$$

When the given mechanical system interacts with an external field B (i.e., by exploiting the approach by R. Penrose, [33], the Hermitian structure on E is deformed as $H \mapsto e^B H$), the coherent state map changes to $\mathscr{K} : \Omega_n \to \mathbb{CP}\big[L^2 H\big(\Omega_n\,,\,e^B\gamma_a\big)\big]$, and while the explicit weighted Bergman kernel $K_{e^B\gamma_a}(\zeta, z)$ isn't available, one may exploit (cf. [3]) the analyticity[4] of the map:

$$K : AW(\Omega_n) \to HA(\Omega_n), \quad \gamma \mapsto K_\gamma\,,$$

[associating to every admissible weight γ the corresponding weighted Bergman kernel K_γ regarded as an element of the complex Fréchet space $HA(\Omega)$], yielding a series development

[4] Discovered by Z. Pasternak-Winiarski, [32].

$$K_{(1+h)\gamma_a} = K_{\gamma_a} + \sum_{k=1}^{\infty} (-1)^k \, K^{(k)}_{1,\,\gamma_a}(h, \, \cdots, \, h),$$
$$h = e^B - 1 \in B_{\frac{1}{2}}(0) \subset L^\infty\big(\Omega_n \, , \, \mathbb{R}\big).$$

As emphasized by A. Odzijewicz (cf. [30]), Z. Pasternak-Winiarski's methods are effective for the case of weak external fields $\epsilon\, B$, $0 < \epsilon << 1$, and one may show (cf. again [3]) that the transition probability amplitude from z to ζ (when identified with coherent states in $\mathbb{CP}\big[L^2 H(\Omega_n \, , \, e^{\epsilon\, B}\, \gamma_a)\big]$) equals

$$\begin{aligned}
&\left[\frac{2\,\rho(z)^{1/2}\,\rho(\zeta)^{1/2}}{i\,(\bar{z}_1 - \zeta_1) - 2\,\langle \zeta' \, , \, z' \rangle}\right]^{n+a+1} \\
&\quad + \frac{8\epsilon}{(c_{n,a})^2}\,\rho(z)^{\frac{n+a+1}{2}}\,\rho(\zeta)^{\frac{n+a+1}{2}} \int_{\Omega_n} \Big\{\Big[\rho(z)^{n+a+1}\,\big|K_{\gamma_a}(w, \, z)\big|^2 \\
&\quad\quad + \rho(\zeta)^{n+a+1}\,\big|K_{\gamma_a}(\zeta, \, w)\big|^2\Big]\,K_{\gamma_a}(\zeta, \, z)] \\
&\quad\quad\quad - \frac{c_{n,a}}{2}\,K_{\gamma_a}(\zeta, \, w)\Big\}\,B(w)\,\rho(w)^a\,d\mu(w) + \mathrm{O}\left(\epsilon^2\right)
\end{aligned}$$

where

$$K_{\gamma_a}(\zeta, \, z) = \frac{2^{n-1+a}\, c_{n,a}}{\big[i\big(\bar{z}_1 - \zeta_1\big) - 2\,\langle \zeta' \, , \, z' \rangle\big]^{n+1+a}} \, ,$$
$$c_{n,a} = \pi^{-n}(a+1)\cdots(a+n).$$

A. Odzijewicz's structural assumptions (I)–(IV) in Sect. 2 were investigated in [3], and one found that for every

$$H \in \big\{H_a \, , \, e^{\epsilon B}\, H_a \; : \; B \in L^\infty(\Omega_n \, , \, \mathbb{R}), \;\; \epsilon > 0, \;\; a > -1\big\}$$

the assumptions (I)–(III) are satisfied, while the assumption (IV) is questionable and indeed only partially satisfied by the model investigated in [3]. Indeed, let $\gamma \in AW(\Omega_n) \cap C^\infty(\Omega_n)$ be a smooth admissible weight. If $H(\sigma_0 \, , \, \sigma_0) = \gamma$, then $(E, \, H)$ is a quantum bundle, and there is yet another Hermitian bundle metric $\hat{H}$ on E given by

$$\hat{H}\big(\sigma_0 \, , \, \sigma_0\big)_\zeta = \frac{(-i)^n\,\Omega_{0\bar{0}}(\gamma)_\zeta}{K_{0\bar{0}}(\zeta, \, \zeta)} \, , \quad \zeta \in \Omega_n \, ,$$
$$\Omega_{0\bar{0}}(\gamma) = \det\big[\omega_{j\bar{k}}\big], \quad \omega = \mathrm{curv}(E, \, H).$$

Both H and $\hat{H}$ are sections in the complex line bundle $E^* \otimes E^*$ (with $H \neq 0$) so their quotient is well defined, providing the map

$$F : AW(\Omega_n) \cap C^\infty(\Omega_n) \to C^\infty(\Omega_n), \quad F(\gamma) = \frac{\hat{H}}{H}.$$

As it turns out, the structural assumption (IV) is equivalent to the requirement that $F(\gamma)_\zeta$ is a constant, both with respect to γ and ζ. Instead the result in [3] is that

$$F\big(\gamma_a\big) = \frac{1}{c_{n,a}} \Big(-\frac{a}{2\pi}\Big)^n, \quad a > -1,$$

so $F(\gamma_a)$ is a constant function, while F is not.[5]

5 Boundary Behavior of Coherent State Maps

Let M be a CR manifold, of CR dimension $n-1$, equipped with the CR structure $T_{1,0}(M)$. Cf. the monograph [12] for the main conventions and notations in CR and pseudohermitian geometry. Given a C^1 map $f : M \to \mathscr{P}$, we adopt the following notations:

$$\begin{array}{ccccc}
T^{0,1}(\mathscr{P}) & \overset{\partial_b \overline{f}}{\longleftarrow} & T_{1,0}(M) & \overset{\partial_b f}{\longrightarrow} & T^{1,0}(\mathscr{P}) \\
\Pi^{0,1}\uparrow & & j_{1,0}\downarrow & & \uparrow \Pi^{1,0} \\
T(\mathscr{P})\otimes\mathbb{C} & \overset{(df)^{\mathbb{C}}}{\longleftarrow} & T(M)\otimes\mathbb{C} & \overset{(df)^{\mathbb{C}}}{\longrightarrow} & T(\mathscr{P})\otimes\mathbb{C} \\
\Pi^{0,1}\downarrow & & j_{0,1}\uparrow & & \downarrow \Pi^{1,0} \\
T^{0,1}(\mathscr{P}) & \overset{\overline{\partial}_b \overline{f}}{\longleftarrow} & T_{0,1}(M) & \overset{\overline{\partial}_b f}{\longrightarrow} & T^{1,0}(\mathscr{P})
\end{array}$$

Let $\mathscr{X}$ be a complex topological vector space, and let $U \subset M$ be an open set. A C^1 function $u : U \to \mathscr{X}$ is a (vector-valued) *CR function* if $\overline{Z}(u) = 0$ for every $Z \in T_{1,0}(M)$. A C^1 mapping $f : M \to \mathscr{P}$ is a *CR mapping* if $(d_z f)^{\mathbb{C}}\, T_{1,0}(M)_z \subset T^{1,0}(\mathscr{P})_z$ for any $z \in M$.

For every C^1 mapping $f : M \to \mathscr{P}$, of a CR manifold M into the infinite dimensional complex projective space $\mathscr{P} = \mathbb{CP}(\mathscr{M})$, the following statements are equivalent:

[5] This however suffices (cf. [3]) for recovering $a_{0\overline{0}}(\zeta, z)$ by averaging $a_{0\overline{0}}(w, z)\, a_{0\overline{0}}(\zeta, w)$ [the transition probability amplitude from z to ζ with simultaneous transition through w] over $w \in \Omega_n$.

(i) f is a CR mapping.
(ii) $\overline{\partial}_b f = 0$.
(iii) For every $z \in M$, the function

$$u : f^{-1}(U_p) \to V_p^{\perp}\,, \quad u = \varphi_{\Phi} \circ f, \quad p := \big[\Phi\big] = f(z),$$

is a (vector valued) CR function.

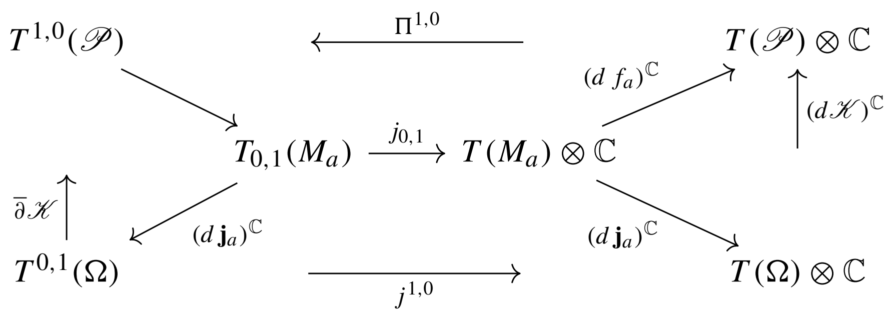
Let $\Omega \subset \mathbb{C}^n$ be a domain with smooth boundary $\partial\Omega$, and let $\mathscr{K} \in C^{\infty}\big(\overline{\Omega}\,,\ \mathscr{P}\big)$ be a solution (smooth up to the boundary) to the Dirichlet problem:

$$\overline{\partial}\,\mathscr{K} = 0 \quad \text{in} \quad \Omega, \qquad \mathscr{K} = f \quad \text{on} \quad \partial\Omega, \tag{14}$$

for a given $f \in C^{\infty}(\partial\Omega,\ \mathscr{P})$. We seek for the compatibility conditions that the boundary data f should satisfy along $\partial\Omega$. We start by looking at the case of the Siegel domain Ω_n (the case where the tangential Cauchy-Riemann equations were first discovered, cf. H. Lewy, [27]). For every $a \in \mathbb{R}$ let $M_a = \rho^{-1}(a)$, where $\rho(z) = \text{Im}(z^n) - |z'|^2$, so that $a \geq 0 \Longrightarrow M_a \subset \overline{\Omega}_n$. In particular, $M_0 = \partial\Omega_n$. Each M_a is a strictly pseudoconvex real hypersurface in $\mathbb{C}^n$, equipped with the CR structure $T_{1,0}(M_a) \equiv \text{span}\,\big\{L_k\,:\,1 \leq k \leq n-1\big\}$ where

$$\big(\mathbf{j}_a\big)_* L_k = \big(Z_k + 2\,i\,\overline{z}^k\,Z_n\big)^{\mathbf{j}_a}\,, \quad 1 \leq k \leq n-1, \quad Z_j \equiv \frac{\partial}{\partial z^j}\,, \quad 1 \leq j \leq n,$$

and $\mathbf{j}_a : M_a \to \mathbb{C}^n$ is the inclusion. Here, $Z^{\mathbf{j}_a} \equiv Z \circ \mathbf{j}_a$. Let $\mathscr{K} : \overline{\Omega}_n \to \mathscr{P}$ be a solution to the Dirichlet problem (14) with $\Omega = \Omega_n$. In particular, $\mathscr{K}$ is smooth up to the boundary, i.e., there is an open set $D \subset \mathbb{C}^n$ with $D \supset \overline{\Omega}_n$ and a C^{∞} map $\mathscr{F} : D \to \mathscr{P}$ such that $\mathscr{F}\big|_{\overline{\Omega}_n} = \mathscr{K}$. For every $a \in [0, +\infty)$, let us set $f_a := \mathscr{K} \circ \mathbf{j}_a$ where $\mathbf{j}_a = \mathbf{j}\big|_{M_a} : M_a \to \Omega_n$ and $\mathbf{j} : D \to \mathbb{C}^n$ is the inclusion (with $f_0 = f$). The base and three out of four faces of the triangular prism[6]



are commutative; hence, so does the fourth face, i.e.,

$$\overline{\partial}_b f_a = \big(\overline{\partial}\,\mathscr{K}\big) \circ \big(d\,\mathbf{j}_a\big)^{\mathbb{C}}. \tag{15}$$

[6] "Collapsed" on its base.

Let $\mathscr{K} : \Omega_n \to \mathscr{P}$ be a C^1 mapping. Let $\Phi \in S$ and $p = \pi(\Phi) \in \mathscr{P}$. Then, for every $z \in \mathscr{K}^{-1}(U_p)$ and any $1 \leq j \leq n$

$$\left(\overline{\partial}\,\mathscr{K}\right)_z \overline{Z}_{j,z} = \frac{1}{\langle\tilde{\mathscr{K}}(z),\ \Phi\rangle^2} \sum_{\nu=1}^{\infty} \left\{ \left\langle \tilde{\mathscr{K}}(z),\ \Phi \right\rangle \left\langle \frac{\partial \tilde{\mathscr{K}}}{\partial \overline{z}_j}(z),\ \Phi_\nu \right\rangle - \left\langle \tilde{\mathscr{K}}(z),\ \Phi_\nu \right\rangle \left\langle \frac{\partial \tilde{\mathscr{K}}}{\partial \overline{z}_j}(z),\ \Phi \right\rangle \right\} W_{\nu,\,\mathscr{K}(z)} \tag{16}$$

where $\tilde{\mathscr{K}} : \mathscr{K}^{-1}(U_p) \to \tilde{U}_p$ is any C^1 map such that $\mathscr{K} = \pi \circ \tilde{\mathscr{K}}$. Only the (infinite-dimensional analog to the) *homogeneous coordinates* $w_\nu : \mathscr{M} \setminus \{0\} \to \mathbb{C}$, $\nu \geq 0$, of a point in $\mathscr{P}$ were used so far. The same symbol will denote the (infinite-dimensional analog to the) *local coordinates*:

$$w_\nu : U_p \to \mathbb{C}, \quad w_\nu(q) = \left\langle \varphi_\Phi(q),\ \Phi_\nu \right\rangle, \quad q \in U_p\,, \quad \nu \geq 1,$$

that is,

$$w_\nu(q) = \frac{\left\langle \Psi\,,\ \Phi_\nu \right\rangle}{\left\langle \Psi\,,\ \Phi \right\rangle}\,, \quad q = [\Psi], \quad \nu \geq 1.$$

For any C^1 map $\mathscr{K} : \Omega_n \to \mathscr{P}$ the *local components* of $\mathscr{K}$ are the functions $\mathscr{K}^\nu : \mathscr{K}^{-1}(U_p) \to \mathbb{C}$ got as the compositions

$$\mathscr{K}^{-1}(U_p) \xrightarrow{\mathscr{K}} U_p \xrightarrow{w_\nu} \mathbb{C}\,.$$

Then

$$\frac{\partial \mathscr{K}^\nu}{\partial \overline{z}_j}(z) = \frac{1}{\langle\tilde{\mathscr{K}}(z),\ \Phi\rangle^2} \left\{ \langle \tilde{\mathscr{K}}(z),\ \Phi \rangle \left\langle \frac{\partial \tilde{\mathscr{K}}}{\partial \overline{z}_j}(z),\ \Phi_\nu \right\rangle - \langle \tilde{\mathscr{K}}(z),\ \Phi_\nu \rangle \left\langle \frac{\partial \tilde{\mathscr{K}}}{\partial \overline{z}_j}(z),\ \Phi \right\rangle \right\}$$

and (16) may be written in the more familiar form

$$\left(\overline{\partial}\mathscr{K}\right)_z \overline{Z}_{j,z} = \sum_{\nu=1}^{\infty} \frac{\partial \mathscr{K}^\nu}{\partial \overline{z}_j}(z)\ W_{\nu,\,\mathscr{K}(z)}\,. \tag{17}$$

Theorem 2 *Let* $\mathscr{K} : \overline{\Omega}_n \to \mathscr{P}$ *a solution smooth up to the boundary to the Dirichlet problem:*

$$\overline{\partial}\mathscr{K} = 0 \quad in \quad \Omega_n\,, \quad \mathscr{K} = f \quad on \quad \partial\Omega_n\,,$$

for some $f \in C^\infty\big(\partial\Omega_n\,,\ \mathscr{P}\big)$. *Then,* f *is a CR mapping, i.e.,* $\overline{\partial}_b f = 0$ *along* $\partial\Omega_n$.

Let $z_0 \in \partial\Omega_n$ be a boundary point, and let $p = \mathscr{K}(z_0) \in \mathscr{P}$. Let $\Phi \in S$ such that $p = [\Phi]$. Let $A := \mathscr{F}^{-1}\big(U_p\big) \subset D$, so that $A \cap \overline{\Omega}_n$ is a neighborhood of z_0 in (the manifold with boundary) $\overline{\Omega}_n$. Let $\{z_m\}_{m\geq 1} \subset A \cap \Omega_n$ be a sequence of points such that $z_m \to z_0$ as $m \to \infty$, and let us set $a_m = \rho(z_m)$, $m \geq 1$. The set $A \cap \Omega_n$ is foliated by (open pieces of) level sets of ρ, so that for every $m \geq 1$ the point z_m belongs to the leaf $A \cap M_{a_m}$. As $\big(\overline{\partial}\mathscr{K}\big)_{z_m} = 0$, it follows [by (17)] that $\dfrac{\partial \mathscr{K}^\nu}{\partial \overline{z}_j}(z_m) = 0$, $m \geq 1$, $\nu \geq 1$, hence [as $\mathscr{K}^\nu$ is smooth up to the boundary, i.e., $\mathscr{K}^\nu \in C^\infty\big(A \cap \overline{\Omega}_n\big)$] $0 = \lim_{m\to\infty} \dfrac{\partial \mathscr{K}^\nu}{\partial \overline{z}_j}(z_m) = \dfrac{\partial \mathscr{K}^\nu}{\partial \overline{z}_j}(z_0)$. Finally, [by (15) with $a = 0$] $\big(\overline{\partial}_b\, f\big)_{z_0} \overline{Z}_{j,\, z_0} = \sum_{\nu=1}^{\infty} \dfrac{\partial \mathscr{K}^\nu}{\partial \overline{z}_j}(z_0)\ W_{\nu\,,\,\mathscr{K}(z_0)} = 0$. Q.e.d.

6 Conclusions and Open Problems

Coherent state maps as considered through this paper are symplectic maps $\mathscr{K} : \mathfrak{M} \to \mathbb{CP}(\mathscr{M})$ of a symplectic manifold $\big(\mathfrak{M},\ \Omega^{\mathfrak{M}}\big)$ into a complex projective Hilbert space $\mathbb{CP}(\mathscr{M})$ with the Fubini-Study symplectic form Ω, i.e., the setting is that in [30] where the claimed novelty is that $\mathscr{K}$ may be used to quantize classical observables, as well, and indeed a quantization procedure (related to the Kostant-Souriau geometric quantization and to $*$-product quantization) is provided. Given a quantum line bundle E over $\mathfrak{M}$, the complex Hilbert space $\mathscr{M}$ customarily adopted is the space of holomorphic sections in E that are square integrable with respect to the Liouville measure (cf. [21]), and the actual novelty in [30] is a construction of $\mathscr{K}$ where $\mathscr{M}$ is the space of all L^2 holomorphic sections in $T^{*\,(n,0)}(\mathfrak{M}) \otimes E$. Such a construction is related to the weighted Bergman kernels $K_{\alpha\overline{\alpha}}(\zeta,\, z)$ on $L^2 H(\Omega_\alpha\,,\ \gamma_\alpha)$, thus allowing for the use of methods in complex analysis of several complex variables (cf. [26]), e.g., reducing the calculation of the transition probability amplitudes $a_{\alpha\overline{\beta}}(\zeta,\, z)$ to the calculation of the weighted Bergman kernels involved. Here [with the notations in Sect. 1], $\Omega_\alpha = \chi_\alpha(U_\alpha)$, $\gamma_\alpha = H\big(\sigma_\alpha\,,\ \sigma_\alpha\big) \circ \chi_\alpha^{-1}$, and $\chi_\alpha = \big(z_\alpha^1\,,\ \cdots\,,\ z_\alpha^n\big) : U_\alpha \to \mathbb{C}^n$. Under a number of structural assumptions [cf. (I)–(IV) in Sect. 4, the coherent state map $\mathscr{K}$ is an anti-holomorphic embedding, so that $\mathscr{K}^* h$ [where h is the Fubini-Study metric on $\mathbb{CP}(\mathscr{M})$] is a Kählerian metric on $\mathfrak{M}$. Consequently, to quantize classical states of a mechanical system, its phase space $\mathfrak{M}$ should be equipped with a complex structure, and the resulting complex manifold $\mathfrak{M}$ must satisfy the topological restrictions allowing for the existence of (globally defined) Kählerian metrics (cf., e.g., [22]). Let $\mathfrak{M}$ be a non-Kählerian locally conformal Kähler (l.c.K.)

manifold (cf. [11]). Let $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$, and let G_λ be the discrete group of holomorphic transformations of $\mathscr{M} \setminus \{0\}$ generated by $\Psi \longmapsto \lambda\,\Psi$. Let $\mathbb{C}H_\lambda(\mathscr{M}) := \big(\mathscr{M} \setminus \{0\}\big)\big/G_\lambda$ be the quotient space (an infinite-dimensional analog to the complex Hopf manifold, cf. [11]). The properties of $\mathbb{C}H_\lambda(\mathscr{M})$ and of the natural projection $\mathbb{C}H_\lambda(\mathscr{M}) \to \mathbb{CP}(\mathscr{M})$ (a principal fibration in tori, in the finite-dimensional case) were not investigated, so far. Is there any useful (anti) holomorphic embedding $\mathscr{K} : \mathfrak{M} \to \mathbb{C}H_\lambda(\mathscr{M})$? If that is the case, does $\mathbb{C}H_\lambda(\mathscr{M})$ admit a metric g similar to the Boothby metric of a complex Hopf manifold (cf. [11]) and is $\mathscr{K}^* g$ a l.c.K. metric on $\mathfrak{M}$?

Let $\Omega \subset \mathbb{C}^n$ be a domain with $\partial\Omega \in C^2$ and let $\mu : \partial\Omega \to \mathbb{R}$ be a Szegö admissible weight (cf. [37], p. 321, and [38]). Let $S_\mu(\zeta, z)$ be the Szegö kernel of $L^2 H(\partial\Omega, \mu)$. The investigation of the properties of the map $\mathscr{K}_S : \overline{\Omega} \to \mathbb{CP}\big(L^2 H(\partial\Omega, \mu)\big)$, $\mathscr{K}_S(\zeta) = \big[\overline{S_\mu(\zeta, \cdot)}\big]$, $\zeta \in \overline{\Omega}$, is an open problem. If $\Omega = \mathbb{B}^n$ (the unit ball), is there a natural metric g_S on $\mathbb{CP}\big(L^2 H(\partial\Omega, \mu)\big)$ such that $\big(\mathscr{K}_S\big)^* g_S$ is the Szegö metric (cf. [4])? See also [24, 25]. Can the construction of $\mathscr{K}_S$ be recovered to the case where Ω is replaced by an arbitrary complex manifold $\mathfrak{M}$ (the base manifold of a given Hermitian line bundle over $\mathfrak{M}$)?

Let $\Omega = \{\varphi < 0\} \subset \mathbb{C}^{n+1}$ be a domain with smooth boundary, and let $\partial\Omega = \{\varphi = 0\}$ be endowed with the CR structure induced by the complex structure on $\mathbb{C}^n$. Let $E \to \partial\Omega$ be a CR-holomorphic complex line bundle (cf. [12]) and let $\Lambda^{n+1,0}(\partial\Omega)$ be the canonical line bundle. Then, $\Lambda^{n+1,0}(\partial\Omega) \otimes E$ is a CR-holomorphic bundle. Let $E \simeq \partial\Omega \times \mathbb{C}$ and $\mu : \partial\Omega \to \mathbb{R}$ a Szegö admissible weight. Let H be the Hermitian structure on E given by $H(\sigma_0\,,\,\sigma_0) = \mu$. If $\theta = \frac{i}{2}\big(\overline{\partial} - \partial\big)\,\varphi$ and $\theta^\alpha = dz^\alpha$, $1 \le \alpha \le n$, then any CR-holomorphic section s in $\Lambda^{n+1,0}(\partial\Omega) \otimes E$ may be represented as $s = \Psi\,\sigma_0 \otimes \theta \wedge \theta^1 \wedge \cdots \wedge \theta^n$ for some CR function $\Psi \in CR^\infty(\partial\Omega)$. Let $\mathscr{M}_b$ consist of all CR-holomorphic sections s that are L^2 in the sense that $\int_{\partial\Omega} H^*(s, s) < \infty$, where $H^*(s, s) = |\Psi|^2\,\mu\,\theta \wedge \theta^1 \wedge \theta^n \wedge \theta^{\overline{1}} \wedge \cdots \wedge \theta^{\overline{n}}$ [with $\theta^{\overline{\alpha}} = \overline{\theta^\alpha}$]. Is there any useful (anti) CR embedding $\mathscr{K}_b : \partial\Omega \to \mathbb{CP}\big(\mathscr{M}_b\big)$, and if that is the case, can $\mathscr{K}_b$ be used to quantize observables on $\partial\Omega$ [leading – when $\partial\Omega$ is nondegenerate – to a theory alternative to that by [16] (on quantization of contact manifolds)]? See also [17, 18]. Can $\mathscr{K}_b$ be realized as the boundary values of a coherent state map $\mathscr{K} : \Omega \to \mathbb{CP}(\mathscr{M})$?

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