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On solving some Cauchy singular integral equations by de la Vallée Poussin filtered approximation

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ABSTRACT

The paper deals with the numerical solution of Cauchy Singular Integral Equations based on some non standard polynomial quasi–projection of de la Vallée Poussin type. Such kind of approximation presents several advantages over classical Lagrange interpolation such as the uniform boundedness of the Lebesgue constants, the near–best order of uniform convergence to any continuous function, and a strong reduction of Gibbs phenomenon. These features will be inherited by the proposed numerical method which is stable and convergent, and provides a near-best polynomial approximation of the sought solution by solving a well conditioned linear system. The numerical tests confirm the theoretical error estimates and, in case of functions subject to Gibbs phenomenon, they show a better local approximation compared with analogous Lagrange projection methods.

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1. Introduction

In the present paper we consider the numerical solution of the following class of Cauchy Singular Integral Equations (CSIEs)

$$Df(y) + \nu Kf(y) = g(y), \qquad -1 < y < 1, \tag{1}$$

where ν denotes a real constant, g a known function and, corresponding to a Jacobi weight of the following kind

$$u(x) = v^{\alpha,-\alpha}(x) := (1-x)^{\alpha}(1+x)^{-\alpha}, \qquad \text{with} \qquad 0 < |\alpha| < 1, \tag{2}$$

the operators D and K are defined as follows

$$Df(y) := \cos(\pi\alpha) f(y) u(y) - \frac{\sin(\pi\alpha)}{\pi} \int_{-1}^{1} \frac{f(x)}{x-y} u(x) dx, \tag{3}$$

$$Kf(y) := \int_{-1}^{1} k(x,y) f(x) u(x) dx, \qquad -1 < y < 1, \tag{4}$$

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where $k(x, y)$ is a known kernel function, and the integral in (3) has to be understood in the Cauchy principal value sense.

These equations have been widely studied in the literature due to their relevance in the applications. There exist several problems leading to this kind of equations like, for instance, the renewed "airfoil equation", described in [13], or the linear elastic crack problems, as shown in [17]. More in general CSIEs are strictly related to boundary value problems arising in various and different fields as, for instance, in potential theory and in hydrodynamics (see for instance [2,14,33]).

The solvability of equation (1) has been investigated in several approximation spaces [25,24,34,4,15,18,21]. In this paper, we consider the equation in spaces of locally continuous functions equipped with suitable weighted uniform norms. Hence, we focus on the case that, in such spaces, (1) has a unique solution, say f^*, which can be approximated by polynomials at the desired precision.

Indeed the "global" approximation approach, that means constructing methods based on the polynomial approximation tools, leads to very efficient and accurate methods, since the approximants behave, more or less, like the best polynomial approximation of the unknown. Moreover, the construction of these approximants turns to be stable and the matrices connected to the methods are well conditioned.

In particular, several polynomial projection methods have been introduced in the literature in order to find good polynomial approximations of the sought solution f^*. The most commonly used polynomial projection for these methods is the Lagrange projection interpolating at suitable Jacobi zeros the choice of which depends on the weight u (see for instance [4,16,22,8] and the references therein). However, it is known that numerical methods based on Lagrange interpolation require a different strategy in the case that the exponent α in (2) does not allow to get optimal Lebesgue constants. Moreover, even using Lagrange projections with optimal Lebesgue constants, the error estimates contain an unavoidable logarithmic factor, dealing with uniform norms.

Here we are going to show how these problems can be overcome by employing some non–standard polynomial quasi–projections of de la Vallée Poussin (shortly VP) type. Such kind of polynomial approximation has been firstly introduced in [36] for Chebyshev weights, and further generalized to more general weights and several multivariate domains, in order to get a valid alternative to classical Lagrange interpolation (see, e.g. [39,37,40,35,30,29,26]). Following the trigonometric pattern, VP approximating polynomials are defined as discretized versions of certain VP means (i.e., delayed arithmetic means) of the Jacobi–Fourier partial sums. Equivalently, such means can be achieved by filtering higher degree Jacobi–Fourier partial sums with a VP filter function [41]. Similarly to Lagrange interpolating polynomial, the VP approximating polynomials are based on the values of the target function at Jacobi zeros but, at such zeros, they generally do not interpolate the function except in some special cases, such as for Chebyshev weights [37,31,32]. In particular, Chebyshev–VP interpolation has been recently used for solving Prandtl's integro–differential equations [10,9] and the generalized airfoil equation [23] that is a particular case of (1). Here we are going to show that even in the general case when the VP approximating polynomials are not interpolating, they constitute an effective tool to get convergent, stable and efficient numerical methods for solving (1). VP approximating polynomials have the peculiarity of depending on two degree–parameters: $n \in \mathbb{N}$, which determines the number of Jacobi nodes, and $m \in \mathbb{N}$ which determines the action range of the VP mean and can be arbitrarily fixed such that $0 < m < n$. It is known that in the limiting case $m = 0$, VP polynomials reduce to Lagrange polynomials interpolating at n Jacobi zeros, but taking the additional parameter $0 < m < n$ the resulting VP approximation presents several advantages over the Lagrange interpolation at the same Jacobi nodes. In particular, by VP approximation, we can get uniformly bounded Lebesgue constants for a larger set of Jacobi weights, which includes those Jacobi weights ensuring the optimal, logarithmic, growth of the Lebesgue constants related to Lagrange interpolation at the same nodes [37,38,31,29].

This feature will allow us to follow a unique strategy, based on the same VP approximating polynomials for all $0 < |\alpha| < 1$, contrarily to the Lagrange case. Moreover, as the number of nodes $n \to \infty$, choosing any degree parameter $0 < m < n$ such that $m \sim n \sim (n - m)$, it is known that the VP approximating polynomials converge with the same rate of the error of best polynomial approximation in suitable weighted uniform norms, providing a near–best approximation for any locally continuous function. On the contrary, the Lagrange polynomials converge to the interpolated function only if it is sufficiently smooth, due to the typical log n factor corrupting the error estimates [20,37].

As expected, we prove that also the numerical method we are proposing inherits this advantage over typical methods based on Lagrange interpolation, presenting near–best error estimates. More precisely, for the numerical solution of (1) we introduce two possible numerical procedures based on VP polynomial approximation. In both the cases, due to well–known mapping properties of the dominant operator D, especially on polynomials, we only take care of approximating the integral operator K and the known function g. We approximate the latter by its VP polynomial associated with u^{-1} while we propose two possible approximations of K. In the case that we know or can easily compute the so called modified moments, i.e. the values of K applied to Jacobi polynomials associated with u, then we can take the VP approximating polynomial of Kf (always associated with u^{-1}) as approximation of Kf. In the general case, we propose to approximate the kernel function $k(x, y)$ by its bivariate VP polynomial deduced via tensor product and associated with suitable Jacobi weights. Contrarily to the previous case, such a procedure yields a discrete approximate equation, which does not require the calculation of modified moments or any other integral but at the cost of introducing an additional sum in the drafting of the associated linear system. On the other hand, it is fully discrete and always applicable, even with weakly singular kernels, due to the good distances between the nodes related to the single variables x and y (see [19]).

We prove that both the resulting numerical methods are stable and convergent. Moreover, they both provide a unique polynomial solution that converges to f^* at the same rate of the error of best polynomial approximation of f^*. Hence, compared with analogous methods based on Lagrange interpolation, we succeed in cutting off that logarithmic factor we have

previously mentioned. Even if this result is relevant from the theoretical point of view, we remark that the absence/presence of an additional log factor is difficult to appreciate from a computational point of view, especially for very smooth functions, if we measure the error in the sup norm. However, if we look at the pointwise errors, the numerical tests show some improvements in the case of functions subject to the Gibbs phenomenon, reflecting the known ability of VP approximation in facing such phenomenon (see e.g. [41,29]).

The paper is organized as follows. In Section 2 some preliminaries are given in three subsections concerning the functional spaces, equation (1), and the VP approximating polynomials we are dealing with, respectively. Section 3 concerns the proposed numerical methods and is also divided in three subsections: the first introduces the approximate discrete equations, the second specifies their associated linear systems, and the third regards the uniform convergence and stability of the numerical solution achieved by solving these systems. Section 4 deals with the numerical experiments and, finally, Section 5 contains the proof of the new stated results.

2. Notation and preliminary results

Throughout the paper we denote by $\mathcal{C}$ a positive constant that may have different values at different occurrences, and we write $\mathcal{C} \neq \mathcal{C}(n, f, \ldots)$ to mean that $\mathcal{C} > 0$ is independent of n, f,... Moreover, if $a, b > 0$ depend on some parameters then $a \sim b$ denotes that there exists a constant $\mathcal{C} > 0$ independent of such parameters and such that $\mathcal{C}^{-1}a \leq b \leq \mathcal{C}a$. Furthermore, $\mathbb{P}_m$ denotes the space of the algebraic polynomials of degrees less than or equal to m.

2.1. Functional spaces

For any Jacobi weight v with nonnegative exponents, we denote by C_v^0 the space of all locally continuous functions on $[-1, 1]$ (i.e. continuous in any compact interval $I \subseteq]-1, 1[$) satisfying

$$\lim_{x\to\pm1} f(x)v(x) = 0, \qquad \text{if } v(\pm1) = 0. \tag{5}$$

C_v^0, equipped with the norm

$$\|f\|_{C_v^0} := \|fv\|_\infty = \max_{x\in[-1,1]} |f(x)|v(x),$$

is a Banach space. Moreover, by virtue of (5), all functions belonging to C_v^0, even those unbounded with algebraic singularities at the endpoints ±1, can be approximated at the desired precision by polynomials, holding the following

$$f \in C_v^0 \Longleftrightarrow \lim_{n\to\infty} E_n(f)_v = 0, \tag{6}$$

where $E_n(f)_v$ denotes the error of best approximation of $f \in C_v^0$ in the space $\mathbb{P}_n$, i.e.

$$E_n(f)_v = \inf_{P\in\mathbb{P}_n} \|(f - P)v\|_\infty.$$

It is well known that, as $n \to \infty$, the rate of convergence of such error depends on the smoothness of the function f and it can be characterized by the following main-part moduli of smoothness introduced in [11] by Z. Ditzian and V. Totik

$$\Omega_\varphi^s(f, t)_v = \sup_{0<h\leq t} \left[\sup_{x\in[-1+2h^2s^2,1-2h^2s^2]} |v(x)\Delta_{h\varphi}^s f(x)| \right], \qquad s \in \mathbb{N},$$

where

$$\Delta_{h\varphi}^s f(x) = \sum_{i=0}^{s} (-1)^i \binom{s}{i} f\left(x + \frac{sh}{2}\varphi(x) - ih\varphi(x)\right), \qquad \varphi(x) := \sqrt{1 - x^2}.$$

In fact, for all $n \in \mathbb{N}$ sufficiently large (say $n \geq n_0$) and $t > 0$ sufficiently small (say $t \leq t_0$), the following Jackson and Stechkin type inequalities hold [11]

$$E_n(f)_v \leq \mathcal{C} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^s(f, \tau)_v}{\tau} d\tau, \qquad \mathcal{C} \neq \mathcal{C}(n, f), \tag{7}$$

$$\Omega_\varphi^s(f, t)_v \leq Ct^s \sum_{n=0}^{[1/t]} (n + 1)^{s-1} E_n(f)_v, \qquad \mathcal{C} \neq \mathcal{C}(f, t), \tag{8}$$

so that for any $0 < r < s \in \mathbb{N}$, we get

$$E_n(f)_v = \mathcal{O}(n^{-r}) \Longleftrightarrow \Omega^s_\varphi(f,t)_v = \mathcal{O}(t^r).$$

In the literature, several approximation spaces have been introduced in order to classify the smoothness of the functions f w.r.t. the decay of $E_n(f)_v$ as $n \to \infty$ (see e.g. [1]). Here we focus on the following subspaces of C^0_v

$$Z_r(v) := \{f \in C^0_v : \sup_{n>0}(n+1)^r E_n(f)_v < \infty\}, \qquad r > 0,$$

equipped with the following norm

$$\|f\|_{Z_r(v)} := \|fv\|_\infty + \sup_{n>0}(n+1)^r E_n(f)_v, \qquad r > 0.$$

These spaces, usually known as Hölder–Zygmund spaces, are Banach spaces compactly embedded into C^0_v. They constitute a particular case of the Besov-type spaces studied in [12]. Moreover, they belong to a larger class of approximation spaces introduced in [15, p.204].

Taking into account (7)–(8), we have the following norm equivalence [12]

$$\|f\|_{Z_r(v)} \sim \|fv\|_\infty + \sup_{t>0} \frac{\Omega^s_\varphi(f,t)_v}{t^r}, \qquad s > r > 0.$$

Moreover, from (7) we deduce

$$E_n(f)_v \le C \frac{\|f\|_{Z_r(v)}}{n^r}, \qquad C \ne C(n,f), \qquad \forall f \in Z_r(v), r > 0. \tag{9}$$

Finally, in some occurrence, in the paper we also consider weighted L^1 spaces. In this setting, the Jacobi weight $v \in L^1$ may have negative exponents and we can repeat all the previous definitions and inequalities by replacing the weighted uniform norm with

$$\|f\|_{L^1_v} := \|fv\|_1 = \int_{-1}^{1} |f(x)|v(x)dx,$$

and the space C^0_v with

$$L^1_v = \{f : \|fv\|_1 < \infty\}.$$

In order to distinguish the L^1 from the infinity case, in L^1_v we are going to use the notation $E_n(f)_{v,1}$, $\Omega^s_\varphi(f,t)_{v,1}$, and $Z^1_r(v)$ for the errors of best approximation, moduli of smoothness, and Hölder–Zygmund subspaces, respectively.

2.2. Mapping properties and solvability of CSIEs

Throughout the paper, we consider any weight satisfying (2) as a ratio of two Jacobi weights with nonnegative exponents, and we use the following standard setting:

$$u(y) = \frac{u_+(y)}{u_-(y)}, \qquad \text{where} \qquad \begin{cases} u_+(y) := (1-y)^{\max\{0,\alpha\}}(1+y)^{\max\{0,-\alpha\}} \\ u_-(y) := (1-y)^{\max\{0,-\alpha\}}(1+y)^{\max\{0,\alpha\}}. \end{cases} \tag{10}$$

We point out that equation (2) defines a class of weights to which belong both a Jacobi weight and its inverse. Consequently, according to the notation introduced in (10), we remark that

$$(u^{-1})_+ = u_- \qquad \text{and} \qquad (u^{-1})_- = u_+. \tag{11}$$

For simplicity, in the sequel we are going to state all the results for a weight u as in (2), but we underline the same holds by replacing u with u^{-1}, taking into account (11).

Firstly, we recall the following fundamental result [21,34]

Theorem 2.1. *For all $r > 0$, the map $D : Z_r(u_+) \to Z_r(u_-)$ defined by (3) is a bounded operator having bounded inverse $D^{-1} = \hat{D}$ given by*

$$\hat{D}f(y) := \cos(\pi\alpha) f(y) u^{-1}(y) + \frac{\sin(\pi\alpha)}{\pi} \int_{-1}^{1} \frac{f(x)}{x-y} u^{-1}(x)dx. \tag{12}$$

As a consequence, by the Fredholm's alternative theorem we get the following result on the unique solvability of equation (1) in Hölder–Zygmund spaces.

Theorem 2.2. *Let us assume that, for some $r > 0$, $g \in Z_r(u_-)$ and the map $K : Z_r(u_+) \to Z_r(u_-)$ defined by (4) is a compact operator. Then equation (1) has a unique, stable, solution $f^* \in Z_r(u_+)$ provided that the associated homogeneous equation has only the null solution, i.e. if* $\ker(D + \nu K) = \{0\}$ *in* $Z_r(u_+)$.

In the remaining part of this subsection, we recall some useful results on the operator K defined by (4). In particular, from the literature we have several sufficient conditions ensuring the compactness of the map $K : Z_r(u_+) \to Z_r(u_-)$. These conditions involve the smoothness of the bivariate kernel function k and are usually expressed in terms of the univariate functions $k_x^{(1)}$ or $k_x^{(2)}$ obtained by fixing equal to $x \in [-1, 1]$ the first or the second variable, respectively, i.e.

$$k_x^{(1)}(t) := k(x, t) \qquad \text{and} \qquad k_x^{(2)}(t) := k(t, x), \qquad -1 \le t \le 1. \tag{13}$$

As regards the case we keep fixed the first variable (that is the integration variable) we recall the following result concerning the function $k_x^{(1)}$ [15, Prop. 4.12]

Proposition 2.3. *Let $r > 0$. If there exists a Jacobi weight v with nonnegative exponents such that $1/(vu_-) \in L^1$, the function $k(x, y)v(x)u_-(y)$ is continuous on $[-1, 1]^2$, and $v(x)k_x^{(1)} \in Z_r(u_-)$ uniformly w.r.t. $x \in [-1, 1]$, then the map $K : C^0_{u_+} \to Z_r(u_-)$ is bounded.*

On the other hand, concerning the case we fix the second variable, we recall the following inequalities involving the function $k_x^{(2)}$ [8, Lemma 2.1]

$$\|u_- Kf\|_\infty \le \Gamma \|f u_+\|_\infty, \qquad \Gamma := \sup_{|x|\le 1} u_-(x) \|k_x^{(2)}\|_{L^1_{1/u_-}}, \tag{14}$$

$$E_n(Kf)_{u_-} \le \Gamma_n \|f u_+\|_\infty, \qquad \Gamma_n := \sup_{|x|\le 1} u_-(x) E_n(k_x^{(2)})_{L^1_{1/u_-}}. \tag{15}$$

From (14) and (15) we easily deduce the next proposition involving Hölder–Zygmund type subspaces of L^1_v.

Proposition 2.4. *Let $r > 0$ and set $v = 1/u_-$. If the kernel k is s.t. $u_-(x)k_x^{(2)} \in L^1_v$ uniformly w.r.t. $x \in [-1, 1]$, then the map $K : C^0_{u_+} \to C^0_{u_-}$ is bounded. Moreover, if for some $r > 0$ we have that $u_-(x)k_x^{(2)} \in Z^1_r(v)$ holds uniformly w.r.t. $x \in [-1, 1]$, then the map $K : C^0_{u_+} \to Z_r(u_-)$ is bounded.*

Recalling the compact embedding

$$Z_s(u_+) \subset C^0_{u_+}, \qquad \forall s > 0, \qquad \text{and} \qquad Z_r(u_-) \subset Z_s(u_-), \qquad \forall s \in]0, r[$$

we remark that under the hypotheses of Proposition 2.3 or 2.4, we also get the following

(*i*) $K : C^0_{u_+} \to Z_s(u_-)$ is a compact operator for all $0 < s < r$,
(*ii*) $K : Z_s(u_+) \to Z_r(u_-)$ is a compact operator for all $s > 0$.

In conclusion let us focus on the case that the function $k(x, y)$ is one of the following weakly singular kernels

$$k_\mu(x, y) := \begin{cases} \ln|x - y|, & \text{if } \mu = 0, \\ |x - y|^\mu, & \text{if } \mu \ne 0 \text{ and } \mu > -1. \end{cases} \tag{16}$$

In this case we recall the following result (e.g., see [22])

Theorem 2.5. *For any $\mu > 1$, let K^μ be the operator defined by (4) with kernel function as in (16). For all $\mu > -1$ with $\mu \ne 0$, the map $K^\mu : C^0_{u_+} \to Z_{1+\mu}(u_-)$ is bounded and, consequently, $K^\mu : Z_r(u_+) \to Z_r(u_-)$ is a compact operator for all $0 < r \le 1 + \mu$. Similarly, in the case $\mu = 0$, for any $0 < r < 1$ we have $K^\mu : C^0_{u_+} \to Z_r(u_-)$ is bounded and $K^\mu : Z_r(u_+) \to Z_r(u_-)$ is compact.*

Finally, we recall that in case of weakly singular kernels as in (16), there exist suitable recurrence relations for computing the modified moments $\int_{-1}^1 k_\mu(x, y)p_j(u, x)u(x)dx$ for any $|y| < 1$ and $j = 0, 1,...$ (see, e.g., [22]).

3. VP filtered polynomial approximation

For an arbitrary Jacobi weight w and any $n \in \mathbb{N}$, let $p_n(w,x)$, or simply $p_n(w)$, denote the orthonormal Jacobi polynomial corresponding to w and having degree n with positive leading coefficient. If a function f is known at the zeros of $p_n(w)$, here denoted by

$$x^w_{n,1} < x^w_{n,2} < \ldots < x^w_{n,n},$$

then we can construct the VP approximating polynomial of f, which depends on an additional degree parameter $m \in \mathbb{N}$, $0 < m < n$, and it is defined as follows

$$V^m_n(w, f, x) := \sum_{i=1}^{n} f(x^w_{n,i})\Phi^m_{n,i}(w,x), \qquad |x| \le 1, \tag{17}$$

with

$$\Phi^m_{n,i}(w,x) = \lambda^w_{n,i} \sum_{j=0}^{n+m-1} \mu^m_{n,j} p_j(w, x^u_{n,i}) p_j(w,x), \qquad i = 1, \ldots, n, \tag{18}$$

where $\lambda^w_{n,i} = [\sum_{j=0}^{n-1} p^2_j(w, x^w_{n,i})]^{-1}$ are the well-known Christoffel numbers w.r.t. w, and $\mu^m_{n,j}$ are filtering coefficients of VP type given by

$$\mu^m_{n,j} := \begin{cases} 1 & \text{if } j = 0, \ldots, n-m, \\ \frac{n+m-j}{2m} & \text{if } n-m+1 \le j \le n+m-1. \end{cases} \tag{19}$$

Note that both the degree parameters $0 < m < n$ determine the general degree of the approximating VP polynomial of f, i.e. $V^m_n(w, f) \in P_{n+m-1}$, and also the degree of the polynomials satisfying the following preserving property [37].

$$V^m_n(w, P) = P, \qquad \forall P \in \mathbb{P}_{n-m}. \tag{20}$$

The mapping properties as well as the weighted approximation provided by VP polynomial quasi-projections $V^m_n(w) : f \to V^m_n(w, f)$ in the spaces C^0_v and L^1_v (critical for Lagrange interpolation) have been studied in the literature for general Jacobi weights w and v (see, e.g., [37,38,31,32,6]).

In this paper, we will focus on the particular case that w is a Jacobi weight as in (2). Moreover, we will use the following notation concerning the degree parameters

$$m \sim n \quad \text{iff} \quad m < n \le \mathcal{C}m \quad \text{holds with } \mathcal{C} \ne \mathcal{C}(n,m), \tag{21}$$

$$m \approx n \quad \text{iff} \quad c_1 m \le n \le c_2 m \quad \text{holds with } c_2 \ge c_1 > 1 \text{ independent of } n, m. \tag{22}$$

The first relation, $m \sim n$ (which agrees with the notation introduced in Section 2) is sufficient to get the following uniform boundedness of VP operator $V^m_n(u)$ in suitable spaces of locally continuous functions [37].

Theorem 3.1. *For all $m \sim n$ and any weight u as in (2), the map $V^m_n(u) : C^0_{u_+} \to C^0_{u_+}$ is uniformly bounded w.r.t. n and m. In particular, we have*

$$\|V^m_n(u, f)u_+\|_\infty \le \mathcal{C} \max_{1 \le i \le n} |(fu_+)(x^u_{i,n})|, \qquad \forall f \in C^0_{u_+}, \qquad \mathcal{C} \ne \mathcal{C}(n, m, f), \tag{23}$$

and the following error estimate holds

$$E_{n+m-1}(f)_{u_+} \le \|[f - V^m_n(u, f)]u_+\|_\infty \le \mathcal{C}E_{n-m}(f)_{u_+}, \qquad \mathcal{C} \ne \mathcal{C}(n, m, f). \tag{24}$$

We point out that, due to (20), the uniform boundedness result (23) is equivalent to the error estimate (24) where $n-m$ can be also constant since the assumption $m \sim n$ does not exclude this case even if both n and m tend to infinity (e.g. when $m = n-1$). The stronger assumption $m \approx n$ overcomes such a problem since it implies that $(n-m) \sim n$ holds too. Consequently, by the previous result we get

$$\lim_{\substack{n\to\infty \\ m \approx n}} \|[f - V^m_n(u, f)]u_+\|_\infty = 0, \qquad \forall f \in C^0_{u_+}, \tag{25}$$

being the convergence order comparable with that one of the errors of best polynomial approximation $E_n(f)_{u_+}$. In particular, for all $m \approx n$, from (7) we deduce

$$\|[f - V_n^m(u,f)]u_+\|_\infty \le \mathcal{C}\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^s(f,t)_{u_+}}{t}dt, \qquad \forall f \in C^0_{u_+}, \qquad \mathcal{C} \ne \mathcal{C}(n,m,f), \tag{26}$$

and from (9), we get

$$\|[f - V_n^m(u,f)]u_+\|_\infty \le \frac{\mathcal{C}}{n^r}\|f\|_{Z_r(u_+)}, \qquad \forall f \in Z_r(u_+), \qquad \mathcal{C} \ne \mathcal{C}(n,m,f). \tag{27}$$

Moreover, from Theorem 3.1 and some recent results proved in [27,28], we can deduce the following theorem which will be useful in the sequel.

Theorem 3.2. *Let u be a Jacobi weight as in (2) and D the associated operator given by (3). For all $f \in C^0_{u_+}$, we have*

$$\|D[V_n^m(u,f)]u_-\|_\infty \le \mathcal{C}\max_{1\le i\le n}|(fu_+)(x_{i,n}^u)|, \qquad \forall m \sim n, \qquad \mathcal{C} \ne \mathcal{C}(n,m,f), \tag{28}$$

and

$$\|D[f - V_n^m(u,f)]u_-\|_\infty \le \mathcal{C}\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^s(f,t)_{u_+}}{t}dt, \qquad \forall m \approx n, \qquad \mathcal{C} \ne \mathcal{C}(n,m,f). \tag{29}$$

With regard to the behavior of VP approximation in Hölder–Zygmund spaces, we recall the following (see, e.g., [5,10,23])

Theorem 3.3. *For all $r > 0$ and any weight u as in (2), if $m \sim n$ then the map $V_n^m(u) : Z_r(u_+) \to Z_r(u_+)$ is uniformly bounded w.r.t. n and m. Moreover, for all $m \approx n$ and any $s \in]0, r]$, we have*

$$\|f - V_n^m(u,f)\|_{Z_s(u_+)} \le \frac{\mathcal{C}}{n^{r-s}}\|f\|_{Z_r(u_+)}, \qquad \forall f \in Z_r(u_+), \qquad \mathcal{C} \ne \mathcal{C}(n,m,f). \tag{30}$$

Finally, concerning the behavior of VP approximation error in weighted L^1 norm, we recall the following result [38,5].

Theorem 3.4. *Suppose that u is a Jacobi weight as in (2), $f \in L^1_{1/u_-}$ is everywhere defined in $]-1,1[$, and $m \sim n$. If f is a function of bounded variation (shortly $f \in \mathcal{BV}$) then we have*

$$\|[f - V_n^m(u,f)]1/u_-\|_1 \le \frac{\mathcal{C}}{n}\int_{-1}^{1} \frac{\varphi(t)}{u_-(t)}|df(t)|, \qquad \mathcal{C} \ne \mathcal{C}(n,m,f). \tag{31}$$

Moreover, in case f is a locally continuous function, we get

$$\|[f - V_n^m(u,f)]1/u_-\|_1 \le \frac{\mathcal{C}}{n}\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^s(f,t)_{1/u_-,1}}{t^2}dt, \qquad \mathcal{C} \ne \mathcal{C}(n,m,f). \tag{32}$$

In conclusion, recalling that the analogous of (7)–(8) holds in weighted L^1 norm, by Theorem 3.4, similarly to (25), we get

$$\lim_{\substack{n\to\infty \\ m\approx n}} \|[f - V_n^m(u,f)]1/u_-\|_1 = 0, \qquad \forall f \in \mathcal{BV}, \tag{33}$$

and the convergence rate depends on the smoothness of $f \in L^1_{1/u_-}$.

In general case of bounded variation functions, (31) yields

$$\|[f - V_n^m(u,f)]1/u_-\|_1 = \mathcal{O}(n^{-1}), \qquad \forall f \in BV, \qquad \mathcal{C} \ne \mathcal{C}(n,m,f). \tag{34}$$

that is the same behavior of the error of best polynomial approximation in $E_n(f)_{1/u_-,1}$. Similarly, in case of smoother functions, the error estimate (32) yields

$$E_n(f)_{1/u_-,1} = \mathcal{O}(n^{-r}) \Longleftrightarrow \|[f - V_n^m(u,f)]1/u_-\|_1 = \mathcal{O}(n^{-r}), \qquad \forall r > 1,$$

and, in particular, for all functions $f \in Z_r^1(1/u_-)$ with $r > 1$, we have

$$\|[f - V_n^m(u,f)]1/u_-\|_1 \le \frac{\mathcal{C}}{n^r}\|f\|_{Z_r^1(1/u_-)}, \qquad \forall m \approx n, \qquad \mathcal{C} \ne \mathcal{C}(n,m,f). \tag{35}$$

4. The numerical method

4.1. The approximate equations

We are going to consider two kinds of approximation of the CSIE (1), which are the following:

$$Df(y) + \nu K_n f(y) = g_n(y), \qquad -1 < y < 1, \tag{36}$$

$$Df(y) + \nu \bar{K}_n f(y) = g_n(y), \qquad -1 < y < 1, \tag{37}$$

where, for simplicity, we use the same symbol f to denote the (generally different) unknowns in different equations.

In both the previous approximate equations, the original right-hand side term g has been approximated by its VP polynomial corresponding to the weight u^{-1} and parameters $m \approx n$, namely we set

$$g_n(y) := V_n^m(u^{-1}, g, y), \qquad |y| \le 1. \tag{38}$$

In (36) the same kind of approximation has been used for the operator K getting the following approximate operator

$$K_n f(y) := V_n^m(u^{-1}, Kf, y), \qquad |y| \le 1. \tag{39}$$

In (37) the approximation operator $\bar{K}_n$ has been obtained by replacing the kernel function $k(x, y)$ with its bivariate VP polynomial deduced via tensor product from the univariate VP polynomials associated with the weight u (w.r.t. the first variable) and u^{-1} (w.r.t. the second variable). More precisely, for any $m \approx n$, we consider the polynomial

$$\mathbf{V}_n^m k(x, y) := \sum_{i=1}^{n} \sum_{j=1}^{n} k(x_{n,i}^u, x_{n,j}^{u^{-1}}) \Phi_{n,i}^m(u, x) \Phi_{n,j}^m(u^{-1}, y), \qquad |x| \le 1, |y| \le 1, \tag{40}$$

that defines

$$\bar{K}_n f(y) := \int_{-1}^{1} \mathbf{V}_n^m k(x, y) f(x) u(x) dx. \tag{41}$$

We point out that all the approximations in (38), (39) and (41) depend on both parameters n and m. However, for simplicity, in our notation we underline only the dependence on n since $m \approx n$ and $\forall m \approx n$ we always get the same estimates. As regards the approximation provided by the operator K_n defined in (39), we state the following

Theorem 4.1. *Let $r > 0$ and suppose $K : C_{u_+}^0 \to Z_r(u_-)$ be a bounded operator. Hence, for all $f \in C_{u_+}^0$ and any $m \approx n$, the operator K_n satisfies the following estimates*

$$\|Kf - K_n f\|_{C_{u_-}^0} \le \frac{\mathcal{C}}{n^r} \|f u_+\|_\infty, \qquad \mathcal{C} \ne \mathcal{C}(n, f), \tag{42}$$

$$\|Kf - K_n f\|_{Z_s(u_-)} \le \frac{\mathcal{C}}{n^{r-s}} \|f u_+\|_\infty, \qquad \forall s \in]0, r], \qquad \mathcal{C} \ne \mathcal{C}(n, f). \tag{43}$$

Under the assumptions of Proposition 2.4 that imply $K : C_{u_+}^0 \to Z_r(u_-)$ is a bounded operator, we get the previous estimates hold for the approximation operator $\bar{K}_n$ too, as specified in the following

Theorem 4.2. *Let $r > 0$ and suppose that $u_-(x) k_x^{(2)} \in Z_r^1(1/u_-)$ holds uniformly w.r.t. $x \in [-1, 1]$. Hence, for all $f \in C_{u_+}^0$ and any $m \approx n$, the operator $\bar{K}_n$ satisfies the following estimates*

$$\|Kf - \bar{K}_n f\|_{C_{u_-}^0} \le \frac{\mathcal{C}}{n^r} \|f u_+\|_\infty, \qquad \mathcal{C} \ne \mathcal{C}(n, f), \tag{44}$$

$$\|Kf - \bar{K}_n f\|_{Z_s(u_-)} \le \frac{\mathcal{C}}{n^{r-s}} \|f u_+\|_\infty, \qquad \forall s \in]0, r], \qquad \mathcal{C} \ne \mathcal{C}(n, f). \tag{45}$$

Finally, another common feature of the approximate equations (36) and (37) is given by the following

Proposition 4.3. *If the equation (36) or (37) has a solution, then it belongs to $\mathbb{P}_{n+m-1}$.*

In the next subsection we will see how equations (36) and (37) differ in the computations required to solve them.

4.2. The linear systems for computing the numerical solution

Throughout this subsection, we assume that there exists a unique solution of the approximate equation (36) or (37) and we focus on its numerical computation. This aspect will suggest us which one of the equations to consider.

Due to Proposition 4.3, in both the approximate equations we look for a polynomial solution in the form

$$f_n^m(y) = \sum_{j=0}^{n+m-1} c_j p_j(u, y), \qquad |y| \le 1, \tag{46}$$

being $\{c_j\}_{j=0,\dots,n+m-1}$ the $(n+m)$ unknown coefficients.

Let us first focus on the case that f_n^m in (46) is the solution of the approximate equation (36).

Note that by (38) and (17)–(18), we get

$$g_n(y) = V_n^m(u^{-1}, g, y) = \sum_{j=0}^{n+m-1} p_j(u^{-1}, y) \left[\mu_{n,j}^m \sum_{l=1}^{n} \lambda_l^{u^{-1}} g(x_{n,l}^{u^{-1}}) p_j(u^{-1}, x_{n,l}^{u^{-1}}) \right]. \tag{47}$$

Moreover, setting

$$m_j(y) := \int_{-1}^{1} p_j(u, x) k(x, y) u(x) dx, \qquad j = 0, 1, \dots, n+m-1, \qquad |y| < 1, \tag{48}$$

we observe that

$$Kf_n^m(y) = \sum_{j=0}^{n+m-1} c_j m_j(y), \qquad |y| < 1,$$

and consequently, by (39) and (17)–(18), we get

$$\begin{aligned} K_n f_n^m(y) = V_n^m(u^{-1}, Kf, y) &= \sum_{j=0}^{n+m-1} p_j(u^{-1}, y) \left[\mu_{n,j}^m \sum_{l=1}^{n} \lambda_l^{u^{-1}} Kf(x_{n,l}^{u^{-1}}) p_j(u^{-1}, x_{n,l}^{u^{-1}}) \right], \\ &= \sum_{j=0}^{n+m-1} p_j(u^{-1}, y) \left[\mu_{n,j}^m \sum_{i=0}^{n+m-1} c_i \sum_{l=1}^{n} \lambda_l^{u^{-1}} p_j(u^{-1}, x_{n,l}^{u^{-1}}) m_i(x_{n,l}^{u^{-1}}) \right]. \end{aligned} \tag{49}$$

On the other hand, recalling the well known invariance property of D [34]

$$Dp_j(u) = p_j(u^{-1}), \ j = 0, 1, 2, \dots,$$

we have

$$Df_n^m(y) = \sum_{j=0}^{n+m-1} c_j p_j(u-1, y). \tag{50}$$

Summing up, if we replace the expression (46) of the solution f_n^m in equation (36) then, by (47), (49), (50), we get

$$\begin{aligned} &\sum_{j=0}^{n+m-1} p_j(u^{-1}, y) \left[c_j + \nu \mu_{n,j}^m \sum_{i=0}^{n+m-1} c_i \sum_{l=1}^{n} \lambda_l^{u-1} p_j(u^{-1}, x_{n,l}^{u^{-1}}) m_i(x_{n,l}^{u^{-1}}) \right] \\ &= \sum_{j=0}^{n+m-1} p_j(u^{-1}, y) \left[\mu_{n,j}^m \sum_{l=1}^{n} \lambda_l^{u^{-1}} g(x_{n,l}^{u^{-1}}) p_j(u^{-1}, x_{n,l}^{u^{-1}}) \right], \end{aligned}$$

and we can compute the coefficients $\{c_j\}_{j=0}^{n+m-1}$ by solving the next linear system

$$\begin{aligned} &c_j + \nu \mu_{n,j}^m \sum_{i=0}^{n+m-1} c_i \sum_{l=1}^{n} \lambda_l^{u^{-1}} p_j(u^{-1}, x_{n,l}^{u^{-1}}) m_i(x_{n,l}^{u^{-1}}) \\ &= \mu_{n,j}^m \sum_{l=1}^{n} \lambda_l^{u^{-1}} g(x_{n,l}^{u^{-1}}) p_j(u^{-1}, x_{n,l}^{u^{-1}}), \qquad j = 0, 1, \dots, n+m-1. \end{aligned} \tag{51}$$

In a more compact matrix form, setting

$$\begin{aligned}
&\mathbf{c} := [c_0, c_1, ..., c_{n+m-1}]^T \in \mathbb{R}^{n+m},\\
&\boldsymbol{M} = \mathrm{diag}(\mu^m_{n,0}, ..., \mu^m_{n,n+m-1}) \in \mathbb{R}^{(n+m)\times(n+m)},\\
&\boldsymbol{P} = (p_{i,j})_{i,j} \in \mathbb{R}^{(n+m)\times n}, \qquad p_{i,j} := p_i(u^{-1}, x^{u^{-1}}_{n,j}), \qquad i = 0, .., n+m-1, \qquad j = 1, .., n,\\
&\boldsymbol{\Lambda}^u = \mathrm{diag}(\lambda^u_{n,1}, ..., \lambda^u_{n,n}) \in \mathbb{R}^{n\times n},\\
&\boldsymbol{R} = \boldsymbol{M}\boldsymbol{P}\boldsymbol{\Lambda}^{u^{-1}} \in \mathbb{R}^{(n+m)\times n},\\
&\mathbf{g} := \left[g(x^{u^{-1}}_{n,1}), ..., g(x^{u^{-1}}_{n,n})\right]^T \in \mathbb{R}^n,\\
&\boldsymbol{v} = \boldsymbol{R}\mathbf{g} \in \mathbb{R}^{n+m},\\
&\boldsymbol{W} = (w_{i,j}) \in \mathbb{R}^{n\times(n+m)}, \qquad w_{i,j} := m_j(x_i^{u^{-1}}), \qquad i = 1, .., n, \qquad j = 0, .., n+m-1.
\end{aligned} \tag{52}$$

linear system (51) becomes

$$(\boldsymbol{I} + \nu\boldsymbol{A})\mathbf{c} = \boldsymbol{v}, \qquad \boldsymbol{A} := \boldsymbol{R}\boldsymbol{W}, \tag{53}$$

where, as usual, $\boldsymbol{I} \in \mathbb{R}^{(n+m)\times(n+m)}$ denotes the identity matrix.

We underline that the above numerical procedure can be used in all the cases that the kernel function $k(x, y)$ is such that the integrals in (48), i.e. its first $n+m$ modified moments are known or easily computable at the zeros of $p_n(u^{-1})$. If this is not the case, we do not know the elements of $\boldsymbol{W}$ and cannot consider (36).

Hence, we are going to take the approximate equation (37) whose solution, for simplicity, we continue to denote as f^m_n given by (46).

In order to find the unknown coefficients $\{c_j\}_{j=0,..,n+m-1}$, in this case we observe that by (40), (41) and (46), we have

$$\begin{aligned}
\bar{K}f^m_n(y) &= \sum_{i,j=0}^{n+m-1} \mu^m_{n,i}\mu^m_{n,j}p_j(u^{-1}, y)\left[\int_{-1}^{1} p_i(u, x)f^m_n(x)u(x)dx\right]\\
&\quad \times \sum_{l,\ell=1}^{n} \lambda^u_\ell\lambda^{u^{-1}}_l p_i(u, x^u_{n,\ell})p_j(u^{-1}, x^{u^{-1}}_{n,l})k(x^u_{n,\ell}, x^{u^{-1}}_{n,l})\\
&= \sum_{j=0}^{n+m-1} p_j(u^{-1}, y)\left[\mu^m_{n,j}\sum_{i=0}^{n+m-1}\mu^m_{n,i}c_i\sum_{l,\ell=1}^{n}\lambda^u_\ell\lambda^{u^{-1}}_l p_i(u, x^u_{n,\ell})p_j(u^{-1}, x^{u^{-1}}_{n,l})k(x^u_{n,\ell}, x^{u^{-1}}_{n,l})\right],
\end{aligned}$$

that is, setting

$$\kappa_{i,j} := \sum_{l,\ell=1}^{n}\lambda^u_\ell\lambda^{u^{-1}}_l p_i(u, x^u_{n,\ell})p_j(u^{-1}, x^{u^{-1}}_{n,l})k(x^u_{n,\ell}, x^{u^{-1}}_{n,l}), \qquad i, j = 0, ..., n+m-1,$$

we get

$$\bar{K}_n f^m_n(y) = \sum_{j=0}^{n+m-1} p_j(u^{-1}, y)\left[\mu^m_{n,j}\sum_{i=0}^{n+m-1}\mu^m_{n,i}c_i\kappa_{i,j}\right]. \tag{54}$$

Hence, if the polynomial in (46) is the solution of (37) then, by (47), (50) and (54), we deduce the identity

$$\begin{aligned}
&\sum_{j=0}^{n+m-1} p_j(u^{-1}, y)\left[c_j + \nu\mu^m_{n,j}\sum_{i=0}^{n+m-1}\mu^m_{n,i}c_i\kappa_{i,j}\right]\\
&= \sum_{j=0}^{n+m-1} p_j(u^{-1}, y)\left[\mu^m_{n,j}\sum_{l=1}^{n}\lambda^{u^{-1}}_l g(x^{u^{-1}}_{n,l})p_j(u^{-1}, x^{u^{-1}}_{n,l})\right]
\end{aligned}$$

that leads to the following linear system

$$c_j + \nu\mu^m_{n,j}\sum_{i=0}^{n+m-1}\mu^m_{n,i}c_i\kappa_{i,j} = \mu^m_{n,j}\sum_{l=1}^{n}\lambda^{u^{-1}}_l g(x^{u^{-1}}_{nl})p_j(u^{-1}, x^{u^{-1}}_{n,l}), \qquad j = 0, .., n+m-1. \tag{55}$$

In order to represent this system in a matrix form, besides the notation in (52), let us introduce the following matrices

$$\boldsymbol{Q} = (p_i, j)_{i,j} \in \mathbb{R}^{(n+m)\times n}, \qquad p_{i,j} := p_i(u, x^u_{n,j}), \qquad i = 0, .., n+m-1, \qquad j = 1, .., n$$
$$\boldsymbol{S} = \boldsymbol{M}\,\boldsymbol{Q}\,\boldsymbol{\Lambda}^{u^{-1}} \in \mathbb{R}^{(n+m)\times n},$$
$$\boldsymbol{K} = (k_{i,j})_{i,j} \in \mathbb{R}^{n\times n}, k_{i,j} := k(x^u_{n,i}, x^{u^{-1}}_{n,j}), \qquad i, j = 1, .., n.$$

Therefore the system (55) can be rewritten as:

$$(\boldsymbol{I} + \nu\boldsymbol{B})\boldsymbol{c} = \boldsymbol{v}, \qquad \boldsymbol{B} := \boldsymbol{S}\boldsymbol{K}\boldsymbol{A}^T. \tag{56}$$

In conclusion, in the case the matrix $\boldsymbol{W}$ of the modified moments is known, we can use this data and solving the linear system (53) that is equivalent to solve approximate equation (36). In the general case, starting from the matrix $\boldsymbol{K}$, we can always set the linear system (56) whose solution allows to compute, by means of (46), the solution of approximate equation (37) at any $|y| \le 1$.

In the next subsection, we are going to see that whatever linear system/ approximate equation we solve, we can get the same degree of approximation with similar error estimates.

4.3. Convergence and stability results

In this subsection, we are going to state the unique solvability of the previous approximate equations (36) and (37) and the associated linear systems. Moreover, we study the stability and convergence providing sharp error estimates for the approximate polynomial solutions.

In order to do this, we first need to prove that operators $D+\nu K_n$ and $D+\nu\bar{K}_n$ are invertible and uniformly bounded. The proof of these properties can be lead in a standard way (see for instance [3]) and the main tool is the strong convergence of the operator sequences $\{K_n\}_n$ and $\{\bar{K}_n\}_n$ to K in the space where we want to consider $D+\nu K_n$ and $D+\nu\bar{K}_n$.

On the other hand Theorem 4.1, together with Proposition 2.4, and Theorem 4.2 state sufficient conditions on the kernel $k(x, y)$ in order to get the strong convergence results of $\{K_n\}_n$ and $\{\bar{K}_n\}_n$ to K.

Therefore we can prove the following stability results, in the statements of which we will omit the subscripts in the operator norms, the meaning being clear.

Theorem 4.4. *Let $K : C^0(u_+) \to Z_r(u_-)$, with $r > 0$ be bounded, and assume* $\ker(D + \nu K) = \{0\}$. *Then operators $D + \nu K_n : Z_s(u_+) \to Z_s(u_-)$, $0 < s < r$, have bounded inverses for all sufficiently large $m \approx n$. Moreover we have*

$$\sup_{m\approx n} \|(D + \nu K_n)^{-1}\| < +\infty, \tag{57}$$

and the condition number of $D + \nu K_n$ tends to the condition number of $D + \nu K$, i.e.

$$\lim_{\substack{n\to\infty \\ m\approx n}} \frac{\|D + \nu K_n\|\|(D + \nu K_n)^{-1}\|}{\|D + \nu K\|\|(D + \nu K)^{-1}\|} = 1. \tag{58}$$

An analogous result, under different additional assumptions for $k(x, y)$, can be obtained for $D + \nu\bar{K}_n$.

Theorem 4.5. *Let $u_-(x)k^{(2)}_x \in Z^1_r(1/u_-)$ with $r > 0$, uniformly w.r.t. $x \in [-1, 1]$, and assume* $\ker(D + \nu K) = \{0\}$. *Then operators $D + \nu\bar{K}_n : Z_s(u_+) \to Z_s(u_-)$, $0 < s < r$, have bounded inverses for all sufficiently large $m \approx n$. Moreover (57) and (58) hold true with $D + \nu K_n$ replaced by $D + \nu\bar{K}_n$.*

We conclude with the convergence estimates of the previous methods. About the first method we get

Theorem 4.6. *Let the assumptions of Theorem 4.4 be fulfilled with $r > 0$. Then for all $m \approx n$, equation (36) has a unique solution $f^m_n \in \mathbb{P}_{n+m-1}$ satisfying*

$$\|(f^* - f^m_n)u_+\|_\infty \le \frac{\mathcal{C}}{n^r}\|g\|_{Z_r(u_-)}, \qquad \mathcal{C} \ne \mathcal{C}(n, f^*) \tag{59}$$

$$\|(f^* - f^m_n)\|_{Z_s(u_+)} \le \frac{\mathcal{C}}{n^{r-s}}\|g\|_{Z_r(u_-)}, \qquad s < r, \qquad \mathcal{C} \ne \mathcal{C}(n, f^*), \tag{60}$$

where f^* denotes the unique solution of equation (1).

About the second method we have

Theorem 4.7. *Let the assumptions of Theorem 4.5 be fulfilled with $r > 0$. Then for all $m \approx n$, equation (37) has a unique solution $\overline{f_n^m} \in \mathbb{P}_{n+m-1}$, and*

$$\|(f^* - \overline{f_n^m})u_+\|_\infty \le \frac{\mathcal{C}}{n^r}\|g\|_{Z_r(u_-)}, \qquad \mathcal{C} \ne \mathcal{C}(n, f^*) \tag{61}$$

$$\|(f^* - \overline{f_n^m})\|_{Z_s(u_+)} \le \frac{\mathcal{C}}{n^{r-s}}\|g\|_{Z_r(u_-)}, \qquad s < r, \qquad \mathcal{C} \ne \mathcal{C}(n, f^*), \tag{62}$$

where f^* denotes the unique solution of equation (1).

Remark. We underline that the convergence and stability results here obtained can be compared with the analogous one proposed in the literature and based on the Lagrange interpolating operator (see for instance [22,8]). The main advantages in our approach are the following.

- The main results, in the cited papers, require the kernel has to fulfill assumptions in the space Z_r w.r.t. the weighted uniform norm, while we succeeded in proving the corresponding results with "minimal" assumptions in L^1 weighted norm.
- In the Lagrange case it is necessary to restrict the values for the exponent α ($\alpha \ge 1/2$), in the definition of the weight u, in order to obtain optimal Lebesgue constant or alternatively one has to use 2 additional interpolation points in order to recover the optimal behavior. Therefore it is necessary to solve different linear systems for different values of α. On the contrary we have no restriction on α.
- Due to the boundedness of the VP operators in the convergence estimates we have no extra log factors as it occurs in the Lagrange case.

5. Numerical tests

We are going to show the performance of the proposed methods by some numerical experiments, making comparisons with the analogous procedures obtained replacing $V_n^m(u^{-1}, f)$ with the Lagrange polynomial $L_{m+n}(u^{-1}, f) \in \mathbb{P}_{n+m-1}$, interpolating f at the zeros of $p_{n+m}(u^{-1})$ (see [22,7]). Such method, referred to as Lag-method in the following, allows us to construct the sequence $\{f_N^{Lag}\}_{N\ge1}$ approximating the solution f^* of (1). We recall that for the Lag-method in [22], the constraint $\alpha \ge \frac{1}{2}$ holds true.

Denoting by Y a sufficiently large uniform mesh of points in $[-1, 1]$, for any $y \in Y$ let

$$e_{n,m}^{VP}(f, y) = |\hat{f}^{VP}(y) - f_n^m(y)|u_+(y), \qquad e_{n+m}^{Lag}(f, y) = |\hat{f}^{Lag}(y) - f_{n+m}^{Lag}(y)|u_+(y),$$

being $\hat{f}^{VP}(y)$, $\hat{f}^{Lag}(y)$ the values achieved by means of higher degree implementation of the VP and Lag-method, respectively, and

$$\begin{aligned} \mathcal{E}_{n+m}^{VP} &:= \max_{y\in Y}\left(e_{n,m}^{VP}(f, y)\right), \\ \mathcal{E}_{n+m}^{Lag} &:= \max_{y\in Y}\left(e_{n+m}^{Lag}(f, y)\right). \end{aligned} \tag{63}$$

In the case the involved functions g and/or k are very smooth, the pointwise absolute errors have been computed by means of

$$\tilde{e}_{n,m}^{VP}(f, y) = |f_n^m(y) - f_{2n}^{2m}(y)|u_+(y), \qquad \tilde{e}_{n+m}^{Lag}(f, y) = |f_{n+m}^{Lag}(y) - f_{2n+2m}^{Lag}(y)|u_+(y),$$

defining in this case

$$\begin{aligned} \tilde{\mathcal{E}}_{n+m}^{VP} &:= \max_{y\in Y}\left(\tilde{e}_{n,m}^{VP}(f, y)\right), \\ \tilde{\mathcal{E}}_{n+m}^{Lag} &:= \max_{y\in Y}\left(\tilde{e}_{n+m}^{Lag}(f, y)\right). \end{aligned} \tag{64}$$

By cond_{n+m}^{VP}, $\mathrm{cond}_{n+m}^{Lag}$we denote the condition numbers of the linear system in infinity matrix norm for the VP and Lag-method, respectively. In all the tests we report the values of n for both VP and Lag methods. About the choice in VP method satisfying $m \approx n$, we have chosen $m = \lfloor \theta n \rfloor$, with $\theta \in \{0.1 : 0.1 : 0.9\}$, and in the tables, for any n, we have reported the value of the selected θ. The solutions of the involved linear systems have been computed by the Gaussian elimination method.

In what follows we will call VP1 the method deriving by the finite dimensional equation (36) and analogously VP2 that deriving by the finite dimensional equation (37).

All the computations were performed in double-machine precision ($eps_D \approx 2.22044e-16$) by using MATLAB R2021a.

Table 1
Example 1: weighted maximum errors.

n	θ	$\mathcal{E}_{n+m}^{VP}$	cond_{n+m}^{VP}	$\mathcal{E}_{n+m}^{Lag}$	cond_{n+m}^{Lag}
17	0.1	4.86e-02	1.50	4.86e-02	8.49
33	0.1	3.36e-02	1.50	3.36e-02	11.1
65	0.1	1.92e-02	1.50	1.93e-02	14.2
129	0.1	1.18e-02	1.50	1.18e-02	17.7
257	0.8	3.94e-03	1.50	7.84e-03	21.5
513	0.1	4.02e-03	1.50	4.00e-03	25.6
1025	0.1	1.83e-03	1.50	1.82e-03	30.3

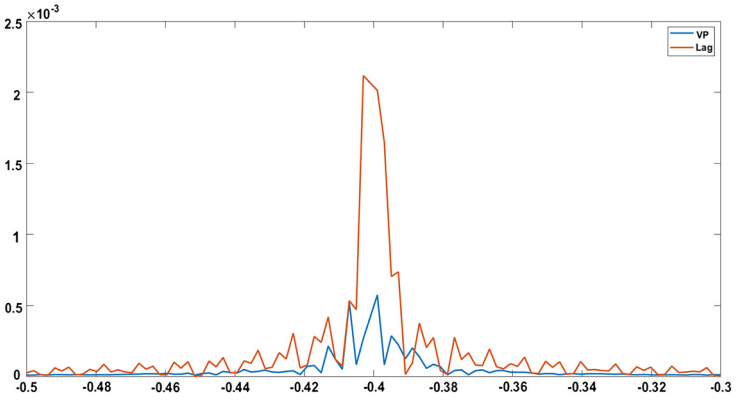

Fig. 1. Example 1: weighted pointwise errors for $n = 1025$, and $\theta = 0.1$.

Table 2
Example 1: weighted pointwise errors.

$y = -0.4543286573146292$		$y = 0.6011422845691383$		$y = 0.8035070140280561$	
$e_{n,m}^{VP}(f,y)$	$e_{n+m}^{Lag}(f,y)$	$e_{n,m}^{VP}(f,y)$	$e_{n+m}^{Lag}(f,y)$	$e_{n,m}^{VP}(f,y)$	$e_{n+m}^{Lag}(f,y)$
9.52e-08	6.38e-05	9.12e-09	2.23e-06	6.34e-09	1.14e-06
$y = -0.3035470941883768$		$y = 0.2043486973947896$		$y = .009919839679358718$	
$e_{n,m}^{VP}(f,y)$	$e_{n+m}^{Lag}(f,y)$	$e_{n,m}^{VP}(f,y)$	$e_{n+m}^{Lag}(f,y)$	$e_{n,m}^{VP}(f,y)$	$e_{n+m}^{Lag}(f,y)$
2.49e-05	2.46e-05	1.10e-06	8.59e-07	2.34e-06	1.36e-06

Example 1. Consider equation (1), with

$$\alpha = 0.5, \qquad \nu = -13\pi, \qquad u_+ = v^{\frac{1}{2},0},$$
$$k(x,y) = \frac{1}{1+10(y-x)^2} + \frac{1}{\sqrt{1+10(y-x)^2}}, \qquad g(y) = |y+0.4|^{0.7}.$$

Here in view of the smoothness of the kernel k we use the method VP2. The function $g \in Z_r(u_-)$ with $r = 0.7$, and the expected order of convergence of our method is $\mathcal{O}(n^{-0.7})$ The experiment has been performed for values of θ varying in the set $\{0.1 : 0.1 : 0.9\}$, with $m = \lfloor \theta n \rfloor$, being θ selected for any n so that the best result for the VP method is achieved. The Lag-method has been implemented so that the order of the final linear is the same of the VP method, i.e. $(n+m)$. The results about the maximum errors as defined in (63) are displayed in Table 1, showing that the convergence results of both the methods are comparable to each other, while the condition numbers of the linear systems, bounded in VP method for n increasing, moderately diverge in the Lag method. The graphic of the pointwise absolute errors in the interval $(-0.5, -0.3)$ (see Fig. 1) evidences a better performance of the VP method around the critical point $y = -0.4$, while the Lag-method presents overshoots and oscillations spilling over the whole interval.

To evidence the local improvement induced by the VP method w.r.t. the Lag method, in Table 2 we report the pointwise errors for 6 points $y \in Y$, selected from those for which VP method or Lag-method gain the best results. To be more precise, for $n = 1025$ and $\theta = 0.1$, we have selected the first three nodes among those the VP2 method achieves the "best results" (see the results in the top of Table 2), while the other three points are those for which the Lag-method gains the "best results" (see the results in the bottom of Table 2). As we can see, a faster improvement is induced by VP method which gains until 3 digits, but the best results obtained by the Lag-method allow at most 2 digits.

Table 3
Example 2: weighted pointwise errors.

	$y=-0.4$		$y=-0.5$		$y=0.5$	
n	$\tilde{e}^{VP}_{n,m}(f,y)$	$\tilde{e}^{Lag}_{n+m}(f,y)$	$\tilde{e}^{VP}_{n,m}(f,y)$	$\tilde{e}^{Lag}_{n+m}(f,y)$	$\tilde{e}^{VP}_{n,m}(f,y)$	$\tilde{e}^{Lag}_{n+m}(f,y)$
16	7.80e-01	1.86e-02	5.49e-01	2.49e-01	4.52e-02	4.88e-02
32	1.44e-01	1.76e-02	4.48e-02	8.66e-03	1.32e-02	4.14e-02
64	9.16e-03	5.23e-03	3.99e-03	2.32e-03	1.09e-04	8.53e-04
128	1.91e-04	1.38e-05	5.43e-05	2.52e-05	8.20e-09	5.50e-06
256	6.62e-10	1.62e-08	8.82e-09	4.51e-09	6.96e-13	1.00e-10
512	5.18e-14	7.78e-13	6.89e-14	8.25e-13	6.07e-13	6.87e-13

Table 4
Example 2.

n	θ	$\tilde{\mathcal{E}}^{VP}_{n+m}$	cond^{VP}_{n+m}	$\tilde{\mathcal{E}}^{Lag}_{n+m}$	$\mathrm{cond}^{Lag}_{n+m}$
16	0.12	1.35e-01	1.15	3.18e-02	1.38e+01
32	0.12	1.44e-01	2.56	1.07e-01	1.72e+01
64	0.12	2.31e-02	2.56	1.69e-02	2.08e+01
128	0.12	3.48e-04	2.56	2.86e-04	2.48e+01
256	0.12	6.31e-07	2.56	7.25e-08	2.52e+01
512	0.12	9.19e-13	2.56	9.95e-13	2.57e+01

Example 2. Consider equation (1), with

$$\alpha = 0.6, \qquad \nu = -\frac{1}{2\pi}, \qquad u_+ = v^{0.6,0}$$
$$k(x,y) = \frac{1}{1+10(y-x)^2} + \frac{1}{\sqrt{1+10(y+x)^2}},$$
$$g(y) = \frac{1}{1+100(0.5-y)^2} + \frac{1}{\sqrt{1+100(0.5+y)^2}}$$

In Table 4 are displayed the maximum errors as defined in (64) for n varying and θ fixed in the VP method, and for the Lag-method of order $n+m$. Analyzing the maximum errors, the VP method seems to have a worse behavior w.r.t. the Lag method. Moreover, the expected machine precision for small n is never gained in both the methods, since eventhough $g \in Z_r(u_-)$ for any $r>0$ it presents two picks at the points $y=\pm 0.5$ (see the graphic of g in Fig. 2), allowing to "large" norms of g in $Z_r(u_-)$ as r is "large". For instance, with $r=7$ it is $\|g\|_{Z_r(u_+)} \sim 109$, while for $r=11$ it is $\|g\|_{Z_r(u_+)} \sim 1.5 \times 1016$. In the same Table are given the conditioning of the final linear systems. As we can see, they are bounded in the VP method, while moderately diverge in the Lag-method. In particular in Table 3 are reported the weighted pointwise errors for some selected points, and the results evidence a better local performance of the VP method versus the Lag-method at the "critical" points $y=\pm 0.5$. Moreover, in Fig. 3 there is the graphic of the absolute pointwise errors provided by both the methods, and it is confirmed that the VP2 method provides almost everywhere better results than the Lag method, along the whole interval.

Example 3. *Consider equation* (1), *with*

$$\alpha = 0.9, \nu = 1, u_+ = v^{0.9,0},$$
$$k(x,y) = |\cos(y-\frac{\pi}{4})|^{4.5} + |\sin(x)|^{3.5},$$
$$g(y) = |x|^{5.5}.$$

Here the function g belongs to $Z_{5.5}(u_+)$, and in view of the poor regularity of the kernel, the order of convergence is $\mathcal{O}\left(\frac{1}{n^{3.5}}\right)$ for our method and $\mathcal{O}\left(\frac{\log(n+m)}{(n+m)^{3.5}}\right)$ for the Lag-method. The maximum errors as defined in (64) are displayed in Table 5. They appear almost comparable to each other, with a little bit improvement gained by the VP method.

Nevertheless, to compute f_n^m are required n values of g and n^2 values of the kernel k, while to compute f_{n+m}^{Lag} are needed $n+m$ and $(n+m)^2$ evaluations of g and k, respectively. Hence, we can conclude that a saving in computing the samples is realized by using the VP method to achieve a comparable approximation, and this represents another interesting point of view in the comparison between the computational costs.

Next two test deals with VP1 method, and involve kernels of the type $k(x,y)=|x-y|^\lambda, \lambda > -1$. In this case the functions $m_j(y)$ defined in (48) can be computed by means of suitable recurrence relations given in [22].

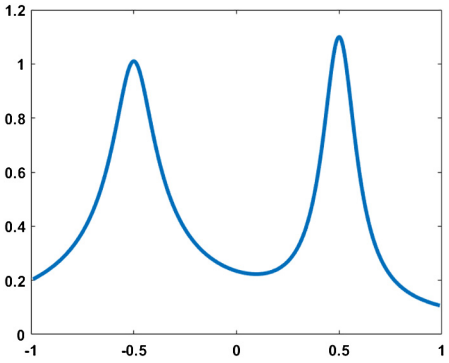

Fig. 2. Example 2: Graphic of the right hand-side function g.

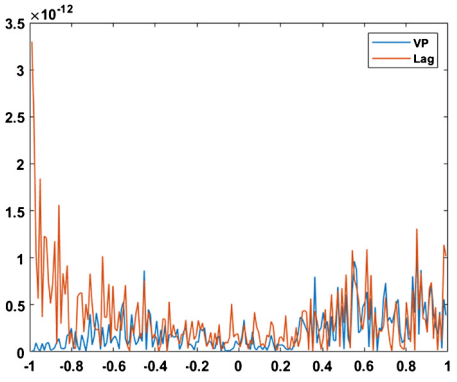

Fig. 3. Pointwise weighted errors $n = 1024$, $\theta = 0.12$.

Table 5
Example 3.

n	θ	$\tilde{\mathcal{E}}_{n+m}^{VP}$	cond_{n+m}^{VP}	$\tilde{\mathcal{E}}_{n+m}^{Lag}$	cond_{n+m}^{Lag}
32	0.3	1.21e-04	3.54e+01	4.81e-05	4.05e+01
64	0.3	8.65e-06	3.54e+01	5.84e-06	4.31e+01
128	0.3	6.69e-07	3.54e+01	5.41e-07	4.55e+01
256	0.3	5.52e-08	3.54e+01	2.52e-08	4.80e+01
512	0.3	4.59e-09	3.54e+01	2.17e-09	5.05e+01
1024	0.3	4.15e-10	3.54e+01	3.09e-10	5.31e+01
2048	0.3	2.94e-11	3.54e+01	5.33e-11	5.57e+01

Example 4. This test can be found in [22]. Let be in (1)

$$\alpha = \frac{2}{3}, \qquad \nu = 1, \qquad u_+ = v^{\frac{2}{3},0},$$
$$k(x,y) = |x-y|^{-\frac{1}{8}}, g(y) = 1 + y^2.$$

The theoretical order of convergence is $\mathcal{O}\left(\frac{1}{n^{\frac{7}{8}}}\right)$ for our method and $\mathcal{O}\left(\frac{\log(n+m)}{(n+m)^{\frac{7}{8}}}\right)$ for the Lag-method. We start comparing the values attained for $y = 0.5$ and $y = 0.9$ by the approximant $f_n^m(y)$ with those achieved by $f_n^{Lag}(y)$ (instead of f_{n+m}^{Lag},) since these are provided in [22, Ex.1, p.323].

As we can see from Table 6, the results are almost comparable, since the same numbers of fixed digits are attained by both the methods.

Now, similarly to the previous tests, the maximum weighted errors and the condition numbers of the linear systems are shown in Table 7, where we have computed f_{n+m}^{Lag} to obtain final systems of the same orders. The convergence results are comparable to each other, while the condition numbers of the linear systems are bounded in the VP method, but moderately diverge in the Lag-method.

Example 5. *Consider the equation* (1), *with*

Table 6
Example 4: approximate values for $y = 0.5$ and $y = 0.9$.

		$y = 0.5$		$y = 0.9$	
n	θ	$f_n^m(y)$	$f^{Lag}(y)$	$f_n^m(y)$	$f_n^{Lag}(y)$
16	0.4	1.00	1.00099	6.346e-01	6.3465e-01
32	0.3	1.00099	1.000993	6.34659e-01	6.346596e-01
64	0.1	1.000993	1.0009932	6.34659e-01	6.346596e-01
128	0.4	1.00099326	1.0009932	6.3465960e-01	6.3465960e-01
256	0.1	1.000993269	1.000993269	6.34659602e-01	6.34659602e-01
512	0.4	1.0009932693	1.0009932693	6.34659602e-01	6.34659602e-01

Table 7
Example 4.

n	θ	$\mathcal{E}_{n+m}^{VP}$	cond_{n+m}^{VP}	$\mathcal{E}_n^{Lag}$	cond_n^{Lag}
16	0.1	3.93e-04	7.57	3.53e-04	11.705
32	0.1	3.59e-05	7.58	3.07e-05	12.310
64	0.4	3.98e-06	7.59	1.76e-06	12.724
128	0.2	3.12e-07	7.59	3.28e-07	12.855
256	0.1	1.99e-08	7.59	2.86e-08	12.947
512	0.2	2.14e-09	7.59	1.56e-09	13.015
1024	0.1	1.73e-10	7.59	2.21e-10	13.045

Table 8
Example 5:maximum errors.

n	θ	$\mathcal{E}_{n+m}^{VP}$	cond_{n+m}^{VP}	$\mathcal{E}_n^{Lag}$	cond_n^{Lag}
16	0.1	3.7007e-01	26.3	4.3809e-01	44.48
32	0.2	3.6803e-01	26.3	3.4120e-01	46.58
64	0.1	2.3635e-01	26.3	2.3080e-01	47.51
128	0.1	2.2164e-01	26.3	2.1441e-01	48.15
256	0.1	1.0897e-01	26.3	1.4869e-01	48.53
512	0.1	9.8079e-02	26.3	9.0089e-02	48.76

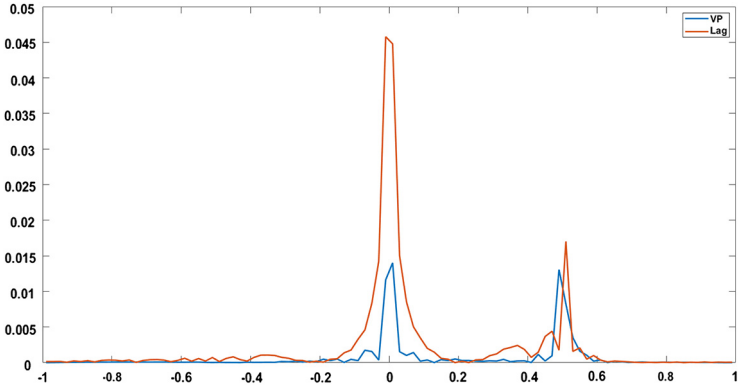

Fig. 4. Example 5: Pointwise weighted errors for $n = 256$, and $\theta = 0.1$.

$$\alpha = 0.89,\ \nu = 1,\ u_+ = v^{0.89,0},$$
$$k(x, y) = |x - y|^{\frac{1}{3}},$$
$$g(y) = |x|^{\frac{1}{11}} + |x - 0.5|^{\frac{1}{7}}.$$

In this case we use VP1 and the Lag-method as implemented in [22]. The function $g \in Z_{\frac{1}{11}}(u_-)$ and the theoretical order of convergence is $\mathcal{O}\left(\frac{1}{n^{\frac{1}{11}}}\right)$ for our method and $\mathcal{O}\left(\frac{\log(n+m)}{(n+m)^{\frac{1}{11}}}\right)$ for the Lag-method. See Table 8 and Fig. 4.

6. Proofs of the main results

Proof of Theorem 3.2. Inequality (28) has been already stated in [28, Lemma 4.3].

In order to prove (29), we note that by (3), (10), (24) and (7), we get

$$\begin{aligned}
&\|D[f - V_n^m(u,f)]u_-\|_\infty \\
&\le \mathcal{C}\|[f - V_n^m(u,f)]u_+\|_\infty + \mathcal{C}\sup_{|y|\le 1}\left|\int_{-1}^{1}\frac{f(x)-V_n^m(u,f,x)}{x-y}u(x)dx\right|u_-(y) \\
&\le \mathcal{C}\int_0^{\frac{1}{n}}\frac{\Omega_\varphi^s(f,t)_{u_+}}{t}dt + \mathcal{C}\sup_{|y|\le 1}\left|\int_{-1}^{1}\frac{f(x)-V_n^m(u,f,x)}{x-y}u(x)dx\right|u_-(y)
\end{aligned}$$

where $\mathcal{C}\neq\mathcal{C}(n,m,f)$. Hence, the statement follows recalling that by [28, Eq. (50)] we have

$$\left|\int_{-1}^{1}\frac{f(x)-V_n^m(u,f,x)}{x-y}u(x)dx\right|u_-(y) \le \mathcal{C}\int_0^{\frac{1}{n}}\frac{\Omega_\varphi^s(f,t)_{u_+}}{t}dt, \qquad \mathcal{C}\neq\mathcal{C}(n,m,f,y). \quad \diamond$$

Proof of Theorem 4.1. The statements (42) and (43) follow by applying estimates (27) and (30) with f, u, u_+ replaced by Kf, u^{-1}, u_-, respectively, and taking into account that we assumed

$$\|Kf\|_{Z_r(u_-)} \le C\|fu_+\|_\infty, \qquad \mathcal{C}\neq\mathcal{C}(f). \quad \diamond$$

Proof of Theorem 4.2. Let us first prove (44). Taking into account that

$$Kf - \bar{K}_n f = (Kf - K_n f) - (\bar{K}_n f - K_n f),$$

and observing that Proposition 2.4 allows us to apply Theorem 4.1 for estimating the norm of $Kf - K_n f$, it is sufficient to prove that

$$\|K_n f - \bar{K}_n f\|_{C^0_{u_-}} \le \frac{\mathcal{C}}{n^r}\|fu_+\|_\infty, \qquad \mathcal{C}\neq\mathcal{C}(n,f). \tag{65}$$

To this aim we observe that, using the notation introduced in (13) and the univariate VP polynomial, if we set

$$K_n^* f(y) = \int_{-1}^{1} V_n^m(u, k_y^{(2)}, x) f(x)u(x)dx, \tag{66}$$

then the operator $\bar{K}_n$ can be equivalently written as follows

$$\bar{K}_n f(y) = V_n^m(u^{-1}, K_n^* f, y) \tag{67}$$

Consequently, applying Theorem 3.1 (with f, u replaced by $Kf - K_n^* f, u^{-1}$ resp.) we get

$$\|K_n f - \bar{K}_n f\|C^0_{u_-} = \|V_n^m(u^{-1}, Kf - K_n^* f)\|C^0_{u_-} \le \|Kf - K_n^* f\|C^0_{u_-},$$

and to prove (65) we are going to state

$$\|Kf - K_n^* f\|C^0_{u_-} \le \frac{\mathcal{C}}{n^r}\|fu_+\|_\infty. \tag{68}$$

This can be proved by applying (35) (with f replaced by $k_y^{(2)}$), as follows

$$\begin{aligned}
\|Kf - K_n^* f\|_{C^0_{u_-}} &\le \max_{|y|\le 1} u_-(y)\int_{-1}^{1}\left|k_y^{(2)}(x) - V_n^m(u,k_y^{(2)},x)\right||f(x)|\frac{u_+(x)}{u_-(x)}dx \\
&\le \|fu_+\|_\infty \max_{|y|\le 1} u_-(y)\left\|[k_y^{(2)} - V_n^m(u,k_y^{(2)})]1/u_-\right\|_1 \\
&\le \mathcal{C}\|fu_+\|_\infty \max_{|y\le 1} u_-(y)\frac{\|k_y\|_{Z_r^1(1/u_-)}}{n^r} \le \frac{\mathcal{C}}{n^r}\|fu_+\|_\infty
\end{aligned}$$

Now let us prove (45). For any $s\in]0,r]$, by using (44) we have

$$\|Kf-\bar K_n f\|_{Z_s(u_-)}=\|Kf-\bar K_n f\|_{C^0_{u_-}}+\sup_{i\in N_0}(1+i)^s E_i(Kf-\bar K_n f)_{u_-}$$
$$\le\frac{\mathcal{C}}{n^r}\|fu_+\|_\infty+\sup_{i\in N_0}(1+i)^s E_i(Kf-\bar K_n f)_{u_-}. \tag{69}$$

In order to estimate the last addendum, we distinguish two cases.

For all integers $i\ge n+m-1$, taking into account that $\bar K_n f\in\mathbb{P}_{n+m-1}$ for any $f\in C^0_{u_+}$, and using (9), we have

$$E_i(Kf-\bar K_n f)_{u_-}=E_i(Kf)_{u_-}\le\mathcal{C}\frac{\|Kf\|_{Z_r(u_-)}}{(1+i)^r}\le\mathcal{C}\frac{\|f\|_{C^0_{u_+}}}{(1+i)^r}.$$

Thus, recalling that $s\le r$ we get

$$\sup_{i\ge n+m-1}(1+i)^s E_i(Kf-\bar K_n f,t)_{u_-}\le\mathcal{C}\sup_{i\ge n+m-1}(1+i)^{s-r}\|f\|_{C^0_{u_+}}\le\mathcal{C}\frac{\|f\|_{C^0_{u_+}}}{n^{r-s}}. \tag{70}$$

On the other hand, by using (42) and (65), for all integers $0\le i<n+m-1$ we get

$$E_i(Kf-\bar K_n f,t)_{u_-}\le\|Kf-\bar K_n f)_{u_-}\|_\infty\le\frac{\mathcal{C}}{n^r}\|fu_+\|_\infty,$$

and hence

$$\sup_{i<n+m-1}(1+i)^s E_i(Kf-\bar K_n f)_{u_-}\le\mathcal{C}\frac{\|fu_+\|_\infty}{n^{r-s}} \tag{71}$$

In conclusion, (45) follows by (69)–(71). ◇

Proof of Proposition 4.3. Let us write the equation as follows

$$Df=g_n-\nu T_n f,\qquad T_n\in\{K_n,\bar K_n\}$$

and apply the operator $\hat D$ to both the members. By Theorem 2.1 we get

$$f=\hat D[g_n-\nu T_n f]$$

where $g_n-\nu T_n f\in\mathbb{P}_{n+m-1}$ holds for all f satisfying the above identity. Hence, the statement follows by taking into account $\hat D$ maps polynomials into polynomials of the same degree [34]. ◇

Proofs of Theorems 4.4 and 4.5. The proofs can be lead in a standard way starting by the strong convergence of the sequences $\{K_n\}_n$ and $\{\bar K_n\}_n$ to $K:Z_r(u_+)\to Z_r(u_-))$ that can be easily deduced by (43) and (45) respectively. The interested reader can consult for instance the proof of Theorem 4.2 in [23]. ◇

Proof of Theorem 4.6. First of all we recall that the operator $\hat D$ defined in (12) is the inverse operator of D. Thus instead of (1) and (36) we can consider the corresponding "regularized" equations

$$(I+\nu\hat D K)f=\hat D g\qquad\text{and}\qquad(I+\nu\hat D K_n)f=\hat D g_n \tag{72}$$

Note that, by our assumptions, these equations are equivalent to (1) and (36) but they can be considered also in the larger spaces $C^0_{u_+}$ and $Z_s(u_+)$ with $0<s<r$. This fact can be proved following standard arguments (see, e.g. [22]) by taking into account that

$$\lim_{\substack{n\to\infty\\ m\approx n}}\|\hat D K-\hat D K_n\|_{C^0_{u_+}\to C^0_{u_+}}=0=\lim_{\substack{n\to\infty\\ m\approx n}}\|\hat D K-\hat D K_n\|_{Z_s(u_+)\to Z_s(u_+)}$$

holds since, from the boundedness of $\hat D$ and (43), we get for all $f\in C^0 u_+$ the following

$$\|(\hat D K-\hat D K_n)f\|_{C^0_{u_+}}\le\|(\hat D K-\hat D K_n)f\|_{Z_s(u_+)}\le\mathcal{C}\|(K-K_n)f\|_{Z_s(u_-)}\le\frac{\mathcal{C}}{n^{r-s}}\|f\|_{C^0_{u_+}}.$$

Hence, in particular, our assumptions ensure that for all $m\approx n$, the operators $(I+\hat D K_n)$ are uniformly bounded and invertible with uniformly bounded inverses if we take both domain and range equal to $C^0_{u_+}$ or $Z_s(u_+)$ with $0<s<r$.

Moreover, starting from (72), it is possible to deduce the following identity concerning the solutions f^* and f_n^m of equation (1) and (36), respectively

$$f^* - f_n^m = (I + \nu \hat{D} K_n)^{-1} \left[\hat{D}(g - g_n) + \nu \hat{D}(K_n f^* - K f^*) \right]. \tag{73}$$

In order to prove (59), we use that $(I + \nu \hat{D} K_n)^{-1} : C^0_{u_+} \to C^0_{u_+}$ is uniformly bounded w.r.t. n, and apply (29) with D, u, u_+ replaced by $\hat{D}, u^{-1}, u_-$, respectively. In this way, by (73) we get

$$\begin{aligned} \|(f^* - f_n^m) u_+\|_\infty &\le \mathcal{C} \left[\|\hat{D}(g - g_n) u_+\|_\infty + \|\hat{D}(K_n f^* - K f^*) u_+\|_\infty \right] \\ &\le \mathcal{C} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi(g, t)_{u_-}}{t} dt + \mathcal{C} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi(K f^*, t)_{u_-}}{t} dt \\ &\le \mathcal{C} \frac{\|g\|_{Z_r(u_-)}}{n^r} + \mathcal{C} \frac{\|K f^*\|_{Z_r(u_-)}}{n^r} \\ &\le \mathcal{C} \frac{\|g\|_{Z_r(u_-)}}{n^r} + \mathcal{C} \frac{\|f^*\|_{C^0_{u_+}}}{n^r} \end{aligned} \tag{74}$$

and (59) follows taking into account that

$$\|f^*\|_{C^0_{u_+}} \le \|f^*\|_{Z_r(u_+)} = \|(D + \nu K)^{-1} g\|_{Z_r(u_+)} \le \mathcal{C} \|g\|_{Z_r(u_-)}.$$

Similarly, in order to prove (60) we start once again from (73). Hence, using the uniform boundedness of the operators $(I + \nu \hat{D} K_n)^{-1} : Z_s(u_+) \to Z_s(u_+)$ and $\hat{D} : Z_s(u_-) \to Z_s(u_+)$, applying Theorem 3.3 with u, u_+ replaced by u^{-1}, u_- and recalling (43), we deduce

$$\begin{aligned} \|(f^* - f_n^m) u_+\|_{Z_s(u_+)} &\le \mathcal{C} \left[\|\hat{D}(g - g_n)\|_{Z_s(u_+)} + \|\hat{D}(K_n f^* - K f^*)\|_{Z_s(u_+)} \right] \\ &\le \mathcal{C} \left[\|g - g_n\|_{Z_s(u_-)} + \|K_n f^* - K f^*\|_{Z_s(u_-)} \right] \\ &\le \mathcal{C} \frac{\|g\|_{Z_r(u_-)}}{n^{r-s}} + \mathcal{C} \frac{\|f^*\|_{C^0_{u_+}}}{n^{r-s}} \le \mathcal{C} \frac{\|g\|_{Z_r(u_-)}}{n^{r-s}}. \quad \diamond \end{aligned}$$

Proof of Theorem 4.7. In order to prove (61) we can proceed as in the proof of Theorem 4.6 starting now from

$$(I + \nu \hat{D} \bar{K}_n) \overline{f_n^m} = \hat{D} g_n.$$

Thus by (73) we get

$$\|(f^* - \overline{f_n^m}) u_+\|_\infty \le \mathcal{C} \left[\|\hat{D}(g - g_n) u_+\|_\infty + \|\hat{D}(\bar{K}_n f^* - K f^*) u_+\|_\infty \right]. \tag{75}$$

In order to estimate the second norm at the right–hand side we remark that

$$\|\hat{D}(\bar{K}_n f^* - K f^*) u_+\|_\infty = \|\hat{D}(\bar{K}_n f^* - K_n f^*) u_+\|_\infty + \|\hat{D}(K_n f^* - K f^*) u_+\|_\infty$$

and the second term is the same as in (74). On the other hand, recalling the definition of K_n and (67) we get

$$\|\hat{D}(\bar{K}_n f^* - K_n f^*) u_+\|_\infty = \|\hat{D} V_n^m(u^{-1}, K_n^* f^* - K f^*)) u_+\|_\infty.$$

By (28) applied to $f = K^* f^* - K f^*$, with $\hat{D}, u^{-1}, u_+$ in place of D, u, u_- respectively, it follows that

$$\|\hat{D} V_n^m(u^{-1}, K_n^* f - K f)) u_+\|_\infty \le \mathcal{C} \|(K_n^* f^* - K f^*) u_-\|_\infty.$$

Recalling the definition of K_n^* given in (66), the right hand side can be easily bounded by using the L^1 convergence estimate (35) and we finally get

$$\|\hat{D}(\bar{K}_n f^* - K_n f^*) u_+\|_\infty \le \mathcal{C} \frac{\|f u_+\|_\infty}{n^r} \max_{x \in [-1,1]} \|k_x^{(2)}\|_{Z_r(u_-^{-1})}. \tag{76}$$

Therefore (61) follows by (75), using (76) and (29) applied to $f = g$, with $\hat{D}, u^{-1}, u_+$ in place of D, u, u_-, respectively.

Finally, in order to prove (62) we can repeat word by word the proof of (60) in Theorem 4.6 and using (45) instead of (43). $\diamond$

7. Conclusion

We present a new numerical method for solving CSIE in order to improve the local accuracy provided by classical numerical methods based on Lagrange interpolation at Jacobi zeros.

Indeed, it is well–known that Lagrange–based methods provide optimal error estimates w.r.t. weighted L^2–norms. Apart from a logarithmic factor, the same holds with weighted L^∞–norms, getting a convergence rate comparable with the error of best polynomial uniform approximation of the unknown function f. The latter, in turn, depends on the smoothness of the known functions in the CSIE. However, in case of smooth data functions with isolated singularities, Lagrange interpolation is not always able to provide a better approximation in the smoother parts, suffering of the Gibbs phenomenon.

To overcome this problem and improve the local accuracy, here, we propose to use VP polynomial approximation at Jacobi zeros.

The resulting numerical method has been deeply investigated from both the theoretical and computational/numerical sides.

In theory, similarly to Lagrange–based method, we prove the convergence order is comparable with that of the best uniform polynomial approximation of f, but without any logarithmic factor. Moreover, we get stability and well–conditioning results not achievable by Lagrange–based methods.

In practice, the absence/presence of the logarithmic factor is not revealed by the numerical experiments, that show almost comparable values of the maximum numerical error. However, compared with Lagrange–based methods, the linear system associated to our method displays condition numbers that do not increase with the dimension of the system. Moreover, in case of isolated singularities in the data, we obtain a much better behavior of the pointwise error in the more regular parts, generally interested by the Gibbs phenomenon.

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