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A DILATION QUADRATURE FORMULA FOR HYPERSINGULAR AND HIGHLY OSCILLATORY INTEGRALS ON THE POSITIVE HALF-LINE*

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Abstract. The aim of this paper is to introduce a new quadrature rule for approximating integrals with highly oscillatory and hypersingular integrands defined on the positive half-line. After the integration interval is split into the subintervals $[0, M]$ and $[M, +\infty)$, so that the part on $[M, +\infty)$ is negligible, the interval $[0, M]$ is suitably dilated and decomposed into a sum of integrals, where each of them is approximated by a Gaussian quadrature rule. We prove that the formula is convergent when the function f is bounded on $\mathbb{R}^+$ together with a certain number of its derivatives. Numerical tests compare the performance of the proposed rule with other formulas available in the literature.

Key words. Hadamard integrals, Cauchy integrals, hypersingular integrals, highly oscillatory functions, dilation rule, Gaussian rule

AMS subject classifications. 65D30, 65R10, 41A05, 42B20

1. Introduction.

This paper addresses the numerical computation of integrals of the type

$$H_p^{\omega,\gamma}(f,t) = \text{⨍}_0^{+\infty} \frac{f(x)x^\gamma e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx, \tag{1.1}$$

where $t > 0$, $\omega \gg 1$, $\mathrm{i}^2 = -1$, $|\gamma| < 1$, and where f is a sufficiently smooth function and the parameter p is a nonnegative integer. If $p = 0$, then the integral is understood in the Cauchy principal value sense, otherwise in the finite-part Hadamard sense.

The integrals $H_p^{\omega,\gamma}(f)$ play an important role in many areas of physics and engineering, such as fluid mechanics, optics, electromagnetics, and seismology image processing. To our knowledge, extensive literature only exists for the numerical evaluation of singular or hypersingular integrals defined on finite or infinite intervals (see, for instance, [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 17, 19, 21, 25, 26, 27, 28, 29, 30, 31, 33, 36] and the references therein). The same applies to integrals with highly oscillatory integrands (see, for instance, [3, 14, 18, 20, 22, 23, 32, 37] and the references therein). However, very few papers exist addressing integrals that are both singular and oscillatory on infinite intervals (see, for instance, [2, 15, 35]).

Recently, in [15] we introduced a quadrature rule of product type to approximate (1.1) in the case $\gamma \geq 0$ and where the function f has an exponential decay at infinity, i.e., it can be written as $f(x) = g(x)e^{-\frac{x}{2}}$. It is constructed by approximating the function g by the truncated Lagrange polynomial based on the zeros of the Laguerre polynomials and an additional point. The rule is stable and convergent when the function g belongs to suitable weighted Sobolev-type spaces.

Here, we introduce a new method for computing (1.1) that is based on the dilation formula introduced in [14]. First, the integral is split into a sum of two integrals defined on the intervals $[0, M]$ and $[M, +\infty)$, respectively, where $M \geq 0$ is chosen such that the value of the latter integral is negligible. Then, the finite interval $[0, M]$ is dilated in order to reduce the oscillation of the function $e^{\mathrm{i}\omega x}$. The resulting integral is written as sum of a certain number of integrals,

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ETNA
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where each of them is approximated by a Gaussian rule. The proposed quadrature rule is convergent for functions f that are bounded on $\mathbb{R}^+$ and have a sufficient number of bounded derivatives of order larger than p. Its efficiency is confirmed by the numerical tests. Comparing its performance with that of the product rule in [15], we can observe that it is more efficient when the function f does not exhibit an exponential decay and is not very smooth. Moreover, unlike the product rule, it can also be applied when the parameter γ is negative.

The outline of the paper is as follows. In the next section, we recall some definitions and preliminary results. In Section 3, we describe the proposed quadrature rule, giving its computational details in Section 4 and in the Appendix A. In Section 5 some numerical tests confirming its efficiency will be shown, and comparisons with the results in [2, 15, 35] are presented. Finally, to conclude, Section 6 is dedicated to the proof of the theorem stated in Section 3.

2. Basic definitions and preliminary results. Throughout the paper we denote by $\mathcal{C}$ a positive constant which may take different values in different formulas, and we write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ to indicate that $\mathcal{C}$ does not depend on the variables $a, b \dots$

2.1. Function spaces. We denote by $C^0(\mathcal{I})$ the space of all bounded and continuous functions on $\mathcal{I} \subset \mathbb{R}^+ = [0, +\infty)$, endowed with the usual uniform norm

$$\|f\|_{\mathcal{I}} := \max_{x \in \mathcal{I}} |f(x)|.$$

In particular, we write $\|f\|_\infty := \max_{x \in \mathbb{R}^+} |f(x)|$. Moreover, we denote by $C^r(\mathcal{I}), r \geq 1$, the space of all functions f such that $f^{(r)} \in C^0(\mathcal{I})$. We also consider the following Sobolev-type subspace of $C^0([-1, 1])$ of order $r \in \mathbb{N}, r \geq 1$,

$$W_r = \left\{ f \in C^0([-1,1]) \,:\, f^{(r-1)} \in AC(-1,1) \text{ and } \|f^{(r)}\varphi^r\|_{[-1,1]} < +\infty \right\},$$

where $\varphi(x) = \sqrt{1-x^2}$ and $AC(-1, 1)$ denotes the set of absolutely continuous functions on $(-1, 1)$. Furthermore, we denote by

$$E_m(f)_\infty := \inf_{P \in \mathbb{P}_m} \|f - P\|_{[-1,1]}$$

the error of the best approximation of a function $f \in C^0([-1, 1])$ by polynomials of degree at most m. For functions belonging to W_r, the following estimate holds true (see, for example, [24, p. 172 (2.5.22)]):

$$E_m(f)_\infty \leq \frac{\mathcal{C}}{m^r} \|f^{(r)}\varphi^r\|_{[-1,1]}, \qquad \mathcal{C} \neq \mathcal{C}(m, f). \tag{2.1}$$

2.2. Gauss–Jacobi quadrature rule. We recall the well-known Gauss–Jacobi quadrature rule defined as

$$\int_{-1}^{1} f(x) v^{\alpha,\beta}(x) dx = \sum_{j=1}^{m} f(x_j^{\alpha,\beta}) \lambda_j^{\alpha,\beta} + e_m^{\alpha,\beta}(f) =: \Phi_m^{\alpha,\beta}(f) + e_m^{\alpha,\beta}(f),$$

where $x_j^{\alpha,\beta} := x_{m,j}^{\alpha,\beta}$ and $\lambda_j^{\alpha,\beta} := \lambda_{m,j}^{\alpha,\beta}$, $j = 1, \dots, m$, denote the nodes and the weights of the rule $\Phi_m^{\alpha,\beta}(f)$ and $e_m^{\alpha,\beta}(f)$ is the remainder term. In the sequel, in the case of the Gauss–Legendre quadrature rule, we set $x_j := x_j^{0,0}, \lambda_j := \lambda_j^{0,0}$, $\Phi_m(f) := \Phi_m^{0,0}(f)$, and $e_m(f) := e_m^{0,0}(f)$.

For all $f \in C^0([-1, 1])$, the quadrature error satisfies the following bound (see, for example, [24, Theorem 5.1.6]):

$$|e_m^{\alpha,\beta}(f)| \leq 2 \left(\int_{-1}^{1} v^{\alpha,\beta}(x) dx \right) E_{2m-1}(f)_\infty. \tag{2.2}$$

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3. The numerical method: a dilation-type rule. In this section we describe the proposed numerical method for the computation of the integral (1.1) for an arbitrary but fixed $t > 0$. As a first step we suppose that there exists a real number M such that for some $\eta > 0$ and $d \in (0, \omega]$ the following condition holds:

$$\left| \frac{f(x)x^\gamma}{(x-t)^{p+1}} \right| < \frac{1}{x^{\eta+1}}, \qquad \forall x > M, \quad x \notin I_t, \tag{3.1}$$

where

$$I_t = \begin{cases} [t - \frac{d}{\omega}, t + \frac{d}{\omega}] & \text{if } t > \frac{d}{\omega}, \\ [0, t + \frac{d}{\omega}] & \text{if } 0 < t \le \frac{d}{\omega}. \end{cases}$$

We then consider the following decompositions of the integral (1.1):

[A] If $0 < t \le \frac{2d}{\omega}$, then

$$H_p^{\omega,\gamma}(f,t) = \left\{ \int\!\!\!\!\!\!-_0^{t+\frac{d}{\omega}} + \int_{t+\frac{d}{\omega}}^{M} + \int_M^{+\infty} \right\} \frac{f(x)x^\gamma e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\, dx.$$

[B] If $\frac{2d}{\omega} < t < M - \frac{d}{\omega}$, then

$$H_p^{\omega,\gamma}(f,t) = \left\{ \int_0^{t-\frac{d}{\omega}} + \int\!\!\!\!\!\!-_{t-\frac{d}{\omega}}^{t+\frac{d}{\omega}} + \int_{t+\frac{d}{\omega}}^{M} + \int_M^{+\infty} \right\} \frac{f(x)x^\gamma e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\, dx.$$

[C] If $M - \frac{d}{\omega} \le t \le M + \frac{d}{\omega}$, then

$$H_p^{\omega,\gamma}(f,t) = \left\{ \int_0^{t-\frac{d}{\omega}} + \int\!\!\!\!\!\!-_{t-\frac{d}{\omega}}^{t+\frac{d}{\omega}} + \int_{t+\frac{d}{\omega}}^{+\infty} \right\} \frac{f(x)x^\gamma e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\, dx.$$

[D] If $t > M + \frac{d}{\omega}$, then

$$H_p^{\omega,\gamma}(f,t) = \left\{ \int_0^{M} + \int_M^{t-\frac{d}{\omega}} + \int\!\!\!\!\!\!-_{t-\frac{d}{\omega}}^{t+\frac{d}{\omega}} + \int_{t+\frac{d}{\omega}}^{+\infty} \right\} \frac{f(x)x^\gamma e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\, dx.$$

Let

$$J_1 = [M, +\infty), \qquad J_2 = \left[t + \frac{d}{\omega}, +\infty \right), \qquad J_3 = \left[M, t - \frac{d}{\omega} \right).$$

Under assumption (3.1), we have

$$\left| \int_{J_i} \frac{f(x)x^\gamma}{(x-t)^{p+1}} e^{\mathrm{i}\omega x}\, dx \right| < \frac{1}{\eta M^\eta},$$

and if M is sufficiently large, then we can neglect the corresponding integrals and only approximate the remaining integrals:

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[A] If $0 < t \le \frac{2d}{\omega}$, then

$$H_{p,M}^{\omega,\gamma}(f,t) := \left\{ \text{⨍}_0^{t+\frac{d}{\omega}} + \int_{t+\frac{d}{\omega}}^{M} \right\} \frac{f(x)x^\gamma e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\, dx.$$

[B] If $\frac{2d}{\omega} < t < M - \frac{d}{\omega}$, then

$$H_{p,M}^{\omega,\gamma}(f,t) := \left\{ \int_0^{t-\frac{d}{\omega}} + \text{⨍}_{t-\frac{d}{\omega}}^{t+\frac{d}{\omega}} + \int_{t+\frac{d}{\omega}}^{M} \right\} \frac{f(x)x^\gamma e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\, dx.$$

[C] If $M - \frac{d}{\omega} \le t \le M + \frac{d}{\omega}$, then

$$H_{p,M}^{\omega,\gamma}(f,t) := \left\{ \int_0^{t-\frac{d}{\omega}} + \text{⨍}_{t-\frac{d}{\omega}}^{t+\frac{d}{\omega}} \right\} \frac{f(x)x^\gamma e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\, dx.$$

[D] If $t > M + \frac{d}{\omega}$, then

$$H_{p,M}^{\omega,\gamma}(f,t) := \left\{ \int_0^{M} + \text{⨍}_{t-\frac{d}{\omega}}^{t+\frac{d}{\omega}} \right\} \frac{f(x)x^\gamma e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\, dx.$$

Following an idea introduced in [14], in order to reduce the oscillations of the function $e^{\mathrm{i}\omega x}$, we make the change of variable $y = \omega x$. Thus, we obtain:

[A] If $0 < \omega t \le 2d$, then

$$H_{p,M}^{\omega,\gamma}(f,t) := \frac{1}{\omega^{\gamma+1}} \left\{ \text{⨍}_0^{\omega t+d} + \int_{\omega t+d}^{\omega M} \right\} \frac{f\left(\frac{y}{\omega}\right)}{\left(\frac{y}{\omega}-t\right)^{p+1}} y^\gamma e^{\mathrm{i}y} dy.$$

[B] If $2d < \omega t < \omega M - d$, then

$$H_{p,M}^{\omega,\gamma}(f,t) := \frac{1}{\omega^{\gamma+1}} \left\{ \int_0^{\omega t-d} + \text{⨍}_{\omega t-d}^{\omega t+d} + \int_{\omega t+d}^{\omega M} \right\} \frac{f\left(\frac{y}{\omega}\right)}{\left(\frac{y}{\omega}-t\right)^{p+1}} y^\gamma e^{\mathrm{i}y} dy.$$

[C] If $\omega M - d \le \omega t \le \omega M + d$, then

$$H_{p,M}^{\omega,\gamma}(f,t) := \frac{1}{\omega^{\gamma+1}} \left\{ \int_0^{\omega t-d} + \text{⨍}_{\omega t-d}^{\omega t+d} \right\} \frac{f\left(\frac{y}{\omega}\right)}{\left(\frac{y}{\omega}-t\right)^{p+1}} y^\gamma e^{\mathrm{i}y} dy.$$

[D] If $\omega t > \omega M + d$, then

$$H_{p,M}^{\omega,\gamma}(f,t) := \frac{1}{\omega^{\gamma+1}} \left\{ \int_0^{\omega M} + \text{⨍}_{\omega t-d}^{\omega t+d} \right\} \frac{f\left(\frac{y}{\omega}\right)}{\left(\frac{y}{\omega}-t\right)^{p+1}} y^\gamma e^{\mathrm{i}y} dy.$$

Letting

$$I_1 = [\omega t + d, \omega M], \qquad I_2 = [0, \omega t - d], \qquad I_3 = [0, \omega M]$$

and

$$I_1^* = [0, \omega t + d], \qquad I_2^* = [\omega t - d, \omega t + d],$$

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we note that the integrals over I_i, $i = 1, 2, 3$, are nonsingular since $|\omega t - y| > d$ for any $y \in I_i$, $i = 1, 2, 3$. Concerning the singular integrals over I_j^*, $j = 1, 2$, and denoting by

$$T_p(f, x, t) := \sum_{r=0}^{p} \frac{f^{(r)}(t)}{r!}(x - t)^r$$

the Taylor polynomial of degree p of the function f at the point t, we use the standard decomposition

$$\begin{aligned} H_{p,I_j^*}^{\omega,\gamma}(f,t) &:= \frac{1}{\omega^{\gamma+1}} \oint_{I_j^*} \frac{f\left(\frac{y}{\omega}\right)}{\left(\frac{y}{\omega} - t\right)^{p+1}} y^\gamma e^{\mathrm{i}y} dy \\ &= \frac{1}{\omega^{\gamma+1}} \left(\int_{I_j^*} \frac{f\left(\frac{y}{\omega}\right) - T_p\left(f, \frac{y}{\omega}, t\right)}{\left(\frac{y}{\omega} - t\right)^{p+1}} y^\gamma e^{\mathrm{i}y}\, dy \right. \\ &\qquad \left. + \sum_{r=0}^{p} \frac{f^{(r)}(t)}{r!} \oint_{I_j^*} \frac{y^\gamma e^{\mathrm{i}y}}{\left(\frac{y}{\omega} - t\right)^{p-r+1}}\, dy \right) \\ &=: \frac{1}{\omega^{\gamma+1}} \left(\int_{I_j^*} \frac{f\left(\frac{y}{\omega}\right) - T_p\left(f, \frac{y}{\omega}, t\right)}{\left(\frac{y}{\omega} - t\right)^{p+1}} y^\gamma e^{\mathrm{i}y}\, dy + \sum_{r=0}^{p} \frac{f^{(r)}(t)}{r!} H_{p-r,I_j^*}^{\omega,\gamma}(t) \right) \\ &=: \frac{1}{\omega^{\gamma+1}} \left(\int_{I_j^*} \frac{f\left(\frac{y}{\omega}\right) - T_p\left(f, \frac{y}{\omega}, t\right)}{\left(\frac{y}{\omega} - t\right)^{p+1}} y^\gamma e^{\mathrm{i}y}\, dy + \mathcal{T}_{p,I_j^*}^{\omega,\gamma}(f,t) \right) \end{aligned} \tag{3.2}$$

in order to write the singular integral $H_{p,I_j^*}^{\omega,\gamma}(f,t)$ as the sum of a Riemann integral and a combination of the integrals $H_{p-r,I_j^*}^{\omega,\gamma}(t)$ that are independent of the function f and can be computed with high accuracy for every choice of t, ω, and p (see Section 4).

Moreover, denoting by a_i and b_i the endpoints of the intervals I_i, $i = 1, 2, 3$, we consider the following partition of the intervals I_i into $S_i + 1$ subintervals of size less or equal to d:

$$[a_i, b_i] = \bigcup_{k=0}^{S_i} A_{i,k}, \qquad \begin{aligned} A_{i,k} &:= [a_i + kd, a_i + (k+1)d], \quad k = 0, \ldots, S_i - 1, \\ A_{i,S_i} &= [a_i + S_i d, b_i], \end{aligned}$$

where $S_i = \left\lfloor \frac{b_i - a_i}{d} \right\rfloor$ ($\lfloor x \rfloor$ denotes the largest integer less than or equal to x). Then, we write

$$H_{p,I_i}^{\omega,\gamma}(f,t) = \frac{1}{\omega^{\gamma+1}} \sum_{k=0}^{S_i} \int_{A_{i,k}} \frac{f\left(\frac{y}{\omega}\right)}{\left(\frac{y}{\omega} - t\right)^{p+1}} y^\gamma e^{\mathrm{i}y} dy, \qquad i = 1, 2, 3.$$

Now, we transform the intervals over I_j^*, $j = 1, 2$, and $A_{i,k}$, $i = 1, 2, 3$, $k = 0, \ldots, S_i$, to the interval $[-1, 1]$. With the notation

$$G_p(f, x, t) := \frac{f(x)}{(x - t)^{p+1}}, \qquad R_p(f, x, t) := \frac{f(x) - T_p(f, x, t)}{(x - t)^{p+1}},$$

we get

$$H_{p,I_j^*}^{\omega,\gamma}(f,t) = \begin{cases} \displaystyle\int_{-1}^{1} F_j^*(x,t) v^{0,\gamma}(x) dx, & j = 1, \\ \displaystyle\int_{-1}^{1} F_j^*(x,t) dx, & j = 2, \end{cases} \tag{3.3}$$

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where

$$F_1^*(x,t) = \left(\frac{\omega t + d}{2\omega}\right)^{\gamma+1} R_p\left(f, \frac{\delta_1^*(x)}{\omega}, t\right) e^{\mathrm{i}\delta_1^*(x)},$$
$$F_2^*(x,t) = \frac{d}{\omega^{\gamma+1}} R_p\left(f, \frac{\delta_2^*(x)}{\omega}, t\right) e^{\mathrm{i}\delta_2^*(x)} \delta_2^*(x)^\gamma,$$

with

$$\delta_1^*(x) = \frac{(\omega t + d)(x+1)}{2}, \qquad \delta_2^*(x) = dx + \omega t,$$

and

$$H_{p,I_i}^{\omega,\gamma}(f,t) = \begin{cases} \displaystyle\int_{-1}^{1} F_{i,0}(x,t) v^{0,\gamma}(x)\,dx + \sum_{k=1}^{S_i} \int_{-1}^{1} F_{i,k}(x,t)\,dx, & i = 2,3, \\ \displaystyle\sum_{k=0}^{S_i} \int_{-1}^{1} F_{i,k}(x,t)\,dx, & i = 1, \end{cases}$$

where

$$F_{i,0}(x,t) = \left(\frac{d}{2\omega}\right)^{\gamma+1} G_p\Big(f, \frac{\delta_{i,0}(x)}{\omega}, t\Big) e^{\mathrm{i}\delta_{i,0}(x)}, \qquad i = 2,3,$$

$$F_{i,k}(x,t) = \frac{d}{2\omega^{\gamma+1}} G_p\Big(f, \frac{\delta_{i,k}(x)}{\omega}, t\Big) \delta_{i,k}(x)^\gamma e^{\mathrm{i}\delta_{i,k}(x)}, \qquad \begin{array}{l} i = 1,2,3, \\ k = 1, \ldots, S_i - 1, \end{array}$$

$$F_{i,S_i}(x,t) = \frac{b_i - a_i - S_i d}{2\omega^{\gamma+1}} G_p\Big(f, \frac{\delta_{i,S_i}(x)}{\omega}, t\Big) \delta_{i,S_i}(x)^\gamma e^{\mathrm{i}\delta_{i,S_i}(x)}, \quad i = 1,2,3,$$

with

$$\begin{array}{l} \delta_{i,k}(x) = \dfrac{d}{2}(x+1) + a_i + kd, \qquad k = 0, \ldots, S_i - 1, \\ \delta_{i,S_i}(x) = \Big(\dfrac{b_i - a_i - S_i d}{2}\Big)(x+1) + a_i + S_i d, \end{array} \qquad i = 1,2,3.$$

Summing up, we obtain:

[A] If $0 < \omega t \le 2d$, then

$$H_{p,M}^{\omega,\gamma}(f,t) = \int_{-1}^{1} F_1^*(x,t) v^{0,\gamma}(x)\,dx + \sum_{k=0}^{S_1} \int_{-1}^{1} F_{1,k}(x,t)\,dx + \frac{1}{\omega^{\gamma+1}} \mathcal{T}_{p,I_1^*}^{\omega,\gamma}(f,t).$$

[B] If $2d < \omega t < \omega M - d$, then

$$\begin{aligned} H_{p,M}^{\omega,\gamma}(f,t) = {} & \int_{-1}^{1} F_{2,0}(x,t) v^{0,\gamma}(x)\,dx + \sum_{k=1}^{S_2} \int_{-1}^{1} F_{2,k}(x,t)\,dx \\ & + \int_{-1}^{1} F_2^*(x,t)\,dx + \sum_{k=0}^{S_1} \int_{-1}^{1} F_{1,k}(x,t)\,dx + \frac{1}{\omega^{\gamma+1}} \mathcal{T}_{p,I_2^*}^{\omega,\gamma}(f,t). \end{aligned}$$

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[C] If $\omega M - d \le \omega t \le \omega M + d$, then

$$H_{p,M}^{\omega,\gamma}(f,t) = \int_{-1}^{1} F_{2,0}(x,t)v^{0,\gamma}(x)dx + \sum_{k=1}^{S_2}\int_{-1}^{1} F_{2,k}(x,t)dx + \int_{-1}^{1} F_2^*(x,t)dx + \frac{1}{\omega^{\gamma+1}}\mathcal{T}_{p,I_2^*}^{\omega,\gamma}(f,t).$$

[D] If $\omega t > \omega M + d$, then

$$H_{p,M}^{\omega,\gamma}(f,t) = \int_{-1}^{1} F_{3,0}(x,t)v^{0,\gamma}(x)dx + \sum_{k=1}^{S_3}\int_{-1}^{1} F_{3,k}(x,t)dx + \int_{-1}^{1} F_2^*(x,t)dx + \frac{1}{\omega^{\gamma+1}}\mathcal{T}_{p,I_2^*}^{\omega,\gamma}(f,t).$$

Finally, we arrive at an approximation of the integral $H_{p,M}^{\omega,\gamma}(f,t)$ for each of the four cases [A]–[D] by applying the Gauss–Jacobi quadrature rule with respect to the Jacobi weight $v^{0,\gamma}$ to the integrals containing $v^{0,\gamma}(x)$ and the Gauss–Legendre rule to the other integrals:

$$H_{p,M}^{\omega,\gamma}(f,t) := \mathcal{Q}_{p,m}^{\omega,\gamma}(f,t) + \mathcal{E}_{p,m}^{\omega,\gamma}(f,t),$$

with the following quadrature rules and the corresponding remainder terms:

[A] If $0 < \omega t \le 2d$, then

$$\mathcal{Q}_{p,m}^{\omega,\gamma}(f,t) := \Phi_m^{0,\gamma}(F_1^*(\cdot,t)) + \sum_{k=0}^{S_1}\Phi_m(F_{1,k}(\cdot,t)) + \frac{1}{\omega^{\gamma+1}}\mathcal{T}_{p,I_1^*}^{\omega,\gamma}(f,t),$$
$$\mathcal{E}_{p,m}^{\omega,\gamma}(f,t) := e_m^{0,\gamma}(F_1^*(\cdot,t)) + \sum_{k=0}^{S_1} e_m(F_{1,k}(\cdot,t)).$$

[B] If $2d < \omega t < \omega M - d$, then

$$\mathcal{Q}_{p,m}^{\omega,\gamma}(f,t) := \Phi_m^{0,\gamma}(F_{2,0}(\cdot,t)) + \sum_{k=1}^{S_2}\Phi_m(F_{2,k}(\cdot,t)) + \Phi_m(F_2^*(\cdot,t)) + \sum_{k=0}^{S_1}\Phi_m(F_{1,k}(\cdot,t)) + \frac{1}{\omega^{\gamma+1}}\mathcal{T}_{p,I_2^*}^{\omega,\gamma}(f,t),$$
$$\mathcal{E}_{p,m}^{\omega,\gamma}(f,t) := e_m^{0,\gamma}(F_{2,0}(\cdot,t)) + \sum_{k=1}^{S_2} e_m(F_{2,k}(\cdot,t)) + e_m(F_2^*(\cdot,t)) + \sum_{k=0}^{S_1} e_m(F_{1,k}(\cdot,t)).$$

[C] If $\omega M - d \le \omega t \le \omega M + d$, then

$$\mathcal{Q}_{p,m}^{\omega,\gamma}(f,t) := \Phi_m^{0,\gamma}(F_{2,0}(\cdot,t)) + \sum_{k=1}^{S_2}\Phi_m(F_{2,k}(\cdot,t)) + \Phi_m(F_2^*(\cdot,t)) + \frac{1}{\omega^{\gamma+1}}\mathcal{T}_{p,I_2^*}^{\omega,\gamma}(f,t),$$

$$\mathcal{E}_{p,m}^{\omega,\gamma}(f,t) := e_m^{0,\gamma}(F_{2,0}(\cdot,t)) + \sum_{k=1}^{S_2} e_m(F_{2,k}(\cdot,t)) + e_m(F_2^*(\cdot,t)).$$

[D] If $\omega t > \omega M + d$, then

$$\mathcal{Q}_{p,m}^{\omega,\gamma}(f,t) := \Phi_m^{0,\gamma}(F_{3,0}(\cdot,t)) + \sum_{k=1}^{S_3} \Phi_m(F_{3,k}(\cdot,t)) + \Phi_m(F_2^*(\cdot,t)) + \frac{1}{\omega^{\gamma+1}} \mathcal{T}_{p,I_2^*}^{\omega,\gamma}(f,t),$$

$$\mathcal{E}_{p,m}^{\omega,\gamma}(f,t) := e_m^{0,\gamma}(F_{3,0}(\cdot,t)) + \sum_{k=1}^{S_3} e_m(F_{3,k}(\cdot,t)) + e_m(F_2^*(\cdot,t)).$$

Note that in the approximation of the integrals $H_{p,I_j^*}^{\omega,\gamma}(f,t), j = 1,2$, in (3.3), numerical cancellation can occur when t is very close to one of the values $\frac{\delta_j^*(x_i)}{\omega}$ in the computation of

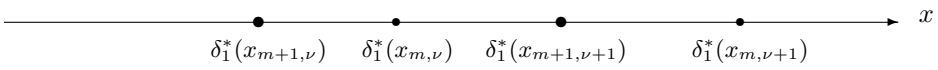
$$R_p\left(f, \frac{\delta_j^*(x_i)}{\omega}, t\right) = \frac{f\left(\frac{\delta_j^*(x_i)}{\omega}\right) - T_p\left(f, \frac{\delta_j^*(x_i)}{\omega}, t\right)}{\left(\frac{\delta_j^*(x_i)}{\omega} - t\right)^{p+1}}, \qquad j = 1,2,$$

which appears in the ith addendum of the Gaussian rules $\Phi_m^{0,\gamma}(F_1^*(\cdot,t))$ if $j = 1$ and in $\Phi_m(F_2^*(\cdot,t))$ if $j = 2$. This phenomenon becomes more severe as p becomes "large". Thus, to avoid a loss of precision in the computation of this term, we apply the above Gaussian rules together with the control algorithm that was firstly introduced in [5].

For the reader's convenience, we briefly describe this control algorithm. For $j = 1$, we have $0 \le \delta_1^*(x) \le \omega t + d$ for $|x| \le 1$, and for any fixed t such that $\omega t \in [0, \omega t + d]$, there exists an index $\nu \in \{1, 2, \ldots, m-1\}$ such that

$$\delta_1^*(x_{m,\nu}) \le \omega t \le \delta_1^*(x_{m,\nu+1}).$$

Since, according to the interlacing property of the Jacobi zeros $\{x_{m,i}\}_{i=1}^m$ and $\{x_{m+1,i}\}_{i=1}^{m+1}$, we have the following arrangement:



The integral $\int_{-1}^1 F_1^*(x,t) v^{0,\gamma}(x) dx$ is approximated by the Gauss–Jacobi rule $\Phi_{m^*}^{0,\gamma}(F_1^*(\cdot,t))$, where m^* is selected as

$$m^* = \begin{cases} m+1, & \text{if } |\delta_1^*(x_{m+1,\nu+1}) - \omega t| > \min\left\{|\omega t - \delta_1^*(x_{m,\nu})|, |\omega t - \delta_1^*(x_{m,\nu+1})|\right\}, \\ m, & \text{otherwise.} \end{cases}$$

The subsequence $\Phi_{m^*}^{0,\gamma}(F_1^*(\cdot,t))$ of the Gauss–Jacobi sequence $\{\Phi_m^{0,\gamma}(F_1^*(\cdot,t))\}_{m\in\mathbf{N}}$ has the property that the minimal distance of the points $\frac{\delta_1^*(x_i)}{\omega}$ from t is large enough to avoid numerical cancellation [1]. In the case $j = 2$, the control algorithm assures that always an even number of Legendre quadrature knots is used, i.e., $m^* = m$ if m is even and $m^* = m+1$ if m is odd. In fact, 0 is always a zero of the Legendre polynomials of odd degree (see, for

example, [34]) and $\frac{\delta_2^*(0)}{\omega} = t$. The importance of applying the control algorithm to avoid numerical cancellation is illustrated in Table 5.5 of Section 5. The next theorem guarantees the convergence of the proposed quadrature rule.

THEOREM 3.1. *Let $|\gamma| < 1$, let $f \in C^{p+1+r}(\mathbb{R}^+)$, with p, r nonnegative integers, such that for some $a > 0$,*

$$\mathcal{N}_a := \max_{0\le j\le p+1+r} \max_{0\le x\le a} |f^{(j)}(x)| < +\infty,$$

and let

$$\mathcal{M}_a := \max_{0\le j\le p+1+r} \max_{x\in[a,+\infty)} |f^{(j)}(x)x^{\gamma}| < +\infty.$$

Then, for any fixed $t > 0$, $M > 0$, and $0 < d \le \omega$, we have

$$|\mathcal{E}_{p,m}^{\omega,\gamma}(f,t)| \le \mathcal{C}\left(\frac{d+2}{m}\right)^r \left(\frac{\omega}{d}\right)^{p+1} M\left[\mathcal{N}_a + \mathcal{M}_a\right], \qquad \mathcal{C} \ne \mathcal{C}(m, f, t, \omega, d, M). \tag{3.4}$$

4. Computational details. The choice of the parameters M, d, and m plays a fundamental role in the accuracy and the efficiency of the proposed numerical method. In our numerical tests we have used the assumption (3.1) in the following form:

$$\left|\frac{f(x)x^{\gamma}}{(x-t)^{p+1}}\right| < \text{eps}, \qquad \forall x > M, \;\; x \notin I_t, \tag{4.1}$$

where eps $\approx$ 2.22044e-16 is the machine epsilon in double arithmetic and where M is chosen by a suitable routine whose details are reported in the Appendix A. The choice of the interval I_t, with $d < \omega$, does not affect the computed value of M. Since M is smaller for larger values of t and if we are interested in approximating the integral (1.1) for $t \in [a, b] \subset \mathbb{R}^+$, we fix M as the value computed by (4.1) with $t = a$. Smaller values of M can be obtained by replacing eps with $\frac{1}{2}10^{-r}$, $r < 16$, in (4.1), but in this case approximations of the integral (1.1) can only be achieved up to r exact decimal digits.

The computational cost of the proposed method measured in terms of function evaluations (f. e.) is approximately $m\left\lfloor\frac{\omega M}{d}\right\rfloor$. Thus, for fixed ω and M, the parameters m and d are crucial for efficiency. The smaller the value of d is, the larger becomes $\left\lfloor\frac{\omega M}{d}\right\rfloor$. On the other hand, when transforming intervals of size d into $[-1, 1]$, the oscillating parameter of the exponential function becomes $\frac{d}{2}$. Then, increasing d requires increasing m in order to achieve the same number of exact decimal digits in the approximation of each integral by the Gauss–Jacobi quadrature rule. As a consequence, starting from a certain value of d, the execution time increases even if the number of function evaluations continues to decrease. After a certain value of d, the number of function evaluations no longer decreases but increases. This behaviour of the proposed dilation rule is in line with the theoretical expectations (see (3.4)) and is shown in the next section in Table 5.3 for the integral considered in Example 5.1. In particular, if the goal is to minimize execution time, then Table 5.3 suggests an optimal value of d of 60. Finding an a priori criterion for the optimal choice of d is still an open problem.

In the numerical tests of Section 5 we compare with the results obtained by the method in [15], and we limit ourselves to choices of d for which the execution times of our method for integral approximations with machine precision are lower than those of [15] (note that, except for Example 5.1, machine precision is never achieved by the method in [15]). However, we think that for $\omega > 8$, a reasonable initial choice of d could be $d = \min\{\frac{\omega}{4}, 60\}$.

Now, we describe the details concerning the computation of the integrals

$$H_{p,I_j^*}^{\omega,\gamma}(t) = \omega^{p+1} \, \text{⨍}_{I_j^*} \frac{y^\gamma e^{\mathrm{i}y}}{(y-\omega t)^{p+1}} \, dy, \qquad p \geq 0, \quad j = 1, 2,$$

appearing in (3.2). For $\gamma = 0$ we can compute them exactly by the relation

$$H_{p,I_j^*}^{\omega,0}(t) = \frac{1}{p!} \frac{d^p}{dt^p} H_{0,I_j^*}^{\omega,0}(t), \qquad p > 0,$$

and taking into account that

$$H_{0,I_1^*}^{\omega,0}(t) = \omega e^{\mathrm{i}\omega t} \left[(C_i(d) - C_i(\omega t)) + \mathrm{i}(S_i(d) + S_i(\omega t)) \right]$$

and

$$H_{0,I_2^*}^{\omega,0}(t) = 2\omega e^{\mathrm{i}\omega t} S_i(d),$$

where $S_i(z)$ and $C_i(z)$ are the sine integral and the cosine integral functions, respectively.

For $\gamma \neq 0$, since the exact expression for $H_{0,I_j^*}^{\omega,\gamma}(t)$ is not known, we propose an approximation of $H_{p,I_j^*}^{\omega,\gamma}(t)$ following a procedure very similar to the one used for (3.2). In this case, we add and subtract the Taylor polynomial of the function $e^{\mathrm{i}y}$, obtaining

$$\begin{aligned} H_{p,I_j^*}^{\omega,\gamma}(t) = \omega^{p+1} \Bigg[& \int_{I_j^*} \frac{e^{\mathrm{i}y} - e^{\mathrm{i}\omega t} \sum_{r=0}^{p} \frac{\mathrm{i}^r}{r!} (y-\omega t)^r}{(y-\omega t)^{p+1}} y^\gamma \, dy \\ & + e^{\mathrm{i}\omega t} \sum_{r=0}^{p} \frac{\mathrm{i}^r}{r!} \, \text{⨍}_{I_j^*} \frac{y^\gamma}{(y-\omega t)^{p-r+1}} \, dy \Bigg] \\ =: \omega^{p+1} & \left[\int_{I_j^*} R_p(e^{\mathrm{i}\,\cdot}, y, \omega t) y^\gamma dy + e^{\mathrm{i}\omega t} \sum_{r=0}^{p} \frac{\mathrm{i}^r}{r!} c_{p-r,j}(t) \right], \end{aligned}$$

where

$$c_{p-r,j}(t) = \frac{1}{(p-j)!} \frac{d^{p-r}}{dt^{p-r}} c_{0,j}(t),$$

with

$$c_{0,1}(t) := \int_{I_1^*} \frac{y^\gamma}{(y-\omega t)} \, dy = -\mathrm{i}(\omega t)^\gamma \left[\pi - \mathrm{i} B_{1+\frac{d}{\omega t}}(1+\gamma, 0) \right],$$

and

$$c_{0,2}(t) := \text{⨍}_{I_2^*} \frac{y^\gamma}{(y-\omega t)} \, dy = (\omega t)^\gamma \left[-\mathrm{i}\pi + B_{1-\frac{d}{\omega t}}(1+\gamma, 0) - B_{1+\frac{d}{\omega t}}(1+\gamma, 0) \right],$$

and $B_z(a,b)$ denoting the incomplete Beta function. Transforming the interval I_j^*, $j=1,2$, into $[-1,1]$ and approximating the Riemann integral by the m-point Gauss–Jacobi quadrature rule ($\alpha=0$, $\beta=\gamma$ when $j=1$ and $\alpha=\beta=0$ when $j=2$), we obtain

$$\begin{aligned} H^{\omega,\gamma}_{p,I_j^*}(t) &= \Phi_m^{\alpha,\beta}(\widetilde{F}_j^*(\cdot,t)) + \omega^{p+1}e^{\mathrm{i}\omega t}\sum_{r=0}^{p}\frac{\mathrm{i}^r}{r!}c_{p-r,j}(t) + e_m^{\alpha,\beta}(\widetilde{F}_j^*(\cdot,t)) \\ &=: \mathcal{Q}^{\omega,\gamma}_{p,m}(t) + \mathcal{E}^{\omega,\gamma}_{p,m}(t), \end{aligned}$$

where

$$\widetilde{F}_j^*(x,t) = \omega^{p-\gamma}\begin{cases} \left(\dfrac{\omega t+d}{2}\right)^{\gamma+1} R_p(e^{\mathrm{i}\,\cdot},\delta_1^*(x),\omega t), & j=1,\\ d\,R_p(e^{\mathrm{i}\,\cdot},\delta_2^*(x),\omega t)\,\delta_2^*(x)^{\gamma}, & j=2. \end{cases}$$

Also in this case we apply the Gaussian rule together with the control algorithm described earlier in order to avoid numerical cancellation.

Tables 4.1 and 4.2 illustrate the excellent performance of the above quadrature rule for different choices of γ, ω, p, and t and using $d=2.5$. We define the absolute error by

$$r^{\omega,\gamma}_{p,m}(t) = |\mathcal{Q}^{\omega,\gamma}_{p,m}(t) - \mathcal{Q}^{\omega,\gamma}_{p,64}(t)|.$$

Moreover, the symbol "$-$"means that machine precision in double arithmetic is achieved.

TABLE 4.1
$r^{\omega,\frac{1}{2}}_{p,m}(t)$ for different choices of ω, p, and t.

$r^{\omega,\frac{1}{2}}_{0,m}(t)$

	$\omega=10$		$\omega=500$	
m	$t=0.003$	$t=60$	$t=0.003$	$t=60$
4	1.72e-7	8.33e-4	2.00e-5	8.33e-4
8	–	5.46e-12	1.50e-13	5.46e-12
16	–	–	–	–

$r^{\omega,\frac{1}{2}}_{1,m}(t)$

	$\omega=10$		$\omega=500$	
m	$t=0.003$	$t=60$	$t=0.003$	$t=60$
4	1.32e-7	8.51e-4	1.04e-3	4.25e-2
8	–	3.06e-12	1.50e-12	1.53e-10
16	–	–	–	–

$r^{\omega,\frac{1}{2}}_{2,m}(t)$

	$\omega=10$		$\omega=500$	
m	$t=0.003$	$t=60$	$t=0.003$	$t=60$
4	1.58e-7	7.86e-4	4.62e-5	2.33e-3
8	–	1.62e-12	6.04e-14	1.57e-11
16	–	–	–	–

TABLE 4.2
$r_{p,m}^{\omega,\frac{9}{10}}(t)$ for different choices of ω, p, and t.

$r_{0,m}^{\omega,\frac{9}{10}}(t)$

	$\omega = 50$		$\omega = 630$	
m	$t = 1$	$t = 200$	$t = 1$	$t = 200$
4	3.06e-5	3.02e-5	3.02e-5	3.22e-5
8	2.08e-13	1.98e-13	1.98e-13	2.01e-13
16	–	–	–	–

$r_{1,m}^{\omega,\frac{9}{10}}(t)$

	$\omega = 50$		$\omega = 630$	
m	$t = 1$	$t = 200$	$t = 1$	$t = 200$
4	5.28e-6	5.19e-6	5.20e-6	5.19e-6
8	1.97e-14	1.87e-14	1.87e-14	1.90e-14
16	–	–	–	–

$r_{2,m}^{\omega,\frac{9}{10}}(t)$

	$\omega = 50$		$\omega = 630$	
m	$t = 1$	$t = 200$	$t = 1$	$t = 200$
4	1.37e-6	1.35e-6	1.35e-6	2.34e-6
8	2.94e-15	2.77e-15	2.78e-15	2.76e-15
16	–	–	–	–

5. Numerical tests. We dedicate this section to report on the performance of the proposed quadrature rule for approximating the integrals (1.1) for different choices of the parameter p. We also provide some comparisons with results obtained by other existing quadrature rules [2, 15, 35]. Concerning the product quadrature rule proposed in [15], in the tables we provide the values of the parameters m and j representing the number of interpolation knots and the truncation index, respectively. In particular, the parameter j corresponds to the number of functions evaluations required by the rule.

In the following, we denote the absolute errors and the relative errors by

$$r_{p,m}^{\omega,\gamma}(f,t) = |H_p^{\omega,\gamma}(f,t) - \mathcal{Q}_{p,m}^{\omega,\gamma}(f,t)|$$

and

$$\bar{r}_{p,m}^{\omega,\gamma}(f,t) = \frac{|H_p^{\omega,\gamma}(f,t) - \mathcal{Q}_{p,m}^{\omega,\gamma}(f,t)|}{|H_p^{\omega,\gamma}(f,t)|},$$

respectively. Moreover, when the exact value of the integral is unknown, we replace it with the reference value $\mathcal{Q}_{p,m}^{\omega,\gamma}(f,t)$ for $m = 64$.

EXAMPLE 5.1. We consider the following example:

$$H_0^{\omega,0}(f,t) = \int_0^{+\infty} \frac{e^{-x}}{x-t} e^{\mathrm{i}\omega x}\, dx = -e^{-t} e^{\mathrm{i}\omega t} E_i(t - \mathrm{i}\omega t),$$

where $E_i(z)$ represents the exponential integral function. Since the function $f(x) = e^{-x}$ belongs to the space $C^\infty(\mathbb{R}^+)$, in agreement with the theoretical expectations, convergence is fast. We obtain machine precision with $m = 11$ for different choices of the oscillation parameter ω, as shown in Tables 5.1 and 5.2 (upper tables). In Tables 5.1 and 5.2 (bottom tables) we also report the results obtained with the product quadrature rule in [15]. In [15, Example 1],

the authors already confirm superior performance of the product quadrature rule compared to the ones proposed in [2, 35].

As announced in the previous section, in Table 5.3 we demonstrate the behavior of our rule for increasing values of d. As can be observed, m increases with d, and for $d > 50$ the running time no longer decreases but increases. Furthermore, in Table 5.4 we report the running times (measured in seconds) of the proposed method and the one in [15] for approximations up to machine precision for $\omega = 16, 80, 320$. As one can see, only in the case $\omega = 320$, the product quadrature rule in [15] is slightly faster than the dilation rule.

Finally, Table 5.5 emphasizes the importance of applying the dilation rule together with the control algorithm of Section 3 for selecting the optimal number m^* of quadrature nodes to prevent numerical cancellation. In fact, in evaluating the approximation $\mathcal{Q}_{0,15}^{20,0}(f,t)$ at the points $t = 0.0111632703293431, 0.521615436$, which are very close to $\frac{\delta_1^*(x_{15,2})}{\omega}$ and $\frac{\delta_1^*(x_{15,7})}{\omega}$, respectively, the control algorithm selects $m^* = 8$ or $m^* = 10$ in order to avoid loss of precision.

TABLE 5.1

Example 5.1: absolute errors for approximating $H_0^{10,0}(f,t)$ with the dilation rule (upper table) and the product quadrature rule in [15] (bottom table).

$r_{0,m}^{10,0}(f,t) \quad M = 33 \quad d = 2.5$

m	$t = 0.1$	$t = 0.01$	$t = 0.001$	$t = 0.0001$
4	3.29e-6	9.13e-7	9.13e-7	9.09e-7
8	5.33e-13	5.83e-13	5.83e-13	5.89e-13
11	–	–	–	–

$e_{0,m}^{10,0}(f,t)$ [15]

m	j	$t = 0.1$	$t = 0.01$	$t = 0.001$	$t = 0.0001$
4	4	3.04e-3	5.52e-3	1.41e-2	2.23e-2
8	8	4.78e-5	2.91e-5	1.19e-4	1.99e-4
16	16	8.45e-9	4.55e-9	9.78e-9	1.87e-8
32	26	–	–	–	–

TABLE 5.2

Example 5.1: absolute errors for approximating $H_0^{\omega,0}(f,0.02)$ with the dilation rule (upper table) and the product quadrature rule in [15] (bottom table).

$r_{0,m}^{\omega,0}(f,0.02) \quad M = 33 \quad d = 2.5$

m	$\omega = 5$	$\omega = 20$	$\omega = 80$	$\omega = 320$
4	1.08e-6	8.38e-7	1.41e-6	2.31e-6
8	5.76e-13	5.76e-13	5.73e-13	5.39e-13
11	–	–	–	–

$e_{0,m}^{\omega,0}(f,0.02)$ [15]

m	j	$\omega = 5$	$\omega = 20$	$\omega = 80$	$\omega = 320$
4	4	2.83e-3	4.56e-3	9.03e-3	8.83e-3
8	8	4.09e-5	2.38e-5	7.17e-5	6.81e-5
16	16	1.18e-8	3.06e-9	5.31e-9	4.43e-9
32	26	–	–	–	–

TABLE 5.3

Example 5.1: Behaviour of the dilation rule $\mathcal{Q}_{0,m}^{320,0}(f,0.02)$ with $M=33$ for different choices of d.

d	m	$r_{0,m}^{320,0}(f,0.02)$	f. e.	second
2.5	11	1.92e-17	46464	0.958
5	12	2.15e-16	25344	0.497
8	15	6.26e-18	19800	0.393
10	15	4.21e-16	15840	0.322
50	31	2.82e-16	6541	0.150
60	34	1.79e-16	5984	0.146
100	47	2.02e-16	4935	0.168
200	77	2.22e-16	4004	0.214
300	105	1.93e-16	3675	0.320

TABLE 5.4

Example 5.1: comparison between the running times (in seconds) of the dilation rule $\mathcal{Q}_{0,16}^{\omega,0}(f,0.02)$ with $M=33$ and the product quadrature rule $H_{0,32}^{\omega,0}(f,0.02)$ in [15] for achieving machine precision with different choices of ω.

ω	$\texttt{Timing}[\mathcal{Q}_{0,m}^{\omega,0}(f,0.02)]$	$\texttt{Timing}[H_{0,32}^{\omega,0}(f,0.02)]$
16	0.022 s $(d=8,m=12)$	0.082 s
80	0.077 s $(d=10,m=13)$	0.086 s
320	0.187 s $(d=50,m=31)$	0.090 s

TABLE 5.5

Example 5.1: absolute errors for approximating $H_{0,m}^{20,0}(f,t)$ with $M=33$ and $d=2.5$ by the dilation rule without the control algorithm.

m	$r_{0,m}^{20,0}(f,0.0111632703293431)$	$r_{0,m}^{20,0}(f,0.521615436)$
8	5.828e-13	1.345e-13
9	8.000e-03	2.400e-02
10	2.220e-16	3.140e-16

EXAMPLE 5.2. We consider the integrals

$$H_p^{\omega,\frac{3}{5}}(f,t)=\neq\int_0^{+\infty}\frac{1}{(x^7+1)^4(x-t)^{p+1}}x^{\frac{3}{5}}e^{\mathrm{i}\omega x}\,dx,\qquad p=0,1,$$

whose exact values are unknown. Also in this case, $f(x)=\frac{1}{(x^7+1)^4}$ belongs to $C^\infty(\mathbb{R}^+)$, and thus we expect fast convergence, both for the parameters $p=0$ and $p=1$.

The results shown in Tables 5.6 and 5.7 (upper tables) fully meet our expectations: in each simulation, machine precision is always achieved with $m=11$ for $p=0$ and, in the worst case, with $m=12$ for $p=1$. In [15], worse results are obtained for approximating the same integrals with the product quadrature rule. In fact, as one can see in Tables 5.6 and 5.7 (bottom tables), an accuracy of at most 7 exact decimal digits is reached, even when further increasing m.

A direct comparison between the new method (upper tables) and the method in [15] (bottom tables) for targeting roughly the same accuracy is presented in Tables 5.8 and 5.9. As

TABLE 5.6

Example 5.2: absolute errors for approximating $H_0^{\omega,\frac{3}{5}}(f,t)$ with the dilation rule (upper table) and the product quadrature rule in [15] (bottom table).

$r_{0,m}^{\omega,\frac{3}{5}}(f,t)\quad d=2.5$

	$t=0.01\quad M=3.51$			$t=4\quad M=4.1$		
m	$\omega=10$	$\omega=100$	$\omega=1000$	$\omega=10$	$\omega=100$	$\omega=1000$
5	2.60e-7	3.87e-7	3.23e-6	1.83e-7	2.36e-8	2.82e-8
7	1.63e-10	5.86e-11	1.01e-9	5.73e-10	5.38e-12	2.58e-11
9	1.64e-13	3.88e-15	1.05e-13	6.86e-13	–	7.24e-15
11	–	–	–	–	–	–

$e_{0,m}^{\omega,\frac{3}{5}}(f,t)$ [15]

	$t=0.01$				$t=4$			
m	j	$\omega=10$	$\omega=100$	$\omega=1000$	j	$\omega=10$	$\omega=100$	$\omega=1000$
32	11	5.33e-2	3.01e-2	8.77e-2	11	3.27e-2	4.16e-2	4.16e-2
64	15	3.08e-2	4.82e-2	6.84e-2	15	1.24e-2	1.67e-2	1.68e-2
128	21	1.99e-2	8.17e-3	1.82e-2	21	1.19e-2	1.07e-2	1.08e-2
256	29	3.48e-3	3.23e-3	1.65e-4	30	2.75e-3	2.49e-3	2.48e-3
512	41	7.27e-5	2.09e-4	3.86e-4	41	1.89e-4	1.47e-4	1.48e-4
1024	57	5.46e-7	4.46e-7	9.01e-8	58	3.88e-7	2.79e-8	2.97e-8

TABLE 5.7

Example 5.2: absolute errors for approximating $H_1^{\omega,\frac{3}{5}}(f,t)$ with the dilation rule (upper table) and the product quadrature rule in [15] (bottom table).

$r_{1,m}^{\omega,\frac{3}{5}}(f,t)\quad d=2.5$

	$t=0.01\quad M=3.41$			$t=4\quad M=4.2$		
m	$\omega=10$	$\omega=100$	$\omega=1000$	$\omega=10$	$\omega=100$	$\omega=1000$
4	7.77e-6	3.33e-5	1.79e-5	1.01e-4	4.40e-6	3.18e-7
6	5.78e-9	1.20e-8	2.02e-7	8.06e-10	2.60e-9	1.68e-9
8	3.80e-12	1.32e-11	2.15e-10	9.58e-13	3.96e-13	2.09e-12
10	–	–	1.60e-13	–	–	–
12	–	–	–	–	–	–

$e_{1,m}^{\omega,\frac{3}{5}}(f,t)$ [15]

	$t=0.01$				$t=4$			
m	j	$\omega=10$	$\omega=100$	$\omega=1000$	j	$\omega=10$	$\omega=100$	$\omega=1000$
512	46	8.94e-2	8.72e-2	4.99e-1	46	2.84e-3	1.51e-2	1.48e-1
1024	63	1.58e-5	5.73e-5	1.89e-4	65	3.79e-6	6.68e-6	3.03e-5

one can see, except for the case $\omega=10$, the new method requires a larger number of function evaluations. However, the running time of the product rule in [15] is always much higher; this is also the case when the dilation rule is employed for obtaining machine precision accuracy (see Tables 5.10 and 5.11). This shows that the overall computational cost of our method is lower even though it requires a larger number of function evaluations.

EXAMPLE 5.3. As a third test example we approximate the integrals

$$H_p^{\omega,\frac{1}{3}}(f,t)=\neq\int_0^{+\infty}\frac{|x-5|^{\frac{11}{2}}e^{-\frac{x}{2}}}{(x+1)^2(x-t)^{p+1}}x^{\frac{1}{3}}e^{\mathrm{i}\omega x}\,dx,\qquad p=0,1,2.$$

Also in this case the exact values are unknown. The results presented in Tables 5.12, 5.13,

TABLE 5.8

Example 5.2: comparison between the number of function evaluations (f. e.) and the running times (in seconds) of the dilation quadrature rule $\mathcal{Q}_{0,m}^{\omega,\frac{3}{5}}(f,t)$ and the product quadrature rule $H_{0,1024}^{\omega,\frac{3}{5}}(f,t)$ in [15] in correspondence of different choices of ω and t.

Dilation rule

	$t=0.01$			$t=4$		
M	3.51	3.51	3.51	4.1	4.1	4.1
ω	10	100	1000	10	100	1000
d	5	90	450	7	90	350
m	8	34	133	9	33	103
f. e.	56	102	931	45	132	1133
$r_{0,m}^{\omega,\frac{3}{5}}(f,t)$	1.46e-7	2.06e-8	6.61e-8	3.18e-7	1.69e-8	2.83e-8
$\texttt{Timing}[\mathcal{Q}_{0,m}^{\omega,\frac{3}{5}}(f,t)]$	0.013 s	0.106 s	1.235 s	0.013 s	0.095 s	0.802 s

Product rule in [15]

	$t=0.01$			$t=4$		
ω	10	100	1000	10	100	1000
f. e.	57	57	57	58	58	58
$e_{0,m}^{\omega,\frac{3}{5}}(f,t)$	5.46e-7	4.46e-7	9.01e-8	3.88e-7	2.79e-8	2.97e-8
$\texttt{Timing}[H_{0,1024}^{\omega,\frac{3}{5}}(f,t)]$	488.2 s	507.6 s	535.7 s	555.0 s	556.7 s	630.4 s

and 5.15 show absolute errors that are much smaller than our theoretical expectations, having $f(x)=\frac{|x-5|^{\frac{11}{2}}e^{-\frac{x}{2}}}{(x+1)^2}\in C^5(\mathbb{R}^+)$. In fact, machine precision is always attained with at most $m=12$, even at the evaluation point 4.99, which is very close to the point 5, where the sixth derivative of f is singular. For comparison, the results presented in [15, Table 8] show that the product quadrature rule suffers from the low smoothness of the function f, yielding the worst results just at the point 4.99.

Analogous conclusions can be drawn from Tables 5.14 and 5.16, where, due to the large values of the integrals, we report the relative errors both for the dilation rule (upper tables) and for the product rule in [15] (bottom tables).

Moreover, in Table 5.17 we report the running times (in seconds) of the proposed dilation-type rule for an approximation with machine precision and the running times of the product rule [15] for an approximation with at most 7 exact decimal digits (its best performance). As one can see, the new method is always faster than the product rule in [15] since it requires fewer knots, both for $\omega=100$ and for $\omega=500$, for reaching machine precision.

EXAMPLE 5.4. We conclude with the following example:

$$H_p^{\omega,-\frac{1}{3}}(f,t)=\fint_0^{+\infty}\frac{1}{(x^2+5)^4(x-t)^{p+1}}x^{-\frac{1}{4}}e^{\mathrm{i}\omega x}\,dx,\qquad p=0,1,2,$$

where $f(x)=\frac{1}{(x^2+5)^4}\in C^\infty(\mathbb{R}^+)$ and $\gamma=-\frac{1}{4}$. Since γ is negative, the product rule proposed in [15] cannot be applied, hence, in order to approximate the integrals, we only use the dilation-type quadrature formula presented above.

Since the function f is very smooth, we also expect in this case a good performance of the proposed method. In Tables 5.18, 5.19, and 5.20, the results obtained with $p=0$, $p=1$, and $p=2$ are presented. In line with the theoretical estimates, convergence is always fast, and we reach machine precision with $m=10$.

TABLE 5.9

Example 5.2: comparison between the number of function evaluations (f. e.) and the running times (in seconds) of the dilation quadrature rule $\mathcal{Q}_{1,m}^{\omega,\frac{3}{5}}(f,t)$ and the product quadrature rule $H_{1,1024}^{\omega,\frac{3}{5}}(f,t)$ in [15] in correspondence of different choices of ω and t.

Dilation rule

	$t=0.01$			$t=4$		
M	3.41	3.41	3.41	4.2	4.2	4.2
ω	10	100	1000	10	100	1000
d	5	90	350	5	90	400
m	7	32	103	6	30	111
$r_{1,m}^{\omega,\frac{3}{5}}(f,t)$	1.64e-6	1.32e-6	3.83e-6	1.28e-6	1.22e-6	1.74e-6
f. e.	42	96	927	48	120	1110
$\texttt{Timing}[\mathcal{Q}_{1,m}^{\omega,\frac{3}{5}}(f,t)]$	0.011 s	0.095 s	0.801 s	0.009 s	0.085 s	0.930 s

Product rule in [15]

	$t=0.01$			$t=4$		
ω	10	100	1000	10	100	1000
f. e.	63	63	63	65	65	65
$e_{1,m}^{\omega,\frac{3}{5}}(f,t)$	1.58e-5	5.73e-5	$1.89-4$	3.79e-6	6.68e-6	3.03e-5
$\texttt{Timing}[H_{1,1024}^{\omega,\frac{3}{5}}(f,t)]$	517.4 s	514.6 s	545.5 s	562.1 s	610.4 s	640.5 s

TABLE 5.10

Example 5.2: comparison between the running times (in seconds) of the dilation quadrature rule $\mathcal{Q}_{0,11}^{\omega,\frac{3}{5}}(f,t)$ and the product quadrature rule $H_{0,1024}^{\omega,\frac{3}{5}}(f,t)$ in [15] in correspondence of different choices of ω. The dilation rule approximates with machine precision, the product rule in [15] approximates with at most 7 exact decimal digits.

ω	$\texttt{Timing}[\mathcal{Q}_{0,11}^{\omega,\frac{3}{5}}(f,0.01)]$	$\texttt{Timing}[H_{0,1024}^{\omega,\frac{3}{5}}(f,0.01)]$
10	0.022 s	488.2 s
100	0.061 s	507.6 s
1000	0.448 s	535.7 s

TABLE 5.11

Example 5.2: comparison between the running times (in seconds) of the dilation quadrature rule $\mathcal{Q}_{1,12}^{\omega,\frac{3}{5}}(f,t)$ and the product quadrature rule $H_{1,1024}^{\omega,\frac{3}{5}}(f,t)$ in [15] in correspondence of different choices of ω. The dilation rule approximates with machine precision, the product rule in [15] approximates with at most 5 exact decimal digits.

ω	$\texttt{Timing}[\mathcal{Q}_{1,12}^{\omega,\frac{3}{5}}(f,0.01)]$	$\texttt{Timing}[H_{1,1024}^{\omega,\frac{3}{5}}(f,0.01)]$
10	0.024 s	517.4 s
100	0.064 s	514.6 s
1000	0.479 s	545.5 s

TABLE 5.12

Example 5.3: absolute errors for approximating $H_0^{\omega,\frac{1}{3}}(f,t)$.

	$r_{0,m}^{\omega,\frac{1}{3}}(f,4.99)$, $d=2.5$, $M=96.1$		$r_{0,m}^{\omega,\frac{1}{3}}(f,10)$ $d=2.5$, $M=97.7$	
m	$\omega=100$	$\omega=500$	$\omega=100$	$\omega=500$
4	4.67e-6	5.68e-7	8.81e-5	9.04e-5
8	5.79e-14	6.72e-15	5.66e-13	5.94e-13
10	–	–	–	–

TABLE 5.13

Example 5.3: absolute errors for approximating $H_1^{\omega,\frac{1}{3}}(f,4.99)$.

$r_{1,m}^{\omega,\frac{1}{3}}(f,4.99)$ $d=2.5$, $M=86.6$		
m	$\omega=100$	$\omega=500$
4	9.38e-7	1.14e-7
8	1.39e-14	1.36e-15
12	–	–

TABLE 5.14

Example 5.3: relative errors for approximating $H_1^{\omega,\frac{1}{3}}(f,10)$ *with the dilation rule (upper table) and the product rule in [15] (bottom table).*

$\bar{r}_{1,m}^{\omega,\frac{1}{3}}(f,10)$ $d=2.5$ $M=88.4$		
m	$\omega=100$	$\omega=500$
4	5.33e-6	5.20e-6
8	2.04e-14	1.86e-14
12	–	–

$\bar{e}_{1,m}^{\omega,\frac{1}{3}}(f,10)$ [15]			
m	j	$\omega=100$	$\omega=500$
128	85	5.93e-7	2.36e-7
256	126	6.79e-7	6.78e-7
512	116	5.84e-9	5.72e-9

TABLE 5.15

Example 5.3: absolute errors for approximating $H_2^{\omega,\frac{1}{3}}(f,4.99)$.

$r_{2,m}^{\omega,\frac{1}{3}}(f,4.99)$ $d=2.5$ $M=77.6$		
m	$\omega=100$	$\omega=500$
4	1.88e-7	2.39e-8
8	2.20e-13	–
12	–	–

TABLE 5.16

Example 5.3: relative errors obtained for approximating $H_2^{\omega,\frac{1}{3}}(f,10)$ *with the dilation rule (upper table) and the product rule in [15] (bottom table).*

$\bar{r}_{2,m}^{\omega,\frac{1}{3}}(f,10)$ $d=2.5$ $M=79.5$		
m	$\omega=100$	$\omega=500$
4	3.69e-6	1.60e-6
8	1.36e-14	4.21e-15
12	–	–

	$\bar{e}_{2,m}^{\omega,\frac{1}{3}}(f,10)$ [15]		
m	j	$\omega=100$	$\omega=500$
128	88	1.13e-6	3.04e-7
256	129	6.80e-7	6.78e-7
512	114	6.25e-9	5.74e-9

TABLE 5.17

Example 5.3: comparison between the running times (in seconds) of the dilation quadrature rule $\mathcal{Q}_{p,16}^{\omega,\frac{1}{3}}(f,t)$ and the product quadrature rule $H_{p,512}^{\omega,\frac{1}{3}}(f,t)$ in [15] with $p=0,1,2$, in correspondence of different choices of ω. The dilation rule approximates with machine precision, the product rule in [15] approximates with at most 7 exact decimal digits.

ω	$\texttt{Timing}[\mathcal{Q}_{0,12}^{\omega,\frac{1}{3}}(f,4.99)]$	$\texttt{Timing}[H_{0,512}^{\omega,\frac{1}{3}}(f,4.99)]$
100	1.711 s	161.266 s
500	8.483 s	169.5 s
ω	$\texttt{Timing}[\mathcal{Q}_{1,12}^{\omega,\frac{1}{3}}(f,4.99)]$	$\texttt{Timing}[H_{1,512}^{\omega,\frac{1}{3}}(f,4.99)]$
100	1.569 s	175 s
500	7.567 s	178.063 s
ω	$\texttt{Timing}[\mathcal{Q}_{2,12}^{\omega,\frac{1}{3}}(f,4.99)]$	$\texttt{Timing}[H_{2,512}^{\omega,\frac{1}{3}}(f,4.99)]$
100	1.501 s	181.297 s
500	7.127 s	182.313 s

TABLE 5.18

Example 5.4: absolute errors for approximating $H_0^{\omega,-\frac{1}{4}}(f,t)$.

$r_{0,m}^{\omega,-\frac{1}{4}}(f,t)$

	$t=0.4$ $d=2.5$ $M=48.8$			$t=400$ $d=2.5$ $M=38.7$		
m	$\omega=5$	$\omega=25$	$\omega=50$	$\omega=5$	$\omega=25$	$\omega=50$
4	4.16e-7	1.93e-7	1.90e-7	8.40e-12	6.52e-13	5.95e-14
8	7.60e-14	1.43e-15	1.28e-15	–	–	–
10	–	–	–	–	–	–

TABLE 5.19

Example 5.4: absolute errors for approximating $H_1^{\omega,-\frac{1}{4}}(f,t)$.

$r_{1,m}^{\omega,-\frac{1}{4}}(f,t)$

	$t=0.4$ $d=2.5$ $M=33.2$			$t=400$ $d=2.5$ $M=18.7$		
m	$\omega=5$	$\omega=25$	$\omega=50$	$\omega=5$	$\omega=25$	$\omega=50$
4	1.50e-7	4.36e-7	9.31e-7	2.10e-14	1.63e-15	9.02e-16
8	6.26e-14	2.49e-13	5.05e-13	–	–	–
10	–	–	–	–	–	–

TABLE 5.20

Example 5.4: absolute errors for approximating $H_2^{\omega,-\frac{1}{4}}(f,t)$.

$r_{2,m}^{\omega,-\frac{1}{4}}(f,t)$

	$t=0.4$ $d=2.5$ $M=24.2$			$t=400$ $d=2.5$ $M=8.8$		
m	$\omega=5$	$\omega=25$	$\omega=50$	$\omega=5$	$\omega=25$	$\omega=50$
4	3.50e-7	5.09e-6	1.99e-5	–	–	–
8	3.24e-13	6.28e-12	2.52e-11	–	–	–
10	–	–	–	–	–	–

6. Proofs. In order to prove Theorem 3.1, we need the following lemma:

LEMMA 6.1. *Let $t \in \mathcal{I}$, with $\mathcal{I}$ a compact interval of $\mathbb{R}^+$. Let p, r be nonnegative integers, and let $|\gamma| < 1$. If $f \in C^{p+1+r}(\mathcal{I})$, then $R_p(f,t) \in C^r(\mathcal{I})$ and*

$$\max_{x\in\mathcal{I}} \left| \frac{\partial^r}{\partial x^r} R_p(f,x,t) \right| \le \max_{x\in\mathcal{I}} |f^{(p+1+r)}(x)| \tag{6.1}$$

and

$$\max_{x\in\mathcal{I}} \left| \frac{\partial^r}{\partial x^r} R_p(f,x,t)x^\gamma \right| \le \max_{x\in\mathcal{I}} |f^{(p+1+r)}(x)x^\gamma|. \tag{6.2}$$

Proof. Suppose that $t < x$. First of all we recall the integral form of the Taylor's remainder term

$$f(x) - T_p(f,x,t) = \frac{1}{p!}\int_t^x f^{(p+1)}(y)(x-y)^p\,dy.$$

Moreover, using the change of variables $y = t + u(x-t)$, we easily get

$$R_p(f,x,t) = \frac{f(x) - T_p(f,x,t)}{(x-t)^{p+1}} = \frac{1}{p!}\int_0^1 f^{(p+1)}(t+u(x-t))(1-u)^p\,du.$$

Taking into account that, for $r \ge 0$,

$$\begin{aligned}\frac{\partial^r}{\partial x^r} R_p(f,x,t) &= \frac{1}{p!}\int_0^1 \frac{\partial^r}{\partial x^r} f^{(p+1)}(t+u(x-t))(1-u)^p\,du \\ &= \frac{1}{p!}\int_0^1 f^{(p+1+r)}(t+u(x-t))u^r(1-u)^p\,du\end{aligned}$$

and that for $u \in [0,1]$ it holds that $y = t + u(x-t) \in [t,x] \subset \mathcal{I}$, we have

$$\begin{aligned}\left|\frac{\partial^r}{\partial x^r} R_p(f,x,t)\right| &\le \frac{1}{p!}\int_0^1 |f^{(p+1+r)}(t+u(x-t))|u^r(1-u)^p\,du \\ &\le \frac{1}{p!}\max_{x\in\mathcal{I}}|f^{(p+1+r)}(x)|\int_0^1 u^r(1-u)^p\,du \\ &\le \frac{1}{p!}\max_{x\in\mathcal{I}}|f^{(p+1+r)}(x)|\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{\partial^r}{\partial x^r} R_p(f,x,t)\right| x^\gamma \\ &\quad\le \frac{1}{p!}\int_0^1 \frac{x^\gamma}{(t+u(x-t))^\gamma}|f^{(p+1+r)}(t+u(x-t))(t+u(x-t))^\gamma|u^r(1-u)^p\,du \\ &\quad\le \frac{1}{p!}\max_{x\in\mathcal{I}}|f^{(p+1+r)}(x)x^\gamma|\int_0^1 u^{r-\gamma}(1-u)^p\,du \\ &\quad\le \frac{1}{p!}\max_{x\in\mathcal{I}}|f^{(p+1+r)}(x)x^\gamma|,\end{aligned}$$

Then, (6.1) and (6.2) follow. The proof in the case $t \ge x$ is similar. ∎

Now, we can prove Theorem 3.1.

Proof of Theorem 3.1. We start with the proof in the case [B], i.e., $\frac{2d}{\omega} < t < M - \frac{d}{\omega}$. Using (2.2) and (2.1), we have

$$\begin{aligned}
|\mathcal{E}_{p,m}^{\omega,\gamma}(f,t)| &\le |e_m^{0,\gamma}(F_{2,0}(\cdot,t))| + \sum_{k=1}^{S_2} |e_m(F_{2,k}(\cdot,t))| \\
&\quad + |e_m(F_2^*(\cdot,t))| + \sum_{k=0}^{S_1} |e_m(F_{1,k}(\cdot,t))| \\
&\le \frac{2^{\gamma+1}}{\gamma+1} E_{2m-1}(F_{2,0}(\cdot,t))_\infty + 4\sum_{k=1}^{S_2} E_{2m-1}(F_{2,k}(\cdot,t))_\infty \\
&\quad + 4E_{2m-1}(F_2^*(\cdot,t))_\infty + 4\sum_{k=0}^{S_1} E_{2m-1}(F_{1,k}(\cdot,t))_\infty \\
&\le \frac{\mathcal{C}}{m^r}\left[\|F_{2,0}^{(r)}(\cdot,t)\varphi^r\|_{[-1,1]} + \sum_{k=1}^{S_2}\|F_{2,k}^{(r)}(\cdot,t)\varphi^r\|_{[-1,1]} + \|F_2^{*(r)}(\cdot,t)\varphi^r\|_{[-1,1]}\right. \\
&\quad \left. + \sum_{k=0}^{S_1}\|F_{1,k}^{(r)}(\cdot,t)\varphi^r\|_{[-1,1]}\right] \\
&\le \frac{\mathcal{C}}{m^r}\left[\|F_{2,0}^{(r)}(\cdot,t)\|_{[-1,1]} + \sum_{k=1}^{S_2}\|F_{2,k}^{(r)}(\cdot,t)\|_{[-1,1]} + \|F_2^{*(r)}(\cdot,t)\|_{[-1,1]}\right. \\
&\quad \left. + \sum_{k=0}^{S_1}\|F_{1,k}^{(r)}(\cdot,t)\|_{[-1,1]}\right]. \qquad (6.3)
\end{aligned}$$

Recalling the definition of $F_{2,0}(\cdot,t)$, we have

$$\|F_{2,0}^{(r)}(\cdot,t)\|_{[-1,1]} = \left(\frac{d}{2\omega}\right)^{\gamma+1} \max_{|x|\le 1}\left|\frac{\partial^r}{\partial x^r} G_p\Big(f, \frac{\delta_{2,0}(x)}{\omega}, t\Big) e^{\mathrm{i}\delta_0(x)}\right|.$$

Since for $|x| \le 1$, it holds that $z := \frac{\delta_{2,0}(x)}{\omega} \in [0, \frac{d}{\omega}]$, and thus, $|z-t| > \frac{d}{\omega}$, we get

$$\begin{aligned}
&\left|\frac{\partial^r}{\partial x^r} G_p\Big(f, \frac{\delta_{2,0}(x)}{\omega}, t\Big) e^{\mathrm{i}\delta_{2,0}(x)}\right| \qquad (6.4) \\
&\quad \le \sum_{j=0}^{r}\binom{r}{j}\left(\frac{d}{2}\right)^{r-j}\left(\frac{d}{2\omega}\right)^j \left|\frac{\partial^j}{\partial z^j} G_p\Big(f,z,t\Big)\right| \\
&\quad \le \left(\frac{d}{2}\right)^r \sum_{j=0}^{r}\binom{r}{j}\frac{1}{\omega^j}\sum_{i=0}^{j}\binom{j}{i}\frac{|f^{(i)}(z)|}{|z-t|^{p+1+j-i}} \\
&\quad \le \left(\frac{d}{2}\right)^r \sum_{j=0}^{r}\binom{r}{j}\frac{1}{\omega^j}\sum_{i=0}^{j}\binom{j}{i}\left(\frac{\omega}{d}\right)^{p+1+j-i}|f^{(i)}(z)| \qquad (6.5) \\
&\quad \le \left(\frac{d}{2}\right)^r \left(\frac{\omega}{d}\right)^{p+1} \max_{0\le j\le r}\max_{0\le z\le \frac{d}{\omega}}|f^{(j)}(z)| \sum_{j=0}^{r}\binom{r}{j}\frac{1}{\omega^j}\sum_{i=0}^{j}\binom{j}{i}\left(\frac{\omega}{d}\right)^{j-i}
\end{aligned}$$

$$\leq \left(\frac{d}{2}+\frac{d}{2\omega}+\frac{1}{2}\right)^r \left(\frac{\omega}{d}\right)^{p+1} \max_{0\leq j\leq r}\max_{0\leq z\leq \frac{d}{\omega}} |f^{(j)}(z)|.$$

Hence,

$$\|F_{2,0}^{(r)}(\cdot,t)\|_{[-1,1]} \leq \frac{1}{2^{\gamma+1}}\left(\frac{\omega}{d}\right)^{p-\gamma}\left(\frac{d}{2}+\frac{d}{2\omega}+\frac{1}{2}\right)^r \max_{0\leq j\leq r}\max_{0\leq z\leq \frac{d}{\omega}} |f^{(j)}(z)|. \tag{6.6}$$

Now, in view of the definition of $F_{2,k}(\cdot,t)$, for $k=1,\dots,S_2-1$, we have

$$\begin{aligned}|F_{2,k}^{(r)}(x,t)| &= \frac{d}{2\omega}\left|\frac{\partial^r}{\partial x^r}G_p\Big(f,\frac{\delta_{2,k}(x)}{\omega},t\Big)\left(\frac{\delta_{2,k}(x)}{\omega}\right)^{\gamma} e^{\mathrm{i}\delta_{2,k}(x)}\right| \\ &\leq \frac{d}{2\omega}\sum_{s=0}^{r}\binom{r}{s}\left|\frac{\partial^s}{\partial x^s}G_p\Big(f,\frac{\delta_{2,k}(x)}{\omega},t\Big)e^{\mathrm{i}\delta_{2,k}(x)}\right|\left(\frac{d}{2\omega}\right)^{r-s}\left(\frac{\delta_{2,k}(x)}{\omega}\right)^{\gamma-r+s}.\end{aligned}$$

Letting $z:=\dfrac{\delta_{2,k}(x)}{\omega}\in\left[\frac{d}{\omega},\frac{S_2 d}{\omega}\right]$, $k=1,\dots,S_2-1$, and using (6.5), we get

$$\begin{aligned}&|F_{2,k}^{(r)}(z,t)| \\ &\quad\leq \frac{1}{2}\left(\frac{\omega}{d}\right)^p\sum_{s=0}^{r}\binom{r}{s}\left(\frac{d}{2}\right)^s\left(\frac{d}{2\omega}\right)^{r-s}\frac{1}{z^{r-s}} \\ &\qquad\qquad\times\sum_{j=0}^{s}\binom{s}{j}\frac{1}{\omega^j}\sum_{i=0}^{j}\binom{j}{i}\left(\frac{\omega}{d}\right)^{j-i}|f^{(i)}(z)z^{\gamma}| \\ &\quad\leq \frac{1}{2}\left(\frac{\omega}{d}\right)^p\max_{0\leq j\leq r}\max_{\frac{d}{\omega}\leq z\leq\frac{S_2 d}{\omega}}\left\{|f^{(i)}(z)z^{\gamma}|\right. \\ &\qquad\qquad\left.\times\sum_{s=0}^{r}\binom{r}{s}\left(\frac{d}{2}\right)^s\left(\frac{1}{2}\right)^{r-s}\sum_{j=0}^{s}\binom{s}{j}\frac{1}{\omega^j}\sum_{i=0}^{j}\binom{j}{i}\left(\frac{\omega}{d}\right)^{j-i}\right\} \\ &\quad\leq \frac{1}{2}\left(\frac{d}{2}+\frac{d}{2\omega}+1\right)^r\left(\frac{\omega}{d}\right)^p\max_{0\leq j\leq r}\max_{\frac{d}{\omega}\leq z\leq\frac{S_2 d}{\omega}}|f^{(i)}(z)z^{\gamma}|.\end{aligned}$$

and thus,

$$\begin{aligned}&\|F_{2,k}^{(r)}(\cdot,t)\|_{[-1,1]} \\ &\quad\leq \frac{1}{2}\left(\frac{d}{2}+\frac{d}{2\omega}+1\right)^r\left(\frac{\omega}{d}\right)^p\max_{0\leq j\leq r}\max_{z\geq\frac{d}{\omega}}|f^{(i)}(z)z^{\gamma}|, \qquad k=1,\dots,S_2-1.\end{aligned} \tag{6.7}$$

Taking into account that $\frac{\omega t-d-S_2 d}{2}\leq\frac{d}{2}$, it is possible to prove in a similar way that

$$\|F_{2,S_2}^{(r)}(\cdot,t)\|_{[-1,1]} \leq \frac{1}{2}\left(\frac{d}{2}+\frac{d}{2\omega}+1\right)^r\left(\frac{\omega}{d}\right)^p\max_{0\leq j\leq r}\max_{z\geq\frac{d}{\omega}}|f^{(i)}(z)z^{\gamma}|. \tag{6.8}$$

Analogously, one verifies that

$$\begin{aligned}&\|F_{1,k}^{(r)}(\cdot,t)\|_{[-1,1]} \\ &\quad\leq \frac{1}{2}\left(\frac{d}{2}+\frac{d}{2\omega}+1\right)^r\left(\frac{\omega}{d}\right)^p\max_{0\leq j\leq r}\max_{z\geq\frac{d}{\omega}}|f^{(i)}(z)z^{\gamma}|, \qquad k=0,\dots,S_1.\end{aligned} \tag{6.9}$$

Finally, taking into account that for $|x| \leq 1$ it holds that $z =: \dfrac{\delta_2^*(x)}{\omega} \in \left[t - \frac{d}{\omega}, t + \frac{d}{\omega}\right]$ and $\omega t - d > d$, and, using (6.2) with $r > 0$, we obtain

$$
\begin{aligned}
|F_2^{*(r)}(x,t)| &= \frac{d}{\omega}\left|\frac{\partial^r}{\partial x^r} R_p\Big(f, \frac{\delta_2^*(x)}{\omega}, t\Big)\left(\frac{\delta_2^*(x)}{\omega}\right)^{\gamma} e^{\mathrm{i}\delta_2^*(x)}\right| \\
&\leq \frac{d}{\omega}\sum_{s=0}^{r}\binom{r}{s}\left|\frac{\partial^s}{\partial x^s} R_p\Big(f, \frac{\delta_2^*(x)}{\omega}, t\Big) e^{\mathrm{i}\delta_2^*(x)}\right|\left(\frac{d}{\omega}\right)^{r-s}\left(\frac{\delta_2^*(x)}{\omega}\right)^{\gamma-r+s} \\
&\leq \frac{d}{\omega}\sum_{s=0}^{r}\binom{r}{s}\left(\frac{d}{\omega}\right)^{r-s}\left(\frac{1}{z}\right)^{r-s}\sum_{j=0}^{s}\binom{s}{j}\left(\frac{d}{\omega}\right)^{j} d^{s-j}\left|\frac{\partial^j}{\partial z^j} R_p\Big(f, z, t\Big) z^{\gamma}\right| \\
&\leq \frac{d}{\omega}\sum_{s=0}^{r}\binom{r}{s}\left(\frac{d}{\omega t - d}\right)^{r-s}\sum_{j=0}^{s}\binom{s}{j}\left(\frac{d}{\omega}\right)^{j} d^{s-j} \\
&\qquad\qquad \times \max_{t-\frac{d}{\omega}\leq z\leq t+\frac{d}{\omega}}\left|\frac{\partial^j}{\partial z^j} R_p\Big(f, z, t\Big) z^{\gamma}\right| \\
&\leq \frac{d}{\omega}\sum_{s=0}^{r}\binom{r}{s}\left(d + \frac{d}{\omega}\right)^{s}\max_{0\leq j\leq r}\max_{t-\frac{d}{\omega}\leq z\leq t+\frac{d}{\omega}}\left|f^{(p+1+j)}(z) z^{\gamma}\right| \\
&\leq \frac{d}{\omega}\left(d + \frac{d}{\omega} + 1\right)^{r}\max_{0\leq j\leq p+1+r}\max_{z\geq\frac{d}{\omega}}\left|f^{(j)}(z) z^{\gamma}\right|. \qquad (6.10)
\end{aligned}
$$

Substituting (6.6), (6.7), (6.8), (6.9), and (6.10) into (6.3), the estimate (3.4) follows in case [B]. In case [A], i.e., $0 < t \leq \frac{2d}{\omega}$, it holds that

$$
\begin{aligned}
|\mathcal{E}_{p,m}^{\omega,\gamma}(f,t)| &\leq |e_m^{0,\gamma}(F_1^*(\cdot,t))| + \sum_{k=0}^{S_1}|e_m(F_{1,k}(\cdot,t))| \\
&\leq \frac{2^{\gamma+1}}{\gamma+1} E_{2m-1}(F_1^*(\cdot,t))_\infty + 4\sum_{k=0}^{S_1} E_{2m-1}(F_{1,k}(\cdot,t))_\infty \\
&\leq \frac{\mathcal{C}}{m^r}\left[\|F_1^{*(r)}(\cdot,t)\|_{[-1,1]} + \sum_{k=0}^{S_1}\|F_{1,k}^{(r)}(\cdot,t)\|_{[-1,1]}\right]. \qquad (6.11)
\end{aligned}
$$

Recalling (6.9), we only need to estimate the first norm at the right-hand side of (6.11). Proceeding as for (6.10) with $z =: \dfrac{\delta_1^*(x)}{\omega} \in \left[0, t + \frac{d}{\omega}\right]$, we have

$$
\begin{aligned}
\|F_1^{*(r)}(\cdot,t)\|_{[-1,1]} &= \left(\frac{\omega t + d}{2\omega}\right)^{\gamma+1}\max_{|x|\leq 1}\left|\frac{\partial^r}{\partial x^r} R_p\Big(f, \frac{\delta_1^*(x)}{\omega}, t\Big) e^{\mathrm{i}\delta_1^*(x)}\right| \\
&\leq \left(\frac{\omega t + d}{2\omega}\right)^{\gamma+1}\left(\frac{\omega t + d}{2}\right)^{r}\sum_{j=0}^{r}\binom{r}{j}\frac{1}{\omega^j}\max_{0\leq z\leq t+\frac{d}{\omega}}\left|\frac{\partial^j}{\partial z^j} R_p\Big(f, z, t\Big)\right|,
\end{aligned}
$$

and, using (6.1), we obtain

$$\|F_1^{*(r)}(\cdot,t)\|_{[-1,1]} \le \left(\frac{\omega t+d}{2\omega}\right)^{\gamma+1}\left(\frac{\omega t+d}{2}\right)^{r}\left(1+\frac{1}{\omega}\right)^{r}\max_{0\le j\le p+1+r}\max_{0\le z\le t+\frac{d}{\omega}}|f^{(j)}(z)|$$

$$\le \mathcal{C}\left(d+\frac{d}{\omega}\right)^{r}\max_{0\le j\le p+1+r}\max_{0\le z\le \frac{3d}{\omega}}|f^{(j)}(z)|. \tag{6.12}$$

Combining (6.9), (6.12), and (6.11), the estimate (3.4) follows in case [A]. The proofs for the cases [C] and [D] are similar. □

7. Conclusions. We have proposed a new method for approximating the values of the integrals (1.1), which are of interest in several applications. Under a suitable assumption on the function f (see (3.1)), the method is based on an approximation of the integral only on the finite interval $[0, M]$. Following an idea in [14], the latter interval is first dilated into the interval $[0, \omega M]$ in order to reduce the oscillation of the function $e^{\mathrm{i}\omega x}$ and then divided into a number of integrals over subintervals of size d. The integrals in each subinterval are approximated by a Gaussian rule. We prove convergence of this procedure for functions f that are bounded on $\mathbb{R}^+$ and have bounded derivatives up to an order larger than p.

The numerical tests show that, in general, the new method requires a larger number of function evaluations than the method proposed in [15], but the running time of the latter method is much higher. This results in an overall computational cost of our method being lower even though a larger number of function evaluations are required. In particular, the new method is more convenient than the method proposed in [15], both in terms of accuracy and efficiency, for functions f having an algebraic decay at infinity (see Example 5.2) and/or a low degree of smoothness (see Example 5.3). When the function f is very smooth and has an exponential decay at infinity (see Example 5.1) the two methods are both highly accurate, but the method in [15] requires slightly smaller execution times for large values of ω. Moreover, as shown in Example 5.4, unlike the product rule, the new method can be applied also when the parameter γ is negative.

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Appendix A. Algorithm for computing M. Here we report the algorithm for the function `FindM` that is used to compute the value of M.

Algorithm 1: FindM.

Input: $f, d, \omega, \gamma, p, t$
Output: M
if $\left(t \leq \frac{2d}{\omega}\right)$ **then**
  $M = t + \frac{d}{\omega}$;
  while $\left|\frac{f(M)M^\gamma}{(M-t)^{p+1}}\right| > \text{eps}$ **do**
    $M = M + \frac{1}{10}$;
  end
else
  $M = \frac{d}{\omega}$;
  while $\left|\frac{f(M)M^\gamma}{(M-t)^{p+1}}\right| > \text{eps} \,\&\, M < t - \frac{d}{\omega}$ **do**
    $M = M + \frac{1}{10}$;
  end
  if $\left|\frac{f(M)M^\gamma}{(M-t)^{p+1}}\right| \leq \text{eps}$ **then**
    $M = M + \frac{1}{10}$;
  end
  if $M \geq t - \frac{d}{\omega}$ **then**
    $M = t + \frac{d}{\omega}$;
    while $\left|\frac{f(M)M^\gamma}{(M-t)^{p+1}}\right| > \text{eps}$ **do**
      $M = M + \frac{1}{10}$;
    end
  end
end
$M = M - \frac{d}{\omega}$;
while $\left(\left|\frac{f(M)M^\gamma}{(M-t)^{p+1}}\right| > 2\,\text{eps}\right)$ **do**
  $M = M + \frac{1}{10}$;
end

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