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Numerical approximation of Fredholm integral equation by the constrained mock-Chebyshev least squares operator

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ABSTRACT

In this paper, we propose two numerical approaches for approximating the solution of the following kind of integral equation

$$f(y) - \mu \int_{-1}^{1} f(x)k(x,y)w(x)\,dx = g(y), \quad y \in [-1,1],$$

where f is the unknown solution, $\mu \in \mathbb{R}\backslash\{0\}$, k, g are given functions not necessarily known in the analytical form, and w is a Jacobi weight. The proposed projection methods are based on the constrained mock-Chebyshev least squares polynomials, and starting from data known at equally spaced points, provide a fine approximation of the solution. Such peculiarity can be helpful in all cases we deal with experimental data, typically measured at equispaced points. We prove the introduced methods are stable and convergent in some Sobolev subspace of $C[-1,1]$. Several numerical tests confirm the theoretical estimates and numerical effectiveness of the proposed methods.

1. Introduction

Let us consider the following Fredholm integral equation of the second kind

$$f(y) - \mu \int_{-1}^{1} f(x)k(x,y)w(x)\,dx = g(y), \quad y \in [-1,1], \tag{1}$$

where f is the unknown solution, $\mu \in \mathbb{R}\backslash\{0\}$, w is a Jacobi weight, the kernel k is defined on $[-1,1]^2$, and the right hand function g is defined on $[-1,1]$.

Fredholm integral equations of the type (1) are frequently models for problems arising in mathematical physics and engineering, such as, for instance, Love's equation that describes the electrostatic problem of a circular plate condenser in an unbounded perfect

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fluid [1]. Furthermore, many efficient numerical methods to solve this kind of equation are available in the literature (see, e.g. [2–4]), most of them require that the involved functions be known analytically, since are based on the zeros of orthogonal polynomials. However, in many applications, the data are obtained by experiments on the field through instruments sampling the functions at equally spaced points. So, the methods mentioned above do not apply, unfortunately. On the other hand, approaches based on piecewise polynomial approximation usually lead to low-degree approximations [2]. About global methods based on equally spaced nodes, we mention [5], where the authors proposed a Nyström method based on the so-called Generalized Bernstein polynomials (see [6] and the references therein). Here, we introduce projection methods that make use of the constrained mock-Chebyshev least-squares polynomials, an approximation tool introduced in [7], which produces accurate results starting from values attained at equally spaced grids. Such operators were developed to reduce the Runge phenomenon [8,9]. The idea over which mock-Chebyshev least squares approximation relies is to select a subset of nodes that best mimic the behavior of Chebyshev-Lobatto ones, i.e. the so-called *mock-Chebyshev* set of nodes. The remaining nodes of the starting grid are used in a simultaneous regression process to improve the accuracy of the approximation. By tensor product, this approximation operator was generalized to the hypercube $[-1,1]^d$, $d \in \mathbb{N}$ in [10], and applied to develop different numerical methods based on it. For instance, to introduce a stable and accurate quadrature formula with a high degree of exactness [11,12], a numerical differentiation formula [13] and a product integration rule in [14].

In this paper, we use the constrained mock-Chebyshev least squares polynomials to approximate the solution of Eq. (1), developing two different cases: in the first, only the function g is known on a set of equispaced points; in the second, also the bivariate kernel k is known on an equally spaced grid. These approaches carry out well-conditioned linear systems. Stability and convergence of the methods are proved in some Sobolev-type subspaces of $C[-1,1]$.

The outline of the paper is the following. Section 2 contains some preliminary notations and properties of the approximation polynomial tools to introduce the methods. These last are introduced and studied in Section 3. Section 4 collects some numerical tests to show the efficiency of the described methods. Finally, Section 5 states the proofs of the main results.

2. Preliminaries

In the sequel, by C we will denote any positive constant having different meanings at different occurrences, and the notation $C \neq C(a,b,\dots)$ will be used to underline that C does not depend on $a,b,\dots$. Moreover, if $A,B>0$ are quantities depending on some parameters, the writing $A \sim B$, has to be understood as there exists a constant $C \neq C(A,B)$ such that $C^{-1}B \le A \le CB$. Furthermore, $\mathbb{P}_r$ denotes the space of the algebraic polynomials of degrees less than or equal to r. Finally, for any bivariate function $q(x,y)$, we denote the projections of the function $q(x,y)$ on one variable as $q_y(x)$ and $q_x(y)$ respectively.

2.1. Function spaces

Let $C[-1,1]$ be the space of continuous functions in $[-1,1]$ endowed with the norm

$$\|f\|_\infty := \max_{x\in[-1,1]} |f(x)|.$$

The error of best polynomial approximation in $C[-1,1]$ is defined as

$$E_r(f) = \inf_{Q_r \in \mathbb{P}_r} \|f - Q_r\|_\infty,$$

and (see e.g. [15])

$$\lim_{r\to\infty} E_r(f) = 0 \text{ if and only if } f \in C[-1,1].$$

In $C[-1,1]$ we will consider the following Sobolev-type subspace of order $s \in \mathbb{N},\ s \ge 1$,

$$W_s := \left\{ f \in C[-1,1] \,:\, f^{(s-1)} \in \mathcal{AC}(-1,1) \text{ and } \|f^{(s)}\phi^s\|_\infty < \infty \right\}, \quad \phi(x) = \sqrt{1-x^2},$$

where $\mathcal{AC}(-1,1)$ denotes the set of the absolutely continuous functions on every closed subset $[a,b] \subset (-1,1)$, equipped with the norm

$$\|f\|_{W_s} = \|f\|_\infty + \|f^{(s)}\phi^s\|_\infty.$$

The error of the best polynomial approximation for functions in W_s can be estimated by the following Favard-type inequality [16]

$$E_r(f) \le C \frac{\|f\|_{W_s}}{r^s}, \quad C \neq C(r,f). \tag{2}$$

2.2. Orthogonal polynomials and Lagrange interpolating polynomial

Setting $\{T_i(x) := \cos(i \arccos x)\}_{i\ge 0}$, and $\sigma(x) := \frac{1}{\sqrt{1-x^2}}$, let be

$$\left\{ p_i(\sigma,x) := \frac{T_i(x)}{\|T_i\|_{2,\sigma}} \right\}_{i=0}^{r} \tag{3}$$

$$\|T_i\|_{2,\sigma} := \left(\int_{-1}^{1} T_i(x)^2\sigma(x)\,dx\right)^{\frac{1}{2}} = \begin{cases}\sqrt{\pi}, & n = 0,\\ \sqrt{\frac{\pi}{2}}, & n \neq 0,\end{cases}$$

the orthonormal sequence of the first kind of Chebyshev polynomials.

Setting $\{z_\ell\}_{\ell=1}^{r+1}$ the zeros of $p_{r+1}(\sigma)$, let $\mathcal{L}_r(\sigma)$ be the Lagrange operator mapping continuous functions into polynomials of degree r,

$$\mathcal{L}_r(\sigma) : C[-1,1] \to \mathbb{P}_r,$$

s.t.

$$\mathcal{L}_r(\sigma, f, z_\ell) = f(z_\ell), \quad \ell = 1, 2, \ldots, r+1.$$

The polynomial $\mathcal{L}_r(\sigma)$ can be represented in the following form

$$\mathcal{L}_r(\sigma, h, y) = \frac{\pi}{r+1}\sum_{j=0}^{r} p_j(\sigma, y)\sum_{\ell=1}^{r+1} h(z_\ell)p_j(\sigma, z_\ell). \tag{4}$$

In what follows, it will be useful the following result [17]

Theorem 2.1. *Let q be a positive integer. For any $f \in W_q$, $q \geq 1$ and $0 \leq s \in \mathbb{N}$, $s \leq q$ we have*

$$\|f - \mathcal{L}_r(\sigma, f)\|_{W_s} \leq C\frac{\log r}{r^{q-s}}\|f\|_{W_q}, \quad C \neq C(r, f).$$

Finally, for $m \in \mathbb{N}$,

$$\xi_i^{CL} = -\cos\left(\frac{\pi}{m}i\right), \quad i = 0, \ldots, m,$$

is the set of Chebyshev–Lobatto nodes of order m.

2.3. Constrained mock-Chebyshev least squares operator

For $n \in \mathbb{N}$, let X_n be the set of $n+1$ equispaced nodes in the interval $[-1, 1]$, i.e.

$$X_n = \{\xi_0, \ldots, \xi_n\}, \quad \xi_i = -1 + \frac{2}{n}i, \quad i = 0, \ldots, n.$$

Fixed the integers m, p, r, chosen as in [7, Sections 2 and 4],

$$m = \left\lfloor \pi\sqrt{\frac{n}{2}} \right\rfloor, \quad p = \left\lfloor \pi\sqrt{\frac{n}{12}} \right\rfloor, \quad r = m + p + 1, \tag{5}$$

let $\mathcal{B}_r = \{u_0(x), \ldots, u_r(x)\}$ be an orthonormal basis of $\mathbb{P}_r$. Let

$$X'_m = \{\xi'_0, \ldots, \xi'_m\} \subset X_n, \quad \xi'_i \approx -\cos\left(\frac{\pi}{m}i\right), \quad i = 0, \ldots, m,$$

be the subset of *mock-Chebyshev* nodes, i.e. the node ξ'_i satisfies

$$|\xi'_i - \xi_i^{CL}| = \min_{j=0,\ldots,n} |\xi_j - \xi_i^{CL}|, \quad i = 0, \ldots, m.$$

We assume that the set X_n is rearranged so that the first $m+1$ points coincide with those of X'_m, and the basis $\mathcal{B}_r$ so that the first $m+1$ elements span the space $\mathbb{P}_m$. Under the previous assumptions, we set $\boldsymbol{b} = [f(\xi_0), \ldots, f(\xi_n)]^T$ and

$$V = [u_j(\xi_i)]_{\substack{i=0,\ldots,n\\ j=0,\ldots,r}},$$

the Vandermonde-like matrix associated to X_n and $\mathcal{B}_r$. The constrained mock-Chebyshev least squares operator maps continuous functions into polynomials of degree r

$$\hat{P}_{r,n} : C[-1,1] \to \mathbb{P}_r$$

such that

$$\hat{P}_{r,n}(f, x) := \sum_{k=0}^{r} a_k u_k(x),$$

where the coefficients vector $\boldsymbol{a} = [a_0, \ldots, a_r]^T$ of $\hat{P}_{r,n}(f)$ associated to the basis $\mathcal{B}_r$ can be computed by solving a generalization of normal equations [18, Ch. 16]

$$\begin{bmatrix} 2V^TV & C^T \\ C & 0 \end{bmatrix}\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{z}\end{bmatrix} = \begin{bmatrix} 2V^T\boldsymbol{b} \\ \boldsymbol{d}\end{bmatrix}, \tag{6}$$

where $C = [u_j(\xi_i)]_{\substack{i=0,\dots,m \\ j=0,\dots,r}}$, $\boldsymbol{d} = [f(\xi_0),\dots,f(\xi_m)]^T$ and $\boldsymbol{z} = [\hat{z}_1,\dots,\hat{z}_{m+1}]^T$ is the Lagrange multipliers vector. In the following, we denote by $\hat{P}^C_{r,n}(f,x)$ the constrained mock-Chebyshev least squares operator expressed in the basis (3), i.e.

$$\hat{P}^C_{r,n}(f,x) := \sum_{k=0}^{r} a_k p_k(\sigma,x) \in \mathbb{P}_r. \tag{7}$$

For a given operator $H : X \to Y$, the symbol $\|H\|_{X\to Y}$ denotes the norm of H as map of X into Y, i.e. $\|H\| = \sup_{f\in X} \frac{\|Hf\|_Y}{\|f\|_X}$. About the error estimate in the uniform norm, setting for brevity

$$\|\hat{P}^C_{r,n}\|_\infty := \|\hat{P}^C_{r,n}\|_{C[-1,1]\to C[-1,1]}$$

we recall the following result [13].

Theorem 2.2. *For any $f \in C[-1,1]$*

$$\|f - \hat{P}^C_{r,n}(f)\|_\infty \le C(1 + \|\hat{P}^C_{r,n}\|_\infty)E_r(f),$$

where $C \ne C(r,f)$.

Remark 2.3. Since in [13] (see also [19]) it has been proved that

$$\|\hat{P}^C_{r,n}\|_\infty \le B_n \sim Cn^2, \tag{8}$$

in view of (2), for any $f \in W_s$ we have

$$\|f - \hat{P}^C_{r,n}(f)\|_\infty \le CB_n \frac{\|f\|_{W_s}}{r^s}, \qquad C \ne C(r,f). \tag{9}$$

2.4. Bivariate constrained mock-Chebyshev least squares operator

Denoting by $X_n \times X_n$ the equispaced grid of $(n+1)^2$ nodes in $[-1,1]^2$, in [10] the operator $\hat{P}^C_{r,n}$ has been generalized to the bivariate case. Indeed, for any continuous function $h(x,y)$, the *biviariate constrained mock-Chebyshev least squares operator* is defined as follows

$$\hat{\mathbf{P}}^C_{r,n}(h,x,y) := \hat{P}^C_{r,n}(\hat{P}^C_{r,n}(h_y,x),y) = \sum_{j=0}^{r}\sum_{i=0}^{r} h_{i,j} p_i(\sigma,y) p_j(\sigma,x). \tag{10}$$

Given the estimates in the univariate case, we can easily deduce

$$\|h - \hat{\mathbf{P}}^C_{r,n}(h)\|_\infty \le CB_n^2 \frac{\|h\|_{W_s}}{r^s}, \qquad C \ne C(r,f). \tag{11}$$

holding for any function $h \in \mathbf{W}_s$.

2.5. The constrained mock-Chebyshev least squares product integration rule

Based on the previous constrained mock-Chebyshev least squares polynomial $\hat{P}^C_{r,n}(f)$, in [14] the following quadrature rule was introduced

$$\int_{-1}^{1} f(x)k(x,y)w(x)dx = \int_{-1}^{1} \hat{P}^C_{r,n}(f,x)k(x,y)w(x)dx + \hat{e}_{r,n}(f,y) = \sum_{k=0}^{r} a_k M^w_k(y) + \hat{e}_{r,n}(f,y), \tag{12}$$

where $\hat{e}_{r,n}(f,y)$ is the remainder term and $\{M^w_k(y)\}_{k=0}^r$ are the *modified moments* of k in the first kind Chebyshev system, i.e.

$$M^w_k(y) := \int_{-1}^{1} p_k(\sigma,x)k(x,y)w(x)dx.$$

They are usually computed using suitable recurrence relations depending on the nature of the kernel k and related to the three-term recurrence formula for the orthogonal polynomials.

About the error estimate, the following result holds true

Theorem 2.4 (*[14]*). *For any $f \in C[-1,1]$, under the assumption*

$$\sup_{y\in[-1,1]} \|k_y w\|_1 < \infty,$$

then

$$\sup_{y\in[-1,1]} |\hat{e}_{r,n}(f,y)| \le C(1 + B_n)E_r(f), \tag{13}$$

where $C \ne C(r,f)$.

3. Main results

First, we state a result about the estimate of the operator $\hat{P}_{r,n} : W_s \to W_s$.

Theorem 3.1. *For any $f \in W_q$,*

$$\|f - \hat{P}^C_{r,n}(f)\|_{W_s} \le C\, B_n \frac{\|f\|_{W_q}}{r^{q-s}},$$

where $C \neq C(r, f)$.

Remark 3.2. Given estimate (8) the uniform convergence of $\hat{P}^C_{r,n}(f)$ to f is assured when $q - s > 4$.

In particular, setting $q = s$, by Theorem 3.1 we get the following result about the norm of the operator $\hat{P}^C_{r,n} : W_s \to W_s$

Theorem 3.3. *For any $s \ge 1$,*

$$\|\hat{P}^C_{r,n}\|_{W_s \to W_s} := \sup_{\|f\|_{W_s}=1} \|\hat{P}^C_{r,n}(f)\|_{W_s} \le C\, B_n,$$

where $C \neq C(r)$.

Denoted by $\mathcal{I}$ the identity operator and setting

$$\mathcal{K} f(y) := \mu \int_{-1}^{1} f(x) k(x, y) w(x)\, dx. \tag{14}$$

Eq. (1) can be rewritten in the following operatorial form

$$(\mathcal{I} - \mathcal{K}) f = g. \tag{15}$$

As it is known (see, e.g. [2]), if the kernel k is a continuous function the integral operator $\mathcal{K} : C[-1, 1] \to C[-1, 1]$ is compact, and hence, assuming $\ker\{\mathcal{I} - \mathcal{K}\} = \{0\}$, Eq. (15) admits a unique solution $f \in C[-1, 1]$, for any $g \in C[-1, 1]$. If in addition, the kernel k satisfies the assumption

$$\sup_{y \in [-1,1]} \|k_y\|_{W_s} < \infty, \quad s \ge 1, \tag{16}$$

then $\mathcal{K} f \in W_s$ and if also $g \in W_s$, then the solution $f \in W_s$ too.

Now we introduce two different methods: the first focuses on the case when the values of g are known in a discrete set of equally spaced nodes, and the analytical form of the kernel k is given; the second one applies when both the kernel k and the right-hand side term g are sampled only on a grid of equally spaced nodes in $[-1, 1]^2$ and $[-1, 1]$, respectively.

3.1. Method 1

By means of the Lagrange operator $\mathcal{L}_r(\sigma)$ defined in Section 2.2, project Eq. (15) on $\mathbb{P}_r$ and look for a polynomial solution of the form

$$f_{r+1}(y) = \sum_{k=0}^{r} \alpha_k p_k(\sigma, y).$$

Then we consider the following finite-dimensional equation

$$(\mathcal{I} - \hat{\mathcal{K}}_{r+1}) f_{r+1} = \hat{g}_{r+1}, \tag{17}$$

where for any $f \in C[-1, 1]$

$$\hat{\mathcal{K}}_{r+1} f(y) = \mathcal{L}_r(\sigma, \mathcal{K}_{r+1} f, y), \tag{18}$$

with

$$\mathcal{K}_{r+1} f(y) = \mu \int_{-1}^{1} \hat{P}^C_{r,n}(f, x) k(x, y) w(x) dx \tag{19}$$

and, setting $\hat{P}^C_{r,n}(g, x) = \sum_{i=0}^{r} d_i p_i(\sigma, x)$,

$$\hat{g}_{r+1}(y) = \mathcal{L}_r(\sigma, \hat{P}^C_{r,n}(g), y) = \sum_{j=0}^{r} p_j(\sigma, y) d_j. \tag{20}$$

By (12)

$$\mathcal{K}_{r+1} f_{r+1}(z_\ell) = \mu \sum_{k=0}^{r} \alpha_k M^w_k(z_\ell),$$

then the coefficients $\{\alpha_j\}_{j=0}^r$ are solutions of the linear system

$$\alpha_j - \mu\frac{\pi}{r+1}\sum_{k=0}^{r}\alpha_k\sum_{\ell=1}^{r+1}M_k^{w}(z_\ell)p_j(\sigma,z_\ell) = d_j, \quad j=0,1,\dots r. \tag{21}$$

If the system admits the unique solution $\{\alpha_j^*\}_{j=0}^r$, the polynomial

$$f_{r+1}^*(y) = \sum_{k=0}^{r}\alpha_k^* p_k(\sigma,y)$$

is the unique solution of the finite-dimensional Eq. (17).

Theorem 3.4. *Let $\mathcal{K}$ and $\hat{\mathcal{K}}_{r+1}$ the operators defined in (14) and (18), respectively. If for some $q-s>4,\ s,q\in\mathbb{N}$*

$$\mathcal{M}_q := \sup_{y\in[-1,1]}\|k_y\|_{W_q} < +\infty, \tag{22}$$

$$\mathcal{N}_q := \sup_{x\in[-1,1]}\|k_x\|_{W_q} < +\infty, \tag{23}$$

then

$$\lim_r \|\hat{\mathcal{K}}_{r+1}-\mathcal{K}\|_{W_s\to W_s} = 0. \tag{24}$$

Through (24), given Theorem [2, Th.2.1.1], it follows that under the assumptions of Theorem 3.4, the discrete operator $(I-\mathcal{K}_{r+1})$ is invertible, and the inverse is uniformly bounded in r

$$\sup_r \|(I-\mathcal{K}_{r+1})^{-1}\|_{W_s\to W_s} < +\infty,$$

i.e. Method 1 is stable.

About Method 1, we can prove the following

Theorem 3.5. *Under the assumptions of Theorem 3.4, for any $g\in W_q$ and for r sufficiently large (say $r>r_0$), Eq. (17) admits a unique solution $f_{r+1}^*\in\mathbb{P}_r$ and the following estimate holds true*

$$\|f-f_{r+1}^*\|_{W_s} = \mathcal{O}\left(B_n\,\frac{\log r}{r^{q-s}}\right), \tag{25}$$

where the constants in "$\mathcal{O}$" are independent of r.

Remark 3.6. Given estimate (8), the convergence is assured for $q-s>4$. Moreover, under the same assumptions of Theorem 3.5, considering $s=1$, it follows

$$\|f-f_{r+1}^*\|_\infty \le \|f-f_{r+1}^*\|_{W_1} = \mathcal{O}\left(\frac{\log r}{r^{q-5}}\right). \tag{26}$$

3.2. Method 2

Projecting equation (15) by means of the constrained mock-Chebyshev least squares operator onto the space $\mathbb{P}_r$ and looking for a polynomial solution of the form

$$\widetilde{f}_{r+1}(y) = \sum_{k=0}^{r}\beta_k p_k(\sigma,y),$$

we consider the following finite-dimensional equation

$$(\mathcal{I}-\widetilde{\mathcal{K}}_{r+1})f_{r+1} = \widetilde{g}_{r+1}, \tag{27}$$

where for any $f\in C[-1,1]$

$$\widetilde{\mathcal{K}}_{r+1}f(y) = \hat{P}_{r,n}^{C}(\mathcal{K}_{r+1}^{*}f,y), \tag{28}$$

with

$$\mathcal{K}_{r+1}^{*}f(y) = \mu\int_{-1}^{1}\hat{P}_{r,n}^{C}(k_y,x)f(x)w(x)dx \tag{29}$$

and

$$\widetilde{g}_{r+1}(y) = \hat{P}_{r,n}^{C}(g,y).$$

Setting

$$\hat{\mathbf{P}}^C_{r,n}(k,x,y) = \sum_{j=0}^{r}\sum_{i=0}^{r} k_{i,j}\, p_j(\sigma,y)p_i(\sigma,x)$$

it is no hard to see that

$$\begin{aligned}\widetilde{\mathcal{K}}_{r+1} f(y) &= \mu \int_{-1}^{1} \hat{\mathbf{P}}^C_{r,n}(k,x,y) f(x) w(x) dx \\ &= \mu \sum_{j=0}^{r}\sum_{i=0}^{r} k_{i,j}\, p_j(\sigma,y) \int_{-1}^{1} p_i(\sigma,x) f(x) w(x) dx,\end{aligned}$$

and the finite-dimensional equation takes the form

$$\widetilde{f}_{r+1}(y) - \mu \sum_{j=0}^{r} p_j(\sigma,y) \sum_{i=0}^{r} k_{i,j} \int_{-1}^{1} p_i(\sigma,x)\widetilde{f}_{r+1}(x) w(x) dx = \sum_{j=0}^{r} d_j p_j(\sigma,y),$$

by which the coefficients $\{\beta_k\}_{k=0}^r$ defining $\widetilde{f}_{r+1}$ are solution of the following linear system

$$\beta_j - \mu \sum_{k=0}^{r} \beta_k \sum_{i=0}^{r} k_{i,j} A_{i,k} = d_j, \quad j = 0,1,\ldots,r, \tag{30}$$

where

$$A_{i,k} = \int_{-1}^{1} p_i(\sigma,x) p_k(\sigma,x) w(x) dx. \tag{31}$$

Remark 3.7. In case $w = \sigma$, it is $A_{i,k} = \delta_{i,k}$, and system (30) takes the simpler form

$$\beta_j - \mu \sum_{k=0}^{r} \beta_k k_{i,k} = d_j, \quad j = 0,1,\ldots,r.$$

We can prove the following theorem

Theorem 3.8. *Let $\mathcal{K}$ and $\widetilde{\mathcal{K}}_{r+1}$ be the operators defined in (14) and (28), respectively. For $q,s \in \mathbb{N}$ s.t. $q-s>8$*

$$\mathcal{M}_q := \sup_{y\in[-1,1]} \|k_y\|_{W_q} < +\infty, \tag{32}$$

$$\mathcal{N}_q := \sup_{x\in[-1,1]} \|k_x\|_{W_q} < +\infty, \tag{33}$$

then

$$\lim_r \|\widetilde{\mathcal{K}}_{r+1} - \mathcal{K}\|_{W_s \to W_s} = 0.$$

Theorem 3.9. *Under the assumption (32) and (33), for any $g \in W_q$, $q-s>8$ and for r sufficiently large (say $r>r_0$), Eq. (27) admits a unique solution $f^*_{r+1} \in \mathbb{P}_r$ and the following estimate holds true*

$$\|f - \widetilde{f}_{r+1}\|_{W_s} = \mathcal{O}\left(\frac{1}{r^{q-s-8}}\right), \tag{34}$$

where the constants in "$\mathcal{O}$" are independent of r.

The proof follows from Theorems 3.8 and 3.1, by using the same arguments for the error estimation of Method 1.

3.3. Pros and cons

Let us compare our methods with other ones, under the common assumption that the known functions k, g belong to W_q. In [5] it was studied a Nyström type method based on the so-called *Generalized Bernstein polynomials* (shortly GB) $B_{n,\ell}(f) \in \mathbb{P}_r$, constructed on $n+1$ equally spaced nodes in $[-1,1]$. As it is known, in Nyström type methods a quadrature sum approximates the integral, and, under suitable assumptions, the solution of the obtained finite-dimensional equation converges to the exact one, with the same rate of convergence of the employed quadrature rule. Now, coming back to the method in [5] (there implemented in $[0,1]$), if the additional parameter $\ell \geq \frac{q}{2}$, the error of the *GB* method behaves as $\mathcal{O}(n^{-q/2})$, i.e., roughly speaking, as half of the order of the best approximation estimate $E_n(f)$ in the space W_q. From this point of view, by (26) and (34), both our Methods 1 and 2 converge slower than the GB method. However, our method requires $r \sim \sqrt{n/2}$ function's evaluation, while the GB method requires $n+1$ function's evaluations. Another aspect to take into account is the loss of precision of the GB method when ℓ trespasses a certain threshold [5], since for n fixed and $\ell \to \infty$ the sequence $B_{n,\ell}(f)$ uniformly converges to the Lagrange polynomial interpolating f at $n+1$ equally spaced points of $[-1,1]$. It is hardly worth mentioning that interpolation processes on equidistant nodes have exponentially divergent Lebesgue constants, which is why a projection method based on them can lead to disastrous results [3].

Consider now the projection method in [3] based on the Lagrange polynomial $\mathcal{L}_r(\sigma)$, whose rate of convergence is $\mathcal{O}(\frac{\log r}{r^q})$. It provides a better performance w.r.t. our methods, but it requires the samples of g, k at the zeros of the orthonormal polynomial $p_r(\sigma)$, which it is not always possible to deal with experimental data.

4. Computational details and numerical tests

In this section, we provide some numerical details about the construction of the algebraic linear systems we solve in both methods. In the second part we present some tests to numerically prove the accuracy of our methods, comparing them also with the results obtained through the global method proposed in [5].

4.1. The linear system of method 1

The linear system (21) can be rewritten in the following matrix form

$$\left(\mathbf{I}_{r+1} - \mu \frac{\pi}{r+1} \mathbf{PM}\right) \boldsymbol{\alpha}_{r+1} = \boldsymbol{d}_{r+1}, \tag{35}$$

where $\mathbf{I}_{r+1}$ is the identity matrix of order $(r+1)$, $\boldsymbol{\alpha}_{r+1} = [\alpha_0, \ldots, \alpha_r]^T$, $\boldsymbol{d}_{r+1} = [d_0, \ldots, d_r]^T$ and

$$\mathbf{P} = (p_{i,j}) \in \mathbb{R}^{(r+1)\times(r+1)}, \quad \text{with} \quad p_{i,j} = p_i(\sigma, z_j), \tag{36}$$

$$\mathbf{M} = (m_{i,j}) \in \mathbb{R}^{(r+1)\times(r+1)}, \quad \text{with} \quad m_{i,j} = M_j^w(z_i). \tag{37}$$

The entries of the matrix P are

$$p_{i,j} = \cos\left(i \frac{(2j-1)}{2(r+1)} \pi \right),$$

and those of the matrix M can be evaluated by suitable three-term recurrence relations, where the computation of the starting moments $M_0(z_i)$ depends on the nature of the kernel k. Once evaluated them, the successive $\{M_j(z_i)\}_{j=1}^r$ require $\mathcal{O}(r^2)$ long operations (shortly l.o.). The construction of the right-hand side vector $\boldsymbol{d}_{r+1}$ leads to $\mathcal{O}(n^2)$ l.o. [7, Section 7], and the final linear system (35) has been solved by Gauss elimination.

4.2. The linear system of method 2

Denoted by

$$\mathbf{A} = (A_{i,j}) \in \mathbb{R}^{(r+1)\times(r+1)}, \quad i, j = 0, \ldots, r,$$
$$\mathbf{H} = (k_{j,i}) \in \mathbb{R}^{(r+1)\times(r+1)}, \quad i, j = 0, \ldots, r,$$

with $A_{i,j}$ are defined as in (31), the system (30) becomes

$$(\mathbf{I}_{r+1} - \mu \mathbf{HA})\boldsymbol{\beta}_{r+1} = \boldsymbol{d}_{r+1}, \tag{38}$$

where $\boldsymbol{\beta}_{r+1} = [\beta_0, \ldots, \beta_r]^T$, $\boldsymbol{d}_{r+1} = [d_0, \ldots, d_r]^T$. The matrix $\mathbf{H}$ is obtained as solution of the bivariate constrained mock-Chebyshev least squares interpolant KKT system and requires $\mathcal{O}(n^4)$ l.o.. The matrix $\mathbf{A}$ is symmetric and can be computed by applying r^2 Gauss quadrature rules, for a global amount of $\mathcal{O}(r^3)$ l.o.. In particular, when $w = \sigma$, it is $\mathbf{A} = \mathbf{I}_{r+1}$. The construction of the vector $\boldsymbol{d}_{r+1}$ requires $\mathcal{O}(n^2)$ l.o. [7, Section 7], and the linear system (38) has been solved by Gauss elimination.

4.3. Numerical examples

In the first example, the exact solution f^* is known, and hence we compute the relative error

$$\hat{e}_{r,n}(f) = \max_{y_i \in \mathbb{I}} \frac{|f^*(y_i) - f^*_{r+1}(y_i)|}{|f^*(y_i)|}, \tag{39}$$

where $\mathbb{I}$ is a sufficiently dense mesh in $[-1, 1]$. In the remaining examples, the exact solution f^* is unknown and in these cases, we report the relative errors

$$\hat{e}_{r,n}(f) = \max_{y_i \in \mathbb{I}} \frac{|\tilde{f}(y_i) - f^*_{r+1}(y_i)|}{|\tilde{f}(y_i)|}, \tag{40}$$

where $\tilde{f}$ is the solution obtained by the Nyström method in [2]. Indeed, we assume $\tilde{f}$ as the exact solution since the method is stable and convergent (see e.g. [2], Ch.4).

In each table we display the condition number in infinity norm of the matrix (35) or (38), depending on the chosen method. Finally, we point out that all the computations were carried out in Matlab R2022a in double precision on a MacBook Pro laptop with 16Gb RAM under the MacOS system.

Table 1
Numerical results for Example 4.1 by Method 1.

n	m	p	r	$\hat{e}_{r,n}(f)$	cond
50	15	6	22	1.74e−12	1.20e+01
100	22	9	32	3.41e−12	1.20e+01

Table 2
Numerical results for Example 4.1 by Method 2.

n	m	p	r	$\hat{e}_{r,n}(f)$	cond
50	15	6	22	6.26e−11	1.20e+01
100	22	9	32	3.76e−11	1.20e+01

Table 3
Numerical results for Example 4.2 by Method 1.

n	m	p	r	$\hat{e}_{r,n}(f)$	cond
50	15	6	22	2.55e−06	1.27
100	22	9	32	1.33e−08	1.27
250	35	14	50	2.73e−09	1.27
500	49	20	70	4.55e−11	1.27
1000	70	28	99	7.62e−13	1.27

Example 4.1. Let us consider the following equation

$$f(y) - \frac{1}{2\pi}\int_{-1}^{1} \frac{f(x)e^{x+y}}{\sqrt{1-x^2}}\,dx = e^y\left(y - \frac{I_1(2)}{2}\right),$$

whose exact solution is $f(y) = ye^y$. In this case the kernel $k(x,y) = e^{x+y}$ and the right hand side term $g(y) = y - e^{y-1}\,I_1(2)$, where $I_1(y)$ is the modified Bessel function of the first kind, are both smooth functions and $w(x) = v^{-1/2,-1/2}(x)$ is the first kind Chebyshev weight. Table 1 and 2 report the numerical results obtained by Method 1 and Method 2, respectively. About the latter one, we assume that k and g are both sampled only over the equally spaced nodes of the grid. Both methods show comparable results and good conditioning of the associated linear systems. This behavior is motivated by the smoothness of the involved functions.

Example 4.2. Let us consider the following equation

$$f(y) - \frac{1}{12}\int_{-1}^{1} f(x)(x^2+y^2)\log(3-x)(1-x)^2\sqrt{1+x}\,dx = \left|y - \frac{1}{2}\right|^{\frac{11}{2}},$$

whose exact solution is unknown. In this case the kernel $k(x,y) = (x^2+y^2)\log(3-x)$ is smooth, while the right hand side term $g(y) = \left|y-\frac{1}{2}\right|^{\frac{11}{2}}$ belongs to the Sobolev space W_5. Finally, $w(x) = v^{2,1/2}(x)$ is a Jacobi weight. In this case, we expect a slightly slower convergence due to the nature of the right-hand side term g. Our expectations are confirmed in Table 3, where we can observe that we need denser equally spaced grids to obtain a satisfying approximation of the unknown solution f^*. Hence, we can deduce that the method is convergent and leads to a well-conditioned system.

Example 4.3. Let us consider the following equation

$$f(y) - \frac{1}{27}\int_{-1}^{1} f(x)\frac{\sin(50x)}{(x^2+50^{-2})^{\frac{11}{10}}}\,dx = y\sin y,$$

whose exact solution is unknown. In this case, the kernel $k(x,y) = \dfrac{\sin(50x)}{(x^2+50^{-2})^{\frac{11}{10}}}$ contains a nearly singular factor and a highly oscillating one. The right-hand side term $g(y) = y\sin y$ is smooth, and $w(x) = v^{0,0}(x)$ is a Legendre weight. In this example, we decided to compare the performances of our method based on equispaced nodes with the ones achieved by a fast Nyström method based on zeros of orthonormal polynomials introduced in [4]. Considering that when the equispaced grid consists of $n+1 = 1001$ points, we essentially go to solve a linear system of $r+1 = 100$ equations, we get a satisfying approximation of the unknown solution, as reported in Table 4. Nevertheless, the method in [4] shows a faster convergence, but this is due to the use of zeros of orthonormal polynomials. About the condition numbers, we note that only in this particular example they get higher with the increase of the density of the grid, although this circumstance does not affect the convergence of the method.

Example 4.4. Let us consider the following equation

$$f(y) - \frac{1}{10}\int_{-1}^{1} f(x)k(x,y)w(x)\,dx = g(y),$$

Table 4
Numerical results for Example 4.3 by Method 1.

n	m	p	r	$\hat{e}_{r,n}(f)$	cond
50	15	6	22	1.40e−10	2.91e+02
100	22	9	32	7.06e−11	1.22e+03
250	35	14	50	1.06e−11	7.16e+03
500	49	20	70	1.50e−11	2.02e+04
1000	70	28	99	1.14e−11	3.96e+04

Table 5
Numerical results for Example 4.4 by Method 2.

n	m	p	r	$\hat{e}_{r,n}(f)$	cond
50	15	6	22	2.18e−13	1.44
100	22	9	32	2.11e−13	1.44
250	35	14	50	3.10e−13	1.44
500	49	20	70	1.40e−13	1.44

whose exact solution is unknown. To test the effectiveness of our second method we start from a set of values taken on a grid of equally spaced nodes $D = \left\{(x_i, y_j) \mid x_i = -1 + i\frac{2}{n}, y_j = -1 + j\frac{2}{n}, i, j = 0, \dots, n\right\}$. In this case the samples of the kernel k are in a square matrix whose entries are $k_{i,j} = k(x_i, y_j)$, while the samples of the right hand side term g are contained in a vector $\boldsymbol{g} = [g_0, \dots, g_n]^T$ where $g_j = g(y_j)$. Finally, $w(x) = v^{-1/2,-1/2}(x)$ is a Chebyshev weight of the first kind. Table 5 displays a fast convergence of the approximate solution to the exact one, as well as a good condition number of the linear system (38). We point out that Method 2 cannot take into consideration meshes that are too dense, due to the more complicated implementation and the high amount of CPU-RAM required to construct the bivariate constrained mock-Chebyshev least squares operator.

5. The proofs

Proof of Theorem 3.1. Start from

$$\begin{aligned}\|f - \hat{P}^C_{r,n}(f)\|_{W_s} &= \|f - \hat{P}^C_{r,n}(f)\|_{W_s} = \|f - \hat{P}^C_{r,n}(f)\|_\infty + \|(f - \hat{P}^C_{r,n}(f))^{(s)}\varphi^s\|_\infty \\ &=: A + B. \end{aligned} \tag{41}$$

Taking into account Theorem 2.2, estimate (2) we get

$$A \le C\ B_n \frac{\|f\|_{W_q}}{r^q}. \tag{42}$$

To estimate B we recall the following result holding for any polynomial $P_n \in \mathbb{P}_n$, (see e.g. [17, p. 276]),

$$\|(f - P_n)^{(s)}\varphi^s\|_\infty \le C\left[n^s\|(f - P_n)\|_\infty + E_{n-s}(f^{(s)})_{\varphi^s}\right], \tag{43}$$

where

$$E_{n-s}(f^{(s)})_{\varphi^s} := \inf_{P\in\mathbb{P}_{n-s}} \|(P - f^{(s)})\varphi^s\|_\infty, \quad E_{n-s}(f^{(s)})_{\varphi^s} \le \frac{\|f\|_{W_q}}{n^{q-s}}, \ \forall f \in W_q,\ q \ge s, \tag{44}$$

and taking into account (2) we have

$$B \le C\ B_n \frac{\|f\|_{W_q}}{r^{q-s}}. \tag{45}$$

The theorem follows combining estimates (42), (45) with (41). ■

Proof of Theorem 3.4. Start from

$$\|\hat{\mathcal{K}}_{r+1} f - \mathcal{K} f\|_{W_s} \le \|\hat{\mathcal{K}}_{r+1} f - \mathcal{K}_{r+1} f\|_{W_s} + \|\mathcal{K}_{r+1} f - \mathcal{K} f\|_{W_s} =: J_1 + J_2. \tag{46}$$

To estimate J_1, first, we observe that the assumption (23) implies $\mathcal{K}_{r+1} f \in W_q$, and by Theorem 2.1, we have

$$J_1 = \|\mathcal{L}_r(\sigma, \mathcal{K}_{r+1} f) - \mathcal{K}_{r+1} f\|_{W_s} \le C\mathcal{M}_q \frac{\log r}{r^{q-s}} \|\mathcal{K}_{r+1} f\|_{W_q} \tag{47}$$

with

$$\|\mathcal{K}_{r+1} f\|_{W_q} = \|\mathcal{K}_{r+1} f\|_\infty + \|(\mathcal{K}_{r+1} f)^{(q)}\varphi^q\|_\infty =: A_1 + A_2. \tag{48}$$

We have

$$A_1 \le \mu(\|\hat{P}^C_{r,n}(f) - f\|_\infty + \|f\|_\infty) \max_{y\in[-1,1]} \int_{-1}^1 |k(x, y)| w(x) dx \le C\mathcal{M}_q\ B_n\ \|f\|_\infty,$$

$$A_2 \le \mu\, \mathcal{M}_q \int_{-1}^{1} |\hat{P}^C_{r,n}(f,x)|w(x)dx \le C\mathcal{M}_q\ B_n\ \|f\|_\infty,$$

and combining last two estimates with (47), we can conclude

$$J_1 \le C\mathcal{M}_q\ B_n\ \frac{\log r}{r^{q-s}}\|f\|_{W_q}. \tag{49}$$

To estimate J_2 we note that

$$|\mathcal{K}_{r+1}f(y) - \mathcal{K}f(y)| \le \mu \int_{-1}^{1} |f(x) - \hat{P}^C_{r,n}(f,x)||k(x,y)|w(x)dx \tag{50}$$

and by estimate (9) it follows

$$\|\mathcal{K}_{r+1}f - \mathcal{K}f\|_\infty \le C\ B_n\ \frac{\|f\|_{W_q}}{r^q} \sup_{(x,y)\in[-1,1]^2} |k(x,y)|.$$

Under the assumption that the kernel k_x satisfies

$$|(\mathcal{K}_{r+1}f(y) - \mathcal{K}f(y))^{(s)}\varphi^s(y)| \le C\mathcal{N}_q \int_{-1}^{1} |\hat{P}^C_{r,n}(f,x) - f(x)|w(x)dx \le C\ B_n\ \mathcal{N}_q \frac{\|f\|_{W_q}}{r^q}. \tag{51}$$

By (50), (51), we get

$$J_2 \le C\ B_n\ \mathcal{N}_q \frac{\|f\|_{W_q}}{r^q}. \tag{52}$$

Combining (47) and (52) with (46), and taking into account (8), under the assumption $q - s > 4$ Theorem 3.4 follows. ■

Proof of Theorem 3.5. The first part follows by [2, p. 55]. To estimate the error, start from

$$f - f^*_{r+1} = (I - \mathcal{K})^{-1}\left[(g - \hat{g}_{r+1}) + (\mathcal{K} - \mathcal{K}_{r+1})((I - \mathcal{K}_{r+1})^{-1}\hat{g}_{r+1})\right].$$

By standard arguments we get

$$\|f - f^*_{r+1}\|_{W_s} \le C\left(\|(\mathcal{K} - \hat{\mathcal{K}}_{r+1})f\|_{W_s} + \|g - \hat{g}_{r+1}\|_{W_s}\right),$$

and the thesis follows taking into account Theorems 3.1 and 3.4. ■

Proof of Theorem 3.8. We start from

$$\|\widetilde{\mathcal{K}}_{r+1}f - \mathcal{K}f\|_{W_s} \le \|\widetilde{\mathcal{K}}_{r+1}f - \mathcal{K}^*_{r+1}f\|_{W_s} + \|\mathcal{K}^*_{r+1}f - \mathcal{K}f\|_{W_s} =: \widetilde{J}_1 + \widetilde{J}_2. \tag{53}$$

First we estimate $\widetilde{J}_2$. Consider

$$\begin{aligned} |\mathcal{K}^*_{r+1}f(y) - \mathcal{K}f(y)| &\le \int_{-1}^{1} |\hat{P}^C_{r,n}(k_y,x) - k(x,y)|f(x)w(x)dx \\ &\le C\ B_n\ \|f\|_\infty E_r(k_y) \le C\ B_n\ \|f\|_\infty \sup_{y\in[-1,1]} E_r(k_y) \end{aligned}$$

and taking the supremum on $y \in [-1,1]$ in the left hand side, we get

$$\|\mathcal{K}^*_{r+1}f - \mathcal{K}f\|_\infty \le C\ B_n\ \|f\|_\infty \sup_{y\in[-1,1]} E_r(k_y).$$

Then, under the assumption about the kernel $k(x,y)$ we have

$$\|\mathcal{K}^*_{r+1}f - \mathcal{K}f\|_\infty \le C\ B_n\ \mathcal{M}_q \frac{\|f\|_\infty}{r^q}. \tag{54}$$

Consider now

$$\begin{aligned} |(\mathcal{K}^*_{r+1}f(y) - \mathcal{K}f(y))^{(s)}|\varphi^s(y) &= \left|\varphi^s(y)\int_{-1}^{1} \frac{d^s}{dy^s}[\hat{P}^C_{r,n}(k_y,x) - k(x,y)]f(x)w(x)dx\right| \\ &\le \|f\|_\infty \int_{-1}^{1} \varphi^s(y)\left|\frac{d^s}{dy^s}[\hat{P}^C_{r,n}(k_y,x) - k(x,y)]\right| w(x)dx \\ &\le \|f\|_\infty \int_{-1}^{1} \max_{y\in[-1,1]} |\varphi^s(y)(\hat{P}_{r,n}(k_y,x) - k_y(x))^{(s)}|w(x)dx \end{aligned}$$

and by using inequality (43), we get

$$\begin{aligned} &|(\mathcal{K}^*_{r+1}f(y) - \mathcal{K}f)^{(s)}(y)|\varphi^s(y) \le C\|f\|_\infty \\ \times &\int_{-1}^{1}\left\{r^s \max_{y\in[-1,1]} |\hat{P}_{r,n}(k_y,x) - k_y(x)| + E_{r-s}(k_y^{(s)}(x))_{\varphi^s}\right\} w(x)dx \end{aligned}$$

$$\leq\; C\|f\|_\infty \sup_{x\in[-1,1]}\left\{ r^s\|\hat{P}_{r,n}(k_y,x)-k_y(x)\|_\infty \;+\; E_{r-s}(k_y^{(s)}(x))_{\varphi^s}\right\}$$
$$=:\; C\|f\|_\infty\left(H_1+H_2\right). \tag{55}$$

By Theorem 2.2

$$\|\hat{P}_{r,n}(k_y,x)-k_y(x)\|_\infty \leq C\; B_n\; E_r(k_y(x))$$

and by estimate (2), under the assumption $k_y\in W_q\ \forall x\in[-1,1]$, it follows

$$\sup_{x\in[-1,1]}\|\hat{P}_{r,n}(k_y,x)-k_y(x)\|_\infty \leq C\; B_n\; \frac{\mathcal{N}_q}{r^q}$$

and hence

$$H_1\leq C\; B_n\; \frac{\mathcal{N}_q}{r^{q-s}}. \tag{56}$$

To estimate H_2 we use (44) to obtain

$$E_{r-s}(k_y^{(s)}(x))_{\varphi^s}\leq C\frac{\|k_y^{(s)}(x)\|_{W_q}}{r^{q-s}},$$

and taking the supremum on $x\in[-1,1]$, in view of the assumptions (33), we can conclude

$$H_2\leq C\frac{\mathcal{N}_q}{r^{q-s}}. \tag{57}$$

By (56), (55) and (57) it follows

$$\|(\mathcal{K}^*_{r+1}f-\mathcal{K}f)^{(s)}\varphi^s\|_\infty\leq C\; B_n\; \|f\|_{W_s}\frac{\mathcal{N}_q}{r^{q-s}} \tag{58}$$

and by the last estimate combined with (54) it follows

$$\|\mathcal{K}^*_{r+1}f-\mathcal{K}f\|_{W_s}\leq C\; B_n\; \|f\|_{W_s}\frac{\mathcal{N}_q+\mathcal{M}_q}{r^{q-s}}. \tag{59}$$

and hence, under the assumption $q-s>4$, we can conclude

$$\lim_r\|\mathcal{K}^*_{r+1}-\mathcal{K}\|_{W_s\to W_s}=0. \tag{60}$$

Now we estimate $\widetilde{J}_1$. Under the assumption (33) it is $\mathcal{K}^*_{r+1}f\in W_q$, and by Theorem 3.1 it follows

$$\widetilde{J}_1=\|\widetilde{\mathcal{K}}_{r+1}f-\mathcal{K}^*_{r+1}f\|_{W_s}\leq C\; B_n\; \frac{\|\mathcal{K}^*_{r+1}f\|_{W_q}}{r^{q-s}} \tag{61}$$

On the other hand, by

$$\|\mathcal{K}^*_{r+1}f\|_{W_q}\leq\|\mathcal{K}^*_{r+1}f-\mathcal{K}f\|_{W_q}+\|\mathcal{K}f\|_{W_q}$$

and by (59) for $q=s$, for any $f\in W_q$ it follows, under the assumption (33),

$$\|\mathcal{K}^*_{r+1}f\|_{W_q}\leq C\; B_n\;(\mathcal{N}_q+\mathcal{M}_q)\|f\|_{W_q}.$$

Combining the last estimate with (61) it follows

$$\widetilde{J}_1\leq C\; B_n^2\; \frac{\|f\|_{W_q}}{r^{q-s}}$$

As a consequence, taking into account estimate (8), it follows that

$$\lim_r\|(\mathcal{K}^*_{r+1}-\widetilde{\mathcal{K}})\|_{W_s\to W_s}=0,$$

under the assumption $q-s>8$. The thesis is proved by combining the last estimate and (60) with (53). ■

6. Conclusions

In this paper, we proposed two projection methods for approximating the solutions of second-kind Fredholm integral equations. The methods are based on the constrained mock-Chebyshev least squares operator, which for its nature, allows to treating the case of functions g and k, known on a discrete set of equally spaced points. Such a situation arises in the experimental field, and numerical methods over equispaced grids are not so common in literature. In this context, our methods contribute from both a numerical and theoretical point of view. Indeed, we have determined conditions assuring the convergence and stability of the proposed methods, depending on the smoothness of the known functions g, k. However, such theoretical estimates are useful when dealing with integral equations for which analytical properties of g and k are known in the experimental context they live in. From the numerical point of view, the numerical experimentation has given good results, which is a good starting point, encouraging us to investigate further. In particular, we aim to find better estimates for the norm of the operator $\hat{P}^C_{r,n} : C[-1,1] \to \mathbb{P}_r$, which breaks the theoretical rate of convergence, even though the numerical results are definitively better. The second aspect we want to focus on is the study of the conditioning of the final linear systems of our methods. Finally, we are planning to extend such methods in the solution of other kinds of integral equations, also frequently arising in applications [20–22].

Data availability

Data will be made available on request.

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