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The L^p -dissipativity of first order partial differential operators

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ABSTRACT

We find necessary and sufficient conditions for the L^p -dissipativity of the Dirichlet problem for systems of partial differential operators of the first order with complex locally integrable coefficients. As a by-product we obtain sufficient conditions for a certain class of systems of the second order.

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1. Introduction

The goal of the present paper is to find necessary and sufficient conditions for the L^p -dissipativity for systems of partial differential equations of the first order ($1 < p < \infty$).

Previously we have considered a scalar second order partial differential operator whose coefficients are complex-valued measures [1]. For some classes of such operators we have algebraically characterized the L^p -dissipativity. The main result is that the algebraic condition

$$|p - 2| |\langle \operatorname{Im} \mathcal{A} \xi, \xi \rangle| \leq 2\sqrt{p-1} \langle \operatorname{Re} \mathcal{A} \xi, \xi \rangle$$

(for any $\xi \in \mathbb{R}^n$) is necessary and sufficient for the L^p -dissipativity of the Dirichlet problem for the differential operator $\nabla^t(\mathcal{A} \nabla)$, where \mathcal{A} is a matrix whose entries are complex measures and whose imaginary part is symmetric.

We remark that conditions obtained in [1] characterizes the L^p -dissipativity individually, for each p . Previous known results in the literature dealt with the L^p -dissipativity for any $p \in [1, +\infty)$, simultaneously. In the same spirit we have studied the elasticity system and some classes of systems of partial differential operators of the second order in [2,3].

Our results are described and considered in the more general frame of semi-bounded operators in the monograph [4].

The main result of the present paper concerns the matrix operator

$$Eu = \mathcal{B}^h(x) \partial_h u + \mathcal{D}(x)u,$$

where $\mathcal{B}^h(x) = \{b_{ij}^h(x)\}$ and $\mathcal{D}(x) = \{d_{ij}(x)\}$ are matrices with complex locally integrable entries defined in the domain Ω of \mathbb{R}^n and $u = (u_1, \dots, u_m)$ ($1 \leq i, j \leq m$, $1 \leq h \leq n$). It

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states that, if $p \neq 2$, E is L^p -dissipative if, and only if,

$$\mathcal{B}^h(x) = b_h(x)I \text{ a.e.,} \tag{1}$$

$b_h(x)$ being real locally integrable functions, and the inequality

$$\operatorname{Re}\langle (p^{-1}\partial_h \mathcal{B}^h(x) - \mathcal{D}(x))\zeta, \zeta \rangle \geq 0$$

holds for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$. If $p = 2$ condition (1) is replaced by the more general requirement that the matrices $\mathcal{B}^h(x)$ are self-adjoint a.e.

5 On combining this with the results we have previously obtained, we deduce sufficient conditions for the L^p -dissipativity of certain systems of partial differential operators of the second order.

2. Preliminaries

10 Let Ω be a domain of \mathbb{R}^n . By $C_0(\Omega)$ we denote the space of complex valued continuous functions having compact support in Ω . Let $C_0^1(\Omega)$ consist of all the functions in $C_0(\Omega)$ having continuous partial derivatives of the first order. The inner product in \mathbb{C}^m is denoted by $\langle \cdot, \cdot \rangle$ and, as usual, the bar denotes complex conjugation.

In what follows, if \mathcal{G} is a $m \times m$ matrix function with complex valued entries, then \mathcal{G}^* is its adjoint matrix, i.e. $\mathcal{G}^* = \overline{\mathcal{G}^t}$, \mathcal{G}^t being the transposed matrix of \mathcal{G} .

15 Let \mathcal{B}^h and \mathcal{C}^h ($h = 1, \dots, n$) be $m \times m$ matrices with complex-valued entries $b_{ij}^h, c_{ij}^h \in (C_0(\Omega))^*$ ($1 \leq i, j \leq m$). Let \mathcal{D} stand for a matrix whose elements d_{ij} are complex-valued distributions in $(C_0^1(\Omega))^*$.

We adopt the summation convention over repeated indices unless otherwise stated.

We denote by $\mathcal{L}(u, v)$ the sesquilinear form

$$\mathcal{L}(u, v) = \int_{\Omega} \langle \mathcal{B}^h \partial_h u, v \rangle - \langle \mathcal{C}^h u, \partial_h v \rangle + \langle \mathcal{D} u, v \rangle$$

defined in $(C_0^1(\Omega))^m \times (C_0^1(\Omega))^m$, where $\partial_h = \partial/\partial x_h$.

The integrals appearing in this definition have to be understood in a proper way. The entries b_{ij}^h being measures, the meaning of the first term is

$$\int_{\Omega} \langle \mathcal{B}^h \partial_h u, v \rangle = \int_{\Omega} \bar{v}_i \partial_h u_j db_{ij}^h.$$

20 Similar meanings have the terms involving \mathcal{C} . Finally, the last term is the sum of the actions of the distribution $d_{ij} \in (C_0^1(\Omega))^*$ on the functions $u_j \bar{v}_i$ belonging to $C_0^1(\Omega)$.

The form \mathcal{L} is related to the system of partial differential operators of the first order:

$$Eu = \mathcal{B}^h \partial_h u + \partial_h(\mathcal{C}^h u) + \mathcal{D} u$$

Following [4], we say that the form \mathcal{L} is L^p -dissipative if

25
$$\operatorname{Re} \mathcal{L}(u, |u|^{p-2}u) \leq 0 \quad \text{if } p \geq 2; \tag{2}$$

$$\operatorname{Re} \mathcal{L}(|u|^{p-2}u, u) \leq 0 \quad \text{if } 1 < p < 2 \tag{3}$$

for all $u \in (C_0^1(\Omega))^m$.

In the present paper, saying the L^p -dissipativity of the operator E , we mean the L^p -dissipativity of the corresponding form \mathcal{L} , just to simplify the terminology.

Let us start with a technical lemma which is a particular case of a result in [4, p.94]. The proof is mainly included here to keep the exposition as self-contained as possible.

Lemma 1: *The operator E is L^p -dissipative in Ω if, and only if,*

$$\int_{\Omega} \left((1 - 2/p)|v|^{-2} \operatorname{Re}\langle \mathcal{B}^h v, v \rangle \operatorname{Re}\langle v, \partial_h v \rangle - \operatorname{Re}\langle \mathcal{B}^h \partial_h v, v \rangle \right. \\ \left. + (1 - 2/p)|v|^{-2} \operatorname{Re}\langle \mathcal{C}^h v, v \rangle \operatorname{Re}\langle v, \partial_h v \rangle + \operatorname{Re}\langle \mathcal{C}^h v, \partial_h v \rangle - \operatorname{Re}\langle \mathcal{D} v, v \rangle \right) \geq 0 \quad (4)$$

for any $v \in (C_0^1(\Omega))^m$. Here and in the sequel the integrand is extended by zero on the set where v vanishes.

Proof:

Sufficiency. First suppose $p \geq 2$. Let $u \in (C_0^1(\Omega))^m$ and set $v = |u|^{(p-2)/2}u$. We have $v \in (C_0^1(\Omega))^m$ and $u = |v|^{(2-p)/p}v$, $|u|^{p-2}u = |v|^{(p-2)/p}v$. From the identities

$$\langle \mathcal{B}^h \partial_h u, |u|^{p-2}u \rangle = -(1 - 2/p)|v|^{-1} \langle \mathcal{B}^h v, v \rangle \partial_h |v| + \langle \mathcal{B}^h \partial_h v, v \rangle, \\ \langle \mathcal{C}^h u, \partial_h (|u|^{p-2}u) \rangle = (1 - 2/p)|v|^{-1} \langle \mathcal{C}^h v, v \rangle \partial_h |v| + \langle \mathcal{C}^h v, \partial_h v \rangle, \\ \langle \mathcal{D} u, |u|^{p-2}u \rangle = \langle \mathcal{D} v, v \rangle, \quad \partial_h |v| = |v|^{-1} \operatorname{Re}\langle v, \partial_h v \rangle$$

we see that the left hand side in (4) is equal to $-\mathcal{L}(u, |u|^{p-2}u)$. Then (2) is satisfied for any $u \in (C_0^1(\Omega))^m$.

If $1 < p < 2$ we may write (3) as

$$\operatorname{Re} \int_{\Omega} \left(\langle (\mathcal{B}^h)^* u, \partial_h (|u|^{p'-2}u) \rangle - \langle (\mathcal{C}^h)^* \partial_h u, |u|^{p'-2}u \rangle + \langle \mathcal{D}^* u, |u|^{p'-2}u \rangle \right) \leq 0$$

for any $u \in (C_0^1(\Omega))^m$. The first part of the proof shows that

$$\int_{\Omega} \left(-(1 - 2/p')|v|^{-2} \operatorname{Re}\langle (\mathcal{B}^h)^* v, v \rangle \operatorname{Re}\langle v, \partial_h v \rangle - \operatorname{Re}\langle (\mathcal{B}^h)^* v, \partial_h v \rangle \right. \\ \left. - (1 - 2/p')|v|^{-2} \operatorname{Re}\langle (\mathcal{C}^h)^* v, v \rangle \operatorname{Re}\langle v, \partial_h v \rangle + \operatorname{Re}\langle (\mathcal{C}^h)^* \partial_h v, v \rangle - \operatorname{Re}\langle \mathcal{D}^* v, v \rangle \right) \geq 0 \quad (5)$$

for any $v \in (C_0^1(\Omega))^m$. Since $1 - 2/p' = -(1 - 2/p)$, the last inequality coincides with (4).

Necessity. Let $p \geq 2$ and set

$$g_{\varepsilon} = (|v|^2 + \varepsilon^2)^{1/2}, \quad u_{\varepsilon} = g_{\varepsilon}^{2/p-1} v,$$

where $v \in (C_0^1(\Omega))^m$. We have

$$\langle \mathcal{B}^h \partial_h u_{\varepsilon}, |u_{\varepsilon}|^{p-2}u_{\varepsilon} \rangle = -(1 - 2/p)g_{\varepsilon}^{-p}|v|^{p-2} \langle \mathcal{B}^h v, v \rangle \operatorname{Re}\langle v, \partial_h v \rangle \\ + g_{\varepsilon}^{-p+2}|v|^{p-2} \langle \mathcal{B}^h \partial_h v, v \rangle, \\ \langle \mathcal{C}^h u_{\varepsilon}, \partial_h (|u_{\varepsilon}|^{p-2}u_{\varepsilon}) \rangle = g_{\varepsilon}^{-p}|v|^{p-4}((1 - 2/p)(1 - p)|v|^2$$

$$\begin{aligned}
 &+ (p - 2)g_\varepsilon^2 \langle \mathcal{E}^h v, v \rangle \operatorname{Re} \langle v, \partial_h v \rangle + g_\varepsilon^{2-p} |v|^{p-2} \langle \mathcal{E}^h v, \partial_h v \rangle, \\
 \langle \mathcal{D} u_\varepsilon, |u_\varepsilon|^{p-2} u_\varepsilon \rangle &= g_\varepsilon^{-p+2} |v|^{p-2} \langle \mathcal{D} v, v \rangle,
 \end{aligned}$$

5 on the set $F = \{x \in \Omega \mid |v(x)| > 0\}$. The inequality $g_\varepsilon^a \leq |v|^a$ for $a \leq 0$, shows that the right hand sides are majorized by L^1 functions. Since $g_\varepsilon \rightarrow |v|$ pointwise as $\varepsilon \rightarrow 0^+$, an application of dominated convergence theorem gives

$$\begin{aligned}
 &\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \langle \mathcal{B}^h \partial_h u_\varepsilon, |u_\varepsilon|^{p-2} u_\varepsilon \rangle dx \\
 &= \int_{\Omega} (-(1 - 2/p) |v|^{-2} \langle \mathcal{B}^h v, v \rangle \operatorname{Re} \langle v, \partial_h v \rangle + \langle \mathcal{B}^h \partial_h v, v \rangle) dx, \\
 &\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \langle \mathcal{E}^h u_\varepsilon, \partial_h (|u_\varepsilon|^{p-2} u_\varepsilon) \rangle dx \\
 10 &= \int_{\Omega} ((1 - 2/p) |v|^{-2} \langle \mathcal{E}^h v, v \rangle \operatorname{Re} \langle v, \partial_h v \rangle + \langle \mathcal{E}^h v, \partial_h v \rangle) dx, \\
 &\lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \langle \mathcal{D} u_\varepsilon, |u_\varepsilon|^{p-2} u_\varepsilon \rangle dx = \int_{\Omega} \langle \mathcal{D} v, v \rangle dx. \tag{6}
 \end{aligned}$$

These formulas show that the limit

$$\lim_{\varepsilon \rightarrow 0^+} (-\operatorname{Re} \mathcal{L}(u_\varepsilon, |u_\varepsilon|^{p-2} u_\varepsilon))$$

is equal to the left-hand side of (4). The functions u_ε being in $(C_0^1(\Omega))^m$, (2) implies (4).

If $1 < p < 2$, from (6) it follows that the limit

$$\lim_{\varepsilon \rightarrow 0^+} (-\operatorname{Re} \mathcal{L}(|u_\varepsilon|^{p'-2} u_\varepsilon, u_\varepsilon))$$

15 coincides with the left-hand side of (5). This shows that (3) implies (5) and the proof is complete. □

3. A result for a system of ordinary differential equations of the first order

The aim of this section is to obtain an auxiliary result (see Theorem 1 below) concerning a particular system of ordinary differential equations of the first order.

We start with an elementary result, which we prove for the sake of completeness.

20 **Lemma 2:** *Let α, β, γ and δ be real constants such that*

$$\int_I (\alpha \cos^2 x + \beta \cos x \sin x + \gamma \sin^2 x) (\varphi^2(x))' dx = \int_I \delta \cos^2 x \varphi^2(x) dx \tag{7}$$

for any real valued $\varphi \in C_0^1(I)$. Then $\alpha = \gamma$ and $\beta = \delta = 0$.

Proof: Setting

$$A = \alpha \cos^2 x + \beta \cos x \sin x + \gamma \sin^2 x, \quad B = \delta \cos^2 x$$

we may write (7) as

$$\int_I A(\varphi^2)' dx = \int_I B\varphi^2 dx, \quad \forall \varphi \in C_0^1(I).$$

By an integration by parts we get

$$\int_I (B + A')\varphi^2 dx = 0, \quad \forall \varphi \in C_0^1(I).$$

Thanks to the arbitrariness of φ , we find $A' = -B$, i.e.

$$(\gamma - \alpha) \sin(2x) + (\beta + \delta/2) \cos(2x) = -\delta/2$$

for any $x \in I$. This implies $\gamma - \alpha = \beta + \delta/2 = -\delta/2 = 0$ and this gives the result. \square

The next theorem provides a criterion for the L^p -dissipativity of one-dimensional operators with complex constant coefficients and no lower order terms.

Theorem 1: Let $I \subset \mathbb{R}$ be an open interval and \mathcal{B} a constant complex matrix. We have that the operator $Eu = \mathcal{B}u'$ is L^p -dissipative if, and only if,

$$\mathcal{B} = bI, \quad b \in \mathbb{R}, \quad \text{if } p \neq 2 \tag{8}$$

$$\mathcal{B} = \mathcal{B}^*, \quad \text{if } p = 2. \tag{9}$$

Proof:

Sufficiency. Let $p = 2$. We have to show that

$$-\operatorname{Re} \int_I \langle \mathcal{B}v', v \rangle dx \geq 0$$

for any $v \in (C^1(I))^m$. The left hand side vanishes because

$$\int_I \langle \mathcal{B}v', v \rangle dx = \int_I \langle v', \mathcal{B}v \rangle dx = - \int_I \langle v, \mathcal{B}v' \rangle dx = - \int_I \overline{\langle \mathcal{B}v', v \rangle} dx.$$

If $p \neq 2$, in view of Lemma 1 we have to show that

$$\int_I ((1 - 2/p)|v|^{-2} \operatorname{Re} \langle \mathcal{B}v, v \rangle \operatorname{Re} \langle v, v' \rangle - \operatorname{Re} \langle \mathcal{B}v', v \rangle) dx \geq 0 \tag{10}$$

for any $v \in (C^1(I))^m$. Condition (8) implies $\langle \mathcal{B}v, v \rangle = b|v|^2$ and $\langle \mathcal{B}v', v \rangle = b \langle v', v \rangle$, the constant b being real. Therefore the left hand side of (10) is equal to

$$-2b/p \operatorname{Re} \int_I \langle v, v' \rangle dx = b/p \int_I (|v|^2)' dx = 0$$

and the sufficiency is proved.

Necessity. In view of Lemma 1 we have that E is L^p -dissipative if, and only if,

$$\int_I (1 - 2/p)|v|^{-2} \operatorname{Re} \langle \mathcal{B}v, v \rangle \operatorname{Re} \langle v, v' \rangle dx - \int_I \operatorname{Re} \langle \mathcal{B}v', v \rangle dx \geq 0 \tag{11}$$

for any $v \in (C_0^1(I))^m$.

Writing the condition (11) for the function $v(-x)$ we find

$$\int_I (1 - 2/p)|v|^{-2} \operatorname{Re}\langle \mathcal{B} v, v \rangle \operatorname{Re}\langle v, v' \rangle dx - \int_I \operatorname{Re}\langle \mathcal{B} v', v \rangle dx \leq 0$$

and then

$$\int_I (1 - 2/p)|v|^{-2} \operatorname{Re}\langle \mathcal{B} v, v \rangle \operatorname{Re}\langle v, v' \rangle dx - \int_I \operatorname{Re}\langle \mathcal{B} v', v \rangle dx = 0 \quad (12)$$

for any $v \in (C_0^1(I))^m$.

Suppose now $p \neq 2$. Fix $1 \leq j \leq m$ and consider the vector $v = (v_1, \dots, v_m)$ in which $v_k = 0$ for $k \neq j$. Equality (12) reduces to

$$(1/2 - 1/p) \operatorname{Re} b_{jj} \int_I (|v_j|^2)' dx - \int_I \operatorname{Re} (b_{jj} v_j' \bar{v}_j) dx = 0,$$

(without summation convention) and since the first integral vanishes, we get

$$\operatorname{Im} b_{jj} \int_I \operatorname{Im} (v_j' \bar{v}_j) dx = 0.$$

5 The arbitrariness of v_j leads to

$$\operatorname{Im} b_{jj} = 0 \quad (j = 1, \dots, m). \quad (13)$$

Fix $1 \leq h, j \leq m$ with $h \neq j$ and consider the vector $v = (v_1, \dots, v_m)$ in which $v_k = 0$ for $k \neq h, j$. In view of (12) we have (without summation convention)

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$$\begin{aligned} & (1/2 - 1/p) \int_I (|v_h|^2 + |v_j|^2)^{-1} \operatorname{Re} (b_{hh}|v_h|^2 + b_{hj}v_j\bar{v}_h + b_{jh}v_h\bar{v}_j + b_{jj}|v_j|^2) \\ & \times (|v_h|^2 + |v_j|^2)' dx \\ & - \int_I \operatorname{Re} (b_{hh}v_h'\bar{v}_h + b_{hj}v_j'\bar{v}_h + b_{jh}v_h'\bar{v}_j + b_{jj}v_j'\bar{v}_j) dx = 0. \end{aligned} \quad (14)$$

In particular, taking $v_h = \alpha$ and $v_j = \beta$, with α and β real valued functions, integrating by parts in the last integral and taking into account (13), we find

15

$$\begin{aligned} & (1/2 - 1/p) \int_I (\alpha^2 + \beta^2)^{-1} (b_{hh}\alpha^2 + \operatorname{Re} (b_{hj} + b_{jh})\alpha\beta + b_{jj}\beta^2) (\alpha^2 + \beta^2)' dx \\ & - \operatorname{Re} (b_{hj} - b_{jh}) \int_I \alpha\beta' dx = 0 \end{aligned}$$

Taking now $\alpha(x) = \varphi(x) \cos x$, $\beta(x) = \varphi(x) \sin x$, $\varphi \in C_0^1(I)$, we obtain

$$\begin{aligned} & (1/2 - 1/p) \int_I (b_{hh} \cos^2 x + b_{jj} \sin^2 x \\ & \quad + (\Re(b_{hj} + b_{jh}) - p/(p-2) \Re(b_{hj} - b_{jh})) \sin x \cos x) (\varphi^2(x))' dx \\ & = \Re(b_{hj} - b_{jh}) \int_I \cos^2 x \varphi^2(x) dx. \end{aligned}$$

By Lemma 2 we get

$$b_{jj} = b_{hh}, \quad \Re b_{hj} = 0 \quad (h \neq j). \tag{15}$$

Take now $v_h = \alpha$, $v_j = i\beta$ in (14). On account of (13), we have

$$\begin{aligned} & (1/2 - 1/p) \Re(i(b_{hj} - b_{jh})) \int_I (\alpha^2 + \beta^2)^{-1} \alpha \beta (\alpha^2 + \beta^2)' dx \\ & \quad - \Re(i(b_{hj} + b_{jh})) \int_I \alpha \beta' dx = 0. \end{aligned}$$

The same reasoning as before leads to

$$\Im b_{hj} = 0 \quad (h \neq j).$$

Together with (13) and (15), this implies the result for $p \neq 2$.

An inspection of the proof just given, shows that, if $p = 2$, we have only:

$$\begin{aligned} & \Im b_{jj} = 0, \quad (j = 1, \dots, m); \\ & \Re(b_{hj} - b_{jh}) = 0, \quad \Im(b_{hj} + b_{jh}) = 0 \quad (j \neq h), \end{aligned}$$

and (9) is proved. □

4. L^p -dissipativity of systems of partial differential operators of the first order

Let us consider the system of partial differential operators of the first order

$$Eu = \mathcal{B}^h(x) \partial_h u + \mathcal{D}(x)u. \tag{16}$$

From now on $\mathcal{B}^h(x) = \{b_{ij}^h(x)\}$ and $\mathcal{D}(x) = \{d_{ij}(x)\}$ are matrices with complex locally integrable entries defined in the domain Ω of \mathbb{R}^n ($1 \leq i, j \leq m$, $1 \leq h \leq n$). Moreover we suppose that also $\partial_h \mathcal{B}^h$ (where the derivatives are in the sense of distributions) is a matrix with complex locally integrable entries.

Theorem 2: *The operator (16) is L^p -dissipative if, and only if, the following conditions are satisfied:*

(1)

$$\mathcal{B}^h(x) = b_h(x)I, \quad \text{if } p \neq 2, \tag{17}$$

$$\mathcal{B}^h(x) = (\mathcal{B}^h)^*(x), \quad \text{if } p = 2, \tag{18}$$

for almost any $x \in \Omega$ and $h = 1, \dots, n$. Here b_h are real locally integrable functions ($1 \leq h \leq n$).

(2)

$$\operatorname{Re} \langle (p^{-1} \partial_h \mathcal{B}^h(x) - \mathcal{D}(x)) \zeta, \zeta \rangle \geq 0 \tag{19}$$

5 for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$.

Proof:

Sufficiency. In view of Lemma 1 we have to show that

$$\int_{\Omega} \left((1 - 2/p) |v|^{-2} \operatorname{Re} \langle \mathcal{B}^h v, v \rangle \operatorname{Re} \langle v, \partial_h v \rangle - \operatorname{Re} \langle \mathcal{B}^h \partial_h v, v \rangle - \operatorname{Re} \langle \mathcal{D} v, v \rangle \right) dx \geq 0 \tag{20}$$

10

holds for any $v \in (C_0^1(\Omega))^m$.

Let $p = 2$. In view of the self-adjointness of \mathcal{B} , we have

$$\operatorname{Re} \int_{\Omega} \langle \mathcal{B}^h \partial_h v, v \rangle dx = - \operatorname{Re} \int_{\Omega} \langle (\partial_h \mathcal{B}^h) v, v \rangle dx - \operatorname{Re} \int_{\Omega} \langle \mathcal{B}^h \partial_h v, v \rangle dx$$

and then

$$2 \operatorname{Re} \int_{\Omega} \langle \mathcal{B}^h \partial_h v, v \rangle dx = - \operatorname{Re} \int_{\Omega} \langle (\partial_h \mathcal{B}^h) v, v \rangle dx.$$

Consequently

$$\begin{aligned} & - \operatorname{Re} \int_{\Omega} \langle \mathcal{B}^h \partial_h v, v \rangle dx - \operatorname{Re} \int_{\Omega} \langle \mathcal{D} v, v \rangle dx \\ & = \int_{\Omega} (2^{-1} \operatorname{Re} \langle (\partial_h \mathcal{B}^h) v, v \rangle - \operatorname{Re} \langle \mathcal{D} v, v \rangle) dx \end{aligned}$$

15

and the last integral is greater than or equal to zero because of (19).

Let now $p \neq 2$. Keeping in mind (17) we get

$$\begin{aligned} & (1 - 2/p) \int_{\Omega} |v|^{-2} \operatorname{Re} \langle \mathcal{B}^h v, v \rangle \operatorname{Re} \langle v, \partial_h v \rangle dx \\ & - \int_{\Omega} \operatorname{Re} \langle \mathcal{B}^h \partial_h v, v \rangle dx - \int_{\Omega} \operatorname{Re} \langle \mathcal{D} v, v \rangle dx \\ & = p^{-1} \int_{\Omega} (\partial_h b_h) |v|^2 dx - \int_{\Omega} \operatorname{Re} \langle \mathcal{D} v, v \rangle dx. \end{aligned}$$

20

Condition (19) gives the result.

Necessity. Denote by B_1 the open ball $\{y \in \mathbb{R}^n \mid |y| < 1\}$, take $\psi \in (C_0^1(B_1))^m$ and define

$$v(x) = \psi((x - x_0)/\varepsilon)$$

where x_0 is a fixed point in Ω and $0 < \varepsilon < \operatorname{dist}(x_0, \partial\Omega)$.

Putting this particular v in (20) and making a change of variables, we obtain

$$\int_{B_1} (1 - 2/p)|\psi|^{-2} \operatorname{Re}\langle \mathcal{B}^h(x_0 + \varepsilon y)\psi, \psi \rangle \operatorname{Re}\langle \psi, \partial_h \psi \rangle dy - \int_{B_1} \operatorname{Re}\langle \mathcal{B}^h(x_0 + \varepsilon y)\partial_h \psi, \psi \rangle dy - \varepsilon \int_{B_1} \operatorname{Re}\langle \mathcal{D}(x_0 + \varepsilon y)\psi, \psi \rangle dy \geq 0.$$

5 Letting $\varepsilon \rightarrow 0^+$ we find

$$\int_{B_1} (1 - 2/p)|\psi|^{-2} \operatorname{Re}\langle \mathcal{B}^h(x_0)\psi, \psi \rangle \operatorname{Re}\langle \psi, \partial_h \psi \rangle dy - \int_{B_1} \operatorname{Re}\langle \mathcal{B}^h(x_0)\partial_h \psi, \psi \rangle dy \geq 0 \tag{21}$$

for almost any $x_0 \in \Omega$ and for any $\psi \in (C_0^1(B_1))^m$.

Fix now $1 \leq k \leq n$. Take $\alpha \in (C_0^1(\mathbb{R}))^m$ and $\beta \in C_0^1(\mathbb{R}^{n-1})$. Consider

$$\psi_\varepsilon(x) = \alpha((x_k - (x_0)_k)/\varepsilon) \beta(y_k),$$

10 where y_k denotes the $(n - 1)$ -dimensional vector $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$. Choose ε, α and β in such a way $\operatorname{spt} \psi_\varepsilon \subset \Omega$.

We have

$$\begin{aligned} & \sum_{h=1}^n \int_{\Omega} |\psi_\varepsilon|^{-2} \operatorname{Re}\langle \mathcal{B}^h(x_0)\psi_\varepsilon, \psi_\varepsilon \rangle \operatorname{Re}\langle \psi_\varepsilon, \partial_h \psi_\varepsilon \rangle dx \\ &= \int_{\mathbb{R}} |\alpha(t)|^{-2} \operatorname{Re}\langle \mathcal{B}^k(x_0)\alpha(t), \alpha(t) \rangle \operatorname{Re}\langle \alpha(t), \alpha'(t) \rangle dt \int_{\mathbb{R}^{n-1}} |\beta(y_k)|^2 dy_k \\ &+ \varepsilon \sum_{\substack{h=1 \\ h \neq k}}^n \int_{\mathbb{R}} \operatorname{Re}\langle \mathcal{B}^h(x_0)\alpha(t), \alpha(t) \rangle dt \int_{\mathbb{R}^{n-1}} \operatorname{Re}(\beta(y_k) \partial_h \overline{\beta(y_k)}) dy_k, \\ & \int_{\Omega} \operatorname{Re}\langle \mathcal{B}^h(x_0)\partial_h \psi_\varepsilon, \psi_\varepsilon \rangle dx \\ &= \int_{\mathbb{R}} \operatorname{Re}\langle \mathcal{B}^k(x_0)\alpha'(t), \alpha(t) \rangle dt \int_{\mathbb{R}^{n-1}} |\beta(y_k)|^2 dy_k \\ &+ \varepsilon \sum_{\substack{h=1 \\ h \neq k}}^n \int_{\mathbb{R}} \operatorname{Re}\langle \mathcal{B}^h(x_0)\alpha(t), \alpha(t) \rangle dt \int_{\mathbb{R}^{n-1}} \operatorname{Re}(\overline{\beta(y_k)} \partial_h \beta(y_k)) dy_k. \end{aligned}$$

20 Therefore

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\Omega} (1 - 2/p)|\psi_\varepsilon|^{-2} \operatorname{Re}\langle \mathcal{B}^h(x_0)\psi_\varepsilon, \psi_\varepsilon \rangle \operatorname{Re}\langle \psi_\varepsilon, \partial_h \psi_\varepsilon \rangle dx \right. \\ & \quad \left. - \int_{\Omega} \operatorname{Re}\langle \mathcal{B}^h(x_0)\partial_h \psi_\varepsilon, \psi_\varepsilon \rangle dx \right) \\ &= \int_{\mathbb{R}} \left((1 - 2/p)|\alpha(t)|^{-2} \operatorname{Re}\langle \mathcal{B}^k(x_0)\alpha(t), \alpha(t) \rangle \operatorname{Re}\langle \alpha(t), \alpha'(t) \rangle dt \right) \end{aligned}$$

$$- \operatorname{Re} \langle \mathcal{B}^k(x_0) \alpha'(t), \alpha(t) \rangle dt \int_{\mathbb{R}^{n-1}} |\beta(y_k)|^2 dy_k$$

and from (21) it follows

$$(1 - 2/p) \int_{\mathbb{R}} |\alpha(t)|^{-2} \operatorname{Re} \langle \mathcal{B}^k(x_0) \alpha(t), \alpha(t) \rangle \operatorname{Re} \langle \alpha(t), \alpha'(t) \rangle dt - \int_{\mathbb{R}} \operatorname{Re} \langle \mathcal{B}^k(x_0) \alpha'(t), \alpha(t) \rangle dt \geq 0.$$

The arbitrariness of α shows that the operator with constant coefficients $\mathcal{B}^k(x_0)u'$ is L^p -dissipative. Theorem 1 applies and (17)–(18) is satisfied.

As we already proved in the sufficiency part, (17) and (18) implies that inequality (20) can be written as

$$\int_{\Omega} (p^{-1} \operatorname{Re} \langle (\partial_h \mathcal{B}^h)v, v \rangle - \operatorname{Re} \langle \mathcal{D}v, v \rangle) dx \geq 0, \tag{22}$$

for any $v \in (C_0^1(\Omega))^m$. Take

$$v_\varepsilon(x) = \varepsilon^{-n/2} \zeta \varphi((x - x_0)/\varepsilon)$$

where $x_0 \in \Omega$, $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$, φ is a real scalar function in $C_0^1(\mathbb{R}^n)$, $\operatorname{spt} \varphi$ is contained in the unit ball, ε is sufficiently small and

$$\int_{\mathbb{R}^n} \varphi^2(x) dx = 1.$$

Putting v_ε in (22) and letting $\varepsilon \rightarrow 0^+$ we obtain (19) for almost any $x_0 \in \Omega$. □

Let us consider now instead of (16), the operator

$$\mathcal{B}^h(x) \partial_h u + \partial_h (\mathcal{C}^h(x)u) + \mathcal{D}(x)u, \tag{23}$$

where $\mathcal{B}^h, \mathcal{C}^h, \mathcal{D}, \partial_h \mathcal{B}^h$ and $\partial_h \mathcal{C}^h$ are matrices with complex locally integrable entries.

Theorem 3: *The operator (23) is L^p -dissipative if, and only if, the following conditions are satisfied*

(1)

$$\mathcal{B}^h(x) + \mathcal{C}^h(x) = b_h(x)I, \quad \text{if } p \neq 2, \tag{24}$$

$$\mathcal{B}^h(x) + \mathcal{C}^h(x) = (\mathcal{B}^h)^*(x) + (\mathcal{C}^h)^*(x), \quad \text{if } p = 2, \tag{25}$$

for almost any $x \in \Omega$ and $h = 1, \dots, n$. Here b_h are real locally integrable functions ($1 \leq h \leq n$).

(2)

$$\operatorname{Re} \langle (p^{-1} \partial_h \mathcal{B}^h(x) - p'^{-1} \partial_h \mathcal{C}^h(x) - \mathcal{D}(x)) \zeta, \zeta \rangle \geq 0 \tag{26}$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$.

Proof: It is sufficient to write the operator (23) as

$$(\mathcal{B}^h(x) + \mathcal{C}^h(x))\partial_h u + \partial_h(\mathcal{C}^h(x))u + \mathcal{D}(x)u$$

and apply Theorem 2, observing that

$$p^{-1}\partial_h(\mathcal{B}^h + \mathcal{C}^h) - \partial_h \mathcal{C}^h = p^{-1}\partial_h \mathcal{B}^h - p'^{-1}\partial_h \mathcal{C}^h.$$

□

5. Sufficient conditions for the L^p -dissipativity of certain systems of partial differential operators of the second order

As a by-product of the results obtained in the previous section, we obtain now sufficient conditions for the L^p -dissipativity of a class of systems of partial differential equations of the second order.

Theorem 4: Let E be the operator

$$Eu = \partial_h(\mathcal{A}^h(x)\partial_h u) + \mathcal{B}^h(x)\partial_h u + \mathcal{D}(x)u, \tag{27}$$

where $\mathcal{A}^h(x) = \{a_{ij}^h(x)\}$ are $m \times m$ matrices with complex locally integrable entries and the matrices $\mathcal{B}^h(x)$, $\mathcal{D}(x)$ satisfy the hypothesis of Theorem 2. If

$$\begin{aligned} &\operatorname{Re}\langle \mathcal{A}^h(x)\lambda, \lambda \rangle - (1 - 2/p)^2 \operatorname{Re}\langle \mathcal{A}^h(x)\omega, \omega \rangle (\operatorname{Re}\langle \lambda, \omega \rangle)^2 \\ &- (1 - 2/p) \operatorname{Re}\langle \mathcal{A}^h(x)\omega, \lambda \rangle - \langle \mathcal{A}^h(x)\lambda, \omega \rangle \operatorname{Re}\langle \lambda, \omega \rangle \geq 0 \end{aligned} \tag{28}$$

for almost every $x \in \Omega$ and for every $\lambda, \omega \in \mathbb{C}^m$, $|\omega| = 1$, $h = 1, \dots, n$, and conditions (17)–(18) and (19) are satisfied, the operator E is L^p -dissipative.

Proof: Theorem 2 shows that the operator of the first order

$$E_1 = \mathcal{B}^h(x)\partial_h u + \mathcal{D}(x)u$$

is L^p -dissipative. Moreover, inequality (28) is necessary and sufficient for the L^p -dissipativity of the second order operator

$$E_0 = \partial_h(\mathcal{A}^h(x)\partial_h u) \tag{29}$$

(see [4, Theorem 4.20, p.115]). Since $E = E_0 + E_1$, the result follows at once.

□

Consider now the operator (27) in the scalar case (i.e. $m = 1$)

$$\partial_h(a^h(x)\partial_h u) + b^h(x)\partial_h u + d(x)u$$

(a^h, b^h and d being scalar functions). In this case such an operator can be written in the form

$$Eu = \operatorname{div}(\mathcal{A}(x)\nabla u) + \mathcal{B}(x)\nabla u + d(x)u \tag{30}$$

where $\mathcal{A} = \{c_{hk}\}$, $c_{hh} = a^h$, $c_{hk} = 0$ if $h \neq k$ and $\mathcal{B} = \{b^h\}$. For such an operator one can show that (28) is equivalent to

$$\frac{4}{pp'} \langle \operatorname{Re} \mathcal{A}(x)\xi, \xi \rangle + \langle \operatorname{Re} \mathcal{A}(x)\eta, \eta \rangle - 2(1 - 2/p) \langle \operatorname{Im} \mathcal{A}(x)\xi, \eta \rangle \geq 0 \quad (31)$$

for almost any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^n$ (see [4, Remark 4.21, p.115]). Condition (31) is in turn equivalent to the inequality:

$$|p - 2| |\langle \operatorname{Im} \mathcal{A}(x)\xi, \xi \rangle| \leq 2\sqrt{p - 1} \langle \operatorname{Re} \mathcal{A}(x)\xi, \xi \rangle \quad (32)$$

for almost any $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$ (see [4, Remark 2.8, p.42]). We have then

Theorem 5: *Let E be the scalar operator (30) where \mathcal{A} is a diagonal matrix. If inequality (32) and conditions (17)–(18) and (19) are satisfied, the operator E is L^p -dissipative.*

More generally, consider the scalar operator (30) with a matrix $\mathcal{A} = \{a_{hk}\}$ not necessarily diagonal. The following result holds true.

Theorem 6: *Let the matrix $\operatorname{Im} \mathcal{A}$ be symmetric, i.e. $\operatorname{Im} \mathcal{A}^t = \operatorname{Im} \mathcal{A}$. The operator*

$$\operatorname{div}(\mathcal{A}(x)\nabla u)$$

is L^p -dissipative if and only if inequality (32) holds for almost any $x \in \Omega$ and for any $\xi \in \mathbb{R}^n$.

For the proof we refer to [4, Theorem 2.7, p.40], where the result is proved under the more general assumption that the entries of \mathcal{A} are complex measures.

As in the proof of Theorem 4, the last result implies the following theorem.

Theorem 7: *Let the matrix $\operatorname{Im} \mathcal{A}$ be symmetric. If inequality (32) and conditions (17)–(18) and (19) are satisfied, the operator (30) is L^p -dissipative.*

Coming back to system (27), in case the main part of the operator (27) has real coefficients, i.e. the matrices \mathcal{A}^h have real locally integrable entries, we have also

Theorem 8: *Let E be the operator (27) where \mathcal{A}^h are real matrices. Let us suppose $\mathcal{A}^h = (\mathcal{A}^h)^t$ and $\mathcal{A}^h \geq 0$ ($h = 1, \dots, n$). If conditions (17)–(18) and (19) are satisfied and*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\mu_1^h(x) + \mu_m^h(x))^2 \leq \mu_1^h(x) \mu_m^h(x) \quad (33)$$

for almost every $x \in \Omega$, $h = 1, \dots, n$, where $\mu_1^h(x)$ and $\mu_m^h(x)$ are the smallest and the largest eigenvalues of the matrix $\mathcal{A}^h(x)$ respectively, the operator E is L^p -dissipative. In the particular case $m = 2$ condition (33) is equivalent to

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 (\operatorname{tr} \mathcal{A}^h(x))^2 \leq \det \mathcal{A}^h(x)$$

for almost every $x \in \Omega$, $h = 1, \dots, n$.

Proof: Theorem 4.22 in [4, p.116] shows that (33) holds if and only if the operator (29) is L^p -dissipative. Combining this with Theorem 2 gives the result. \square

Results similar to Theorems 4–8 hold for the operator

$$\partial_h(\mathcal{A}^h(x)\partial_h u) + \mathcal{B}^h(x)\partial_h u + \partial_h(\mathcal{C}^h(x)u) + \mathcal{D}(x)u.$$

We have just to replace conditions (17), (18) and (19) by (24), (25) and (26) respectively.

6. The L^p -quasi-dissipativity

The operator E is said to be L^p -quasi-dissipative if the operator $E - \omega I$ is L^p -dissipative for a suitable $\omega \geq 0$. This means that there exists $\omega \geq 0$ such that

$$\operatorname{Re} \int_{\Omega} \langle Eu, |u|^{p-2}u \rangle dx \leq \omega \|u\|_p^p$$

for any u in the domain of E .

The aim of this section is to provide necessary and sufficient conditions for the L^p -quasi-dissipativity of a partial differential operator of the first order.

Lemma 3: *The operator (16) is L^p -quasi-dissipative if, and only if, there exists $\omega \geq 0$ such that*

$$\begin{aligned} (1 - 2/p) \int_{\Omega} |v|^{-2} \operatorname{Re} \langle \mathcal{B}^h v, v \rangle \operatorname{Re} \langle v, \partial_h v \rangle dx - \operatorname{Re} \int_{\Omega} \langle \mathcal{B}^h \partial_h v, v \rangle dx \\ - \operatorname{Re} \int_{\Omega} \langle \mathcal{D} v, v \rangle dx \geq -\omega \int_{\Omega} |v|^2 dx \end{aligned} \quad (34)$$

for any $v \in (C_0^1(\Omega))^m$.

Proof: The result follows immediately from Lemma 1. □

Theorem 9: *Let E be the operator (16), in which the entries of \mathcal{D} and the entries of $\partial_h \mathcal{B}^h$ belong to $L^\infty(\Omega)$. The operator E is L^p -quasi-dissipative if, and only if, condition (17) and (18) is satisfied.*

Proof:

Necessity. Arguing as in the first part of the proof of Theorem 2, we find that (34) implies that the ordinary differential operator $\mathcal{B}^k(x_0)u'$ is L^p -dissipative, for almost any $x_0 \in \Omega$, $k = 1, \dots, m$. As we know, this in turn implies that (17) and (18) is satisfied.

Sufficiency. As in the proof of Theorem 2, condition (17) and (18) gives

$$\begin{aligned} (1 - 2/p) \int_{\Omega} |v|^{-2} \operatorname{Re} \langle \mathcal{B}^h v, v \rangle \operatorname{Re} \langle v, \partial_h v \rangle dx - \operatorname{Re} \int_{\Omega} \langle \mathcal{B}^h \partial_h v, v \rangle dx \\ = p^{-1} \operatorname{Re} \int_{\Omega} \langle (\partial_h \mathcal{B}^h) v, v \rangle dx. \end{aligned}$$

We define ω by setting

$$\omega = \max \left\{ 0, \operatorname{ess\,sup}_{\substack{x \in \Omega \\ \zeta \in \mathbb{C}^m, |\zeta|=1}} \operatorname{Re} \langle (\mathcal{D}(x) - p^{-1} \partial_h \mathcal{B}^h(x)) \zeta, \zeta \rangle \right\}. \quad (35)$$

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The left hand side of (34) being equal to

$$\operatorname{Re} \int_{\Omega} \langle (p^{-1} \partial_h \mathcal{B}^h(x) - \mathcal{D}(x))v, v \rangle dx,$$

inequality (34) follows from (35). □

Remark: It is clear from the proof of Theorem 9 that, if

$$\lambda = \operatorname{ess\,sup}_{\substack{x \in \Omega \\ \zeta \in \mathbb{C}^m, |\zeta|=1}} \operatorname{Re} \langle (\mathcal{D}(x) - p^{-1} \partial_h \mathcal{B}^h(x))\zeta, \zeta \rangle < 0,$$

we have not only the L^p -dissipativity of the operator E , but also the stronger inequality

$$\operatorname{Re} \int_{\Omega} \langle Eu, |u|^{p-2}u \rangle dx \leq \lambda \|u\|_p^p$$

with $\lambda < 0$.

The next result follows from similar arguments to those above.

Theorem 10: Let E be the operator (23), in which the entries of \mathcal{D} , $\partial_h \mathcal{B}^h$ and $\partial_h \mathcal{C}^h$ belong to $L^\infty(\Omega)$. The operator E is L^p -quasi-dissipative if, and only if, condition (24) and (25) is satisfied.

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7. The angle of dissipativity

Let E be the operator (16) and assume it is L^p -dissipative. The aim of this Section is to determine the angle of dissipativity of E . This means to find the set of complex values z such that zE is still L^p -dissipative.

What comes out is that the angle of dissipativity of E is always a zero-sided angle, unless E degenerates to the operator $\mathcal{D}u$.

We start recalling the following lemma

Lemma 4: Let P and Q two real measurable functions defined on a set $\Omega \subset \mathbb{R}^n$. Let us suppose that $P(x) \geq 0$ almost everywhere. The inequality

$$P(x) \cos \vartheta - Q(x) \sin \vartheta \geq 0 \quad (\vartheta \in [-\pi, \pi])$$

holds for almost every $x \in \Omega$ if and only if

$$\operatorname{arccot} \left[\operatorname{ess\,inf}_{x \in \Xi} (Q(x)/P(x)) \right] - \pi \leq \vartheta \leq \operatorname{arccot} \left[\operatorname{ess\,sup}_{x \in \Xi} (Q(x)/P(x)) \right]$$

where $\Xi = \{x \in \Omega \mid P^2(x) + Q^2(x) > 0\}$ and we set

$$Q(x)/P(x) = \begin{cases} +\infty & \text{if } P(x) = 0, Q(x) > 0 \\ -\infty & \text{if } P(x) = 0, Q(x) < 0. \end{cases}$$

Here $0 < \operatorname{arccot} y < \pi$, $\operatorname{arccot} (+\infty) = 0$, $\operatorname{arccot} (-\infty) = \pi$ and

$$\operatorname{ess\,inf}_{x \in \Xi} (Q(x)/P(x)) = +\infty, \quad \operatorname{ess\,sup}_{x \in \Xi} (Q(x)/P(x)) = -\infty$$

if Ξ has zero measure.

For a proof we refer to [2, p.236] (see also [4, p.138]).

Theorem 11: Let E be the operator (16) and suppose it is L^p -dissipative. Set

$$\begin{aligned} P(x, \zeta) &= -\operatorname{Re}(\mathcal{D}(x)\zeta, \zeta), \\ Q(x, \zeta) &= -\operatorname{Im}(\mathcal{D}(x)\zeta, \zeta), \\ \Xi &= \{(x, \zeta) \in \Omega \times \mathbb{C}^m \mid |\zeta| = 1, P^2(x, \zeta) + Q^2(x, \zeta) > 0\}. \end{aligned}$$

If

$$\mathcal{B}^h(x) = 0 \text{ a.e. } (h = 1, \dots, n), \tag{36}$$

the operator zE is L^p -dissipative if, and only if,

$$\vartheta_- \leq \arg z \leq \vartheta_+, \tag{37}$$

where

$$\begin{aligned} \vartheta_- &= \operatorname{arccot} \left(\operatorname{ess\,inf}_{(x, \zeta) \in \Xi} Q(x, \zeta)/P(x, \zeta) \right) - \pi \\ \vartheta_+ &= \operatorname{arccot} \left(\operatorname{ess\,sup}_{(x, \zeta) \in \Xi} Q(x, \zeta)/P(x, \zeta) \right). \end{aligned}$$

If condition (36) is not satisfied and there exist $\zeta \in \mathbb{C}^m$ and $x \in \Omega$ such that

$$\operatorname{Re}((p^{-1}\partial_h \mathcal{B}^h(x) - \mathcal{D}(x))\zeta, \zeta) > 0, \tag{38}$$

the angle of dissipativity of E is zero, i.e. zE is dissipative if, and only if, $\operatorname{Im} z = 0$, $\operatorname{Re} z \geq 0$. Finally, if condition (36) is not satisfied and

$$\operatorname{Re}((p^{-1}\partial_h \mathcal{B}^h(x) - \mathcal{D}(x))\zeta, \zeta) = 0, \tag{39}$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$, the operator zE is L^p -dissipative if, and only if, $\operatorname{Re} z = 0$.

Proof: Suppose (36) holds. It is obvious that zE satisfies condition (17) and (18) for any $z \in \mathbb{C}$. Since E is L^p -dissipative, $P(x, \zeta) \geq 0$ for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$ (see (19)). In view of Lemma 4 we have

$$\operatorname{Re}(-z \mathcal{D}(x)\zeta, \zeta) \geq 0$$

if, and only if, (37) holds.

Assume now that (36) is not satisfied. Condition (17) and (18) is valid for all the matrices $z \mathcal{B}^h$ if, and only if, $\operatorname{Im} z = 0$. Moreover, suppose that there exist $\zeta \in \mathbb{C}^m$ and $x \in \Omega$ such

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that (39) holds; therefore

$$\operatorname{Re} \left(\operatorname{Re} z \left((p^{-1} \partial_h \mathcal{B}^h(x) - \mathcal{D}(x)) \zeta, \zeta \right) \right) \geq 0 \quad (40)$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$ if, and only if, $\operatorname{Re} z \geq 0$.

Assume instead (39) for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$; then condition (40) holds for any $z \in \mathbb{C}$. This means that zE is L^p -dissipative if, and only if, $\operatorname{Im} z = 0$. \square

Remark: Suppose (36) is not satisfied and that (39) holds for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ and for almost any $x \in \Omega$. In this case we have ‘two’ zero sided angles of dissipativity and not only one. This should not surprise. Indeed for such operators we have the L^p -dissipativity of both E and $(-E)$. This is evident, e.g. for the operators with constant coefficients considered in Theorem 1.

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No potential conflict of interest was reported

Please add:
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