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Numerical method for hypersingular integrals of highly oscillatory functions on the positive semiaxis

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Abstract

This paper deals with a quadrature rule for the numerical evaluation of hypersingular integrals of highly oscillatory functions on the positive semiaxis. The rule is of product type and consists in approximating the density function f by a truncated interpolation process based on the zeros of generalized Laguerre polynomials and an additional point. We prove the stability and the convergence of the rule, giving error estimates for functions belonging to weighted Sobolev spaces equipped with uniform norm. We also show how the proposed rule can be used for the numerical solution of hypersingular integral equations. Numerical tests which confirm the theoretical estimates and comparisons with other existing quadrature rules are presented.

1 Introduction

We consider the approximation of the following integrals

$$H_p^{\omega,\gamma}(f,t) = \int_0^{+\infty} \frac{f(x)e^{i\omega x}}{(x-t)^{p+1}} u_\gamma(x)\, dx, \tag{1}$$

where $t > 0, p \geq 0$ is an integer, $i^2 = -1$, $\omega \gg 1$, $u_\gamma(x) = x^\gamma e^{-\frac{x}{2}}, \gamma \geq 0$, is a generalized Laguerre weight and the integral is understood in the Cauchy principal value sense if $p = 0$ and in the finite part Hadamard sense if $p > 0$. Details on the properties fulfilled by finite part integrals on unbounded intervals can be found in [15]. Among them we recall that integrals (1) can also be regarded as the p-th derivative of Cauchy principal value integrals, i.e.

$$H_p^{\omega,\gamma}(f,t) = \frac{1}{p!}\frac{d^p}{dt^p}\int_0^{+\infty} \frac{f(x)e^{i\omega x}}{(x-t)} u_\gamma(x)\, dx. \tag{2}$$

Integrals $H_p^{\omega,\gamma}(f)$ are of interests because of their frequent occurrences in many areas of science ranging from image analysis, optics, electrodynamics, and fluid mechanics. In particular, they appear in boundary element methods and their effectiveness often depends upon the accuracy of the numerical evaluation of the integrals (1) (see [1, 2, 9, 26, 43] and the references there in).

A wide literature dealing with numerical methods for the approximation of singular and hypersingular integrals of non oscillatory functions can be found in both cases of bounded intervals (see, for example, [4, 7, 8, 10, 11, 20, 22, 25, 34, 35, 36, 37, 40, 46])) and unbounded intervals (see, for example, [12, 13, 14, 15, 16, 17, 21, 38]). The same can be said for integrals of highly oscillatory functions (see, for example, [6, 18, 19, 24, 26, 27, 39, 44]).

Concerning integrals of functions presenting both singularity and oscillation, to our knowledge, most of the papers in literature are devoted to the case of bounded intervals (see, for instance, [3, 41, 45, 42] and the references therein) and only a very small numbers of papers deal with the numerical evaluation of (1) with $p = 0$ [5, 43]. In particular, in [43] the authors propose three different quadrature rules depending on the position of the singular point t: $t = \mathcal{O}(1)$ or $t \gg 1$, $0 < t \ll 1$ and $t = 0$. The numerical methods proposed in each regime are based on special reformulations and/or decompositions of the integral and on the application of the Gauss-Laguerre quadrature rules. Two numerical procedures are proposed in [5]. In the first one the integral (1) is approximated by a $s-$step asymptotic quadrature rule requiring the evaluation of the function $F(x) = \frac{f(x)-f(t)}{x-t}e^x$ and its $s-1$ derivatives at the point 0. The second quadrature rule is derived by the first one approximating $F^{(k-1)}(x), k = 0, \dots, s$, with an interpolation formula. All the quadrature rules introduced in [5, 43] share the following two characteristics: the function f has to belong to $C^k(0, +\infty)$ for some $k \geq 1$ and the accuracy of the rule improves when the frequency ω increases.

We propose a quadrature rule of product type based on the truncated Lagrange polynomial interpolating the function f at generalized Laguerre zeros and the additional point $4m$. The stability and convergence of the rule have been proved in weighted uniform spaces of Sobolev type. This quadrature rule has three main advantages: the procedure is always the same for any

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choice of the point t; no evaluations of the derivatives of the function f are required; the stability and the convergence of the rule have been proved for a class of functions wider than the one considered in [5, 43]. The use of truncated interpolation processes is crucial in order to reduce the number of function computations and possible overflow ranges (see, for example, [10, 28, 29, 33]). Moreover, the rate of convergence is always the same regardless the choices of the values of t and ω. Finally, with a computational cost of the same order and no more evaluations of the function f, it is possible to perform the simultaneous approximation of $H_s^{\omega,\gamma}(f,t), s=0,\dots,p$ (we recall that in [5, 43] only the case $p=0$ has been considered).

The paper is organized as follows. In Section 2 we give some preliminary definitions and results and sufficient conditions for the existence of the integrals (1). Section 3 contains the description of the quadrature rule, the details for its implementation, and the results dealing with the stability and the convergence of the rule. Error estimates in uniform norm are also given. In Section 4 we show how the proposed rule can be employed in the construction of a Nyström method for solving some integral equations. Comparisons with other method existing in literature and numerical tests showing the performances of the rule are presented in Section 5. Finally, in Section 6 we give the proofs of the theoretical results.

2 Preliminaries and existence of the integrals $H_p^{\omega,\gamma}(f,t)$

We denote by $C^0(\mathcal{I})$ the space of all continuous functions on the set $\mathcal{I}$ and, with $u(x)=(1+x)^\delta x^\gamma e^{-\frac{x}{2}}, \gamma,\delta\ge 0$, we consider the following set of functions

$$C_u=\begin{cases}\left\{f\in C^0((0,+\infty)) : \lim\limits_{\substack{x\to 0^+\\ x\to+\infty}} f(x)u(x)=0\right\}, & \gamma>0,\\ \left\{f\in C^0([0,+\infty)) : \lim\limits_{x\to+\infty} f(x)u(x)=0\right\}, & \gamma=0,\end{cases}$$

equipped with the norm

$$\|f\|_{C_u}:=\|fu\|=\sup_{x\ge 0}|(fu)(x)|.$$

We also consider the following Sobolev-type subspaces of C_u of order $1\le r\in\mathbb{N}$ [30]

$$W_r(u)=\left\{f\in C_u : f^{(r-1)}\in AC((0,+\infty)) \text{ and } \|f^{(r)}\varphi^r u\|<+\infty\right\},$$

where $AC((0,+\infty))$ is the set of all absolutely continuous functions on every closed subset of $\mathbb{R}^+$ and $\varphi(x)=\sqrt{x}$. These spaces equipped with the norm

$$\|f\|_{W_r(u)}:=\|f\|_{C_u}+\|f^{(r)}\varphi^r u\|$$

are Banach spaces.

In the sequel we will denote by u_γ the weight function u with $\delta=0$ and by $u_{\gamma,\delta}$ the weight function u with $\delta\neq 0$. Moreover, $\mathcal{C}$ will denote a positive constant having different meanings in different formulas. In particular, we will write $\mathcal{C}\neq\mathcal{C}(a,b,\dots)$ to mean that the positive constant $\mathcal{C}$ is independent of the variables $a,b,\dots$.

We denote by $w_\alpha(x)=x^\alpha e^{-x}$, $\alpha>-1$, the generalized Laguerre weight and by $\{p_m(x):=p_m(w_\alpha,x)\}_m$ the sequence of the orthonormal generalized Laguerre polynomials. We recall that they satisfy the following three-term recurrence relation

$$\begin{cases}p_{-1}(x)=0, \qquad p_0(x)=\dfrac{1}{\sqrt{\Gamma(\alpha+1)}},\\ a_{\nu+1}p_{\nu+1}(x)=(x-b_\nu)p_\nu(x)-a_\nu p_{\nu-1}(x),\\ a_\nu=\sqrt{\nu(\nu+\alpha)}, \qquad b_\nu=2\nu+\alpha+1.\end{cases} \tag{3}$$

Denoting by z_k, $k=1,\dots,m$, the zeros of $p_m(w_\alpha)$ and recalling that $0<z_1<z_2<\dots<z_m<4m$, we consider the truncated version of the Lagrange polynomial interpolating a continuous function f at the knots $z_1,\dots,z_m,4m$. It is defined as follows

$$L_{m+1}(w_\alpha,f):=\sum_{k=1}^{j} f(z_k)\ell_{m+1,k}(x), \tag{4}$$

where

$$\ell_{m+1,k}(x)=l_{m,k}(x)\frac{(4m-x)}{(4m-z_k)}, \quad k=1,\dots,j, \tag{5}$$

with $l_{m,k}$ the $k-th$ fundamental Lagrange polynomial and, for $0<\theta<1$ fixed,

$$j=\min_{k=1,\dots,m}\{k : z_k\ge 4\theta m\}. \tag{6}$$

In [12, Theorem 2.2] the following theorem, dealing with the simultaneous approximation of the function f and its derivatives, has been proved.

Theorem 2.1. *Let $1\le r\in\mathbb{N}$, $p\in\mathbb{N}$ and α,γ satisfying*

$$\max\left\{-1,2\gamma-\frac{5}{2}\right\}<\alpha\le 2\gamma-\frac{1}{2}. \tag{7}$$

If $f\in W_{p+r}(u_\gamma)$, for any $0\le k\le p$, we have

$$\|(f-L_{m+1}(w_\alpha,f))^{(k)}\varphi^k u_\gamma\|\le\mathcal{C}\left(\frac{\log m}{(\sqrt{m})^{p+r-k}}\|f\|_{W_{p+r}(u_\gamma)}+e^{-Am}\|fu_\gamma\|\right),$$

where $\mathcal{C}\neq\mathcal{C}(m,f)$ and $A\neq A(m,f)$.

It is easy to prove that the above theorem holds true also if the weight u_γ is replaced by $u_{\gamma,\delta}$.

The following theorem establishes the existence of the integral $H_p^{\omega,\gamma}(f,t)$ for functions belonging to $W_1(u_{\gamma,\delta})$ if $p=0$ and to $W_{p+1}(u_\gamma)$ if $p\geq 1$.

Theorem 2.2. *For all $f\in W_1(u_{\gamma,\delta})$, with $\gamma\geq 0$ and $\delta>0$, and for any $t>0$ we have*

$$\max\left\{1,\log^{-1}t^{-1}\right\}|H_0^{\omega,\gamma}(f,t)|\leq \mathcal{C}\|f\|_{W_1(u_{\gamma,\delta})}$$

and for all $f\in W_{p+1}(u_\gamma)$, $p\geq 1$, $\gamma\geq 0$, and for any $t>0$ we get

$$t^p|H_p^{\omega,\gamma}(f,t)|\leq \mathcal{C}\|f\|_{W_{p+1}(u_\gamma)},$$

where $\mathcal{C}\neq\mathcal{C}(f,t)$.

3 The quadrature rule

The quadrature rule we propose for approximating the integral (1) is of product type and consists in replacing the function f by the truncated Lagrange polynomial $L_{m+1}(w_\alpha,f)$ defined in (4). Thus, we obtain

$$H_p^{\omega,\gamma}(f,t)=H_{p,m}^{\omega,\gamma}(f,t)+e_{p,m}^{\omega,\gamma}(f,t),$$

where

$$H_{p,m}^{\omega,\gamma}(f,t)=H_p^{\omega,\gamma}(L_{m+1}(w_\alpha,f),t)=\sum_{k=1}^{j}f(z_k)⨎_0^{+\infty}\frac{\ell_{m+1,k}(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}u_\gamma(x)\,dx=:\sum_{k=1}^{j}f(z_k)\mathcal{M}_k^{(p)}(t) \tag{8}$$

is the product quadrature rule and

$$e_{p,m}^{\omega,\gamma}(f,t)=H_p^{\omega,\gamma}(f-L_{m+1}(w_\alpha,f),t)$$

is the remainder term. Recalling (5) and that

$$l_{m,k}(x)=\lambda_{m,k}\sum_{\nu=0}^{m-1}p_\nu(z_k)p_\nu(x),$$

we can write

$$\mathcal{M}_k^{(p)}(t)=\frac{1}{4m-z_k}⨎_0^{+\infty}\frac{(4m-x)l_{m,k}(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}u_\gamma(x)\,dx=\frac{\lambda_{m,k}}{4m-z_k}\sum_{\nu=0}^{m-1}p_\nu(z_k)A_\nu^{(p)}(t)$$

where

$$\begin{aligned}A_\nu^{(p)}(t)&:=⨎_0^{+\infty}\frac{(4m-x)p_\nu(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}u_\gamma(x)\,dx\\&=(4m-t)⨎_0^{+\infty}\frac{p_\nu(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}u_\gamma(x)\,dx-⨎_0^{+\infty}\frac{p_\nu(x)e^{\mathrm{i}\omega x}}{(x-t)^{p}}u_\gamma(x)\,dx\\&=:\begin{cases}(4m-t)M_\nu^{(0)}(t)-d_\nu & p=0,\\(4m-t)M_\nu^{(p)}(t)-M_\nu^{(p-1)}(t) & p\geq 1.\end{cases}\end{aligned}$$

The integrals $d_\nu=\displaystyle\int_0^{+\infty}p_\nu(x)x^\gamma e^{\mathrm{i}\omega x-\frac{x}{2}}\,dx$, $\nu=0,\dots,m-1$, are exactly computed using, for $\gamma=0$ [23, p. 809, n° 5, 6]

$$d_\nu=(-1)^\nu\sqrt{\frac{\nu!}{\Gamma(\alpha+1+\nu)}}\begin{cases}(b-1)^\nu-b^{-\nu-1}, & \alpha=0\\\displaystyle\sum_{k=0}^{\nu}\binom{\alpha+k-1}{k}\frac{(b-1)^{\nu-k}}{b^{\nu-k+1}}, & \alpha\neq 0\end{cases},$$

where $b=\frac{1}{2}-\mathrm{i}\omega$, and, for $\gamma\neq 0$ [23, p. 809, n° 7]

$$d_\nu=(-1)^\nu\sqrt{\frac{\nu!}{\Gamma(\alpha+1+\nu)}}\frac{\Gamma(\gamma+1)\gamma(\alpha+\nu+1)}{\nu!\gamma(\alpha+1)}b^{-\gamma-1}{}_2F_1\left(-\nu;\gamma+1;\alpha+1;\frac{1}{b}\right),$$

where ${}_2F_1$ is the hypergeometric function.

Taking into account the recurrence relation (3), a stable recursion scheme can be deduced for the computation of the integrals $\{M_\nu^{(p)}(t)\}_{\nu=0,\dots,m}$. In fact, starting from

$$M_0^{(0)}(t)=\frac{1}{\sqrt{\Gamma(\alpha+1)}}\begin{cases}-e^{\mathrm{i}\omega t-\frac{t}{2}}E_i\left(\frac{t}{2}-\mathrm{i}\omega t\right), & \gamma=0,\\e^{\mathrm{i}\omega t-\frac{t}{2}}t^\gamma\left[\mathrm{i}\pi+\gamma e^{\mathrm{i}\gamma\pi}\Gamma(\gamma)\Gamma\left(-\gamma,-\frac{t}{2}+\mathrm{i}\omega t\right)\right], & \gamma\neq 0,\end{cases}$$

and (see (2))

$$M_0^{(p)}(t) = \frac{1}{p}\frac{d}{dt}M_0^{(p-1)}(t) = \frac{1}{p!}\frac{d^p}{dt^p}M_0^{(0)}(t),$$

where E_i is the exponential integral function, it is easy to deduce that

$$\begin{cases} M_1^{(0)}(t) = \frac{1}{a_1}\left[d_0 + (t-b_0)M_0^{(0)}(t)\right], \\ M_{\nu+1}^{(0)}(t) = \frac{1}{a_{\nu+1}}\left[d_\nu + (t-b_\nu)M_\nu^{(0)}(t) - a_\nu M_{\nu-1}^{(0)}(t)\right], \\ M_1^{(p)}(t) = \frac{1}{a_1}\left[M_0^{(p-1)}(t) + (t-b_0)M_0^{(p)}(t)\right], \\ M_{\nu+1}^{(p)}(t) = \frac{1}{a_{\nu+1}}\left[M_\nu^{(p-1)}(t) + (t-b_\nu)M_\nu^{(p)}(t) - a_\nu M_{\nu-1}^{(p)}(t)\right]. \end{cases}$$

Denoting by $\{M_{\nu,Q}^{(s)}(t)\}_{\nu=0,\dots,m}$ and $\{M_{\nu,D}^{(s)}(t)\}_{\nu=0,\dots,m}$, $s=0,\dots,p$, the above sequences computed in double (eps_D) and quadruple machine precision (eps_Q), respectively, we have studied numerically the stability of the above recursion scheme computing the quantities

$$S(t) := \max_{s=0,\dots,p}\ \max_{\nu=0,\dots,500}\left|\frac{M_{\nu,Q}^{(s)}(t) - M_{\nu,D}^{(s)}(t)}{M_{\nu,Q}^{(s)}(t)}\right|$$

for different values of α,γ,ω and t. Choosing $\alpha=0$, $\gamma=2/3$, $\omega=1000$ and $p=0,1,2$ we get the results presented in Table 1. Since, analogous results have been obtained for many other selections of α,γ,ω and t we can empirically assume the stability of the scheme.

t	10^{-12}	10^{-9}	10^{-7}	10^{-3}	10^{-2}	10^{-1}	10	10^2
$S(t)$	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D

Table 1

The computation of $H_p^{\omega,\gamma}(f,t)$ requires an overall computational cost of about $4mj+3m(p+1)+m+j$ multiplicative operations and j evaluations of the function f. Since the computation of the sequence $\{M_\nu^{(p)}(t)\}_{\nu=0,\dots,m}$ requires the computation of all the sequences $\{M_\nu^{(s)}(t)\}_{\nu=0,\dots,m}, s=0,\dots,p-1$, with an additional computational cost of about $(mj+m+j)p$ multiplicative operations and no more evaluations of the function f, it is possible to perform the simultaneous approximation of $H_s^{\omega,\gamma}(f,t), s=0,\dots,p$.

The following theorems give sufficient conditions for the stability and convergence of the quadrature rules $H_{p,m}^{\omega,\gamma}(f,t)$, $p\geq 0$.

Theorem 3.1. *Let α,γ satisfying (7) and $\delta>0$. For any $f\in W_1(u_{\gamma,\delta})$ and for any $t>0$ we get*

$$\max\{1,\log^{-1}t^{-1}\}|H_{0,m}^{\omega,\gamma}(f,t)| \leq \mathcal{C}\|f\|_{W_1(u_{\gamma,\delta})}\log m, \quad \mathcal{C}\neq\mathcal{C}(m,f,t).$$

Moreover, if $f\in W_r(u_{\gamma,\delta})$ with $r\geq 1$ integer, then for any $t>0$ we obtain

$$\max\{1,\log^{-1}t^{-1}\}|e_{0,m}^{\omega,\gamma}(f,t)| \leq \mathcal{C}\frac{\|f\|_{W_r(u_{\gamma,\delta})}}{m^{\frac{r}{2}}}\log^2 m, \quad \mathcal{C}\neq\mathcal{C}(m,f,t). \tag{9}$$

Theorem 3.2. *Let $p\geq 1$ be an integer and let α,γ satisfying (7). For any $f\in W_{p+1}(u_\gamma)$ and for any $t>0$ we get*

$$t^p|H_{p,m}^{\omega,\gamma}(f,t)| \leq \mathcal{C}\|f\|_{W_{p+1}(u_\gamma)}\log m, \quad \mathcal{C}\neq\mathcal{C}(m,f,t). \tag{10}$$

Moreover, if $f\in W_{p+r}(u_\gamma)$ with $r\geq 1$ integer, then for any $t>0$ we obtain

$$t^p|e_{p,m}^{\omega,\gamma}(f,t)| \leq \mathcal{C}\frac{\|f\|_{W_{p+r}(u_\gamma)}}{m^{\frac{r}{2}}}\log^2 m, \quad \mathcal{C}\neq\mathcal{C}(m,f,t). \tag{11}$$

4 Application of the quadrature rule $H_{p,m}^{\omega,\gamma}(f,t)$ to the resolution of an integral equation

In this section, we show how the above introduced product quadrature rule can be an useful tool for the construction of a Nyström type method for the numerical solution of integral equations of the following kind

$$f(t) + \int_0^{+\infty}\!\!\!\!\!\!\!\!\!\!=\ \ \frac{f(x)e^{i\omega x}}{(x-t)^{p+1}}u_\gamma(x)\,dx = g(t), \quad t>0, \tag{12}$$

where g is a given function and f is the unknown solution.

Since we have proved the convergence of the quadrature rules $H_{p,m}^{\omega,\gamma}(f,t), p\geq 0$, in the space C_u, with $u=u_{\gamma,\delta}$ if $p=0$ and $u=u_\gamma$ if $p\geq 1$, it seems natural to assume that the integral equation (12) admits a unique solution in C_u and, as a first step in the construction of the Nyström method, we multiply both sides of the equation by the weight function u. Then, replacing the integral $H_p^{\omega,\gamma}(f,t)$ by its approximation $H_{p,m}^{\omega,\gamma}(f,t)$, we get the following finite dimensional equation

$$(f_m u)(t) + u(t) \sum_{k=1}^{j} (f_m u)(z_k) \frac{\mathcal{M}_k^{(p)}(t)}{u(z_k)} = (gu)(t). \tag{13}$$

in the unknown $f_m u$. Finally, collocating the above equation at the quadrature knots z_r, $r = 1, \ldots, j$, we get the following linear system

$$\sum_{k=1}^{j} \left[\delta_{r,k} + \frac{u(z_r)}{u(z_k)} \mathcal{M}_k^{(p)}(z_r) \right] a_k = (gu)(z_r), \quad r = 1, \ldots, j, \tag{14}$$

whose unknowns are $a_k = (f_m u)(z_k)$, $k = 1, \ldots, j$.
If $\{\bar{a}_k\}_{k=1,\ldots,j}$ is the unique solution of system (14), we construct the solution of the approximating equation (13) as follows

$$(f_m u)(t) = (gu)(t) - u(t) \sum_{k=1}^{j} \frac{\mathcal{M}_k^{(p)}(t)}{u(z_k)} \bar{a}_k.$$

Under suitable assumptions the so called Nyström interpolating functions $\{f_m\}_m$ converge to the exact solution f of the integral equation (12).

The study of the uniqueness of the solution of (12) and of the stability and convergence of the above described Nyström method is beyond the scope of this work and will be the topic of a forthcoming paper. Here we just show how the introduced quadrature rule can be used in the context of integral equations.

5 Numerical tests

In this section we will show the numerical results obtained approximating some integrals of the type (1) by the proposed product quadrature rule. In particular, in the first example we will make also some comparisons with other numerical methods available in literature [5, 43]. The density functions f considered in Examples 2 and 3 are representatives of the function space $W_r(u_{\gamma,\delta})$ where we want to test the method and to verify the sharpness of the error estimates given in Theorems 3.1 and 3.2. Finally, in the last example we will show the performance of the Nyström method described in Section 4 in solving some integral equations of the type (12).

In the tables that follows we will report the absolute errors

$$e_{p,m}^{\omega,\gamma}(f,t) = |H_{p,m}^{\omega,\gamma}(f,t) - H_p^{\omega,\gamma}(f,t)|.$$

In the examples where the exact solution is unknown, we will show the above errors computed with $H_p^{\omega,\gamma}(f,t)$ replaced by $H_{p,1024}^{\omega,\gamma}(f,t)$. In all the numerical tests the value of j has been dynamically detected according to the following criteria

$$j = \min_{k=1,\ldots,m} \{k \,:\, |\mathcal{M}_k^{(p)}(t)| < \varepsilon\}, \tag{15}$$

where ε is the epsilon machine of the used precision arithmetic. Recalling the definition of $\mathcal{M}_k^{(p)}(t)$ in (8) and that [32] $\lambda_{m,k} \leq C z_k^{\alpha} e^{-z_k}(z_k - z_{k-1})$, the above definition of j is equivalent to (6) in the sense that there exists a $\theta \in (0,1)$ such that $z_{j-1} < 4\theta m < z_j$ with j defined in (15).

The parameter α defining the interpolation process $L_{m+1}(w_\alpha, f)$ has been always chosen inside the interval defined by (7). According to Theorems 3.1 and 3.2, this choice is crucial in order to assure the stability and the convergence of the proposed quadrature scheme, and analogous numerical results are obtained whatever is the selection of α in such interval (see, for example, Tables 4 and 7).
Finally, concerning the parameter δ appearing in the definition of the weight of the space $W_r(u_{\gamma,\delta})$, according to Theorems 3.1 and 3.2, it has to be equal to 0 for $p > 0$ and greater than 0 for $p = 0$. In the latter case its choice strictly depends on the function f appearing under the integral sign. For the reader convenience we recall that if, for some $r \geq 1$, f belongs to $W_r(u_{\gamma,\delta})$ with $\delta > 0$ then it belongs to $W_r(u_\gamma)$, too.

Unless specified otherwise, all the computations have been performed in double-precision arithmetic ($\varepsilon_D = 2.22044e-16$).

Example 5.1. As first example we consider the following integral

$$H_0^{\omega,0}(f,t) = \int_0^{+\infty} \frac{e^{-x}}{x-t} e^{\mathrm{i}\omega x}\, dx = -e^{-t} e^{\mathrm{i}\omega t} E_i(t - \mathrm{i}\omega t), \tag{16}$$

where the exact solution is known. We have approximated it using the proposed quadrature rule with $f(x) = e^{-\frac{x}{2}}$ and, since $\gamma = 0$, according to Theorem 2.1, with $\alpha = -\frac{1}{2}$. We note that $f \in W_r(u_{\gamma,\delta})$ for any $r \geq 1$ and $\delta > 0$. This example has been also considered in [43, 5], where all the computations have been performed in quadruple-precision arithmetic ($\varepsilon_Q = 1.92592e-34$). Then, in order to make comparisons with the numerical results presented in such papers, the approximations of the integral (16) presented in Figure 1 and in Tables 2, 3, 5 and 6 have been computed in quadruple-precision arithmetic.

In Figure 1 and in Tables 2 and 3 we present the approximations of the integral (16) with the same choices of ω and t considered in [43, Fig. 2, p.726], [43, Table 1, p.734] and [43, Table 2, p.734], respectively. One can see that our quadrature rule provides absolute errors of the same order regardless the choices of the values of t and ω. In particular, taking $m = 94$ and $j = 51$ we always get the machine precision eps_Q in quadruple arithmetic. On the contrary, the accuracy of the quadrature rule proposed in [43] strictly depends on the choice of ω and t. More precisely, it increases as both t and ω increase (see [43, Fig. 2

on the right, p.726]), but, even when ω increases, approximations with at most 25 exact decimal digits are obtained for small values of t (see [43, Table 2, p.734]).

Moreover, in Table 4 we report the approximations obtained in double-precision arithmetic for the same integrals considered in Figure 1. One can see that, also varying the parameter $\alpha \in \left(-1, \frac{1}{2}\right]$, in order to obtain absolute errors of the order of the machine precision eps_D it is sufficient to apply our product rule with $m = 32$ and $j = 21$. Analogous results are obtained with other choices of ω, t and α.

In [5] the authors have also made comparisons with the results presented in [43, Table 2, p.734]. In Table 5 we summarize the approximations obtained for the integral (16) with $t = 0.02$ using the quadrature rule $Q_{16,16}(f)$ proposed in [43], the quadrature rule $Q^A_{16}[f]$ proposed in [5, Table 1, p. 180] and our quadrature rule $e^{\omega,0}_{0,94}(f,t)$. As one can see, the accuracies of both the rule $Q^A_{16}[f]$ and the rule $Q_{16,16}(f)$ increase as ω increases, but the rule $Q^A_{16}[f]$ with respect to the rule $Q_{16,16}(f)$ has the advantage of providing approximations of the order of eps_Q for $\omega = 80$ and $\omega = 320$. As already observed above, the absolute errors $e^{\omega,0}_{0,94}(f,t)$ of our rule are always of the order of eps_Q no matter the value of ω is. However, comparing [5, Fig.1 and Fig.3] with Table 6 both the rule $Q^A_{16}[f]$ and the rule $Q^I_{16}[f]$ give larger absolute errors with respect to our rule when ω goes from 1 up to 200 and $t = 1$ or $t = 5$.

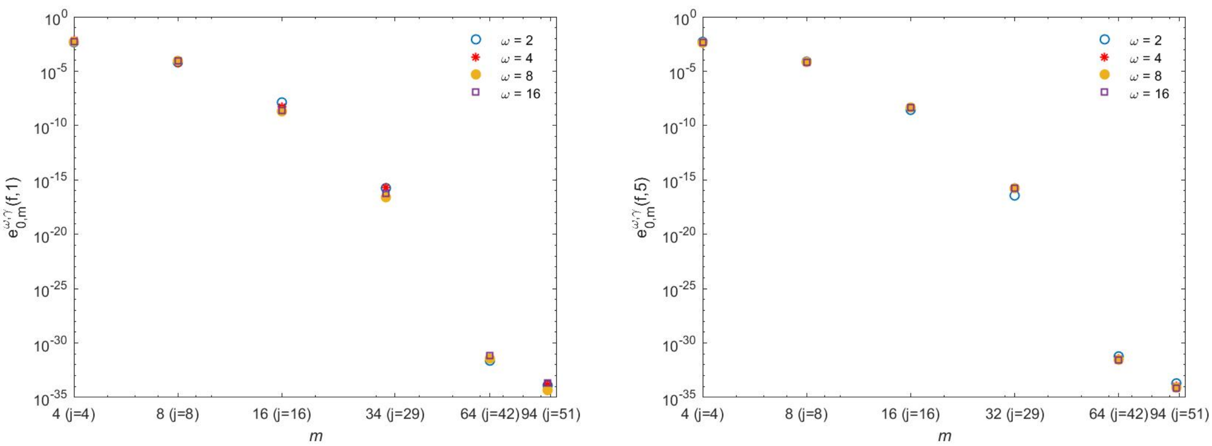

Figure 1: Example 5.1: Absolute errors $e^{\omega,0}_{0,m}(f,t)$ obtained for $t = 1$ (left) and for $t = 5$ (right)

		$e^{10,0}_{0,m}(f,t)$			
m	j	$t = 10^{-1}$	$t = 10^{-2}$	$t = 10^{-3}$	$t = 10^{-4}$
8	8	$4.78e-5$	$2.91e-5$	$1.19e-4$	$1.99e-4$
16	16	$8.45e-9$	$4.55e-9$	$9.78e-9$	$1.87e-8$
32	28	$1.65e-17$	$1.99e-16$	$5.39e-17$	$1.69e-16$
64	42	$9.03e-32$	$5.41e-32$	$1.14e-32$	$7.47e-32$
94	51	eps_Q	eps_Q	eps_Q	eps_Q

Table 2: Example 5.1 with $\omega = 10$

		$e^{\omega,0}_{0,m}(f,0.02)$			
m	j	$\omega = 5$	$\omega = 20$	$\omega = 80$	320
8	8	$4.09e-5$	$2.38e-5$	$7.17e-5$	$6.80e-5$
16	16	$1.18e-8$	$3.06e-9$	$5.31e-9$	$4.43e-9$
32	28	$1.44e-16$	$1.56e-16$	$3.96e-17$	$1.72e-17$
64	42	$4.21e-32$	$7.73e-32$	$4.00e-32$	$5.26e-32$
94	51	eps_Q	eps_Q	eps_Q	eps_Q

Table 3: Example 5.1 with t=0.02

Example 5.2. Now we consider the following integrals

$$H^{\omega,\frac{3}{5}}_p(f,t) = \fint_0^{+\infty} \frac{1}{(x^2+1)^3(x-t)^{p+1}} e^{i\omega x} x^{\frac{3}{5}}\, dx, \quad p = 0, 1,$$

		$\omega=2$		$\omega=4$		$\omega=8$		$\omega=16$	
$e^{\omega,0}_{0,m}(f,t)$, with $\alpha=-\frac{1}{2}$									
m	j	$t=1$	$t=5$	$t=1$	$t=5$	$t=1$	$t=5$	$t=1$	$t=5$
8	8	$6.48e-5$	$7.69e-5$	$5.82e-5$	$7.18e-5$	$8.93e-5$	$6.98e-5$	$8.32e-5$	$6.86e-5$
16	15	$1.36e-8$	$2.75e-9$	$5.48e-9$	$3.91e-9$	$2.01e-9$	$4.55e-9$	$2.29e-9$	$4.57e-9$
32	21	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D
$e^{\omega,0}_{0,m}(f,t)$, with $\alpha=-\frac{2}{3}$									
		$\omega=2$		$\omega=4$		$\omega=8$		$\omega=16$	
m	j	$t=1$	$t=5$	$t=1$	$t=5$	$t=1$	$t=5$	$t=1$	$t=5$
8	8	$4.97e-5$	$7.93e-5$	$6.40e-5$	$7.18e-5$	$8.80e-5$	$7.04e-5$	$8.31e-5$	$6.97e-5$
16	16	$1.11e-8$	$3.53e-9$	$4.90e-9$	$5.46e-9$	$4.08e-9$	$6.01e-9$	$4.29e-9$	$6.08e-9$
32	26	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D
$e^{\omega,0}_{0,m}(f,t)$, with $\alpha=-\frac{3}{4}$									
		$\omega=2$		$\omega=4$		$\omega=8$		$\omega=16$	
m	j	$t=1$	$t=5$	$t=1$	$t=5$	$t=1$	$t=5$	$t=1$	$t=5$
8	8	$4.24e-5$	$7.92e-5$	$6.46e-5$	$7.08e-5$	$8.54e-5$	$6.97e-5$	$8.12e-5$	$6.92e-5$
16	16	$9.86e-9$	$4.01e-9$	$4.72e-9$	$6.11e-9$	$4.83e-9$	$6.62e-9$	$4.96e-9$	$6.69e-9$
32	26	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D	eps_D

Table 4: Example 5.1 with different choices of α.

	$\omega=5$	$\omega=20$	$\omega=80$	$\omega=320$
$Q_{16,16}(f)$	$7.49e-11$	$5.44e-21$	$2.78e-25$	$3.65e-25$
$Q^A_{16}[f]$	$5.53e-14$	$4.32e-24$	$2.56e-34$	$1.49e-44$
$e^{\omega,0}_{0,94}(f,0.02)$ $(j=51)$	$1.23e-34$	$6.84e-35$	$1.62e-34$	$2.75e-34$

Table 5: Example 5.1 with $t=0.02$

		$e^{\omega,0}_{0,m}(f,1)$					$e^{\omega,0}_{0,m}(f,5)$			
m	j	$\omega=50$	$\omega=100$	$\omega=150$	$\omega=200$	j	$\omega=50$	$\omega=100$	$\omega=150$	$\omega=200$
8	8	$8.15e-5$	$8.19e-5$	$8.20e-5$	$8.21e-5$	8	$6.88e-5$	$6.90e-5$	$6.90e-5$	$6.89e-5$
16	16	$2.21e-9$	$2.17e-9$	$2.16e-9$	$2.15e-9$	16	$4.63e-9$	$4.65e-9$	$4.65e-9$	$4.64e-9$
32	28	$6.60e-17$	$6.63e-17$	$6.66e-17$	$6.68e-17$	29	$1.76e-16$	$1.76e-16$	$1.76e-16$	$1.76e-16$
64	42	$6.91e-32$	$6.95e-32$	$6.94e-32$	$6.94e-32$	42	$2.88e-32$	$2.87e-32$	$2.86e-32$	$2.87e-32$
94	51	eps_Q	eps_Q	eps_Q	eps_Q	51	eps_Q	eps_Q	eps_Q	eps_Q

Table 6: Example 5.1 with $t=1$ and $t=5$

where $\gamma = \frac{3}{5}$ and $f(x) = \frac{e^{\frac{x}{2}}}{(x^2+1)^3}$. According to (7) we choose $\alpha = \frac{1}{2}, \frac{1}{5}, -\frac{1}{10}$. Moreover, since $f \in W_8(u_{\frac{3}{5},\delta})$ for $\frac{9}{10} < \delta \leq \frac{7}{5}$, by (9) the order of convergence of the rule $H_{0,m}^{\omega,\frac{3}{5}}(f)$ is $m^{-4}\log^2 m$ and by (11) the proposed quadrature rule $H_{1,m}^{\omega,\frac{3}{5}}(f)$ converge with order $m^{-\frac{7}{2}}\log^2 m$. The corresponding absolute error $e_{p,m}^{\omega,\frac{3}{5}}(f,t)$, $p = 0,1$ are reported in Table 7 for different choices of t, ω and α. As one can see, they agree with the theoretical expectations. In Figure 2 some of the results in Table 7 are graphically shown.

$e_{0,m}^{\omega,\frac{3}{5}}(f,t)$, with $\alpha = \frac{1}{2}$

		$t = 0.01$			$t = 0.3$			$t = 7$		
m	j	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$
$1.72e-4$	$4.41e-4$	$2.98e-4$	$2.96e-4$							
64	64	$1.14e-4$	$3.84e-4$	$5.48e-4$	$4.13e-4$	$3.95e-4$	$3.94e-4$	$4.68e-6$	$1.03e-5$	$1.00e-5$
128	128	$2.11e-6$	$6.21e-7$	$1.46e-6$	$7.49e-6$	$8.83e-7$	$8.63e-7$	$7.38e-7$	$4.62e-7$	$4.64e-7$
256	256	$1.68e-9$	$1.04e-9$	$4.87e-11$	$3.16e-9$	$7.37e-9$	$7.39e-9$	$4.67e-10$	$7.33e-10$	$7.33e-10$
512	512	$1.01e-13$	$5.02e-13$	$1.23e-12$	$2.62e-12$	$1.73e-13$	$8.82e-14$	$3.67e-13$	$4.51e-13$	$4.53e-13$

$e_{1,m}^{\omega,\frac{3}{5}}(f,t)$, with $\alpha = \frac{1}{2}$

		$t = 0.01$			$t = 0.3$			$t = 7$		
m	j	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$
64	64	$4.52e-2$	$6.30e-2$	$5.43e-1$	$7.61e-3$	$4.03e-2$	$3.94e-1$	$9.98e-5$	$1.01e-3$	$1.01e-2$
128	128	$1.85e-4$	$1.86e-4$	$1.44e-3$	$6.91e-5$	$1.54e-4$	$8.73e-4$	$1.01e-5$	$4.72e-5$	$4.64e-4$
256	256	$2.69e-8$	$1.27e-7$	$2.24e-7$	$2.26e-7$	$7.40e-7$	$7.39e-6$	$3.42e-8$	$8.06e-8$	$7.33e-7$
512	512	$2.96e-10$	$2.88e-10$	$1.28e-9$	$2.63e-11$	$1.08e-10$	$1.39e-10$	$8.68e-12$	$4.59e-11$	$4.53e-10$

$e_{0,m}^{\omega,\frac{3}{5}}(f,t)$, with $\alpha = \frac{1}{5}$

		$t = 0.01$			$t = 0.3$			$t = 7$		
m	j	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$
64	64	$1.65e-4$	$1.58e-4$	$2.50e-4$	$5.40e-4$	$1.03e-4$	$1.03e-4$	$3.50e-5$	$2.51e-5$	$2.52e-5$
128	128	$1.29e-6$	$6.61e-8$	$6.25e-8$	$3.62e-6$	$9.61e-7$	$9.57e-8$	$2.00e-6$	$1.82e-6$	$1.82e-6$
256	256	$3.50e-9$	$3.18e-9$	$1.53e-9$	$1.70e-9$	$2.34e-10$	$3.12e-10$	$2.73e-9$	$2.22e-9$	$2.22e-9$
512	512	$2.61e-13$	$2.35e-13$	$2.13e-13$	$1.05e-12$	$1.12e-12$	$1.10e-12$	$9.27e-13$	$9.46e-13$	$9.46e-13$

$e_{1,m}^{\omega,\frac{3}{5}}(f,t)$, with $\alpha = \frac{1}{5}$

		$t = 0.01$			$t = 0.3$			$t = 7$		
m	j	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$
64	64	$2.52e-2$	$2.87e-2$	$2.47e-1$	$6.48e-3$	$1.18e-2$	$1.03e-1$	$1.29e-4$	$2.52e-3$	$2.52e-2$
128	128	$6.40e-6$	$1.08e-5$	$6.26e-5$	$2.87e-5$	$1.15e-4$	$9.59e-4$	$2.08e-5$	$1.83e-4$	$1.82e-3$
256	256	$1.89e-7$	$3.94e-7$	$1.67e-6$	$2.72e-8$	$3.31e-8$	$3.11e-7$	$4.66e-7$	$2.25e-7$	$2.22e-6$
512	512	$5.88e-11$	$6.13e-11$	$2.22e-10$	$4.40e-11$	$1.20e-10$	$1.10e-9$	$1.16e-11$	$9.49e-11$	$9.46e-10$

$e_{0,m}^{\omega,\frac{3}{5}}(f,t)$, with $\alpha = -\frac{1}{10}$

		$t = 0.01$			$t = 0.3$			$t = 7$		
m	j	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$
64	64	$1.30e-4$	$3.86e-5$	$8.27e-5$	$5.62e-4$	$1.63e-4$	$1.63e-4$	$6.95e-5$	$6.43e-5$	$6.43e-5$
128	128	$1.54e-6$	$3.17e-7$	$4.43e-9$	$1.35e-6$	$3.48e-7$	$3.33e-7$	$3.56e-6$	$3.50e-6$	$3.50e-6$
256	256	$2.79e-9$	$3.00e-9$	$2.39e-9$	$5.49e-9$	$1.85e-9$	$1.78e-9$	$6.05e-9$	$5.95e-9$	$5.95e-9$
512	512	$1.21e-13$	$6.27e-14$	$8.90e-14$	$3.91e-13$	$8.76e-13$	$8.84e-13$	$1.48e-12$	$1.40e-12$	$1.40e-12$

$e_{1,m}^{\omega,\frac{3}{5}}(f,t)$, with $\alpha = -\frac{1}{10}$

		$t = 0.01$			$t = 0.3$			$t = 7$		
m	j	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$	$\omega = 10$	$\omega = 100$	$\omega = 1000$
64	64	$1.17e-2$	$1.03e-2$	$8.14e-2$	$5.97e-3$	$1.81e-2$	$1.62e-1$	$6.80e-4$	$6.44e-3$	$6.44e-2$
128	128	$2.50e-5$	$5.29e-5$	$7.54e-5$	$1.97e-6$	$3.61e-5$	$3.33e-4$	$3.58e-5$	$3.50e-4$	$3.50e-3$
256	256	$2.26e-7$	$4.12e-7$	$2.47e-6$	$5.90e-8$	$2.27e-7$	$1.78e-6$	$7.03e-8$	$5.96e-7$	$5.95e-6$
512	512	$1.88e-11$	$1.72e-11$	$9.04e-11$	$3.58e-11$	$8.88e-11$	$8.84e-10$	$1.44e-11$	$1.40e-10$	$1.40e-9$

Table 7: Example 5.2 with different choices of α.

Example 5.3. Let us consider the following integrals

$$H_p^{\omega,\frac{1}{3}}(f,t) = \int_0^{+\infty} \frac{|x-5|^{\frac{11}{2}}}{(x+1)^2(x-t)^{p+1}} e^{i\omega x} x^{\frac{1}{3}} e^{-\frac{x}{2}}\, dx, \quad p = 0,1,2,$$

having $\gamma = \frac{1}{3}$ and $f(x) = \frac{|x-5|^{\frac{11}{2}}}{(x+1)^2} \in W_5(u_{\frac{1}{3},\delta})$, for any $\delta \geq 0$. Taking into account (7) we apply the quadrature rules choosing $\alpha = 0$. In Table 8 and Figure 3 we display the obtained absolute errors for some values of ω and t. In agreement with the theoretical expectations (see Theorems 3.1 and 3.2) the convergence orders are $m^{-\frac{5-p}{2}}\log^2 m$, $p = 0,1,2$, and, when the values of t are far from the critical point 5, the errors become smaller.

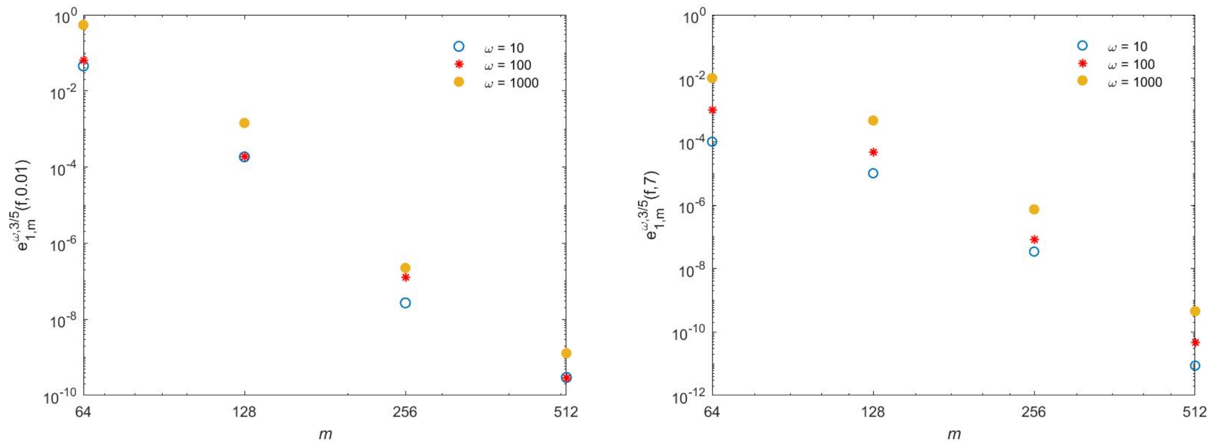

Figure 2: Example 5.2: Absolute errors $e^{\omega,\frac{3}{5}}_{1,m}(f,t)$ obtained for $t = 0.01$ (left) and for $t = 7$ (right)

		$e^{\omega,\frac{1}{3}}_{0,m}(f,4.99)$			$e^{\omega,\frac{1}{3}}_{0,m}(f,10)$	
m	j	$\omega = 100$	$\omega = 500$	j	$\omega = 100$	$\omega = 500$
64	57	$2.34e-2$	$2.35e-2$	57	$1.17e-2$	$1.17e-2$
128	85	$5.08e-6$	$4.86e-6$	82	$6.18e-7$	$5.50e-7$
256	119	$1.37e-6$	$1.37e-6$	122	$1.79e-6$	$1.79e-6$
512	174	$2.71e-7$	$2.71e-7$	174	$1.51e-8$	$1.51e-8$

		$e^{\omega,\frac{1}{3}}_{1,m}(f,4.99)$			$e^{\omega,\frac{1}{3}}_{1,m}(f,10)$	
m	j	$\omega = 100$	$\omega = 500$	j	$\omega = 100$	$\omega = 500$
128	87	$4.92e-4$	$2.44e-3$	85	$1.56e-4$	$3.11e-4$
256	124	$1.38e-4$	$6.84e-4$	126	$1.79e-4$	$8.93e-4$
512	179	$2.78e-5$	$1.37e-4$	116	$1.54e-6$	$7.53e-6$

		$e^{\omega,\frac{1}{3}}_{2,m}(f,4.99)$			$e^{\omega,\frac{1}{3}}_{2,m}(f,10)$	
m	j	$\omega = 100$	$\omega = 500$	j	$\omega = 100$	$\omega = 500$
128	88	$2.50e-2$	$6.11e-1$	88	$1.49e-2$	$1.00e-1$
256	129	$7.00e-3$	$1.71e-1$	129	$8.96e-3$	$2.23e-1$
512	114	$1.50e-3$	$3.40e-2$	137	$8.23e-5$	$1.89e-3$

Table 8: Example 5.3 with t=4.99 and $t = 10$

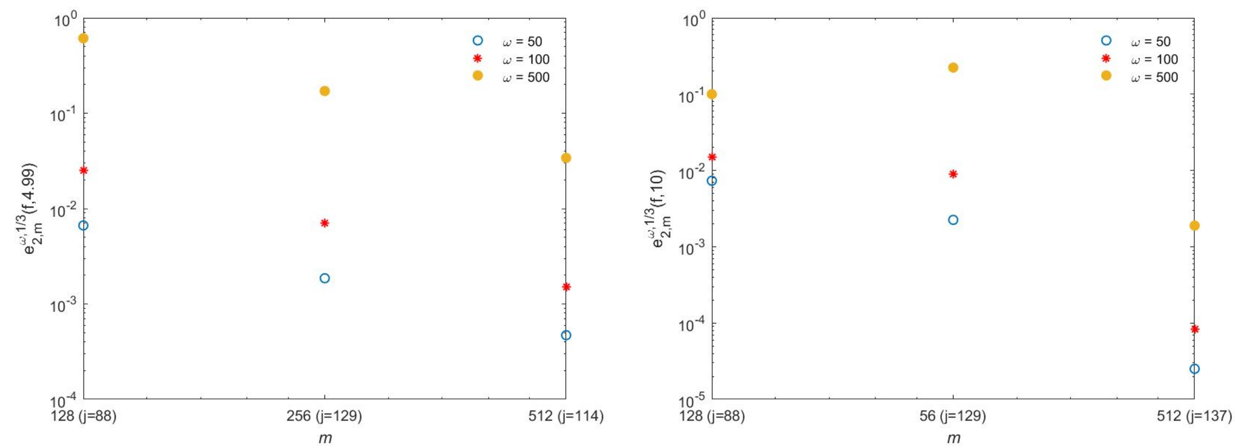

Figure 3: Example 5.3: Absolute errors $e^{\omega,\frac{1}{3}}_{2,m}(f,t)$ obtained for $t = 4.99$ (left) and for $t = 10$ (right)

Example 5.4. Finally we consider the following integral equation of the type (12)

$$f(t)+\int_0^{+\infty}\frac{f(x)}{(x-t)^{p+1}}e^{\mathrm{i}100x}e^{-\frac{x}{2}}\,dx=g(t),\quad p=0,1,$$

where

$$g(t)=\begin{cases}\dfrac{2}{40001}\dfrac{400\mathrm{i}}{40001}+t-e^{(-\frac{1}{2}+100\mathrm{i})t}\,tE_i\left(\dfrac{1}{2}t-100\mathrm{i}t\right) & \text{if } p=0\\ -1+x+\dfrac{1}{2}e^{(-\frac{1}{2}+100\mathrm{i})t}(-2+(1-200\mathrm{i})t)E_i\left(\dfrac{1}{2}t-100\mathrm{i}\right) & \text{if } p=1.\end{cases}$$

Both have $f(t)=t$ as exact solution. Since $\gamma=0$ we have applied the Nyström method described in Section 4 with $\alpha=-\frac{1}{2}$. Moreover, since of course $f\in W_r(u_{0,\delta})$ for any $r\geq 1$ and $\delta\geq 0$, in the case $p=0$ we have chosen $\delta=1$. In Table 9 we report the absolute errors

$$\bar{e}^{\omega,\gamma}_{p,m}(f,t)=|(f(t)-f_m(t))u_\gamma(t)|.$$

As one can see, solving linear systems of order only 40 the method gives approximations with 14 exact decimal digits for $p=0$ and with 13 exact decimal digits for $p=1$.

m	j	$\bar{e}^{100,0}_{0,m}(f,4.99)$	$\bar{e}^{100,0}_{1,m}(f,10)$
8	6	$2.36e-3$	$2.59\,e-1$
16	11	$4.19e-5$	$9.80\,e-4$
32	21	$1.80e-9$	$4.72\,e-13$
64	40	$9.61e-16$	$3.55\,e-14$

Table 9: Example 5.4 with t=4.99 and $t=10$

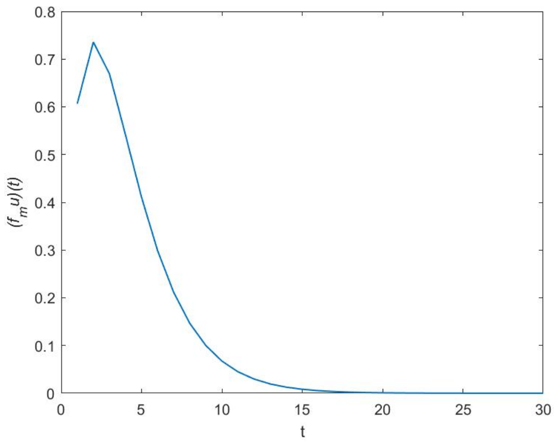

Figure 4: Example 5.4: Weighted Nyström interpolating function obtained for $\omega=100$ and $p=0,1$

6 Proofs

In order to prove Theorem 2.2 we need the following lemmas.

Lemma 6.1. *If $f\in C_{u_{\gamma,\delta}}$, with $\delta>0,\gamma\geq 0$ and $t>0$, then*

$$\left|\int_{\substack{|x-t|\geq 1\\ x\in\mathbb{R}^+}}\frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)}\,dx\right|\leq \mathcal{C}\|fu_{\gamma,\delta}\|,$$

where $\mathcal{C}\neq\mathcal{C}(f,t)$.

Proof. In the case $0<t<1$ we have

$$\begin{aligned}\left|\int_{\substack{|x-t|\geq 1\\ x\in\mathbb{R}^+}}\frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)}\,dx\right| &= \left|\int_{t+1}^{+\infty}\frac{f(x)u_{\gamma,\delta}(x)e^{\mathrm{i}\omega x}}{(x-t)(1+x)^\delta}\,dx\right|\\ &\leq \mathcal{C}\|fu_{\gamma,\delta}\|\int_{t+1}^{+\infty}\frac{dx}{(x-t)(1+x)^\delta}\\ &\leq \mathcal{C}\|fu_{\gamma,\delta}\|. \end{aligned}\tag{17}$$

While, in the case $t \geq 1$ it results

$$\left|\int_{\substack{|x-t|\geq 1\\ x\in\mathbb{R}^+}} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)}\,dx\right| \leq \left|\int_0^{t-1} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(t-x)}\,dx\right| + \left|\int_{t+1}^{+\infty} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)}\,dx\right|.$$

Then, taking into account (17) and

$$\left|\int_0^{t-1} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)}\,dx\right| \leq \mathcal{C}\|fu_{\gamma,\delta}\| \int_0^{t-1} \frac{dx}{(x-t)} \leq \mathcal{C}\log t^{-1}\|fu_{\gamma,\delta}\| \leq \mathcal{C}\|fu_{\gamma,\delta}\|,$$

the lemma easily follows. □

Lemma 6.2. *If $f \in C_{u_\gamma}$, $\gamma \geq 0$, $p \geq 1$ and $t > 0$, then*

$$\left|\int_{\substack{|x-t|\geq 1\\ x\in\mathbb{R}^+}} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| \leq \mathcal{C}\|fu_\gamma\|,$$

where $\mathcal{C} \neq \mathcal{C}(f,t)$.

Proof. Let $0 < t < 1$. Letting $x - t = yt$ we have

$$\begin{aligned}\left|\int_{\substack{|x-t|\geq 1\\ x\in\mathbb{R}^+}} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| &= \left|\int_{t+1}^{+\infty} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| \\ &= \left|\frac{1}{t^p}\int_{\frac{1}{t}}^{+\infty} \frac{(fu_\gamma)(t+yt)e^{\mathrm{i}\omega(t+yt)}}{y^{p+1}}\,dy\right| \\ &\leq \mathcal{C}\frac{\|fu_\gamma\|_\infty}{t^p}\int_{\frac{1}{t}}^{+\infty} y^{-p-1}\,dy \\ &\leq \mathcal{C}\|fu_\gamma\|. \end{aligned} \tag{18}$$

If $t \geq 1$ we get

$$\left|\int_{\substack{|x-t|\geq 1\\ x\in\mathbb{R}^+}} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| \leq \left|\int_0^{t-1} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(t-x)^{p+1}}\,dx\right| + \left|\int_{t+1}^{+\infty} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right|.$$

Taking into account (18) and

$$\left|\int_0^{t-1} \frac{f(x)u_\gamma(x)e^{\mathrm{i}\omega x}}{(t-x)^{p+1}}\,dx\right| \leq \mathcal{C}\frac{\|fu_\gamma\|_\infty}{t^p} \leq \mathcal{C}\|fu_\gamma\|,$$

the lemma follows. □

Lemma 6.3. *Let $\gamma \geq 0$ and $p \geq 0$ then, for $0 < t \leq 1$, we have*

$$\left|\fint_{\substack{|x-t|>1\\ x\in\mathbb{R}^+}} \frac{u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| \leq \mathcal{C}u_\gamma(t),$$

while, for $t > 1$, we get

$$\left|\fint_{\substack{|x-t|>1\\ x\in\mathbb{R}^+}} \frac{u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| \leq \mathcal{C}u_\gamma(t)\begin{cases}\max\{1,\log t^{-1}\}, & p=0\\ t^{-p}, & p\geq 1\end{cases},$$

where $\mathcal{C} \neq \mathcal{C}(f,t)$.

Proof. We first consider the case $0 < t < 1$. We can write

$$\left|\fint_{\substack{|x-t|>1\\ x\in\mathbb{R}^+}} \frac{u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| = \left|\fint_0^{t+1} \frac{u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| \tag{19}$$

$$\begin{aligned} &\leq \left|\fint_0^{2t} \frac{u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| + \left|\fint_{2t}^{t+1} \frac{u_\gamma(x)e^{\mathrm{i}\omega x}}{(x-t)^{p+1}}\,dx\right| \\ &:= I_1(t) + I_2(t). \end{aligned} \tag{20}$$

Since

$$I_1(t) \le \left| \fint_0^{2t} \frac{u_\gamma(x)\cos(\omega x)}{(x-t)^{p+1}}\,dx \right| + \left| \fint_0^{2t} \frac{u_\gamma(x)\sin(\omega x)}{(x-t)^{p+1}}\,dx \right|,$$

denoting by $g(x)$ the function $\cos(\omega x)$ or $\sin(\omega x)$, we get

$$\begin{aligned}\left| \fint_0^{2t} \frac{u_\gamma(x)g(x)}{(x-t)^{p+1}}\,dx \right| \le{}& \left| \int_0^{2t} \frac{g(x)-\sum_{k=0}^{p}\frac{g^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} u_\gamma(x)\,dx \right| \\ &+ \sum_{k=0}^{p} \frac{|g^{(k)}(t)|}{k!} \left| \fint_0^{2t} \frac{u_\gamma(x)}{(x-t)^{p-k+1}}\,dx \right|.\end{aligned}$$

Proceeding as done in the proof of [12, Lemma 6.1], we obtain

$$\begin{aligned}\left| \fint_0^{2t} \frac{u_\gamma(x)g(x)}{(x-t)^{p+1}}\,dx \right| &\le \int_0^{2t} |g^{(p+1)}(\xi)| u_\gamma(x)dx + \sum_{k=0}^{p} \frac{|g^{(k)}(t)|}{k!} \fint_0^{2t} \frac{u_\gamma(x)}{(x-t)^{p-k+1}}\,dx \\ &\le \mathcal{C}\int_0^{2t} x^\gamma e^{-\frac{x}{2}}\,dx + \mathcal{C}\sum_{k=0}^{p} \fint_0^{2t} \frac{u_\gamma(x)}{(x-t)^{p-k+1}}\,dx \\ &\le \mathcal{C}t^{\gamma+1} + \mathcal{C}\sum_{k=0}^{p-1} t^{\gamma-p+k} \\ &\le \mathcal{C}t^\gamma + \mathcal{C}t^{\gamma-p} \le \mathcal{C}u_\gamma(t)t^{-p}.\end{aligned}$$

Consequently

$$I_1 \le \mathcal{C}u_\gamma(t)t^{-p}, \quad p \ge 0. \tag{21}$$

Moreover, for $p=0$ we have

$$I_2(t) \le \mathcal{C}t^\gamma \int_{2t}^{t+1} \frac{dx}{x-t} \le \mathcal{C}t^\gamma \log t^{-1} \tag{22}$$

and for $p \ge 1$ we get

$$I_2(t) \le \mathcal{C}u_\gamma(t)\int_{2t}^{t+1} \frac{dx}{(x-t)^{p+1}} \le \mathcal{C}(t^{\gamma-p} - t^\gamma) \le \mathcal{C}t^\gamma \le \mathcal{C}u_\gamma(t)t^{-p}. \tag{23}$$

Substituting (22)-(23) and (21) into (19), we get

$$\left| \fint_{\substack{|x-t|>1\\ x\in\mathbb{R}^+}} \frac{u_\gamma(x)e^{i\omega x}}{(x-t)^{p+1}}\,dx \right| \le \mathcal{C}u_\gamma(t) \begin{cases} \max\{1, \log t^{-1}\} & p=0 \\ t^{-p} & p \ge 1 \end{cases}$$

The lemma is proved for $0<t<1$. Concerning the case $t \ge 1$, we write

$$\begin{aligned}\left| \fint_{\substack{|x-t|>1\\ x\in\mathbb{R}^+}} \frac{u_\gamma(x)e^{i\omega x}}{(x-t)^{p+1}}\,dx \right| &= \left| \fint_{t-1}^{t+1} \frac{u_\gamma(x)e^{i\omega x}}{(x-t)^{p+1}}\,dx \right| \\ &\le \left| \fint_{t-1}^{t+1} \frac{u_\gamma(x)\cos(\omega x)}{(x-t)^{p+1}}\,dx \right| + \left| \fint_{t-1}^{t+1} \frac{u_\gamma(x)\sin(\omega x)}{(x-t)^{p+1}}\,dx \right|.\end{aligned}$$

Proceeding also in this case as done in the proof of [12, Lemma 6.1], we obtain

$$\begin{aligned}\left| \fint_{\substack{|x-t|>1\\ x\in\mathbb{R}^+}} \frac{u_\gamma(x)g(x)}{(x-t)^{p+1}}\,dx \right| \le{}& \left| \int_{t-1}^{t+1} \frac{g(x)-\sum_{k=0}^{p}\frac{g^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} u_\gamma(x)\,dx \right| \\ &+ \sum_{k=0}^{p} \frac{|g^{(k)}(t)|}{k!} \left| \fint_{t-1}^{t+1} \frac{u_\gamma(x)}{(x-t)^{p-k+1}}\,dx \right| \\ \le{}& \int_{t-1}^{t+1} |g^{(p+1)}(\xi)| u_\gamma(x)dx + \sum_{k=0}^{p} \frac{|g^{(k)}(t)|}{k!} \fint_{t-1}^{t+1} \frac{u_\gamma(x)}{(x-t)^{p-k+1}}\,dx \\ \le{}& \mathcal{C}\int_{t-1}^{t+1} x^\gamma e^{-\frac{x}{2}}\,dx + \mathcal{C}\sum_{k=0}^{p} \fint_{t-1}^{t+1} \frac{u_\gamma(x)}{(x-t)^{p-k+1}}\,dx \\ \le{}& \mathcal{C}u_\gamma(t).\end{aligned}$$

Then the lemma is also proved for $t \ge 1$. □

Now we can prove Theorem 2.2.

Proof of Theorem 2.2. For $p \geq 0$ we have

$$\begin{aligned} |H_p^{\omega,\gamma}(f,t)| &\leq \left| \int_{|x-t|\geq 1} \frac{f(x)u_\gamma(x)e^{i\omega x}}{(x-t)^{p+1}}\, dx \right| + \left| \int_{|x-t|<1} \frac{f(x)-\sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} u_\gamma(x)e^{i\omega x}\, dx \right| \\ &+ \sum_{k=0}^{p} \frac{|f^{(k)}(t)|}{k!} \left| \fint_{|x-t|<1} \frac{u_\gamma(x)e^{i\omega x}}{(x-t)^{p-k+1}}\, dx \right| \\ &=: A_1(t)+A_2(t)+A_3(t). \end{aligned} \tag{24}$$

Using Lemmas 6.1 and 6.2 we get

$$A_1(t) \leq \mathcal{C} \begin{cases} \|f u_{\gamma,\delta}\| & p=0 \\ \|f u_\gamma\| & p\geq 1 \end{cases}. \tag{25}$$

In order to estimate $A_3(t)$ we use Lemma 6.3 and [16, (17)]

$$\begin{aligned} A_3(t) &\leq \sum_{k=0}^{p-1} \frac{|f^{(k)}(t)|}{k!} \left| \int_{|x-t|<1} \frac{u_\gamma(x)e^{i\omega x}}{(x-t)^{p-k+1}}\, dx \right| + \frac{|f^{(p)}(t)|}{p!} \left| \int_{|x-t|<1} \frac{u_\gamma(x)e^{i\omega x}}{(x-t)}\, dx \right| \\ &\leq \mathcal{C} \sum_{k=0}^{p-1} |f^{(k)}(t)\varphi^k(t)u_\gamma(t)| t^{-p+k-\frac{k}{2}} + |f^{(p)}(t)\varphi^p(t)u_\gamma(t)| t^{-\frac{p}{2}} \max\{1, \log t^{-1}\} \\ &\leq \mathcal{C} \max\{t^{-\frac{p}{2}}, t^{-p}\} \sum_{k=0}^{p-1} \|f^{(k)}\varphi^k u_\gamma\| + \mathcal{C}\|f^{(p)}\varphi^p u_\gamma\| t^{-\frac{p}{2}} \max\{1, \log t^{-1}\} \\ &\leq \mathcal{C} \begin{cases} \max\{1, \log t^{-1}\} \|f u_{\gamma,\delta}\| & p=0 \\ \max\{t^{-p}, t^{-\frac{p}{2}}\} \|f\|_{W_p(u_\gamma)} & p\geq 1 \end{cases}. \end{aligned} \tag{26}$$

It remains to estimate $A_2(t)$. For $0<t<1$ we write

$$\begin{aligned} A_2(t) &\leq \left| \int_0^{2t} \frac{f(x)-\sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} u_\gamma(x)e^{i\omega x}\, dx \right| + \left| \int_{2t}^{t+1} \frac{f(x)u_\gamma(x)e^{i\omega x}}{(x-t)^{p+1}}\, dx \right| \\ &+ \left| \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \int_{2t}^{t+1} \frac{u_\gamma(x)e^{i\omega x}}{(x-t)^{p-k+1}}\, dx \right| \\ &:= A_2'(t)+A_2''(t)+A_2'''(t). \end{aligned}$$

Using (22)-(23) and [16, (17)] we get

$$A_2'''(t) \leq \mathcal{C} \begin{cases} \max\{1, \log t^{-1}\} \|f u_{\gamma,\delta}\| & p=0 \\ t^{-p}\|f\|_{W_p(u_\gamma)} & p\geq 1 \end{cases}.$$

Moreover,

$$A_2''(t) \leq \mathcal{C} \begin{cases} \log t^{-1} \|f u_{\gamma,\delta}\| & p=0 \\ t^{-p}\|f u_\gamma\| & p\geq 1 \end{cases}.$$

In order to estimate $A_2'(t)$ we use an argument in [16, Proof of Lemma 5.4], obtaining

$$A_2'(t) \leq \mathcal{C} \begin{cases} \displaystyle\int_0^1 \frac{\Omega_\varphi(f,y)_{u_{\gamma,\delta}}}{y}\, dy & p=0 \\ \displaystyle t^{-\frac{p}{2}} \int_0^1 \frac{\Omega_\varphi(f^{(p)},y)_{u_\gamma \varphi^p}}{y}\, dy & p\geq 1 \end{cases},$$

where $\Omega_\varphi(f,y)_u$ is the main part of the first φ−modulus of smoothness (see [16, p. 2527] for the details of the definition). Summing up, for $0<t<1$, we get

$$A_2(t) \leq \mathcal{C} \begin{cases} \displaystyle\int_0^1 \frac{\Omega_\varphi(f,y)_{u_{\gamma,\delta}}}{y}\, dy + \max\{1, \log t^{-1}\}\|f u_{\gamma,\delta}\|, & p=0, \\ \displaystyle t^{-\frac{p}{2}} \int_0^1 \frac{\Omega_\varphi(f^{(p)},y)_{u_\gamma \varphi^p}}{y}\, dy + t^{-p}\|f\|_{W_p(u_\gamma)}, & p\geq 1. \end{cases} \tag{27}$$

By similar arguments, for $t \geq 1$, we obtain

$$A_2(t) = \left| \int_{t-1}^{t+1} \frac{f(x) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} u_\gamma(x) e^{i\omega x}\, dx \right| \leq C \begin{cases} \displaystyle\int_0^1 \frac{\Omega_\varphi(f,y)_{u_{\gamma,\delta}}}{y}\, dy & p = 0 \\ t^{-\frac{p}{2}} \displaystyle\int_0^1 \frac{\Omega_\varphi(f^{(p)},y)_{u_\gamma \varphi^p}}{y}\, dy & p \geq 1 \end{cases}. \tag{28}$$

Finally, combining (25), (27)-(28) and (26) with (24), we obtain

$$\max\left\{1, \log^{-1} t^{-1}\right\} |H_0^{\omega,\gamma}(f,t)| \leq C \int_0^1 \frac{\Omega_\varphi(f,y)_{u_{\gamma,\delta}}}{y}\, dy + C\|f u_{\gamma,\delta}\|,$$

and

$$t^p |H_p^{\omega,\gamma}(f,t)| \leq C \int_0^1 \frac{\Omega_\varphi(f^{(p)},y)_{u_\gamma \varphi^p}}{y}\, dy + C\|f\|_{W_p(u_\gamma)}. \tag{29}$$

Taking into account that [31, pp. 175-176]

$$\Omega_\varphi^r(f,y)_u \leq C y^r \|f^{(r)} \varphi u\|, \quad \forall f \in W_r(u), \tag{30}$$

the thesis follows.

□

Finally, we prove Theorem 3.2. The proof of Theorem 3.1 is similar.

Proof of Theorem 3.2. We first prove (10). Using (29) we get

$$\begin{aligned} t^p |H_{p,m}^{\omega,\gamma}(f,t)| &= t^p |H_p^{\omega,\gamma}(L_{m+1}(w_\alpha,f),t)| \\ &\leq C \left(\int_0^1 \frac{\Omega_\varphi(L_{m+1}^{(p)}(w_\alpha,f),y)_{u_\gamma\varphi^p}}{y}\, dy + \|L_{m+1}(w_\alpha,f)\|_{W_p(u_\gamma)} \right) \\ &\leq C \left(\int_0^1 \frac{\Omega_\varphi(L_{m+1}^{(p)}(w_\alpha,f),y)_{u_\gamma\varphi^p}}{y}\, dy + \|L_{m+1}(w_\alpha,f)u_\gamma\| + \|[f - L_{m+1}(w_\alpha,f)]^{(p)}\varphi^p u_\gamma\| + \|f^{(p)}\varphi^p u_\gamma\| \right). \end{aligned}$$

Recalling [17, Lemma 6.2]

$$\int_0^1 \frac{\Omega_\varphi(L_{m+1}^{(p)}(w_\alpha,f),y)_{u_\gamma\varphi^p}}{y}\, dy \leq C\|f\|_{W_{p+1}(u_\gamma)} \log m,$$

and [30, (2.6)]

$$\|L_{m+1}(w_\alpha,f)u_\gamma\| \leq C\|f u_\gamma\| \log m,$$

and Theorem 2.1, the inequality (10) follows.
Concerning (11), by (29) we get

$$\begin{aligned} t^p |e_{p,m}(f,t)| &= t^p |H_p(f - L_{m+1}(w_{\alpha,f}),t)| \\ &\leq C \left(\int_0^1 \frac{\Omega_\varphi((f - L_{m+1}(w_\alpha,f))^{(p)},y)_{u_\gamma\varphi^p}}{y}\, dy + \|f - L_{m+1}(w_\alpha,f)\|_{W_p(u_\gamma)} \right) \\ &\leq C \left(\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi((f - L_{m+1}(w_\alpha),f))^{(p)},y)_{u_\gamma\varphi^p}}{y}\, dy + \int_{\frac{1}{\sqrt{m}}}^1 \frac{\Omega_\varphi((f - L_{m+1}(w_\alpha,f))^{(p)},y)_{u_\gamma\varphi^p}}{y}\, dy + \|f - L_{m+1}(w_\alpha,f)\|_{W_p(u_\gamma)} \right). \end{aligned}$$

Moreover, using [17, Lemma 6.1] we deduce

$$t^p |e_{p,m}(f,t)| \leq C \left(\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi^r(f^{(p)},y)_{u_\gamma\varphi^p}}{y}\, dy + \|(f - L_{m+1}(w_\alpha,f))^{(p)} u_\gamma \varphi^p\| \log m + \|(f - L_{m+1}(w_\alpha,f))u_\gamma\| \right).$$

Finally, applying (30) and Theorem 2.1 with $k = p$ and $k = 0$, the thesis follows. □

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