Pseudo-ovals of elliptic quadrics as Delsarte designs of association schemes

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Abstract

A *pseudo-oval* of a finite projective space over a finite field of odd order q is a configuration of equidimensional subspaces that is essentially equivalent to a translation generalised quadrangle of order (q^n, q^n) and a Laguerre plane of order q^n (for some n). In setting out a programme to construct new generalised quadrangles, Shult and Thas [17] asked whether there are pseudo-ovals consisting only of lines of an elliptic quadric $Q^-(5,q)$, non-equivalent to the *classical example*, a so-called *pseudo-conic*. To date, every known pseudo-oval of lines of $Q^-(5,q)$ is projectively equivalent to a pseudo-conic. Thas [18] characterised pseudo-conics as pseudo-ovals satisfying the *perspective* property, and this paper is on characterisations of pseudo-conics from an algebraic combinatorial point of view. In particular, we show that pseudo-ovals in $Q^-(5,q)$ and pseudo-conics can be characterised as certain Delsarte designs of an interesting five-class association scheme. These association schemes are introduced and explored, and we provide a complete theory of how pseudo-ovals of lines of $Q^-(5,q)$ can be analysed from this viewpoint.

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1 Introduction

A *pseudo-oval* of the finite projective space $\mathrm{PG}(3n-1,q)$ is a set of q^n+1 subspaces, each of dimension $n-1$, such that any three distinct elements of the set span the whole space. Such configurations are essentially *equivalent* to translation generalised quadrangles of order (q^n, q^n) [13]. For q odd, a pseudo-oval of $\mathrm{PG}(3n-1,q)$ is equivalent to a Laguerre plane of order q^n.

The *classical example* can be constructed in the following way. If we consider the $\mathrm{GF}(q^n)$-vector space underlying $\mathrm{PG}(2,q^n)$ as a $\mathrm{GF}(q)$-vector space, each point of $\mathrm{PG}(2,q^n)$ becomes an $(n-1)-$subspace of $\mathrm{PG}(3n-1,q)$. In particular, the q^n+1 subspaces corresponding to the points of a non-degenerate conic of $\mathrm{PG}(2,q^n)$ form a pseudo-oval, known as *pseudo-conic*. Thas [18] characterised pseudo-conics as pseudo-ovals of $\mathrm{PG}(3n-1,q)$, q odd, satisfying the *perspective* property (see Section 3 for more details).

Let q be odd. For n even any pseudo-conic belongs to an elliptic quadric $Q^-(3n-1,q)$, and for n odd any pseudo-conic belongs to a non-degenerate parabolic quadric of $\mathrm{PG}(3n-1,q)$ [17]. For q even a pseudo-oval is never contained in a non-degenerate quadric [18]. In the quest to construct new generalised quadrangles, Shult and Thas [17] asked whether there are pseudo-ovals consisting only of lines of an elliptic quadric $Q^-(5,q)$, non-equivalent to the classical example. To date, every known pseudo-oval of lines of $Q^-(5,q)$ is projectively equivalent to a pseudo-conic. Indeed, the discovery of a new pseudo-oval of $Q^-(5,q)$ would result in a new generalised quadrangle and new Laguerre plane.

Under the Klein correspondence, pseudo-ovals contained in $Q^-(5,q)$ are mapped onto *special sets* of $H(3,q^2)$ [16]. A *special set* of the Hermitian surface $H(3,q^2)$ is a set $\mathcal{S}$ of q^2+1 points such that any point of $H(3,q^2)$ not in $\mathcal{S}$ is orthogonal to 0 or 2 points of $\mathcal{S}$ [16]. From a result by De Soete and Thas [8], q is necessarily odd. Bader, O'Keefe, Penttila in [1] and, independently, Shult in [16] constructed an example of a special set of $H(3,q^2)$. This consists of the q^2+1 points of an elliptic quadric over $\mathrm{GF}(q)$ which is the complete intersection of $H(3,q^2)$ with a hyperbolic quadric of $\mathrm{PG}(3,q^2)$ whose polarity commutes with the given unitary one. The special sets in this class are called *of CP-type* [4]. Theorem 3.1 in [4] gives a characterisation of special sets of CP-type in terms of the unitary form defining $H(3,q^2)$. By [5, Theorem 2.1], a special set of CP-type corresponds to a pseudo-conic.

This paper is on characterisations of pseudo-conics from an algebraic combinatorial point of view. In particular, we will show (see Theorem 5.6) that pseudo-ovals and pseudo-conics in $Q^-(5,q)$ can be characterised as certain Delsarte designs of an interesting five-class association scheme. These association schemes are introduced and explored, and we provide a complete theory of how pseudo-ovals of lines of $Q^-(5,q)$ can be analysed from this viewpoint.

The paper is organised as follows. Section 2 contains some notation and introductory material on projective geometry, classical polar geometries and association schemes. In Section 3 we investigate the perspective property for lines of $Q^-(5,q)$. By representing subspaces of $\mathrm{PG}(5,q)$ in a matrix form, we give an algebraic characterisation of being in perspective for a triple of lines of $Q^-(5,q)$ (Proposition 3.1). This allows us to translate the above algebraic condition in terms of a (local) geometric property involving certain configurations arising from non-degenerate hyperplanes (Proposition 3.5). In Section 4 an imprimitive five-class association scheme is constructed on certain points of $H(3,q^2)$. The relations of the scheme are defined by considering a function, introduced by Shult in [16], associated with the hermitian form of $H(3,q^2)$. As a by-product, the study of its quotient scheme produces a strongly regular graph isomorphic to the bilinear forms graph $\mathrm{Bil}_2(q)$ (Proposition 4.10). Section 5 is the real core of the whole paper: the opening theorem, providing the link between Shult's function and the property to be in perspective for the lines of $Q^-(5,q)$, allows us to consider pseudo-ovals of $Q^-(5,q)$, as well as pseudo-conics, as subsets of a five-class association scheme on certain lines of $Q^-(5,q)$, isomorphic to the scheme explored in Section 4. In this setting, from comparing the characteristic vector of a pseudo-oval with the common eigenspaces of the scheme, we provide a characterisation of pseudo-conics in terms of the configurations introduced in Section 3.1. Finally, Section 6 contains some computational results leading to the conjecture that every pseudo-oval in $Q^-(5,q)$ is a pseudo-conic, for all q odd.

2 Background theory

For any given $n-$dimensional vector space $V = V(n,F)$ over the field F, the *projective geometry* defined by V is the partially ordered set of all subspaces of V, and it will be denoted by $\mathrm{PG}(V)$. Two elements of $\mathrm{PG}(V)$ are said to be *disjoint* or *skew* if they intersect in the zero vector. In order to simplify notation, for each proper subspace U of V, that is an element of $\mathrm{PG}(V)$, we will use the same letter for the projective geometry defined by U. If $S \subset V$, we use $\langle S \rangle$ to denote the subspace spanned by S.

If F is the finite field $\mathrm{GF}(q)$ with q elements, then we may write $V = V(n,q)$ and $\mathrm{PG}(n-1,q)$ instead of $\mathrm{PG}(V)$. The 1-dimensional subspaces are called *points*, the 2-dimensional subspaces are called *lines*, the 3-dimensional subspaces are called *planes*, and the $(n-1)-$dimensional subspaces are called *hyperplanes* of $\mathrm{PG}(V)$. If V is endowed with a non-degenerate alternating, quadratic or Hermitian form of Witt index m, the set of totally isotropic (or totally singular, in case of a quadratic form) subspaces of V is a *polar geometry of rank* m of $\mathrm{PG}(V)$, which is called *symplectic, orthogonal* or *unitary*, respectively. When $n = 2r$, the vector space V has precisely two (non-degenerate) quadratic forms, and they differ by their Witt index. It can be $r-1$ or r, and the quadratic form is *elliptic* or *hyperbolic*, respectively. It is customary to set $\mathrm{sgn}(Q) = -$ in the former case, and $\mathrm{sgn}(Q) = +$ in the latter. In terms of the associated projective geometry $\mathrm{PG}(V)$, the orthogonal polar geometry arising from an elliptic (resp. hyperbolic) quadratic form is known as an *elliptic* (resp. *hyperbolic*) *quadric* of $\mathrm{PG}(V)$, and it is denoted by $Q^-(n-1,q)$ (resp. $Q^+(n-1,q)$). Our principal reference on projective geometries and polar geometries is [19].

Association schemes are important objects in algebraic combinatorics that generalise distance-regular graphs, linear codes, and combinatorial designs. As we shall see, the theory of association schemes can be a powerful tool when applied to some problems in finite geometry. An association scheme $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ is a set of *vertices* X and binary *relations* R_i on X satisfying the following:

1. R_0 is the diagonal relation, that is, $R_0 = \{(x,x) : x \in X\}$.

2. $\{R_i\}$ is closed under taking the opposite relation; that is, $R_j^* := \{(x,y)\colon (y,x) \in R_j\}$ is in $\{R_i\}$, for each j.

3. For each $i,j,k \in \{0,\ldots,d\}$, there exist constants $p_{i,j}^k$, such that if $(x,y) \in R_k$, then there are $p_{i,j}^k$ vertices z such that $(x,z) \in R_i$ and $(z,y) \in R_j$. The $p_{i,j}^k$ are called *intersection numbers*.

We will say that the association scheme is *symmetric* if each relation is equal to its opposite. Let $\mathfrak{X} = (X, \{R_i\}_{0\leqslant i\leqslant d})$ be an association scheme with d classes. For $0 \leqslant i \leqslant d$, let A_i be the adjacency matrix of the relation R_i, and E_i the $i-$th primitive idempotent of the Bose-Mesner algebra of $\mathfrak{X}$ which projects onto the $i-$th maximal common eigenspace of $A_0,\ldots,A_d$. The matrices $\mathcal{P}$ and $\mathcal{Q}$ defined by

$$(A_0\ A_1\ \ldots\ A_d) = (E_0\ E_1\ \ldots\ E_d)\mathcal{P}$$

and

$$(E_0\ E_1\ \ldots\ E_d) = |X|^{-1}(A_0\ A_1\ \ldots\ A_d)\mathcal{Q}$$

are the *first* and the *second eigenmatrix* of $\mathfrak{X}$, respectively. The reader is referred to [2, 3, 7] for additional information on association schemes.

3 Investigating the perspective property

In $V = V(6,q^2)$ consider the 6-dimensional $\mathrm{GF}(q)-$subspace

$$\widehat{V} = \{(x,x^q,y,y^q,z,z^q) : x,y,z \in \mathrm{GF}(q^2)\}.$$

Let $\mathrm{PG}(\widehat{V})$ be the projective geometry defined by $\widehat{V}$. For any vector $(x,x^q,y,y^q,z,z^q) \in \widehat{V}$ we will use the short-hand notation $(x,y,z)_2$.

We consider the hyperbolic quadric $Q^+(5,q^2)$ of $\mathrm{PG}(5,q^2)$ (known as *Klein quadric*) defined by the (non-degenerate) quadratic form $Q(\mathbf{X}) = -X_1X_6 - X_2X_5 + X_3X_4$ on $V(6,q^2)$. For any given $v = (x,y,z)_2 \in \widehat{V}$,

$$\widehat{Q}(v) = Q|_{\widehat{V}}(v) = -xz^q - x^qz + y^{q+1}. \tag{1}$$

It turns out that $\widehat{Q}$ is a non-degenerate quadratic form of rank 2 on $\widehat{V}$ with associated symmetric form

$$\widehat{\mathbf{b}}(v,v') = -xz'^q - x^qz' + yy'^q + y^qy' - zx'^q - z^qx'.$$

Therefore, $\widehat{Q}$ gives rise to an elliptic quadric $Q^-(5,q)$ of $\mathrm{PG}(\widehat{V})$ embedded in $Q^+(5,q^2)$. For any subspace W of $\widehat{V}$, set

$$W^\perp = \{v \in \widehat{V} : \widehat{\mathbf{b}}(v,u) = 0,\ \text{for all } u \in W\}.$$

In the following, $\mathrm{Tr}_{q^2/q}$ and $\mathrm{N}_{q^2/q}$ will denote the *relative trace* and *norm* functions from $\mathrm{GF}(q^2)$ onto $\mathrm{GF}(q)$. Let F be a $\mathrm{GF}(q)-$linear transformation from $\mathrm{GF}(q^2)$ to itself. Then, F can be represented by a unique polynomial over $\mathrm{GF}(q^2)$ of type $F(x) = ax + bx^q$. Such a polynomial is called a *$q-$polynomial over* $\mathrm{GF}(q^2)$ [12, Chapter 3]. The trivial

$q-$polynomial will be denoted by I. The *adjoint* of a linearised polynomial $F(x) = ax + bx^q$, with respect to the symmetric bilinear form $(a, b) \to \mathrm{Tr}_{q^2/q}(ab)$, is given by $F^*(x) = ax + b^q x^q$.

In $\widehat{V}$, any line l is written as

$$l = \{(F_0(x), F_1(x), F_2(x))_2 : x \in \mathrm{GF}(q^2)\}$$

where F_0, F_1, F_2 are $q-$polynomials over $\mathrm{GF}(q^2)$; for short, we will write $l = L(F_0, F_1, F_2)$. The triple (F_0, F_1, F_2) is determined by l up to a right factor of proportion, which is a non-singular $q-$polynomial. Since a 4-dimensional subspace of $\widehat{V}$ is a 2-dimensional subspace in the dual space $\widehat{V}^*$, any such a subspace T can be represented by three $q-$polynomials H_0, H_1, H_2 over $\mathrm{GF}(q^2)$. A way to write equations for T is the following. Fix an element $\theta \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$. Let $\mathcal{H}_i$ be the 2×2 Dickson matrix[1] associated with H_i, $i = 1, 2, 3$, then T has equations

$$\begin{pmatrix} 1 & 1 \\ \theta & \theta^q \end{pmatrix} \begin{pmatrix} \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 \end{pmatrix} \begin{pmatrix} x \\ x^q \\ y \\ y^q \\ z \\ z^q \end{pmatrix} = 0 \ .$$

For short, write $T = \pi(H_0, H_1, H_2)$. The triple (H_0, H_1, H_2) is determined by T up to a left factor of proportion, which is a non-singular $q-$polynomial. It is easy to check that a line $L(F_0, F_1, F_2)$ is contained in the subspace $\pi(H_0, H_1, H_2)$ if and only if

$$H_0 \circ F_0 + H_1 \circ F_1 + H_1 \circ F_1 = 0;$$

where $H \circ F$ is the $q-$polynomial $H(F(x)) \bmod (x^{q^2} - x)$.

The line $l = L(F_0, F_1, F_2)$ is totally singular with respect to the symmetric form $\widehat{\mathbf{b}}$ if and only if

$$F_2^* \circ K \circ F_0 - F_1^* \circ K \circ F_1 + F_0^* \circ K \circ F_2 = 0, \tag{2}$$

where $K(x) = x^q$ (note that $K^* = K$). Let $F_i(x) = f_i x + g_i x^q$, $i = 0, 1, 2$. Then, Eq. (2) is equivalent to

$$\begin{cases} f_2 g_0^q + f_0 g_2^q & = & f_1 g_1^q \\ f_0 f_2^q + f_0^q f_2 + g_0 g_2^q + g_0^q g_2 & = & f_1^{q+1} + g_1^{q+1} \end{cases} . \tag{3}$$

Let l_1, l_2, l_3 be mutually skew lines of $\mathrm{PG}(\widehat{V})$, T_i be a 4-dimensional space containing l_i but skew to l_j and l_k, and $s_k = T_i \cap T_j$, with $\{i, j, k\} = \{1, 2, 3\}$. The space spanned by s_i and l_i will be denoted by Σ_i, with $i = 1, 2, 3$. If Σ_1, Σ_2 and Σ_3 have non-trivial intersection, then $\{l_1, l_2, l_3\}$ and $\{T_1, T_2, T_3\}$ are said to be *in semi-perspective*; if Σ_1, Σ_2 and Σ_3 share a line, then $\{l_1, l_2, l_3\}$ and $\{T_1, T_2, T_3\}$ are said to be *in perspective*. For our aims, if l_1, l_2, l_3 are lines of $Q^-(5, q)$, we set $T_i = l_i^\perp$ and we will simply say that l_1, l_2, l_3 are in semi-perspective or perspective.

[1] The *Dickson matrix* of the $q-$polynomial $\sum_{i=0}^{n-1} a_i x^{q^i} \in \mathrm{GF}(q^n)[x]$ is $\begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1}^q & a_0^q & \cdots & a_{n-2}^q \\ \vdots & \vdots & \vdots & \vdots \\ a_1^{q^{n-1}} & a_2^{q^{n-1}} & \cdots & a_0^{q^{n-1}} \end{pmatrix}$.

Since for q even pseudo-ovals are never contained in an orthogonal polar geometry [18], from now on we assume q is odd.

Let $\theta \in \mathrm{GF}(q^2) \setminus \mathrm{GF}(q)$ be taken such that $\theta^2 = \xi$ with ξ a non-square in $\mathrm{GF}(q)$, i.e., $\theta^q = -\theta$.

The following result translates [18, Theorem 5.1] in terms of the projective geometry $\mathrm{PG}(\widehat{V})$.

Proposition 3.1. *Consider the three lines of the $Q^-(5,q)$, arising from $\widehat{Q}$,*

$$l = L(I,0,0), \quad m = L(0,0,I), \quad n = L(F_0,F_1,F_2),$$

with $F_i(x) = f_i x + g_i x^q$, $i = 0,1,2$, spanning the whole space. Then, l,m,n are in perspective if and only if $f_0^q f_2 + g_0 g_2^q \in \mathrm{GF}(q)$.

Proof. As $\langle l,m\rangle = \{(x,0,z)_2 : x,z \in \mathrm{GF}(q^2)\}$ and n trivially intersects $\langle l,m\rangle$, then F_1 is invertible. We set $T_1 = l^\perp$, $T_2 = m^\perp$, $T_3 = n^\perp$. Straightforward calculation yields

$$T_1 = \pi(0,0,I), \quad T_2 = \pi(I,0,0), \quad T_3 = \pi(F_2^* \circ K, -F_1^* \circ K, F_0^* \circ K).$$

Further,

$$\begin{aligned} s_3 &= T_1 \cap T_2 = L(0,I,0),\\ s_2 &= T_1 \cap T_3 = L(I,(K\circ F_2 \circ F_1^{-1} \circ K)^*,0),\\ s_1 &= T_2 \cap T_3 = L(0,(K\circ F_0 \circ F_1^{-1} \circ K)^*,I). \end{aligned}$$

Now we want to write $\Sigma_1 = \langle l, s_1\rangle$ and $\Sigma_2 = \langle m, s_2\rangle$ in the form $\pi(H_0,H_1,H_2)$. To do this, we solve the linear system

$$\begin{pmatrix} 1 & 1 \\ \theta & \theta^q \end{pmatrix} \begin{pmatrix} \mathcal{H}_0 & \mathcal{H}_1 & \mathcal{H}_2 \end{pmatrix} \begin{pmatrix} x \\ x^q \\ y \\ y^q \\ z \\ z^q \end{pmatrix} = 0 \ ,$$

where, in turn, we substitute in the coordinates of four linearly independent vectors of Σ_i, $i = 1,2$. Consequently,

$$\Sigma_1 = \pi(0,I,-(K\circ F_0\circ F_1^{-1}\circ K)^*), \quad \Sigma_2 = \pi(-(K\circ F_2 \circ F_1^{-1}\circ K)^*, I, 0).$$

Since $\Sigma_3 = \langle n, s_3\rangle = \{(F_0(x), y, F_2(x))_2 : x,y \in \mathrm{GF}(q^2)\}$, by imposing that the generic point of Σ_3 belongs to Σ_1 as to Σ_2, the points of $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ are obtained by solving the following system of linear equations

$$\begin{cases} y - (F_2^* \circ K \circ F_0 \circ F_1^{-1} \circ K)^*(x) = 0 \\ y - (F_0^* \circ K \circ F_2 \circ F_1^{-1} \circ K)^*(x) = 0. \end{cases}$$

This yields

$$(F_2^* \circ K \circ F_0 - F_0^* \circ K \circ F_2)(x) = 0.$$

From Eq. (2), we get

$$(2F_2^* \circ K \circ F_0 - F_1^* \circ K \circ F_1)(x) = 0. \tag{4}$$

Note that $(F_2^* \circ K \circ F_0)(x) = (f_0 g_2^q + f_2 g_0^q)x + (f_0^q f_2 + g_0 g_2^q)x^q$ and $(F_1^* \circ K \circ F_1)(x) = 2f_1 g_1^q x + (f_1^{q+1} + g_1^{q+1})x^q$. Therefore,

$$\begin{aligned}(2F_2^* \circ K \circ F_0 - F_1^* \circ K \circ F_1)(x) &= 2(f_0 g_2^q + f_2 g_0^q - f_1 g_1^q)x + [2(f_0^q f_2 + g_0 g_2^q) - f_1^{q+1} - g_1^{q+1}]x^q \\ &= (f_0^q f_2 - f_0 f_2^q + g_0 g_2^q - g_0^q g_2)x^q,\end{aligned}$$

by Eq. (3). Consequently, $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3 = \{0\}$ if and only if 0 is the unique solution of (4) if and only if $f_0^q f_2 + g_0 g_2^q \notin \mathrm{GF}(q)$. Similarly, $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ is a line if and only if $f_0^q f_2 + g_0 g_2^q \in \mathrm{GF}(q)$. $\square$

Remark 3.2. In the proof of the previous result we used that F_1 is invertible. This property holds also for F_0 and F_2, because n trivially intersects m and l.

3.1 Construction of the subsets of type $\mathcal{U}_{p_1,p_2}$

As above, let $Q^-(5,q)$ be the elliptic quadric of $\mathrm{PG}(\widehat{V})$ defined by $\widehat{Q}$, and fix a totally singular line l. For any given non-degenerate hyperplane Π not containing l, let $B = l \cap \Pi$ and σ be the line $\langle l, \Pi^\perp \rangle \cap \Pi$. As $l \subset B^\perp$ and $B \in \Pi$ then $\langle l, \Pi^\perp \rangle \subset B^\perp$, hence $\sigma \subset B^\perp \cap \Pi$. In particular, σ corresponds to an internal point for the non-singular conic $(B^\perp \cap \Pi \cap Q^-(5,q))/B$ of the quotient space $(B^\perp \cap \Pi)/B$. To see this we observe that $\sigma^\perp = \langle l^\perp \cap \Pi, \Pi^\perp \rangle$ shares with the quadratic cone $B^\perp \cap \Pi \cap Q^-(5,q)$ just the point B. Therefore, if σ corresponded to an external point, $\sigma^\perp$ would have two generators in common with the cone, which is a contradiction. Then, for any given totally singular line p_1 lying in Π and passing through B, the plane $\langle p_1, \sigma \rangle$ meets $Q^-(5,q)$ in a further line p_2. Let $\mathcal{O}_i$, $i = 1,2$, be the totally singular lines in Π intersecting p_i, but not at B. We set $\mathcal{U}_{p_1,p_2} = \mathcal{O}_1 \cup \mathcal{O}_2$, and $\mathcal{U}_{p_1,p_2}$ is said to be *constructed on the flag* (B,l).

By the reasoning above, it is now evident that the line σ induces an involution $\tilde{\sigma}$ on the generators of the quadratic cone $B^\perp \cap \Pi \cap Q^-(5,q)$.

Lemma 3.3. *Let* $B = \langle (1,0,0)_2 \rangle$ *and* $l = L(I,0,0)$.

(i) A non-degenerate hyperplane Π *through* B *not containing* l *has equation*

$$\Pi : \theta(X - X^q) + \beta^q Y + \beta Y^q - \alpha^q Z - \alpha Z^q = 0$$

for all $\alpha, \beta \in \mathrm{GF}(q^2)$, *such that* $\beta^{q+1} - \theta(\alpha^q - \alpha) \neq 0$;

(ii) the generators of the quadratic cone $B^\perp \cap \Pi \cap Q^-(5,q)$ *have the form* $l_y = L(F_0, F_1, F_2)$, *where*

$$F_0(x) = (2\xi - y^{q+1})x + (2\xi + y^{q+1})x^q, \quad F_1(x) = 2\theta y(x - x^q), \quad F_2(x) = 2\xi(x - x^q),$$

for all $y \in \mathrm{GF}(q^2)$ *such that*

$$y^{q+1} - (\beta^q y + \beta y^q) + \theta(\alpha^q - \alpha) = 0. \tag{5}$$

(iii) the involution induced by $\sigma = \langle l, \Pi^\perp \rangle \cap \Pi$ on the l_y's is

$$l_y \overset{\tilde{\sigma}}{\longmapsto} l_{2\beta - y}\,.$$

Proof. (i) Under the polarity $\perp$ of $\mathrm{PG}(\widehat{V})$ associated with $\widehat{\mathbf{b}}$, a non-degenerate hyperplane Π not containing l corresponds to a non-singular point $P \in B^\perp \setminus l^\perp$. Such a point has the form $P = \langle (\alpha, \beta, \theta)_2 \rangle$, with $\alpha, \beta \in \mathrm{GF}(q^2)$, such that $\beta^{q+1} - \theta(\alpha^q - \alpha) \neq 0$.

(ii) In order to find the totally singular lines through B, we consider the restriction of $\widehat{Q}$ on the 4-dimensional subspace $\Sigma = \{(\theta a, y, \theta b)_2 : a, b \in \mathrm{GF}(q), y \in \mathrm{GF}(q^2)\}$ of $B^\perp$ not on B. We get that these totally singular lines, apart from l, have the form $l_y = L(F_0, F_1, F_2)$, where

$$F_0(x) = (2\xi - y^{q+1})x + (2\xi + y^{q+1})x^q, \quad F_1(x) = 2\theta y(x - x^q), \quad F_2(x) = 2\xi(x - x^q),$$

for all $y \in \mathrm{GF}(q^2)$. In particular, l_y is in Π if and only if Eq. (5) holds.

(iii) By definition, $\sigma = \langle l, \Pi^\perp \rangle \cap \Pi$ is $L(F_0, F_1, F_2)$ where

$$F_0(x) = \left(\frac{2\beta^{q+1}}{\theta} + \alpha - 2\alpha^q \right) x - \alpha x^q, \quad F_1(x) = -\beta(x + x^q), \quad F_2(x) = \theta(x + x^q),$$

Fix the points $R = \langle (\alpha^q - \frac{\beta^{q+1}}{\theta}, \beta, \theta)_2 \rangle$ of σ and $R_1 = \langle (-\theta y_1^{q+1}, 2\xi y_1, 2\xi\theta)_2 \rangle$ of $p_1 = l_{y_1}$ with y_1 satisfying Eq. (5). Since σ corresponds to an internal point for the non-singular conic $(B^\perp \cap \Pi \cap Q^-(5, q))/B$ of the quotient space $(B^\perp \cap \Pi)/B$, the line p_2 is the unique totally singular line l_y, $y \neq y_1$, intersecting the line $\langle R, R_1 \rangle$. Thus, we are required to determine the triples $(y, x, \lambda) \in \mathrm{GF}(q^2) \times \mathrm{GF}(q^2) \times \mathrm{GF}(q)^*$, satisfying the system

$$\left\{ \begin{array}{rcl} (2\xi - y^{q+1})x + (2\xi + y^{q+1})x^q & = & (\alpha^q - \frac{\beta^{q+1}}{\theta}) - \theta y_1^{q+1}\lambda \\ 2\theta y(x - x^q) & = & \beta + 2\xi y_1 \lambda \\ 2\xi(x - x^q) & = & \theta + 2\xi\theta\lambda \end{array} \right. , \qquad (6)$$

together with the condition Eq. (5). By plugging $x = x_0 + \theta x_1$, $\alpha = a_0 + \theta a_1$, $x_i, a_i \in \mathrm{GF}(q)$, into (6) (note that $x_1 \neq 0$ otherwise the intersection point would coincide with B), we rewrite (6) in the equivalent form

$$\left\{ \begin{array}{rcl} 4\xi x_0 & = & a_0 \\ 2\xi x_1 y^{q+1} & = & a_1 \xi + \beta^{q+1} + \xi y_1^{q+1} \lambda \\ 4\xi x_1 y & = & \beta + 2\xi y_1 \lambda \\ 4\xi x_1 & = & 1 + 2\xi\lambda \end{array} \right. . \qquad (7)$$

Hence,

$$x = \frac{a_0}{4\xi} + \theta \frac{\beta - y_1}{4\xi(y - y_1)}, \qquad \lambda = \frac{\beta - y}{2\xi(y - y_1)}.$$

By using Eq. (5) in the second equation of (7), we come to

$$\beta(\beta^q(y - y_1) - \beta(y^q - y_1^q) + (y^q y_1 - y y_1^q)) = 0.$$

Assume $\beta = 0$. From (5), it follows that $y = y_1 c$, for some $c \neq 1$ with $N(c) = 1$. As $x_1 = -1/(4\xi(c - 1)) \in \mathrm{GF}(q)$, $c \in \mathrm{GF}(q)$ with $c^{q+1} = c^2 = 1$, that is, $c = -1$. Assume

$\beta^q(y-y_1)-\beta(y^q-y_1^q)+(y^qy_1-yy_1^q)=0$, i.e., $(\beta^q-y_1^q)(y-y_1))-(\beta-y_1)(y^q-y_1^q)=0$, where $\beta^q-y_1^q\neq 0$ (as $\Pi^\perp$ is non-singular). Then,

$$y=\frac{a+(\beta^q-y_1^q)y_1}{\beta^q-y_1^q} \tag{8}$$

for some $a\in \mathrm{GF}(q)^*$. Substituting (8) into (5) yields $a=2(\beta-y_1)^{q+1}$, whence $y=2\beta-y_1$. This concludes the proof. □

Remark 3.4. The line $(B^\perp\cap\Pi)^\perp$ contains precisely q non-singular points, one of which is $\Pi^\perp$. Under the polarity defined by $Q^-(5,q)$, the corresponding hyperplanes share $B^\perp\cap\Pi$. For any such a hyperplane H, the line $\langle l,H^\perp\rangle\cap H$ coincides with the line σ constructed from Π [2]. Therefore, these hyperplanes define the same involution on the generators of the quadratic cone $B^\perp\cap\Pi\cap Q^-(5,q)$.

Proposition 3.5. *Let l_1,l_2,l_3 be three distinct lines of $Q^-(5,q)$ spanning the whole space. Then, l_1,l_2,l_3 are in perspective if and only if, for some flag (B,l_i), the totally singular lines through B concurrent with l_j and l_k, $i\neq j\neq k\neq i$, correspond under the map $\tilde\sigma$ defined by the hyperplane containing B, l_j and l_k.*

Proof. Fix $B\in l_i$. By [19, Theorem 10.12], up to the Klein correspondence, we may choose coordinates such that $l_i=l=L(I,0,0)$, $l_j=m=L(0,0,I)$ and $l_k=n=L(F_0,F_1,I)$, with $B=\langle(1,0,0)_2\rangle$. Let Π be the (non-degenerate) hyperplane spanned by B, m and n. Let p_1 and p_2 be the two totally singular lines on B concurrent with m and n, respectively. Since $m^\perp=\pi(I,0,0)$, by Lemma 3.3(i) Π has an equation of the form

$$\Pi:\theta(X-X^q)+\beta^qY+\beta Y^q=0,$$

for some $\beta\in\mathrm{GF}(q^2)^*$.

Since the line p_1 has the form given by Lemma 3.3(ii), $p_1=l_0$. Lemma 3.3(ii) and (iii) imply $p_1^{\tilde\sigma}=l_{2\beta}=L(G_0,G_1,G_2)$, with

$$G_0(x)=(2\xi-4\beta^{q+1})x+(2\xi+4\beta^{q+1})x^q,\quad G_1(x)=4\theta\beta(x-x^q),\quad G_2(x)=2\xi(x-x^q).$$

By Remark 3.2, we may assume $F_2=I$, that is, $n=L(F_0,F_1,I)$, with $F_0(x)=f_0x+g_0x^q$, $F_1(x)=f_1x+g_1x^q$. The condition that n belongs to Π is equivalent to have

$$\beta g_1^q+\beta^qf_1+\theta(f_0-g_0^q)=0. \tag{9}$$

Therefore, n is concurrent with $l_{2\beta}$ if and only if there exist $x,\bar x\in\mathrm{GF}(q^2)^*$ such that

$$\left\{\begin{array}{rcl} f_0x+g_0x^q &=& (2\xi-4\beta^{q+1})\bar x+(2\xi+4\beta^{q+1})\bar x^q\\ f_1x+g_1x^q &=& 4\theta\beta(\bar x-\bar x^q)\\ x &=& 2\xi(\bar x-\bar x^q)\end{array}\right. . \tag{10}$$

Write $\bar x=\bar x_0+\theta\bar x_1$, $\bar x_i\in\mathrm{GF}(q)$. Then, $x=4\xi\theta\bar x_1\neq 0$.

[2]To see this, just note that these hyperplanes are the "perp" of the non-singular points on the line $\langle B,\Pi^\perp\rangle$, and have the form $\langle(\lambda+\alpha,\beta,\theta)_2\rangle$, for all $\lambda\in\mathrm{GF}(q)$. Straightforward calculations show that the corresponding line $\sigma_\lambda=\langle l,H^\perp\rangle\cap H$ coincides with σ.

From the second equation of (10), we get $2\beta = \theta(f_1 - g_1)$. This, together with Eq. (9), yields

$$2f_1g_1^q + 2(f_0 - g_0^q) - (f_1^{q+1} + g_1^{q+1}) = 0. \tag{11}$$

The equations (3) with $f_2 = 1$ and $g_2 = 0$, applied to (11), give $f_0 \in \mathrm{GF}(q)$, and Proposition 3.1 leads to the result. □

Remark 3.6. Note that if Proposition 3.5 holds for one point $B \in l_i$, then it holds for all points of l_i.

4 A five-class association scheme on $H(3, q^2)$

Let $V = V(4, q^2)$ equipped with a non-degenerate Hermitian form $h : V \times V \to \mathrm{GF}(q^2)$. As usual, $H(3, q^2)$ denotes the unitary polar geometry of rank 2 defined by h, and it is called a *Hermitian surface* of $\mathrm{PG}(3, q^2)$. A *point* (resp. *line*) of $H(3, q^2)$ is a 1-dimensional (resp. 2-dimensional) subspace in $H(3, q^2)$, that is, totally isotropic with respect to h. A pair of vectors $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}$ and $\mathbf{y}$ are isotropic with $h(\mathbf{x}, \mathbf{y}) = 1$ is called a *hyperbolic pair*; in this case, $\langle \mathbf{x}, \mathbf{y} \rangle$ in $\mathrm{PG}(3, q^2)$ is said to be a *hyperbolic line*. Any hyperbolic line intersects $H(3, q^2)$ in $q + 1$ points. Two distinct points $P = \langle \mathbf{p} \rangle$ and $Q = \langle \mathbf{q} \rangle$ of $H(3, q^2)$ are said to be *orthogonal* or *collinear* if $h(\mathbf{p}, \mathbf{q}) = 0$; in other words, they span a totally isotropic line.

Since all non-degenerate Hermitian forms on V are isometric, we may take an ordered basis $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ for V such that

$$h(\mathbf{x}, \mathbf{y}) = x_0y_3^q - x_1y_1^q - x_2y_2^q + x_3y_0^q, \tag{12}$$

where $\mathbf{x} = x_0\mathbf{v}_0 + x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3$ and $\mathbf{y} = y_0\mathbf{v}_0 + y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + y_3\mathbf{v}_3$.

In [16], Shult introduced the following function on $H(3, q^2)$. For any three distinct points $P = \langle \mathbf{p} \rangle$, $Q = \langle \mathbf{q} \rangle$ and $R = \langle \mathbf{r} \rangle$ of $H(3, q^2)$, let

$$z(P, Q, R) = h(\mathbf{p}, \mathbf{q})h(\mathbf{q}, \mathbf{r})h(\mathbf{r}, \mathbf{p})\mathrm{GF}(q)^*,$$

where $\mathrm{GF}(q)^*$ denotes the multiplicative group of non-zero elements of $\mathrm{GF}(q)$. Then, $z(P, Q, R)$ is well-defined and

$$\begin{aligned} z(P, Q, R) &= z(R, Q, P) = z(Q, P, R); \\ z(P, Q, R) &= z(Q, P, R)^q. \end{aligned}$$

In the multiplicative group $T = \mathrm{GF}(q^2)^*/\mathrm{GF}(q)^* \simeq Z_{(q+1)}$, with identity $e = \mathrm{GF}(q)^*$, the element $t = \theta\mathrm{GF}(q)^*$ is the unique involution.

Lemma 4.1 ([16]). *Let P, Q, R be three pairwise non-collinear points of $H(3, q^2)$. Then, the span of P, Q, R is a degenerate plane if and only if $z(P, Q, R) = t$.*

Let $\Gamma = T \setminus \{e, t\}$. Fix a point P of $H(3, q^2)$ and consider the set $\mathcal{X}$ of all the points of $H(3, q^2)$ that are not collinear with P. On the set $\mathcal{X}$, which consists of q^5 points, we define the following relations:

$$R_1 = \{(Q, R) : z(P, Q, R) = 0\},$$

$$R_2 = \{(Q,R) : \langle P,Q,R\rangle \text{ is a (hyperbolic) line}\},$$

$$R_3 = \{(Q,R) : z(P,Q,R) = t\},$$

$$R_4 = \{(Q,R) : z(P,Q,R) \in \Gamma\},$$

$$R_5 = \{(Q,R) : z(P,Q,R) = e\}.$$

Note that $(Q,R) \in R_1$ if and only if Q is collinear with R and $(Q,R) \in R_3$ if and only if P,Q,R span a degenerate plane (see Lemma 4.1).

Set $\mathcal{R} = \{R_0, R_1, \ldots, R_5\}$, where R_0 is the diagonal relation. We are going to prove that $\mathfrak{X}_P = (\mathcal{X}, \mathcal{R})$ is a symmetric, hence commutative, imprimitive association scheme. Clearly all the above relations are symmetric. We now show that all of the intersection numbers p_{ij}^k are well defined. Note that if p_{ij}^k is well defined then so too is $p_{ji}^k = p_{ij}^k$. We will be aided by the fact that the projective unitary group $\mathrm{PGU}(4,q^2)$ is transitive on the set of pairs of non-collinear points of $H(3,q^2)$ [9, Corollary 11.12]. Thus, in the computations of the parameters we may assume $P = \langle(0,0,0,1)\rangle$, $Q = \langle(1,0,0,0)\rangle$, and $R = \langle(1,r_1,r_2,r_3)\rangle$, with $r_1^{q+1}+r_2^{q+1} = r_3+r_3^q$, since $R \in \mathcal{X}$. Note that $z(P,Q,R) = h(\mathbf{q},\mathbf{r})\mathrm{GF}(q)^* = r_3^q\mathrm{GF}(q)^*$.

Lemma 4.2. *The valencies $\eta_k = p_{kk}^0$ are as follows: $\eta_1 = (q^2-1)(q+1)$, $\eta_2 = q-1$, $\eta_3 = (q^2-1)^2$, $\eta_4 = (q^3-q)(q-1)^2$, $\eta_5 = (q^3-q)(q-1)$.*

Proof. We calculate $\eta_1, \eta_2, \eta_3, \eta_5$ directly, obtaining η_4 by subtraction. First,

$$\begin{aligned}
\eta_1 &= |\{R \in \mathcal{X} : (Q,R) \in R_1\}| \\
&= |\{R \in \mathcal{X} : r_3 = 0\}| \\
&= |\{(1,r_1,r_2,0) : r_1^{q+1} + r_2^{q+1} = 0\}|.
\end{aligned}$$

Note that $r_1, r_2 \neq 0$, otherwise $R = Q$. Fix $r_1 \in \mathrm{GF}(q^2)^*$. There exist $q+1$ elements $r_2 \in \mathrm{GF}(q^2)^*$ satisfying $r_2^{q+1} = -r_1^{q+1}$. Therefore, $\eta_1 = (q+1)(q^2-1)$. Next,

$$\begin{aligned}
\eta_2 &= |\{R \in \mathcal{X} : (Q,R) \in R_2\}| \\
&= |\{R \in \mathcal{X} : R \in \langle P,Q\rangle\}| \\
&= |\{R \in \mathcal{X} : r_1 = r_2 = 0\}| \\
&= |\{(1,0,0,r_3) : r_3 + r_3^q = 0\}|,
\end{aligned}$$

where $r_3 \neq 0$, otherwise $R = Q$. Since there exist q elements $r_3 \in \mathrm{GF}(q^2)^*$ satisfying $\mathrm{Tr}_{q^2/q}(r_3) = 0$, we have $\eta_2 = q-1$.

$$\begin{aligned}
\eta_3 &= |\{R \in \mathcal{X} : (Q,R) \in R_3\}| \\
&= |\{R \in \mathcal{X} : r_3 \in \theta\mathrm{GF}(q)^*\}| \\
&= |\{(1,r_1,r_2,\theta a) : a \in \mathrm{GF}(q)^*, r_1^{q+1} + r_2^{q+1} = 0\}|.
\end{aligned}$$

Note that $r_1, r_2 \neq 0$, otherwise $(Q,R) \in R_2$. For fixed $r_1 \in \mathrm{GF}(q^2)^*$, there are $q+1$ elements $r_2 \in \mathrm{GF}(q^2)^*$ such that $r_2^{q+1} = -r_1^{q+1}$. Therefore, as $r_3 = \theta a, a \in \mathrm{GF}(q)^*$, $\eta_3 = (q+1)(q^2-1)(q-1)$. Finally,

$$\begin{aligned}
\eta_5 &= |\{R \in \mathcal{X} : (Q,R) \in R_5\}| \\
&= |\{R \in \mathcal{X} : r_3 \in \mathrm{GF}(q)^*\}| \\
&= |\{(1,r_1,r_2,r_3) : r_3 \in \mathrm{GF}(q)^*, r_1^{q+1} + r_2^{q+1} = 2r_3\}|.
\end{aligned}$$

Fix $r_3 \in \mathrm{GF}(q)^*$. Then, for any $r_1 \in \mathrm{GF}(q^2)$ such that $r_1^{q+1} \neq 2r_3$, we find $q+1$ non-zero elements $r_2 \in \mathrm{GF}(q^2)$ which satisfy $r_2^{q+1} = 2r_3 - r_1^{q+1}$; for any $r_1 \in \mathrm{GF}(q^2)$ with $r_1^{q+1} = 2r_3$, $r_2 = 0$ necessarily. Therefore, $\eta_5 = ((q+1)(q^2-q-1)+q+1)(q-1)$.

Finally, $\eta_4 = |\mathcal{X}| - (1+\eta_1+\eta_2+\eta_3+\eta_5) = q(q^2-1)(q-1)^2$. □

Lemma 4.3. *The intersection numbers p_{1j}^k are well defined. They are collected in the following intersection matrix L_1 whose (k,j)−entry is p_{1j}^k:*

$$L_1 = \begin{pmatrix} 0 & (q^2-1)(q+1) & 0 & 0 & 0 & 0 \\ 1 & q^2-2 & 0 & q(q-1) & q(q-1)^2 & q(q-1) \\ 0 & 0 & 0 & (q-1)(q+1)^2 & 0 & 0 \\ 0 & q & 1 & 2(q^2-q-1) & q(q-1)^2 & q(q-1) \\ 0 & q+1 & 0 & q^2-1 & q^3-q^2-2q & q^2-1 \\ 0 & q+1 & 0 & q^2-1 & (q+1)(q-1)^2 & (q-2)(q+1) \end{pmatrix}$$

Proof. To check that p_{1j}^k is well defined, for any pair $(X,Q) \in R_k$ we count the number of points R collinear with Q and j−related with X. As $R = \langle(1,r_1,r_2,r_3)\rangle$ is collinear with $Q = \langle(1,0,0,0)\rangle$, we have $r_3 = 0$, so $r_1^{q+1} + r_2^{q+1} = 0$.

Assume $k=1$, and let X be collinear with Q. Then, $X = \langle(1,x_1,x_2,0)\rangle$ and

$$z(P,R,X) = (r_1x_1^q + r_2x_2^q)\mathrm{GF}(q)^*.$$

Any pair $(r_1,r_2) \in \mathrm{GF}(q^2)^2 \setminus \{(0,0),(x_1,x_2)\}$ such that $r_1x_1^q + r_2x_2^q = 0$ is of type $(-(x_2/x_1)^q a, a)$, for some $a \in \mathrm{GF}(q^2)^* \setminus \{x_2\}$. This implies $p_{11}^1 = q^2-2$. When $R \in \langle P,Q\rangle$ it is easy to check that $p_{12}^1 = 0$.

Let $(r_1,r_2) \in \mathrm{GF}(q^2)^2 \setminus \{(0,0),(x_1,x_2)\}$ such that $r_1x_1^q + r_2x_2^q \in \theta\mathrm{GF}(q)^*$. Then, $\theta(r_1x_1^q + r_2x_2^q) \in \theta^2\mathrm{GF}(q)^* = \mathrm{GF}(q)^*$. Since $(\theta x_1)^{q+1} + (\theta x_2)^{q+1} = \theta^{q+1}(x_1^{q+1} + x_2^{q+1}) = 0$, counting the elements 3-related with $X = \langle(1,x_1,x_2,0)\rangle$ is equivalent to counting the elements 5-related with $\langle(1,\theta x_1,\theta x_2,0)\rangle$, that is, $p_{13}^1 = p_{15}^1$. So we assume $r_1x_1^q + r_2x_2^q \in \mathrm{GF}(q)^*$. For any fixed $a \in \mathrm{GF}(q)^*$, we have $r_2 = (a - r_1x_1^q)/x_2^q$. From $r_1^{q+1} + r_2^{q+1} = 0$, it follows that $\mathrm{Tr}_{q^2/q}(r_1x_1^q) = a$. As for any given a there are q elements $x \in \mathrm{GF}(q^2)$ such that $\mathrm{Tr}_{q^2/q}(x) = a$, we see that $p_{13}^1 = p_{15}^1 = q(q-1)$. Finally $p_{14}^1 = \eta_1 - (p_{10}^1 + p_{11}^1 + p_{12}^1 + p_{13}^1 + p_{15}^1) = q(q-1)^2$.

Assume $k=2$, and let $X \in \langle P,Q\rangle \cap H(3,q^2)$. Then, $X = \langle(1,0,0,\theta a)\rangle$, for some $a \in \mathrm{GF}(q)^*$, and $z(P,R,X) = \theta^q\mathrm{GF}(q)^* = t$. This implies $p_{11}^2 = p_{12}^2 = p_{14}^2 = p_{15}^2 = 0$ and $p_{13}^2 = \eta_1 = (q-1)(q+1)^2$.

Assume $k=3$, and let $X \in \mathcal{X}$ such that $\langle P,Q,X\rangle$ is a degenerate plane. Then, $X = \langle(1,x_1,x_2,\theta a)\rangle$ for some $a \in \mathrm{GF}(q)^*$, with $x_1^{q+1} + x_2^{q+1} = 0$, $x_1 \neq 0 \neq x_2$. We have $z(P,R,X) = (\theta a + r_1x_1^q + r_2x_2^q)\mathrm{GF}(q)^*$.

It is easy to see that $p_{11}^3 = p_{13}^1 = q(q-1)$ and $p_{12}^3 = 1$.

Let $(r_1,r_2) \in \mathrm{GF}(q^2)^2 \setminus \{(0,0),(x_1,x_2)\}$ such that $r_1x_1^q + r_2x_2^q = \theta(b-a)$ for some $b \in \mathrm{GF}(q)^*$. For any given $b \neq a$, we have q such pairs, and this gives $q(q-2)$ pairs as b varies in $\mathrm{GF}(q)^* \setminus \{a\}$. Let $b=a$. Then, the number of pairs $(r_1,r_2) \in \mathrm{GF}(q^2)^2 \setminus \{(0,0),(x_1,x_2)\}$ with $r_1^{q+1} + r_2^{q+1} = 0$ and $r_1x_1^q + r_2x_2^q = 0$ is $q^2-2 = p_{11}^1$. Therefore, $p_{13}^3 = 2(q^2-q-1)$.

Let $(r_1,r_2) \in \mathrm{GF}(q^2)^2 \setminus \{(0,0),(x_1,x_2)\}$ such that $r_1x_1^q + r_2x_2^q = b - \theta a$, for some $b \in \mathrm{GF}(q)^*$. For any fixed $b \in \mathrm{GF}(q)^*$, we have $r_2 = (b - \theta a - r_1x_1^q)/x_2^q$. By plugging this into $r_1^{q+1} + r_2^{q+1} = 0$, we get $\mathrm{Tr}_{q^2/q}((b+\theta a)r_1x_1^q) = b^2 - \xi a^2 = \mathrm{N}_{q^2/q}(b-\theta a)$. As there are q elements $x \in \mathrm{GF}(q^2)$ such that $\mathrm{Tr}_{q^2/q}(x) = c$, for any given $c \in \mathrm{GF}(q)$, we get $p^3_{15} = q(q-1)$. Finally $p^3_{14} = \eta_1 - (p^3_{10} + p^3_{11} + p^3_{12} + p^3_{13} + p^3_{15}) = q(q-1)^2$.

Assume $k = 4$, and let $X \in \mathcal{X}$ such that $z(P, Q, X) \in \Gamma$. Then, $X = \langle(1, x_1, x_2, w^i a)\rangle$ for some $i \neq 0, (q+1)/2$ and $a \in \mathrm{GF}(q)^*$, with $x_1^{q+1} + x_2^{q+1} = a\mathrm{Tr}_{q^2/q}(w^i)$. We have

$$z(P, R, X) = (w^{iq}a - r_1 x_1^q - r_2 x_2^q)\mathrm{GF}(q)^*.$$

Let $w^{iq}a - r_1 x_1^q - r_2 x_2^q = 0$. Assume $x_2 = 0$. Then, $x_1 \neq 0$ (otherwise $(X, Q) \in R_2$), $r_1 = w^{iq}a/x_1^q$ and $r_2^{q+1} = -\mathrm{N}_{q^2/q}(r_1) = -a\mathrm{N}_{q^2/q}(w^i)/\mathrm{Tr}_{q^2/q}(w^i)$. Therefore, in this case there are $q + 1$ pairs (r_1, r_2) which satisfy the above properties.

Assume $x_2 \neq 0$. Then, $r_2 = (w^{iq}a - r_1 x_1^q)/x_2^q$. By plugging this into $r_1^{q+1} + r_2^{q+1} = 0$, we get

$$\mathrm{Tr}_{q^2/q}(w^i) r_1^{q+1} - \mathrm{Tr}_{q^2/q}(w^i r_1 x_1^q) + a\mathrm{N}_{q^2/q}(w^i) = 0,$$

or

$$(r_1, 1)\begin{pmatrix} \mathrm{Tr}_{q^2/q}(w^i) & -w^i x_1^q \\ -w^{iq} x_1 & a\mathrm{N}_{q^2/q}(w^i) \end{pmatrix}\begin{pmatrix} r_1^q \\ 1 \end{pmatrix} = 0,$$

with $\mathrm{N}_{q^2/q}(w^i)(a\mathrm{Tr}_{q^2/q}(w^i) - x_1^{q+1}) \neq 0$, as $x_2 \neq 0$. Since $\mathrm{Tr}_{q^2/q}(w^i) \neq 0$, the above non-singular Hermitian matrix defines a unitary form of $V(2, q^2)$ not admitting the point $\langle(1, 0)\rangle$ as a totally isotropic point. Therefore, there are $q + 1$ values for r_1, whence $q + 1$ pairs (r_1, r_2) satisfying the above properties. Hence, $p^4_{11} = q + 1$.

Let $R \in \langle P, X\rangle \cap H(3, q^2)$, i.e., $R = \langle(1, r_1, r_2, 0)\rangle$. Since $r_1^{q+1} + r_2^{q+1} = a\mathrm{Tr}_{q^2/q}(w^i) \neq 0$, we get $R \notin H(3, q^2)$, whence $p^4_{12} = 0$. By arguing as we did for p^4_{11}, we find $p^4_{13} = p^4_{15} = q^2 - 1$.

Finally, $p^4_{14} = \eta_1 - (p^4_{10} + p^4_{11} + p^4_{12} + p^4_{13} + p^4_{15}) = q^3 - q^2 - 2q$.

Assume $k = 5$, and let $X \in \mathcal{X}$ such that $z(P, Q, X) = e$. Then, $X = \langle(1, x_1, x_2, a)\rangle$ for some $a \in \mathrm{GF}(q)^*$, with $x_1^{q+1} + x_2^{q+1} = 2a$. We have

$$z(P, R, X) = (a - r_1 x_1^q - r_2 x_2^q)\mathrm{GF}(q)^*.$$

We may argue as above to show that $p^5_{11} = q+1$, $p^5_{12} = 0$, $p^5_{13} = q^2-1$, $p^5_{15} = (q-2)(q+1)$, and $p^5_{14} = (q+1)(q-1)^2$. □

Lemma 4.4. *The intersection numbers p^k_{2j} are well defined. They are collected in the following intersection matrix L_2 whose $(k, j)-$entry is p^k_{2j}:*

$$L_2 = \begin{pmatrix} 0 & 0 & q-1 & 0 & 0 & 0 \\ 0 & 0 & 0 & q-1 & 0 & 0 \\ 1 & 0 & q-2 & 0 & 0 & 0 \\ 0 & 1 & 0 & q-2 & 0 & 0 \\ 0 & 0 & 0 & 0 & q-2 & 1 \\ 0 & 0 & 0 & 0 & q-1 & 0 \end{pmatrix}$$

Proof. To check that p^k_{2j} is well defined, for any pair $(X, Q) \in R_k$ we count the number of points $R \in \mathcal{X}$ such that R is on the hyperbolic line spanned by P and Q with $(R, X) \in R_j$. It is easily seen that $R = \langle(1, 0, 0, a\theta)\rangle$, for some $a \in \mathrm{GF}(q)^*$. We have $p^k_{21} = p^k_{12}$, for $k = 0, \ldots, 5$.

Assume $k = 1$. Then, $X = \langle(1, x_1, x_2, 0)\rangle$, with $x_1 \neq 0 \neq x_2$. Therefore, there is no point R on the hyperbolic line spanned by P and X giving $p^1_{22} = 0$. In addition,

$$z(P, R, X) = \theta a \mathrm{GF}(q)^* = t.$$

This implies $p^1_{24} = p^1_{25} = 0$ and $p^1_{23} = q - 1$.

Assume $k = 2$, and let $X = \langle(1, 0, 0, b\theta)\rangle$, for some $b \in \mathrm{GF}(q)^*$. Since $R \in \langle P, Q\rangle \setminus \{Q, X\}$, we have $p^2_{22} = q - 2$. This also implies $p^2_{23} = p^2_{24} = p^2_{25} = 0$.

Assume $k = 3$, and let $X = \langle(1, x_1, x_2, b\theta)\rangle$ for some $b \in \mathrm{GF}(q)^*$, with $x_1^{q+1} + x_2^{q+1} = 0$, $x_1 \neq 0 \neq x_2$. Then, $p^3_{22} = 0$. In addition,

$$z(P, R, X) = \theta(a + b)\mathrm{GF}(q)^*.$$

For $a = -b$, $(R, X) \in R_1$. For $a \neq -b$, $z(P, R, X) = t$. Therefore, $p^3_{24} = p^3_{25} = 0$ and $p^3_{23} = q - 2$.

Assume $k = 4$, and let $X = \langle(1, x_1, x_2, x_3)\rangle$ for some $x_3 \notin \mathrm{GF}(q)^* \cup \theta\mathrm{GF}(q)^*$. Then, $p^4_{22} = 0$, otherwise $x_3 \in \theta\mathrm{GF}(q)^*$. In addition,

$$z(P, R, X) = (a\theta + x_3^q)\mathrm{GF}(q)^*.$$

Therefore, $p^4_{23} = 0$. To calculate p^4_{25} we see that $a\theta + x_3^q \in \mathrm{GF}(q)^*$ if and only if $a = (x_3 - x_3^q)/2\theta$. Therefore, $p^4_{25} = 1$ and $p^4_{24} = q - 2$.

Assume $k = 5$, and let $X = \langle(1, x_1, x_2, b)\rangle$ for some $b \in \mathrm{GF}(q)^*$. Then, $p^5_{22} = 0$, otherwise $b = 0$. In addition,

$$z(P, R, X) = (a\theta + b)\mathrm{GF}(q)^*.$$

Therefore, there are no $a \in \mathrm{GF}(q)^*$ such that $a\theta + b \in \mathrm{GF}(q)^*$ or $a\theta + b \in \theta\mathrm{GF}(q)^*$. This implies, $p^5_{25} = p^5_{23} = 0$ and $p^5_{24} = q - 1$. □

Lemma 4.5. *The intersection numbers p^k_{3j} are well defined. They are collected in the following intersection matrix L_3 whose $(k, j)-$entry is p^k_{3j}:*

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & (q^2-1)^2 & 0 & 0 \\ 0 & q(q-1) & q-1 & 2(q-1)(q^2-q-1) & q(q-1)^3 & q(q-1)^2 \\ 0 & (q-1)(q+1)^2 & 0 & (q^2-1)(q^2-q-2) & 0 & 0 \\ 1 & 2(q^2-q-1) & q-2 & 2q^3-5q^2+q+4 & q(q-1)^3 & q(q-1)^2 \\ 0 & q^2-1 & 0 & (q^2-1)(q-1) & q^4-2q^3-q^2+3q+1 & q(q+1)(q-2) \\ 0 & q^2-1 & 0 & (q^2-1)(q-1) & q(q^2-1)(q-2) & (q^2-1)(q-1) \end{pmatrix}$$

Proof. To check that p^k_{3j} is well defined, for any pair $(X, Q) \in R_k$ we count the number of points R which are $3-$related with Q and $j-$related with X. It is easily seen that $R = \langle(1, r_1, r_2, a\theta)\rangle$, for some $a \in \mathrm{GF}(q)^*$, with $r_1^{q+1} + r_2^{q+1} = 0, r_1 \neq 0 \neq r_2$ (otherwise $(Q, R) \in R_2$). From the previous calculations, we already have $p^k_{31} = p^k_{13}$, $p^k_{32} = p^k_{23}$, for $k = 0, \ldots, 5$.

Assume $k = 1$. Then, $X = \langle(1, x_1, x_2, 0)\rangle$, with $x_1^{q+1} + x_2^{q+1} = 0$, $x_1 \neq 0 \neq x_2$. Therefore, $(R, X) \in R_3$ if and only if there exists $b \in \mathrm{GF}(q)^*$ such that $r_1 x_1^q + r_2 x_2^q = \theta(a - b)$. First of all, suppose $a \neq b$. As $\theta(r_1 x_1^q + r_2 x_2^q) \in \mathrm{GF}(q)^*$ and $(\theta x_1)^{q+1} + (\theta x_2)^{q+1} = \theta^{q+1}(x_1^{q+1} + x_2^{q+1}) = 0$, we may consider $(r_1, r_2) \in \mathrm{GF}(q^2)^* \times \mathrm{GF}(q^2)^*$ such that $r_1 x_1^q + r_2 x_2^q \in \mathrm{GF}(q)^*$

(see the calculation of p^1_{15}). Let $c \in \mathrm{GF}(q)^*$, and write $r_2 = x_2^{-q}(c - r_1 x_1^q)$. By using $r_1^{q+1} + r_2^{q+1} = 0$, it follows that $\mathrm{Tr}_{q^2/q}(r_1 x_1^q) = c$, which is true for exactly q elements $r_1 \in \mathrm{GF}(q)^*$. Therefore, for $b \in \mathrm{GF}(q)^* \setminus \{a\}$, $q(q-1)(q-2)$ is the number of the triples (r_1, r_2, a), c being one-to-one with b. Now, consider $a = b$. Then, $r_1 x_1^q + r_2 x_2^q = 0$, for $(r_1, r_2) \in \mathrm{GF}(q^2)^2 \setminus \{(0,0),(x_1,x_2)\}$. So, in the case $a = b$, the number of the triples (r_1, r_2, a) is equal to $(q-1)(q^2-2)$. Finally, by summing the previous two quantities, we have $p^1_{33} = 2(q-1)(q^2-q-1)$.

We show now that p^1_{35} is well-defined, by explicitly calculating it. Thus, $(R, X) \in R_5$ if and only if there exists $b \in \mathrm{GF}(q)^*$ such that $r_1 x_1^q + r_2 x_2^q = \theta a - b$. By deriving r_2 from the previous expression and considering $r_1^{q+1} + r_2^{q+1} = 0$ as usual, we obtain $\mathrm{Tr}_{q^2/q}((\theta a + b)(r_1 x_1^q)) = \theta^2 a^2 - b^2$, which is satisfied by exactly q elements r_1. As $a, b \in \mathrm{GF}(q)^*$, $p^1_{35} = q(q-1)^2$.

Finally, $p^1_{34} = \eta_3 - (p^1_{30} + p^1_{31} + p^1_{32} + p^1_{33} + p^1_{35}) = q(q-1)^3$.

Assume $k = 2$. Then, $X = \langle(1,0,0,\theta b)\rangle$ with $b \in \mathrm{GF}(q)^*$, and

$$z(P, R, X) = \theta(a+b)\mathrm{GF}(q)^*.$$

This means that, for $a + b \neq 0$ (otherwise $p^2_{31} = p^2_{13}$), $(R, X) \in R_3$, from which $p^2_{3j} = 0$, for $j = 0, 2, 4, 5$, and $p^2_{33} = \eta_3 - p^2_{31} = (q^2-1)(q^2-q-2)$.

Take $k = 3$. Then, $X = \langle(1, x_1, x_2, \theta b)\rangle$, with $x_1^{q+1} + x_2^{q+1} = 0$, $x_1 \neq 0 \neq x_2$, for some $b \in \mathrm{GF}(q)^*$. Here, $(R, X) \in R_3$ if and only if $r_1 x_1^q + r_2 x_2^q = \theta(c - b + a)$ for some $c \in \mathrm{GF}(q)^*$. We need, at this point, to distinguish different cases. First of all, let $c \neq b$ and $a \neq b - c$. Once multiplied the right hand-side by θ, in order to simplify the calculation, we may equivalentely look at the triples (r_1, r_2, a) such that $r_1 x_1^q + r_2 x_2^q = \theta^2(c - b + a)$, i.e., $r_2 = x_2^{-q}(\theta^2(c-b+a) - r_1 x_1^q)$. By using $r_1^{q+1} + r_2^{q+1} = 0$, we find $\mathrm{Tr}_{q^2/q}(r_1 x_1^q) = \theta^2(c-b+a)$, which provides q values for r_1. Hence, for $c \in \mathrm{GF}(q)^* \setminus \{b\}, a \in \mathrm{GF}(q)^* \setminus \{b-c\}$, the number of the triples (r_1, r_2, a) is $q(q-2)^2$. Consider now the subcase $c - b + a = 0$. Then, $r_2 = x_2^{-q} r_1 x_1^q$ for $r_1 \in \mathrm{GF}(q^2)^* \setminus \{x_1\}$ (otherwise $r_2 = x_2$ and $(R, X) \in R_2$). So, for $a = b - c \neq 0$, the number of the triples (r_1, r_2, a) is equal to $(q-2)(q^2-2)$. Finally, we explore the case $c = b$. As before, the computation may be reduced to considering the triples (r_1, r_2, a) such that $r_1 x_1^q + r_2 x_2^q = \theta^2 a$, i.e., $r_2 = x_2^{-q}(\theta^2 a - r_1 x_1^q)$. Since $r_1^{q+1} + r_2^{q+1} = 0$ yields $\mathrm{Tr}_{q^2/q}(r_1 x_1^q) = \theta^2 a$, the number of all possible choices for r_1 is q. To conclude, by putting together all the previous results, we have $p^3_{33} = q(q-2)^2 + (q-2)(q^2-2) + q(q-1) = 2q^3 - 5q^2 + q + 4$.

We count now $R = \langle(1, r_1, r_2, a\theta)\rangle$ such that $(R, X) \in R_5$. This condition means $r_1 x_1^q + r_2 x_2^q = c + \theta(a - b)$, for some $c \in \mathrm{GF}(q)^*$. By plugging this into $r_1^{q+1} + r_2^{q+1} = 0$, we get $\mathrm{Tr}_{q^2/q}((c - \theta(a-b)) r_1 x_1^q) = c^2 - \theta^2(a-b)^2$. Thus, there are q elements r_1 satisfying this equation, and $p^3_{35} = q(q-1)^2$, as $c, a \in \mathrm{GF}(q)^*$.

Finally, in order to conclude the study of the case $k = 3$, write $p^3_{34} = \eta_3 - (p^3_{30} + p^3_{31} + p^3_{32} + p^3_{33} + p^3_{35}) = q(q-1)^3$.

Take $k = 4$. Therefore, $X = \langle(1, x_1, x_2, \omega^i b)\rangle$, where $x_1^{q+1} + x_2^{q+1} = \mathrm{Tr}_{q^2/q}(\omega^i b)$, $b \in \mathrm{GF}(q)^*$. Suppose R is $3-$related with X. This means there exists $c \in \mathrm{GF}(q)^*$ such that $r_1 x_1^q + r_2 x_2^q = \omega^{iq} b + \theta(a - c)$. At this point, we distinguish when x_2 is zero and when it is not. Suppose $x_2 = 0$. As $x_1 \neq 0$, we have $r_1 = x_1^{-q}(\omega^{iq} b + \theta(a - c))$, hence $r_2^{q+1} = -r_1^{q+1} = -x_1^{-(q+1)} \mathrm{N}_{q^2/q}(\omega^{iq} b + \theta(a-c))$, which is satisfied by exactly $q+1$ elements r_2. Since $a, c \in \mathrm{GF}(q)^*$, the number of the triples (r_1, r_2, a) is $(q+1)(q-1)^2$. Now suppose $x_2 \neq 0$. Then, $r_2 = (\theta(a-c) + \omega^{iq} b - r_1 x_1^q)/x_2^q$. By plugging this into $r_1^{q+1} + r_2^{q+1} = 0$, we get

$$r_1^{q+1}\mathrm{Tr}_{q^2/q}(\omega^i b) - \mathrm{Tr}_{q^2/q}((\omega^{iq} b + \theta(a-c))x_1 r_1^q) + \mathrm{N}_{q^2/q}(w^{iq} b + \theta(a-c)) = 0,$$

or

$$(r_1, 1)\begin{pmatrix} \mathrm{Tr}_{q^2/q}(\omega^i b) & -(\omega^i b - \theta(a-c))x_1^q \\ -(\omega^{iq} b + \theta(a-c))x_1 & \mathrm{N}_{q^2/q}(\omega^{iq} b + \theta(a-c)) \end{pmatrix}\begin{pmatrix} r_1^q \\ 1 \end{pmatrix} = 0,$$

with $\mathrm{N}_{q^2/q}(\omega^{iq} b + \theta(a-c))(\mathrm{Tr}_{q^2/q}(\omega^i b) - x_1^{q+1}) \neq 0$, as $x_2 \neq 0$. Since $\mathrm{Tr}_{q^2/q}(\omega^i) \neq 0$, the above non-singular Hermitian matrix defines a unitary form of $V(2, q^2)$ not admitting the point $\langle(1, 0)\rangle$ as a totally isotropic point. Therefore, there are $q + 1$ values for r_1. As $a, c \in \mathrm{GF}(q)^*$, the number of the triples (r_1, r_2, a) is again $(q + 1)(q - 1)^2$. Hence, we may write $p^4_{33} = (q + 1)(q - 1)^2$.

Suppose now $(R, X) \in R_5$. This is equivalent to having, for some $c \in \mathrm{GF}(q)^*$, $r_1 x_1^q + r_2 x_2^q = \omega^{iq} b + \theta a + c$. The second member is zero if and only if the pair (a, c) concides with the unique pair (a', c') such that $-\omega^{iq} b = \theta a' + c'$, $\{1, \theta\}$ being a basis for $\mathrm{GF}(q^2)$ over $\mathrm{GF}(q)$. Anyway, for this pair of values, we would have $r_1 x_1^q + r_2 x_2^q = 0$, from which $r_1 = 0$ (or $r_2 = 0$), a contradiction. Then, let $(a, c) \neq (a', c')$. At this point, by proceeding as before for the computation of p^4_{33}, we distinguish the case $x_2 = 0$ from $x_2 \neq 0$, and $p^4_{35} = (q + 1)((q - 1)^2 - 1) = (q + 1)q(q - 2)$ is obtained.

Finally, $p^4_{34} = \eta_3 - (p^4_{30} + p^4_{31} + p^4_{32} + p^4_{33} + p^4_{35}) = q^4 - 2q^3 - q^2 + 3q + 1$.

Assume $k = 5$, and let $X = \langle(1, x_1, x_2, b)\rangle$ for some $b \in \mathrm{GF}(q)^*$, with $x_1^{q+1} + x_2^{q+1} = 2b$. Then, $(R, X) \in R_3$ if and only if there exists $c \in \mathrm{GF}(q)^*$ such that $r_1 x_1^q + r_2 x_2^q = \theta(c - a) - b$. First of all, take $x_2 = 0$. Here, $r_1 = x_1^{-q}(\theta(c - a) - b)$ $(x_1 \neq 0)$ and $r_2^{q+1} = -r_1^{q+1} = -x_1^{-(q+1)}\mathrm{N}_{q^2/q}(\theta(c - a) - b)$. Since for any $a, c \in \mathrm{GF}(q)^*$ there are $q + 1$ values of r_2 solving the previous equation, we have $(q + 1)(q - 1)^2$ triples (r_1, r_2, a) when $x_2 = 0$. Now, take $x_2 \neq 0$. Then, $r_2 = (\theta(c - a) - b - r_1 x_1^q)/x_2^q$. By plugging this into $r_1^{q+1} + r_2^{q+1} = 0$, we obtain

$$r_1^{q+1} 2b - \mathrm{Tr}_{q^2/q}((\theta(c-a) - b)x_1 r_1^q) + \mathrm{N}_{q^2/q}(\theta(c-a) - b) = 0,$$

or

$$(r_1, 1)\begin{pmatrix} 2b & -(-\theta(c-a) - b)x_1^q \\ -(\theta(c-a) - b)x_1 & \mathrm{N}_{q^2/q}(\theta(c-a) - b) \end{pmatrix}\begin{pmatrix} r_1^q \\ 1 \end{pmatrix} = 0,$$

with $\mathrm{N}_{q^2/q}(\theta(c - a) - b)(2b - x_1^{q+1}) \neq 0$, as $x_2 \neq 0$. Since $b \neq 0$, the above non-singular Hermitian matrix defines a unitary form of $V(2, q^2)$ not admitting the point $\langle(1, 0)\rangle$ as a totally isotropic point. Therefore, there are $q + 1$ values for r_1. As $a, c \in \mathrm{GF}(q)^*$, the number of the triples (r_1, r_2, a) is again $(q + 1)(q - 1)^2$. Thus, we may write $p^5_{33} = (q + 1)(q - 1)^2$.

In order to complete the entries of L_3, the case $(R, X) \in R_5$ $(k = 5)$ remains to be studied, as p^5_{34} will be obtained by taking a difference. Considering $(R, X) \in R_5$ is equivalent to having $r_1 x_1^q + r_2 x_2^q = c - b - \theta a$, for some $c \in \mathrm{GF}(q)^*$. Since $x_1^{q+1} + x_2^{q+1} = 2b$, with $b \in \mathrm{GF}(q)^*$, we may distinguish the two cases $x_2 = 0$ and $x_2 \neq 0$, and then proceed exactly as before in the computation of p^5_{33}, so getting $p^5_{35} = p^5_{33} = (q + 1)(q - 1)^2$.

Finally, $p^5_{34} = \eta_3 - (p^5_{30} + p^5_{31} + p^5_{32} + p^5_{33} + p^5_{35}) = (q^2 - 1)q(q - 2)$. □

Lemma 4.6. *The intersection numbers p^k_{5j} are well defined. They are collected in the following intersection matrix L_5 whose $(k, j)-$entry is p^k_{5j}:*

$$L_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & (q^3-q)(q-1) \\ 0 & q(q-1) & 0 & q(q-1)^2 & q(q-1)^3 & q(q-2)(q-1) \\ 0 & 0 & 0 & 0 & (q^3-q)(q-1) & 0 \\ 0 & q(q-1) & 0 & q(q-1)^2 & q^2(q-1)(q-2) & q(q-1)^2 \\ 0 & q^2-1 & 1 & q(q+1)(q-2) & q^4-3q^3+q^2+4q-1 & q^3-2q^2-q+1 \\ 1 & (q-2)(q+1) & 0 & (q^2-1)(q-1) & q^4-3q^3+q^2+2q-1 & q^3-2q^2+q+1 \end{pmatrix}$$

Proof. To check that p^k_{5j} is well defined, for any pair $(X, Q) \in R_k$ we count the number of points R that are $5-$related with Q and $j-$related with X. It is easy to see that $R = \langle(1, r_1, r_2, a)\rangle$, for some $a \in \mathrm{GF}(q)^*$, with $r_1^{q+1} + r_2^{q+1} = 2a$. From the previous calculations, we already have $p^k_{51} = p^k_{15}$, $p^k_{52} = p^k_{25}$, $p^k_{53} = p^k_{35}$, for $k = 0, \ldots, 5$.

Let $k = 1$. Thus, $X = \langle(1, x_1, x_2, 0)\rangle$, with $x_1^{q+1} + x_2^{q+1} = 0$, $x_1 \neq 0 \neq x_2$. Therefore, $(R, X) \in R_5$ if and only if there exists $b \in \mathrm{GF}(q)^*$ such that $r_1x_1^q + r_2x_2^q = a - b$. Note that for $a = b$, there is no triple (r_1, r_2, a) satysfing the previous equation because of the conditions on (x_1, x_2). For this reason, set $a \neq b$. By writing $r_2 = x_2^{-q}(a - b - r_1x_1^q)$ and substituting it into $r_1^{q+1} + r_2^{q+1} = 2a$, we get $\mathrm{Tr}_{q^2/q}(r_1x_1^q) = (a - b)^{-1}((a - b)^2 - 2ax_2^{q+1})$, that is satisfied by q elements r_1. Therefore, as $b \in \mathrm{GF}(q)^*, a \in \mathrm{GF}(q)^* \setminus \{b\}$, we obtain $p^1_{55} = (q - 2)(q - 1)q$.

Then, $p^1_{54} = \eta_5 - (p^1_{50} + p^1_{51} + p^1_{52} + p^1_{53} + p^1_{55}) = q(q - 1)^3$.

Assume $k = 2$, and let $X = \langle(1, 0, 0, \theta b)\rangle$ with $b \in \mathrm{GF}(q)^*$. Note that

$$z(P, R, X) = (a - \theta b)\mathrm{GF}(q)^* \in \Gamma,$$

i.e., $j = 4$, from which $p^2_{55} = 0$, $p^2_{54} = \eta_5$.

Assume $k = 3$, and let $X = \langle(1, x_1, x_2, \theta b)\rangle$, with $x_1^{q+1} + x_2^{q+1} = 0$, $x_1 \neq 0 \neq x_2$. Thus, $(R, X) \in R_5$ if and only if there exists $c \in \mathrm{GF}(q)^*$ such that $r_1x_1^q + r_2x_2^q = c + a - \theta b$, i.e., $r_2 = x_2^{-q}(c + a - \theta b - r_1x_1^q)$. By using $r_1^{q+1} + r_2^{q+1} = 2a$, it follows that $\mathrm{Tr}_{q^2/q}((c + a + \theta b)r_1x_1^q) = (c + a)^2 - \theta^2b^2 - 2a$. Since the latter equation in the unknown r_1 admits q solutions, for any $a, c \in \mathrm{GF}(q)^*$, we have $q(q - 1)^2$ triples (r_1, r_2, a), i.e., $p^3_{55} = q(q - 1)^2$.

Therefore, $p^3_{54} = \eta_5 - (p^3_{50} + p^3_{51} + p^3_{52} + p^3_{53} + p^3_{55}) = q^2(q - 1)(q - 2)$.

Take $k = 4$. Thus, $X = \langle(1, x_1, x_2, \omega^i b)\rangle$, where $x_1^{q+1} + x_2^{q+1} = \mathrm{Tr}_{q^2/q}(\omega^i b)$, $b \in \mathrm{GF}(q)^*$. Suppose R is $5-$related with X. This means there exists $c \in \mathrm{GF}(q)^*$ such that $r_1x_1^q + r_2x_2^q = \omega^{iq}b + a - c$. At this point, we study separately when x_2 is zero and when it is not. Suppose $x_2 = 0$. As $x_1 \neq 0$, we find $r_1 = x_1^{-q}(\omega^{iq}b + a - c)$, hence $r_2^{q+1} = -r_1^{q+1} + 2a = -x_1^{-(q+1)}\mathrm{N}_{q^2/q}(\omega^{iq}b + a - c) + 2a$. The latter equation is satisfied by exactly $q + 1$ elements r_2 if $-r_1^{q+1} + 2a \neq 0$, otherwise $r_2 = 0$. So it is necessary to study the case $-r_1^{q+1} + 2a = 0$. Consider $r_1^{q+1} = 2a$, i.e., $x_1^{-(q+1)}\mathrm{N}_{q^2/q}(\omega^{iq}b + a - c) = 2a$. By writing this expression explicitly, we find that the elements $a \in \mathrm{GF}(q)^*$ for which $r_1^{q+1} = 2a$ are the solutions of the equation

$$X^2 + (\mathrm{Tr}_{q^2/q}(\omega^i b) - 2x_1^{q+1} - 2c)X + c^2 - c\mathrm{Tr}_{q^2/q}(\omega^i b) + \mathrm{N}_{q^2/q}(\omega^i b) = 0, \qquad (13)$$

whose discriminant is

$$\Delta = 8cx_1^{q+1} - 4\mathrm{N}_{q^2/q}(\omega^i b) + (\mathrm{Tr}_{q^2/q}(\omega^i b) - 2x_1^{q+1})^2$$

$$= 8cx_1^{q+1} + b^2(\omega^{iq} - \omega^i)^2.$$

For $\overline{c} = -b^2(\omega^{iq} - \omega^i)^2/8x_1^{q+1}$, $\Delta = 0$ holds, i.e., there is a unique $X = \overline{a}$ satisfying (13), to which $\overline{r}_1 = x_1^{-q}(\omega^{iq}b + \overline{a} - \overline{c})$ and $\overline{r}_2 = 0$ correspond. Therefore, for $c = \overline{c}$, we get the triple $(\overline{r_1}, 0, \overline{a})$ and further $(q+1)(q-2)$ triples (r_1, r_2, a) with $r_1 = x_1^{-q}(\omega^{iq}b + a - \overline{c})$, $a \neq \overline{a}$, $r_2 \neq 0$.

Assume Δ is a square in $\mathrm{GF}(q)^*$, i.e., Δ is a $\frac{q-1}{2}$–th root of the unity. Precisely one element c corresponds with any such a root, and for any such a c, there are two values of X satisfying (13). Therefore, for every fixed c among the previous ones, we get two triples of type $(r_1, 0, a)$ and further $(q+1)(q-3)$ triples with $r_2 \neq 0$, i.e., in total we have $\frac{q-1}{2}(2 + (q+1)(q-3))$.

For the remaining $q - 1 - (1 + \frac{q-1}{2}) = \frac{q-3}{2}$ values of $c \in \mathrm{GF}(q)^*$, Δ is a non-square, i.e., there is no $a \in \mathrm{GF}(q)^*$ making $-r_1^{q+1} + 2a$ equal to zero. This means that, here, in the light of the initial considerations on both r_1 and r_2, we have $\frac{q-3}{2}(q-1)(q+1)$. To sum up, for $x_2 = 0$, the number we seek is $1 + (q-2)(q+1) + \frac{q-1}{2}(2 + (q+1)(q-3)) + \frac{q-3}{2}(q-1)(q+1) = q^3 - 2q^2 - q + 1$.

Now suppose $x_2 \neq 0$. Then, $r_2 = (a - c + \omega^{iq}b - r_1x_1^q)/x_2^q$. By plugging this into $r_1^{q+1} + r_2^{q+1} = 2a$, we get

$$r_1^{q+1}\mathrm{Tr}_{q^2/q}(\omega^i b) - \mathrm{Tr}_{q^2/q}((\omega^i b + a - c)r_1x_1^q) + \mathrm{N}_{q^2/q}(w^i b + a - c) - 2ax_2^{q+1} = 0,$$

or

$$(r_1, 1)\begin{pmatrix} \mathrm{Tr}_{q^2/q}(\omega^i b) & -(\omega^i b + a - c)x_1^q \\ -(\omega^{iq}b + a - c)x_1 & \mathrm{N}_{q^2/q}(\omega^i b + a - c) - 2ax_2^{q+1} \end{pmatrix}\begin{pmatrix} r_1^q \\ 1 \end{pmatrix} = 0,$$

whose determinant is $\mathrm{N}_{q^2/q}(\omega^i b + a - c)(\mathrm{Tr}_{q^2/q}(\omega^i b) - x_1^{q+1}) - 2a\mathrm{Tr}_{q^2/q}(\omega^i b)x_2^{q+1}$, or better $(\mathrm{N}_{q^2/q}(\omega^i b + a - c) - 2a\mathrm{Tr}_{q^2/q}(\omega^i b))x_2^{q+1}$. If $\mathrm{N}_{q^2/q}(\omega^i b + a - c) \neq 2a\mathrm{Tr}_{q^2/q}(\omega^i b)$, since $\mathrm{Tr}_{q^2/q}(\omega^i) \neq 0$, the above Hermitian matrix is a non-singular matrix which defines a unitary form of $V(2, q^2)$ not admitting the point $\langle(1, 0)\rangle$ as a totally isotropic point. Therefore, there are $q+1$ values for r_1 if the determinant is non-zero, otherwise there is a unique r_1 satisfying the above sesquilinear form. So $\mathrm{N}_{q^2/q}(\omega^i b + a - c) - 2a\mathrm{Tr}_{q^2/q}(\omega^i b) = 0$ needs to be studied. In this case, by writing the expression explicitly, we find that the elements $a \in \mathrm{GF}(q)^*$ making the determinant equal to zero are the solutions of the equation

$$X^2 - (\mathrm{Tr}_{q^2/q}(\omega^i b) + 2c)X + c^2 - c\mathrm{Tr}_{q^2/q}(\omega^i b) + \mathrm{N}_{q^2/q}(\omega^i b) = 0,$$

whose discriminant is

$$\Delta = 8c\mathrm{Tr}_{q^2/q}(\omega^i b) - 4\mathrm{N}_{q^2/q}(\omega^i b) + (\mathrm{Tr}_{q^2/q}(\omega^i b))^2.$$

Here, for $x_2 \neq 0$, by arguing exactly as before, the number of triples obtained is again $q^3 - 2q^2 - q + 1$, and so we may denote it by p_{55}^4.

Finally, we have $p_{54}^4 = \eta_5 - (p_{50}^4 + p_{51}^4 + p_{52}^4 + p_{53}^4 + p_{55}^4) = q^4 - 3q^3 + q^2 + 4q - 1$.

Assume $k = 5$. Then, $X = \langle(1, x_1, x_2, b)\rangle$ for some $b \in \mathrm{GF}(q)^*$, with $x_1^{q+1} + x_2^{q+1} = 2b$. Therefore, $(R, X) \in R_5$ if and only if there is $c \in \mathrm{GF}(q)^*$ such that $r_1x_1^q + r_2x_2^q = a + b - c$. As b is fixed, we may set $d = b - c$ with $d \in \mathrm{GF}(q) \setminus \{b\}$. Assume $x_2 = 0$, i.e., $x_1^{q+1} = 2b$. Thus $r_1 = x_1^{-q}(a + d)$, from which

$$r_2^{q+1} = -r_1^{q+1} + 2a = -x_1^{-(q+1)}(a + d)^2 + 2a.$$

This equation gives $r_2 = 0$ whenever the latter quantity is zero, otherwise it gives $q+1$ non-zero values for r_2. Assume first $d = 0$. Then, $a^2 - 2x_1^{q+1}a = 0$ if and only if $a = 2x_1^{q+1} = 4b$, thus $r_1 = 4b/x_1^q$. Therefore, for $d = 0$, we get only one triple with $r_2 = 0$ and $(q-2)(q+1)$ triples with $r_2 \neq 0$. Now assume $d \neq 0$. The values of $a \in \mathrm{GF}(q)^*$ making $-r_1^{q+1} + 2a = -x_1^{-(q+1)}(a+d)^2 + 2a$ equal to zero are the solutions of the equation $X^2 + 2(d - x_1^{q+1})X + d^2 = 0$, whose discriminant is $\Delta = x_1^{q+1}(x_1^{q+1} - 2d) = 4bc \neq 0$. Thus Δ is either a non-zero square or a non-square in $\mathrm{GF}(q)$. When Δ is a non-zero square, $x_1^{q+1}(x_1^{q+1} - 2d)$ is a $\frac{q-1}{2}-$th root of unity, and for such a root, and hence for the corresponding d, there are two values of a satisfying the previous quadratic equation in X. Note that for $d = 0$, $x_1^{2(q+1)}$ is evidently a $\frac{q-1}{2}-$th root of unity (but $d = 0$ has already been analysed before), while Δ would be 0 for $d = b$. Therefore, for any $\frac{q-1}{2}-$th root of unity but $x_1^{2(q+1)}$, we find two triples with $r_2 = 0$ and $(q-3)(q+1)$ triples with $r_2 \neq 0$. For the remaining $\frac{q-1}{2}$ non-zero elements of $\mathrm{GF}(q)$, Δ is a non-square in $\mathrm{GF}(q)$. Thus, for any such an element we get $(q-1)(q+1)$ triples (r_1, r_2, a).

By taking into account all the above results, for $x_2 = 0$, the following number of triples is obtained:

$$1 + (q-2)(q+1) + \frac{q-3}{2}(2 + (q-3)(q+1)) + \frac{q-1}{2}(q-1)(q+1) = q^3 - 2q^2 + q + 1.$$

In order to conclude the study of the case $k = 5$, consider $x_2 \neq 0$. Therefore, by deriving r_2 from $r_1x_1^q + r_2x_2^q = a + d$, where $d \in \mathrm{GF}(q) \setminus \{b\}$, and plugging it into $r_1^{q+1} + r_2^{q+1} = 2a$, we get

$$r_1^{q+1}2b - (a+d)\mathrm{Tr}_{q^2/q}(r_1x_1^q) + (a+d)^2 - 2ax_2^{q+1} = 0,$$

or

$$(r_1, 1)\begin{pmatrix} 2b & -(a+d)x_1^q \\ -(a+d)x_1 & (a+d)^2 - 2ax_2^{q+1} \end{pmatrix}\begin{pmatrix} r_1^q \\ 1 \end{pmatrix} = 0, \tag{14}$$

whose determinant is $((a+d)^2 - 4ab)x_2^{q+1}$. If $(a+d)^2 \neq 4ab$, the above Hermitian matrix is a non-singular matrix which defines a unitary form of $V(2, q^2)$ not admitting the point $\langle(1,0)\rangle$ as a totally isotropic point. Therefore, if $(a+d)^2 - 4ab = 0$, there is a unique r_1 satisfying (14), otherwise there are $q+1$ values for r_1 satisfying (14). It is evident that the $a \in \mathrm{GF}(q)^*$ making $(a+d)^2 - 4ab$ equal to zero are the solutions of the equation $X^2 + 2(d-2b)X + d^2 = 0$, whose discriminant is $\Delta = 4bc \neq 0$. By arguing exactly as in the case $x_2 = 0$, we find that the number of triples obtained is again $q^3 - 2q^2 + q + 1$, and so we may denote it by p_{55}^5.

Finally, we get $p_{54}^5 = \eta_5 - (p_{50}^5 + p_{51}^5 + p_{52}^5 + p_{53}^5 + p_{55}^5) = q^4 - 3q^3 + q^2 + 2q - 1$. $\square$

Lemma 4.7. *The intersection numbers p_{4j}^k are well defined. They are collected in the following intersection matrix L_4 whose $(k,j)-$entry is p_{4j}^k*

$$L_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & (q^3-q)(q-1)^2 & 0 \\ 0 & q(q-1)^2 & 0 & q(q-1)^3 & q^2(q-1)^2(q-2) & q(q-1)^3 \\ 0 & 0 & 0 & 0 & (q^3-q)(q-1)(q-2) & (q^3-q)(q-1) \\ 0 & q(q-1)^2 & 0 & q(q-1)^3 & q(q-1)(q^3-3q^2+2q+1) & q^2(q-1)(q-2) \\ 1 & q^3-q^2-2q & q-2 & q^4-2q^3-q^2+3q+1 & q^5-4q^4+4q^3+3q^2-7q+1 & q^4-3q^3+q^2+4q-1 \\ 0 & (q+1)(q-1)^2 & q-1 & q(q^2-1)(q-2) & q^5-4q^4+4q^3+3q^2-5q+1 & q^4-3q^3+q^2+2q-1 \end{pmatrix}$$

Proof. To check that p_{4j}^k is well defined, for any pair $(X, Q) \in R_k$ we would have to count the number of points R that are $4-$related with Q and $j-$related with X. However, thanks to the previous Lemmas 4.3, 4.4, 4.5, 4.6, we already have all the entries of L_4 except for p_{44}^k, with $k = 1, \ldots, 5$. These can be derived from the well-known relations $\eta_i = \sum_j p_{ij}^k$, i.e., $p_{44}^k = \eta_4 - (p_{40}^k + p_{41}^k + p_{42}^k + p_{43}^k + p_{45}^k)$, for $k = 1, \ldots, 5$. $\square$

Set $I = \{0, 2\}$. For $i, j \in I$ and $k \in \{0, \ldots, 5\} \setminus I$, we have $p^k_{i,j} = 0$. By Theorem 9.3 (iii) in [2], the scheme $\mathfrak{X}_P$ is imprimitive.

Summarising, we have the following result.

Theorem 4.8. *For any point P of $H(3, q^2)$, $\mathfrak{X}_P = (\mathcal{X}, \mathcal{R})$ is a symmetric and imprimitive association scheme, whose first and second eigenmatrices are*

$$\mathcal{P} = \begin{pmatrix} 1 & (q^2-1)(q+1) & q-1 & (q^2-1)^2 & (q^3-q)(q-1)^2 & (q^3-q)(q-1) \\ 1 & q^2-q-1 & q-1 & q^3-2q^2+1 & -q(q-1)^2 & -q(q-1) \\ 1 & q^2-1 & -1 & -q^2+1 & 0 & 0 \\ 1 & -q-1 & q-1 & -q^2+1 & q(q-1) & q \\ 1 & -q-1 & -1 & q+1 & -q(q+1) & q(q+1) \\ 1 & -q-1 & -1 & q+1 & q(q-1) & -q(q-1) \end{pmatrix}$$

$$\mathcal{Q} = \begin{pmatrix} 1 & (q-1)(q+1)^2 & q^3(q-1) & q(q-1)^2(q+1) & \frac{1}{2}q^2(q-1)^3 & \frac{1}{2}q^2(q+1)(q-1)^2 \\ 1 & q^2-q-1 & \frac{q^3(q-1)}{q+1} & -q(q-1) & -\frac{q^2(q-1)^2}{2(q+1)} & -\frac{1}{2}q^2(q-1) \\ 1 & (q-1)(q+1)^2 & -q^3 & q(q-1)^2(q+1) & -\frac{1}{2}q^2(q-1)^2 & -\frac{1}{2}q^2(q+1)(q-1) \\ 1 & q^2-q-1 & -\frac{q^3}{q+1} & -q(q-1) & \frac{q^2(q-1)}{2(q+1)} & \frac{q^2}{2} \\ 1 & -q-1 & 0 & q & -\frac{q^2}{2} & \frac{q^2}{2} \\ 1 & -q-1 & 0 & q & \frac{1}{2}q^2(q-1) & -\frac{1}{2}q^2(q-1) \end{pmatrix}$$

Remark 4.9. We point out that $\mathfrak{X}_P = (\mathcal{X}, \mathcal{R})$ is a fusion of a Schurian scheme. Let G be the collineation group of $H(3, q^2)$ and let G_P be the stabiliser of the point P. Then, it is not difficult to see that the action of G_P on the set $\mathcal{X}$ of points not collinear with P is generously transitive. Hence, the permutation group $G_P^{\mathcal{X}}$ induced by G_P acting on $\mathcal{X}$ yields a Schurian scheme. In fact, this association scheme has $4 + (q-1)/2$ classes, since the relation R_4 splits up under the action of $G_P^{\mathcal{X}}$. Hence, an alternative proof of Theorem 4.8 could be given that would rely on understanding the irreducible constituents of the permutation character $G_P^{\mathcal{X}}$ and computation of the intersections of double cosets $HgH \cap k^{G_P}$, where H is a point stabiliser in $G_P^{\mathcal{X}}$ and $k^{G_P^{\mathcal{X}}}$ is a conjugacy class of elements of $G_P^{\mathcal{X}}$.

A quotient scheme

We now explore the quotient scheme, say $\mathfrak{B}$, arising from the imprimitivity of $\mathfrak{X}_P$. Since $R_0 \cup R_2$ is an equivalence relation on $\mathcal{X}$, the vertices of $\mathfrak{B}$ are the hyperbolic lines through P; this set will be denoted by Σ. Let $\sim$ be the equivalence relation on $\{0, \ldots, 5\}$ defined by

$$i \sim j \quad \text{if and only if} \quad p^j_{i\alpha} \neq 0, \qquad \text{for some } \alpha \in I.$$

The equivalence classes are $I = \{0, 2\}$, $1' = \{1, 3\}$, $2' = \{4, 5\}$. This yields that the non-trivial relations of $\mathfrak{B}$ are

$$\begin{aligned} R_{1'} &= \{([x], [y]) \in \Sigma \times \Sigma : (x, y) \in R_1 \cup R_3\}, \\ R_{2'} &= \{([x], [y]) \in \Sigma \times \Sigma : (x, y) \in R_4 \cup R_5\}, \end{aligned}$$

i.e., $(\Sigma, R_{1'})$ is a strongly regular graph (and the same holds for its complementary graph $(\Sigma, R_{2'})$).

Since $\sim$ is an equivalence relation, the set $R_1(x) \cup R_3(x)$ is partitioned into hyperbolic lines, for any $x \in \mathcal{X}$. Then, the valency of the graph is $k = (\eta_1 + \eta_3)/q = (q^2-1)(q+1)$. Similarly, the set

$$\bigcup_{a,b=1,3} \mathcal{P}_{a,b}^{(x,y)},$$

where $\mathcal{P}_{a,b}^{(x,y)} = \{z \in \mathcal{X} : (x,z) \in R_a, (y,z) \in R_b\}$, for $(x,y) \in R_i$, $i \notin I$, is partitioned into hyperbolic lines. This implies that the other parameters of $(\Sigma, R_{1'})$ are

$$\lambda = \frac{p_{11}^1 + 2p_{13}^1 + p_{33}^1}{q} = \frac{p_{11}^3 + 2p_{13}^3 + p_{33}^3}{q} = 2q^2 - q - 2$$

and

$$\mu = \frac{p_{11}^4 + 2p_{13}^4 + p_{33}^4}{q} = \frac{p_{11}^5 + 2p_{13}^5 + p_{33}^5}{q} = q(q+1).$$

Let $\mathrm{Bil}_2(q)$ be the graph defined on the set of bilinear forms from $V(2,q) \times V(2,q)$ to $\mathrm{GF}(q)$, with two forms f and g being adjacent if and only if $\mathrm{rank}(f-g) = 1$. By [6, Proposition 2.6], $\mathrm{Bil}_2(q)$ is isomorphic to the matrix algebra $\mathcal{D}_2(q^2)$ consisting of all 2×2 Dickson matrices over $\mathrm{GF}(q^2)$, where a 2×2 Dickson matrix over $\mathrm{GF}(q^2)$ has the form

$$D_{(a,b)} = \begin{pmatrix} a & b \\ b^q & a^q \end{pmatrix},$$

with $a, b \in \mathrm{GF}(q^2)$.

Proposition 4.10. *The graph* $(\Sigma, R_{1'})$ *is isomorphic to* $\mathrm{Bil}_2(q)$.

Proof. Any point $\langle(1, r_1, r_2, r_3)\rangle$ with $r_3 + r_3^q = r_1^{q+1} + r_2^{q+1}$, together with $P = \langle(0,0,0,1)\rangle$, spans a hyperbolic line of $H(3,q^2)$, denoted by $l_{(r_1,r_2)}$. Fix $\delta \in \mathrm{GF}(q^2)$ with $\mathrm{N}_{q^2/q}(\delta) = -1$, and define the bijection

$$\varphi : \begin{array}{ccc} \Sigma & \to & \mathrm{Bil}_2(\mathrm{GF}(q)) \\ l_{(r_1,r_2)} & \mapsto & D_{(r_1,\delta r_2)} \end{array}.$$

Let $m = l_{(m_1,m_2)}, n = l_{(r_1,r_2)} \in \Sigma$. For any $Q \in m$ and $R \in n$, $z(P,Q,R) = h(\mathbf{q},\mathbf{r})\mathrm{GF}(q)^*$. Straightforward calculations show that $(m,n) \in R_{1'}$ if and only if $\mathrm{Tr}_{q^2/q}(h(\mathbf{q},\mathbf{r})) = 0$, i.e., $\det(\varphi(m) - \varphi(n)) = 0$. □

5 Pseudo-ovals as subsets in the scheme $\mathfrak{X}_P$ on $Q^-(5,q)$

Via the Klein correspondence κ, the lines of $\mathrm{PG}(3,q^2)$ are mapped to the points of a hyperbolic quadric $Q^+(5,q^2)$ of $\mathrm{PG}(5,q^2)$. In particular, the lines of a unitary polar geometry of rank 2 of $\mathrm{PG}(3,q^2)$ are mapped to the points of an elliptic quadric $Q^-(5,q)$ of a $\mathrm{PG}(5,q)$ embedded in $\mathrm{PG}(5,q^2)$. The reader is referred to [11] for more details on the Klein correspondence.

Let

$$\tau : \begin{array}{ccc} V(6,q^2) & \longrightarrow & V(6,q^2) \\ (X_1,X_2,X_3,X_4,X_5,X_6) & \mapsto & (X_1, -\mu^q X_2, X_3, -\mu^q X_4, X_5, \mu^q X_6), \end{array}$$

for a fixed $\mu \in \mathrm{GF}(q^2)$ with $\mathrm{N}_{q^2/q}(\mu) = -1$, and set $\rho = \tau \circ \kappa$. It turns out that the lines of $H(3, q^2)$ defined by (12) are mapped by ρ to the points of the $Q^-(5, q)$ defined by $\widehat{Q}$ in (1). For any point $P \in H(3, q^2)$, by abuse of notation, we write $\rho(P)$ to denote the totally singular line $\{\rho(r) : r$ is a totally isotropic line on $P\}$.

Proposition 5.1. *Let P, Q and R be three distinct points of the $H(3, q^2)$ defined by (12). Then, $z(P, Q, R) = e$ if and only if $l = \rho(P)$, $m = \rho(Q)$, $n = \rho(R)$ are in perspective.*

Proof. Let P, Q and R such that $z(P, Q, R) = e$. Since $\langle P, Q, R\rangle$ is a non-degenerate plane, coordinates can be chosen such that $P = \langle(1, 0, 0, 0)\rangle$, $Q = \langle(0, 0, 0, 1)\rangle$ and $R = \langle(1, t_1, t_2, t_3)\rangle$, with $t_3 = \frac{1}{2}(t_1^{q+1} + t_2^{q+1}) \neq 0$.

The totally isotropic lines on P have the form $\langle(1, 0, 0, 0), (0, x, \mu x^q, 0)\rangle$, for some non-zero $x \in \mathrm{GF}(q^2)$. Thus, $\rho(P) = L(I, 0, 0)$. Similarly, $\rho(Q) = L(0, 0, I)$.

The totally isotropic lines on R have the form $\langle(1, t_1, t_2, t_3), (0, x, \mu x^q, t_1^q x + t_2^q \mu x^q)\rangle$, for some non-zero $x \in \mathrm{GF}(q^2)$. Thus, $\rho(R) = L(I, F_1, F_2)$, with $F_1(x) = t_1^q x + t_2 \mu x^q$ and $F_2(x) = \frac{1}{2}(t_1^{q+1} - t_2^{q+1})x + t_1 t_2^q \mu x^q$. By using Proposition 3.1, it is immediate that l, m and n are in perspective.

Let l, m, n be three lines of the $Q^-(5, q)$ arising from $\widehat{Q}$ which are in perspective. Set $P = \rho^{-1}(l)$, $Q = \rho^{-1}(m)$, $R = \rho^{-1}(n)$. We proceed to show that $z(P, Q, R) = e$.

Coordinates in $\widehat{V}$ can be chosen such that

$$l = L(I, 0, 0), \quad m = L(0, 0, I), \quad n = L(I, F_1, F_2),$$

with F_1, F_2 non-singular. Let $\overline{n}$ be the extension of n in $V(6, q^2)$, that is

$$\overline{n} = \{(x, y, f_1 x + g_1 y, g_1^q x + f_1^q y, f_2 x + g_2 y, g_2^q x + f_2^q y) : x, y \in \mathrm{GF}(q^2)\}.$$

Note that $P = \rho^{-1}(l) = \langle(1, 0, 0, 0)\rangle$, $Q = \rho^{-1}(m) = \langle(0, 0, 0, 1)\rangle$. Furthermore, we find that $\rho^{-1}(\overline{n})$ actually consists of $q^2 + 1$ coplanar lines of $\mathrm{PG}(3, q^2)$ through R, $q + 1$ of them are totally isotropic. Since m and n are not concurrent, then we may write $R = \langle(1, t_1, t_2, t_3\rangle$, with $t_3^q + t_3 = t_1^{q+1} + t_2^{q+1}$. Straightforward calculations show that these lines have the form $\langle(1, t_1, t_2, t_3), (0, x_1, x_2, t_1^q x_1 + t_2^q x_2)\rangle$, for some $x_1, x_2 \in \mathrm{GF}(q^2)$. For $x_1 = 1$ and $x_2 = 0$, the $\rho-$image of the corresponding line is $\langle(1, 0, t_1^q, \mu^q t_2, t_1^{q+1} - t_3, \mu^q t_1^q t_2)\rangle \in \overline{n}$. On the other hand, the unique point of $\overline{n}$ of this form is $\langle(1, 0, f_1, g_1^q, f_2, g_2^q)\rangle$, whence

$$\begin{cases} t_1^q & = & f_1 \\ \mu^q t_2 & = & g_1^q \\ t_1^{q+1} - t_3 & = & f_2 \\ \mu^q t_1^q t_2 & = & g_2^q \\ t_3^q + t_3 & = & t_1^{q+1} + t_2^{q+1} \end{cases}.$$

Eqs. (3) with $F_0 = I$ make the system compatible, and so we get the unique solution $t_1 = f_1^q$, $t_2 = -\mu g_1^q$ and $t_3 = f_1^{q+1} - f_2$. By Proposition 3.1, $f_2 \in \mathrm{GF}(q)$ providing $t_3 \in \mathrm{GF}(q)$. Then, $z(P, Q, R) = t_3 \mathrm{GF}(q)^* = e$, which is the desired conclusion. □

Proposition 5.1 allows us to view the association scheme $\mathfrak{X}_P$ in the dual setting. Fix a line l of $Q^-(5, q)$ and consider the set $\mathcal{X}'$ of all lines of $Q^-(5, q)$ that are disjoint from $l^\perp$, that is, from l. There are q^5 such lines and we equip this set with the following five non-trivial relations:

$$R'_1 = \{(m,n) : m \text{ and } n \text{ are concurrent}\},$$

$$R'_2 = \{(m,n) : \dim \langle l,m,n\rangle = 4\},$$

$$R'_3 = \{(m,n) : m \text{ and } n \text{ are disjoint and } \dim \langle l,m,n\rangle = 5\},$$

$$R'_4 = \{(m,n) : l,m,n \text{ span the whole space and they are not in perspective}\},$$

$$R'_5 = \{(m,n) : l,m,n \text{ span the whole space and they are in perspective}\}.$$

By using the well-known correspondences given by the Klein map from $H(3,q^2)$ to $Q^-(5,q)$, we get that $\{(\rho(P),\rho(Q)) : (P,Q) \in R_i\} = R'_i$, for $i = 1,2,3$. Proposition 5.1 provides the equivalence between R_5 and R'_5, hence the one between R_4 and R'_4. The transitivity of the unitary group on the points of $H(3,q^2)$, as well as the transitivity of the orthogonal group on the lines of $Q^-(5,q)$, leads to the following result.

Theorem 5.2. *Set $\mathcal{R}' = \{R'_0, R'_1, \ldots, R'_5\}$, where R'_0 is the diagonal relation. $\mathfrak{X}_l = (\mathcal{X}', \mathcal{R}')$ is a symmetric, imprimitive association scheme, isomorphic to $\mathfrak{X}_P = (\mathcal{X}, \mathcal{R})$, for any line l of $Q^-(5,q)$ and any point P of $H(3,q^2)$.*

Let $\{A_i\}_{0\leqslant i\leqslant d}$ be the adjacency matrices for a $d-$class association scheme $\mathfrak{X} = (\mathcal{X}, \{R_i\}_{0\leqslant i\leqslant d})$, and let $\{E_i\}_{0\leqslant i\leqslant d}$ be the set of minimal idempotents for $\mathfrak{X}$. For any subset Y of $\mathcal{X}$, χ_Y will denote the characteristic vector of Y.

The *inner distribution* of a non-empty subset Y of $\mathcal{X}$ is the array $\mathbf{a} = (a_0, \ldots, a_d)$ of the non-negative rational numbers a_i given by

$$a_i = |Y|^{-1}|R_i \cap Y^2| = |Y|^{-1}\chi_Y A_i \chi_Y^\top.$$

Let M be a subset of $\{0,\ldots,d\}$ with $0 \in M$. A non-empty subset Y of $\mathcal{X}$ is an *$M-$clique* of $\mathfrak{X}$ if it satisfies

$$R_i \cap Y^2 = \varnothing, \qquad \text{for all } i \in \{0,\ldots,d\} \setminus M,$$

or equivalently, the $i-$th entry of the inner distribution $\mathbf{a}$ of Y is zero, for all $i \in \{0,\ldots,d\} \setminus M$. Let T be a subset of $\{1,\ldots,d\}$. A non-empty subset Y of $\mathcal{X}$ is a *$T-$design* of $\mathfrak{X}$ if its inner distribution $\mathbf{a}$ satisfies

$$\sum_i a_i \mathcal{Q}(i,j) = 0, \qquad \text{for all } j \in T,$$

where $\mathcal{Q}$ is the second eigenmatrix of the scheme. Equivalently, Y is a $T-$design if and only if $\chi_Y E_j = 0$, for all $j \in T$.

The *dual degree set* of a vector $v \in \mathbb{R}^{|\mathcal{X}|}$ is the set of indices $j \in \{1,\ldots,d\}$ such that $vE_j \neq 0$. Two vectors of $\mathbb{R}^{|\mathcal{X}|}$ are *design-orthogonal* if their dual degree sets are disjoint.

Recall that a *pseudo-oval* of $\mathrm{PG}(5,q)$ is a set $\mathcal{S}$ of q^2+1 lines, such that any three distinct elements of $\mathcal{S}$ span the whole space. We consider pseudo-ovals consisting only of lines of $Q^-(5,q)$.

By transferring on $H(3,q^2)$ the characterization of pseudo-conics of $Q^-(5,q)$ by Thas [18, Theorem 6.4], the characterization of Cossidente, King and Marino [4] is obtained as a corollary of Proposition 5.1.

Corollary 5.3 ([4, Theorem 3.1]). *A special set $\tilde{\mathcal{S}}$ of $H(3,q^2)$ is of CP-type if and only if $z(P,Q,R)=e$, for all triples of distinct points P,Q,R of $\tilde{\mathcal{S}}$.*

Proof. By Theorem 2.1 in [5], a special set $\tilde{\mathcal{S}}$ of CP-type corresponds to a pseudo-conic $\mathcal{S}$ of $Q^-(5,q)$ under the Klein map ρ. By [18, Theorem 6.4], this means that any three distinct elements l,m,n of $\mathcal{S}$ are in perspective, that is, by Proposition 5.1, $z(P,Q,R)=e$, where $P=\rho^{-1}(l)$, $Q=\rho^{-1}(m)$ and $R=\rho^{-1}(n)$. We note that $z(P,Q,R)=e$ is equivalent to the fact that the Segre invariant of (P,Q,R) defined in [4] is equal to 1. □

Proposition 5.4. *Let $\mathcal{S}$ be a set of q^2+1 lines of $Q^-(5,q)$. Then, $\mathcal{S}$ is a pseudo-oval if and only if every non-degenerate hyperplane contains either 0 or 2 elements of $\mathcal{S}$.*

Proof. Let $\mathcal{S}$ be a pseudo-oval. There are $q^2(q^3+1)$ non-degenerate hyperplanes in a $\mathrm{PG}(5,q)$, among which $q^2(q+1)$ contain a given totally singular line. A simple double count shows that the number of non-degenerate hyperplanes containing a pair of disjoint totally singular lines is $q+1$. Now count the triples (l,m,Π) where l and m are distinct totally singular lines of $\mathcal{S}$, Π is a non-degenerate hyperplane, under the conditions that l and m are disjoint and Π contains $\langle l,m\rangle$. For any non-degenerate hyperplane Π_i, let μ_i be the number of elements of $\mathcal{S}$ contained in Π_i.

Then, we have

$$\begin{aligned}\sum_i \mu_i(\mu_i-1) &= |\mathcal{S}|(|\mathcal{S}|-1)(q+1)\\ &= (q^2+1)q^2(q+1).\end{aligned}$$

On the other hand, the number of pairs (l,Π_i), with $l\in\mathcal{S}$ contained in Π_i, is

$$\begin{aligned}\sum_i \mu_i &= |\mathcal{S}|q^2(q+1)\\ &= (q^2+1)q^2(q+1).\end{aligned}$$

Since the two sums are equal, it follows that

$$\sum_i \mu_i(2-\mu_i) = \sum_i \mu_i - \sum_i \mu_i(\mu_i-1) = 0.$$

Every three elements of $\mathcal{S}$ span the whole space, so $\mu_i \leqslant 2$ for each i. Therefore, each term of the left-most sum is positive, hence $\mu_i(2-\mu_i)=0$ for each i, i.e., $\mu_i\in\{0,2\}$.

Conversely, let l,m,n be three lines of $\mathcal{S}$. Assume that l and m intersect. Simple geometric arguments show that $\langle l,m,n\rangle$ is a 4-dimensional subspace which is contained in some non-degenerate hyperplane. This contradicts the property of $\mathcal{S}$. Assume that l and m are disjoint and n intersects $\langle l,m\rangle$ in a point. Then, $\langle l,m,n\rangle$ is a non-degenerate hyperplane, and we have again a contradiction. Therefore, $\mathcal{S}$ is a pseudo-oval. □

Remark 5.5. The " if " part of Proposition 5.4 was already proved in [13], see result 8.7.2. To check this, note that a hyperplane containing $l^\perp$, for some totally singular line l, is degenerate; and conversely.

Theorem 5.6. *Let $\mathcal{S}$ be a pseudo-oval of $Q^-(5,q)$. Then*

(a) $\mathcal{S}\setminus\{l\}$ is a $\{0,4,5\}$−clique of $\mathfrak{X}_l$, and a $\{1\}$−design of $\mathfrak{X}_l$, for each $l\in\mathcal{S}$.

(b) The following are equivalent:

(i) $\mathcal{S} \setminus \{l\}$ *is a* $\{0,5\}-$*clique, for each* $l \in \mathcal{S}$*;*

(ii) $\mathcal{S} \setminus \{l\}$ *is a* $\{1,5\}-$*design, for each* $l \in \mathcal{S}$*;*

(iii) $\mathcal{S}$ *is a pseudo-conic;*

Proof. Let l be any line of $\mathcal{S}$ and set $\mathcal{S}' = \mathcal{S} \setminus \{l\}$. By the definition of pseudo-oval and the scheme $\mathfrak{X}_l$, we find that $\mathcal{S}'$ is a $\{0,4,5\}-$clique of $\mathfrak{X}_l$.

Let $\mathbf{a}$ be the inner distribution of $\mathcal{S}'$ (note that $|\mathcal{S}'| = q^2$):

$$\mathbf{a} = \frac{1}{q^2}\left(\chi_{\mathcal{S}'} A_i \chi_{\mathcal{S}'}^\top\right)_{i=0}^5 = (1,0,0,0,x,q^2-x-1),$$

where x is undetermined. The MacWilliams Transform $\mathbf{a}\mathcal{Q}$ of $\mathbf{a}$ is

$$\mathbf{a}\mathcal{Q} = \left(q^2, 0, q^3(q-1), q^2(q^2-1), \frac{1}{2}q^3(2q^2-4q+2-x), \frac{q^3x}{2}\right). \tag{15}$$

Therefore, $\mathcal{S}'$ is a $\{1\}-$design, and (i) and (ii) are equivalent in (b). To see the equivalence with (iii) in (b), note that $\mathcal{S}'$ is a $\{0,5\}-$clique with respect to every l in $\mathcal{S}$ if and only if l,m,n are in perspective, for every triple l,m,n of distinct lines of $\mathcal{S}$, i.e., $\mathcal{S}$ is a pseudo-conic by [18, Theorem 6.4]. $\square$

Let $\mathcal{U}_{p_1,p_2} = \mathcal{O}_1 \cup \mathcal{O}_2$ be any set constructed as in Section 3.1, and Π the hyperplane containing it. Let $\chi_{\mathcal{O}_i}$ be the characteristic vectors of $\mathcal{O}_i$, , $i=1,2$. We will see (Proposition 5.9) that $v = \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2}$ and the characteristic vector of a pseudo-conic are design-orthogonal.

We introduce the following subsets of $\mathcal{X}$ referred to $\mathcal{U}_{p_1,p_2}$:

- V is the set of lines of $\mathcal{X}$ contained in Π and not intersecting $p_1 \cup p_2$;
- J_i is the set of lines of $\mathcal{X}$ not contained in Π and intersecting p_i, $i=1,2$;
- W is the set of lines of $\mathcal{X}$ not contained in Π and intersecting $(B^\perp \cap \Pi) \setminus (p_1 \cup p_2)$;
- Z is the set of lines of $\mathcal{X}$ not contained in Π and not intersecting $B^\perp \cap \Pi$.

Lemma 5.7.

$$\chi_{\mathcal{O}_1} A_1 = \mathbf{j} + (q-2)\chi_{\mathcal{O}_1} + (q-1)(\chi_{\mathcal{O}_2 \cup V} + \chi_{J_1}) - \chi_{J_2 \cup W};$$

$$\chi_{\mathcal{O}_1} A_2 = \mathbf{j} - \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2 \cup V} - \chi_{J_2 \cup W} - \chi_Z;$$

$$\chi_{\mathcal{O}_1} A_3 = \mathbf{j} + (q^2-q-1)\chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2 \cup V} + (q^2-q-2)\chi_{J_1} + (q-1)\chi_{J_2 \cup W} + (q-2)\chi_Z;$$

$$\chi_{\mathcal{O}_1} A_4 = (q^2-1)(\mathbf{j} - \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2} - \chi_{J_1}) - (q-1)(\chi_{J_2} + \chi_V + 2\chi_Z) - (2q-1)\chi_W;$$

$$\chi_{\mathcal{O}_1} A_5 = \mathbf{j} - \chi_{\mathcal{O}_1} + (q^2-q-1)\chi_{\mathcal{O}_2} - \chi_{J_1} - \chi_{J_2} + (q-1)\chi_W + (q-2)\chi_Z,$$

where $\mathbf{j}$ *is the all-ones vector. Similarly, for* $\chi_{\mathcal{O}_2}$.

Proof. We calculate $\chi_{\mathcal{O}_1} A_1$. This is equivalent to counting how many lines of $\mathcal{O}_1$ are concurrent with a fixed line $n \in \mathcal{X}$.

Assume $n \in \mathcal{O}_1$. Then, there are $q-1$ lines of $\mathcal{O}_1$ concurrent with n. Assume $n \in \mathcal{O}_2 \cup V$. For each point of $p_1 \setminus \{B\}$ there is exactly one line of $\mathcal{O}_1$ intersecting n. So, we find q lines of $\mathcal{O}_1$ concurrent with n. Assume $n \in J_1$. Set $R = n \cap p_1$. Then, the unique lines of $\mathcal{O}_1$ concurrent with n are those through R, which are q. Assume $n \in J_2 \cup W$. Set $R = n \cap B^\perp$. The unique line joining R and p_1 is $\langle B, R \rangle$, that is not in $\mathcal{X}$. In this case n contributes 0. Assume $n \in Z$. Set $R = n \cap \Pi$. There is a unique line joining R and p_1, and it is in $\mathcal{O}_1$. In this case n contributes 1. Finally,

$$\begin{aligned} \chi_{\mathcal{O}_1} A_1 &= (q-1)\chi_{\mathcal{O}_1} + q\chi_{\mathcal{O}_2 \cup V} + q\chi_{J_1} + (\mathbf{j} - \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2 \cup V} - \chi_{J_1} - \chi_{J_2 \cup W}) \\ &= \mathbf{j} + (q-2)\chi_{\mathcal{O}_1} + (q-1)(\chi_{\mathcal{O}_2 \cup V} + \chi_{J_1}) - \chi_{J_2 \cup W}. \end{aligned}$$

We now compute $\chi_{\mathcal{O}_1} A_2$. This is equivalent to counting how many lines of $\mathcal{O}_1$ are contained in the 4-dimensional subspace $\langle l, n \rangle$, for a fixed $n \in \mathcal{X}$.

Assume $n \in \mathcal{O}_1$. Since the plane $\langle l, n \rangle \cap \Pi$, containing n and p_1, is degenerate, there are no lines of $\mathcal{O}_1$ different from n satisfying the property. Assume $n \in \mathcal{O}_2 \cup V$. By arguing as above, it is easy to see that there are no lines of $\mathcal{O}_1$ different from n satisfying the property in this case too. Assume $n \in J_1$. The plane $\langle l, n \rangle \cap \Pi$ is degenerate containing p_1. Thus, it contains exactly a further totally singular line which is necessarily in $\mathcal{O}_1$. Assume $n \in J_2 \cup W$. By arguing as above, the plane $\langle l, n \rangle \cap \Pi$ is degenerate as it contains a totally singular line p on B. Then, the plane contains exactly a further totally singular line intersecting p, which is not in $\mathcal{O}_1$. Assume $n \in Z$. Since the plane $\langle l, n \rangle \cap \Pi$ is non-degenerate, there are no lines of $\mathcal{O}_1$ satisfying the property. Summarising,

$$\chi_{\mathcal{O}_1} A_2 = \chi_{J_1} = \mathbf{j} - \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2 \cup V} - \chi_{J_2 \cup W} - \chi_Z.$$

We now compute $\chi_{\mathcal{O}_1} A_3$. This is equivalent to counting how many lines of $\mathcal{O}_1$ share a point with the 4-dimensional subspace $\langle l, n \rangle$ which is not on n, for a fixed $n \in \mathcal{X}$.

Assume $n \in \mathcal{O}_1$. Since the plane $\langle l, n \rangle \cap \Pi$, containing n and p_1, is degenerate, there are $q(q-1)$ lines of $\mathcal{O}_1$ sharing one point with p_1, different from $n \cap p_1$. Assume $n \in \mathcal{O}_2 \cup V$. By arguing as above, it is easy to see that there are no lines of $\mathcal{O}_1$ satisfying the property. Assume $n \in J_1$. The plane $\langle l, n \rangle \cap \Pi$, containing p_1, is degenerate. Then, it contains exactly a further totally singular line which is necessarily in $\mathcal{O}_1$. Hence, there are $(q-2)q + q - 1 = q^2 - q - 1$ lines of $\mathcal{O}_1$ that intersect p_1. Assume $n \in J_2 \cup W$. The plane $\langle l, n \rangle \cap \Pi$ is degenerate as it contains a totally singular line p on B and a further totally singular line, say s, intersecting p. For any $R \in p_1 \setminus \{B\}$, there is a unique line of $\mathcal{O}_1$ on R concurrent with s. Hence, there are q lines of $\mathcal{O}_1$ that intersect $\langle l, n \rangle$ in a point not in n. Assume $n \in Z$. The plane $\langle l, n \rangle \cap \Pi$ is non-degenerate, and, together with p_1, it spans a 4-dimensional subspace intersecting $Q^-(5, q)$ in either a hyperbolic quadric or a quadratic cone projecting the conic in the plane from a point of $p_1 \setminus \{B\}$. Anyway, the number of lines of $\mathcal{O}_1$ that meet $\langle l, n \rangle$ in exactly one point not in n is $q-1$. Therefore,

$$\begin{aligned} \chi_{\mathcal{O}_1} A_3 &= q(q-1)\chi_{\mathcal{O}_1} + (q^2 - q - 1)\chi_{J_1} + q\chi_{J_2 \cup W} + (q-1)\chi_Z \\ &= \mathbf{j} + (q^2 - q - 1)\chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2 \cup V} + (q^2 - q - 2)\chi_{J_1} + (q-1)\chi_{J_2 \cup W} + (q-2)\chi_Z. \end{aligned}$$

We now compute $\chi_{\mathcal{O}_1} A_5$. This is equivalent to counting how many lines m of $\mathcal{O}_1$ span the whole space together with l and a fixed line $n \in \mathcal{X}$, such that l, m, n are in perspective.

For any $n \in \mathcal{O}_1$, there is no line of $\mathcal{O}_1$ satisfying the property. Assume $n \in \mathcal{O}_2$. By the arguments used to calculate $\chi_{\mathcal{O}_1} A_1$, there are q lines of $\mathcal{O}_1$ which are concurrent with n. All the other $q^2 - q$ lines of $\mathcal{O}_1$ satisfy the property by Lemma 3.5. If $n \in V$, there is no line of $\mathcal{O}_1$ satisfying the property by Lemma 3.5. For any $n \in J_1$, there is no line of $\mathcal{O}_1$ satisfying the property. Assume $n \in J_2$. For any given line $m \in \mathcal{O}_1$, by Lemmas 3.3 (iii) and 3.5, the lines in $\mathcal{X}$ that satisfy the property are those contained in the unique hyperplane Π such that p_1 and p_2 correspond under the involution $\tilde{\sigma}$. Therefore, for any line $n \in J_2$, there are no lines in $\mathcal{O}_1$ satisfying the property. Assume $n \in W$. Let $p \neq p_2$ the unique totally singular line on B concurrent with n. By Lemma 3.5, $m \in \mathcal{O}_1$ satisfies the property if and only if p corresponds to p_1 under the involution arising from some non-degenerate hyperplane containing p_1 and p but not l. Let Λ be such a hyperplane. The 4-dimensional subspace $\Lambda \cap \Pi$, containing p and p_1, meets $Q^-(5, q)$ in either a quadratic cone or a hyperbolic quadric. If the former case occurred, Λ and Π would define the same line σ by Remark 3.4, and then $p = p_2$. Hence, the intersection is necessarily a hyperbolic quadric. This implies that the number of lines of $\mathcal{O}_1$ satisfying the property is q. Assume $n \in Z$. Let p the unique totally singular line on B concurrent with n. By Lemma 3.5, $m \in \mathcal{O}_1$ satisfies the property if and only if p corresponds to p_1 under the involution arising from some non-degenerate hyperplane containing p_1 and p but not l. Let Λ be such a hyperplane. The 4-dimensional subspace $\Lambda \cap \Pi$, containing p_1, meets $Q^-(5, q)$ in the line p_1, a quadratic cone or a hyperbolic quadric. If the former case occurred, $\Lambda \cap \Pi = p_1^\perp$ from which Λ and Π would be degenerate (as $\Lambda^\perp, \Pi^\perp \in p_1$). Hence, the intersection is necessarily a quadratic cone or a hyperbolic quadric. In each case, there is exactly one line of $\mathcal{O}_1$ concurrent with n, and so there are $q - 1$ lines satisfying the property.

Finally,

$$\begin{aligned} \chi_{\mathcal{O}_1} A_5 &= (q^2 - q)\chi_{\mathcal{O}_2} + q\chi_W + (q-1)\chi_Z \\ &= \mathbf{j} - \chi_{\mathcal{O}_1} + (q^2 - q - 1)\chi_{\mathcal{O}_2} - \chi_{J_1} - \chi_{J_2} + (q-1)\chi_W + (q-2)\chi_Z. \end{aligned}$$

We will now calculate $\chi_{\mathcal{O}_1} A_4$, using the fact that the sum of the adjacency matrices is the all-ones matrix J:

$$\begin{aligned} \chi_{\mathcal{O}_1} A_4 &= \chi_{\mathcal{O}_1} J - (\chi_{\mathcal{O}_1} I + \chi_{\mathcal{O}_1} A_1 + \chi_{\mathcal{O}_1} A_2 + \chi_{\mathcal{O}_1} A_3 + \chi_{\mathcal{O}_1} A_5) \\ &= (q^2 - 1)(\mathbf{j} - \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2} - \chi_{J_1}) - (q-1)(\chi_{J_2} + \chi_V + 2\chi_Z) - (2q-1)\chi_W. \end{aligned}$$

The same arguments work for the characteristic vector $\chi_{\mathcal{O}_2}$. □

Corollary 5.8. *Let $v = \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2}$. Then:*

$$vA_1 = -v + q(\chi_{J_1} - \chi_{J_2});$$

$$vA_2 = \chi_{J_1} - \chi_{J_2};$$

$$vA_3 = q(q-1)v + (q^2 - 2q - 1)(\chi_{J_1} - \chi_{J_2});$$

$$vA_4 = -(q^2 - q)(\chi_{J_1} - \chi_{J_2});$$

$$vA_5 = -(q^2 - q)v.$$

Proposition 5.9. *The dual degree set of $v = \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2}$ is $\{1, 5\}$.*

Proof. By using Theorem 4.8, we express each idempotent matrix E_j, $j = 0, \ldots, 5$, of $\mathfrak{X}_l$ in terms of adjacency matrices as $E_i = \frac{1}{q^5}\sum_{j=0}^{5} \mathcal{Q}(j,i)A_j$, where $\mathcal{Q}(j,i)$ is the $(j,i)-$entry of $\mathcal{Q}$. From Corollary 5.8, we have:

$$
\begin{aligned}
q^5 vE_0 &= vJ = 0;\\
q^5 vE_1 &= v\left[(q-1)(q+1)^2(I+A_2) + (q^2-q-1)(A_1+A_3) - (q+1)(A_4+A_5)\right]\\
&= q^4(v + \chi_{J_1} - \chi_{J_2}) \neq 0;\\
q^5 vE_2 &= q^3 v\left[(q-1)I + \frac{(q-1)}{q+1}A_1 - A_2 - \frac{1}{q+1}A_3\right] = 0;\\
q^5 vE_3 &= q\,v\left[(q-1)^2(q+1)(I+A_2) - (q-1)(A_1+A_3) + A_4 + A_5\right] = 0;\\
q^5 vE_4 &= \frac{1}{2}q^2 v\left[(q-1)^3 I - \frac{(q-1)^2}{(q+1)}A_1 - (q-1)^2 A_2 + \frac{(q-1)}{(q+1)}A_3 - A_4 + (q-1)A_5\right] = 0;\\
q^5 vE_5 &= \frac{1}{2}q^2 v\left[(q+1)(q-1)^2 I - (q-1)(A_1+A_5) - (q^2-1)A_2 + A_3 + A_4\right]\\
&= -\,q^2(\chi_{J_1} - \chi_{J_2}) \neq 0.
\end{aligned}
$$

□

Fix a totally singular line l in $Q^-(5,q)$. For any given B on l, two vectors are associated with each $\mathcal{U}_{p_1,p_2} = \mathcal{O}_1 \cup \mathcal{O}_2$ constructed on (B,l), namely, $v = \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2}$ and $-v = \chi_{\mathcal{O}_2} - \chi_{\mathcal{O}_1}$. Let $\mathcal{V}_l$ be the set of all such vectors as B varies on l.

Lemma 5.10.

(a) The number of $\mathcal{U}_{p_1,p_2}$ constructed on the flag (B,l) is $\frac{q+1}{2}q^3(q^2-1)$.

(b) The number of $\mathcal{U}_{p_1,p_2}$ constructed on the flag (B,l) sharing a fixed line disjoint from l is $(q+1)(q^2-1)$.

Proof. To prove (a), by using the polarity associated with $Q^-(5,q)$, it suffices to count the number of non-singular points in $B^\perp \setminus l^\perp$, for all $B \in l$. Secondly, (b) follows from the standard double counting of the pairs $(\mathcal{U}_{p_1,p_2}, m)$, with $m \in \mathcal{U}_{p_1,p_2}$, by considering (a) and the fact that the number of lines of each $\mathcal{U}_{p_1,p_2}$ is $2q^2$. □

Proposition 5.11. *The size of $\mathcal{V}_l$ is* $\dim(V_1 \perp V_5) = q^3(q-1)(q+1)^2$.

Proof. This follows from Lemma 5.10, and taking into account that $v = \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2} \neq -v$. □

Each of the minimal idempotents E_i, $i = 0, \ldots, 5$, of $\mathfrak{X}_l$ projects onto a common eigenspace V_i of the adjacency matrices of the scheme. The vector space $\mathbb{R}^{|\mathcal{X}|}$, endowed with the standard inner product $\cdot$, decomposes as $V_0 \perp \cdots \perp V_5$, and a basis for it is the set of the characteristic vectors χ_m with $m \in \mathcal{X}$. As usual, V_0 is the space spanned by the all-ones vector $\mathbf{j}$. Therefore, the set $\{\chi_m E_i : m \in \mathcal{X}\}$ forms a basis for V_i, for $0 \leqslant i \leqslant 5$, that is, $V_i = \mathrm{row}(E_i)$.

Proposition 5.12. *$\mathcal{V}_l$ spans $V_1 \perp V_5$.*

Proof. Let A be the matrix whose rows are the vectors $\chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2}$ in $\mathcal{V}_l$, and columns are indexed by the elements of the scheme. Let $M = A^\top A$. Note that it consists of the standard scalar products of columns of A.

For any line m, $\mathbf{m}$ will denote the column of A pertaining to m. Index the elements of $\mathcal{V}_l$ by v_i, where $i \in \{1, \ldots, q^3(q-1)(q+1)^2\}$. Then, $\mathbf{m}_i = (v_i)_m$ and, by writing $v_i = \chi_{\mathcal{O}_1} - \chi_{\mathcal{O}_2}$, we have $\mathbf{m}_i = 1$ if m lies in $\mathcal{O}_1$, $\mathbf{m}_i = -1$ if m lies in $\mathcal{O}_2$, $\mathbf{m}_i = 0$ otherwise.

First we calculate what the diagonal entries of M are. Note that $\mathbf{m} \cdot \mathbf{m} = \sum_i \mathbf{m}_i^2$ equals the number of elements of $\mathcal{V}_l$ whose support contains the line m. By using the standard double counting argument on pairs $(\mathcal{O}_1, m)$, with $m \in \mathcal{O}_1$, we get $\mathbf{m} \cdot \mathbf{m} = 2(q-1)(q+1)^2$.

Now suppose n is a line disjoint from l, not equal to m. To evaluate $\mathbf{m} \cdot \mathbf{n}$, we take into account the equalities

$$\mathbf{m}_i \mathbf{n}_i = \begin{cases} 1 & \text{if} \quad m, n \in \mathcal{O}_1 \quad \text{or} \quad m, n \in \mathcal{O}_2 \\ -1 & \text{if} \quad m \in \mathcal{O}_1, n \in \mathcal{O}_2 \quad \text{or} \quad m \in \mathcal{O}_2, n \in \mathcal{O}_1 \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

and how m and n are related in the association scheme. We will use the calculations done in the proof of Lemma 5.7.

Assume $(m, n) \in R_1'$. We first count in two different ways the number of triples $(\mathcal{O}_1, m, n)$ with $m, n \in \mathcal{O}_1$. We obtain

$$c_1 \eta_1 = (q-1)(q-1)(q+1)^2,$$

where c_1 is the number of the sets of type $\mathcal{O}_1$ containing both m and n. Hence, $c_1 = q-1$. Similarly, for $m, n \in \mathcal{O}_2$.

We now count in different ways the number of triples $(\mathcal{O}_1, m, n)$ with $m \in \mathcal{O}_1$ and $n \in \mathcal{O}_2$. It follows that

$$c_2 \eta_1 = q(q-1)(q+1)^2,$$

where c_2 is the number of the sets of type $\mathcal{O}_1 \cup \mathcal{O}_2$ such that $m \in \mathcal{O}_1$ and $n \in \mathcal{O}_2$. Hence, $c_2 = q$. Similarly, for $m \in \mathcal{O}_2$ and $n \in \mathcal{O}_1$. This yields $\mathbf{m} \cdot \mathbf{n} = 2(q-1) - 2q = -2$.

Assume $(m, n) \in R_2'$. Consider the triples $(\mathcal{O}_1, m, n)$ with $m, n \in \mathcal{O}_1$. From the proof of Lemma 5.7, we see that the number of such triples is zero. Similarly, for all other cases in (16). Hence, $\mathbf{m} \cdot \mathbf{n} = 0$.

Assume $(m, n) \in R_3'$. We first count in different ways the number of triples $(\mathcal{O}_1, m, n)$ with $m, n \in \mathcal{O}_1$. We obtain

$$c_3 \eta_3 = q(q-1)(q-1)(q+1)^2,$$

where c_3 is the number of the sets of type $\mathcal{O}_1$ containing both m and n. Hence, $c_3 = q$. Similarly, for $m, n \in \mathcal{O}_2$.

We now count in different ways the number of triples $(\mathcal{O}_1, m, n)$ with $m \in \mathcal{O}_1$ and $n \in \mathcal{O}_2$. From the proof of Lemma 5.7, this number is zero. Hence, $\mathbf{m} \cdot \mathbf{n} = 2q$.

Assume $(m, n) \in R_4'$. Consider the triples $(\mathcal{O}_1, m, n)$ with $m, n \in \mathcal{O}_1$. From the proof of Lemma 5.7, we see that the number of such triples is zero. Similarly, for all other cases in (16). Hence, $\mathbf{m} \cdot \mathbf{n} = 0$.

Assume $(m,n) \in R'_5$. Then, the number of triples $(\mathcal{O}_1, m, n)$ with $m, n \in \mathcal{O}_1$ is zero. Similarly for $m, n \in \mathcal{O}_2$.

We now count in different ways the number of triples $(\mathcal{O}_1, m, n)$ with $m \in \mathcal{O}_1$ and $n \in \mathcal{O}_2$. It follows that

$$c_5\eta_5 = (q^2 - q)(q-1)(q+1)^2,$$

where c_5 is the number of the sets of type $\mathcal{O}_1 \cup \mathcal{O}_2$ such that $m \in \mathcal{O}_1$ and $n \in \mathcal{O}_2$. Hence, $c_5 = q+1$. Similarly, for $m \in \mathcal{O}_2$ and $n \in \mathcal{O}_1$. This yields $\mathbf{m} \cdot \mathbf{n} = -2(q+1)$.

Therefore,

$$M = 2\left((q-1)(q+1)^2 I - A_1 + qA_3 - (q+1)A_5\right)$$

and, from the first eigenmatrix $\mathcal{P}$, we see that

$$M = 2q^2(q^2 E_1 + 2(q+1)E_5).$$

It is well known (c.f., [7, Eq. (2.10)]) that there exists an orthogonal matrix U which simultaneously diagonalises each of the minimal idempotents of the scheme, i.e.,

$$U^{-1}E_iU = \operatorname{diag}(0, \ldots, 0, \underbrace{1, \ldots, 1}_{\dim V_i}, 0, \ldots, 0),$$

for $i = 0, \ldots, 5$. This implies M itself takes a diagonal form with respect to the basis of the eigenvectors of E_i, so that

$$\operatorname{row}(M) = \operatorname{row}(E_1) \perp \operatorname{row}(E_5) = V_1 \perp V_5.$$

Therefore, $V_1 \perp V_5 = \operatorname{row}(M) \leqslant \operatorname{row}(A) = \langle \mathcal{V}_l \rangle$. By Proposition 5.9, $V_1 \perp V_5 = \langle \mathcal{V}_l \rangle$. □

Theorem 5.13. *Let $\mathcal{S}$ be a pseudo-oval of $Q^-(5,q)$. Then, the following are equivalent:*

(a) $\mathcal{S}$ is a pseudo-conic;

(b) for any l in $\mathcal{S}$ and B in l, each set $\mathcal{U}_{p_1,p_2}$ constructed on (B,l) meets $\mathcal{S}\backslash\{l\}$ in 0 or 2 elements.

Proof. By Theorem 5.6, $\mathcal{S}$ is a pseudo-conic if and only if $\mathcal{S}' = \mathcal{S} \setminus \{l\}$ is a $\{1,5\}-$design of $\mathfrak{X}_l$, for any l in $\mathcal{S}$. Fix $l \in \mathcal{S}$. By Proposition 5.12, $\mathcal{V}_l$ spans $V_1 \perp V_5$. Hence, $\mathcal{S}'$ is a $\{1,5\}-$design of $\mathfrak{X}_l$ if and only if $\chi_{\mathcal{S}'} \cdot v = 0$, for all $v \in \mathcal{V}_l$. On the other hand,

$$\chi_{\mathcal{S}'} \cdot v = \chi_{\mathcal{S}'} \cdot \chi_{\mathcal{O}_1} - \chi_{\mathcal{S}'} \cdot \chi_{\mathcal{O}_2} = |\mathcal{S} \cap \mathcal{O}_1| - |\mathcal{S} \cap \mathcal{O}_2|.$$

Since $\mathcal{S}$ is a pseudo-oval, we have $|\mathcal{S} \cap \mathcal{O}_1|, |\mathcal{S} \cap \mathcal{O}_2| \leqslant 1$. Furthermore,

$$|\mathcal{S} \cap (\mathcal{O}_1 \cup \mathcal{O}_2)| = |\mathcal{S} \cap \mathcal{O}_1| + |\mathcal{S} \cap \mathcal{O}_2|,$$

because $\mathcal{O}_1$ and $\mathcal{O}_2$ are disjoint sets of lines. Hence, $\chi_{\mathcal{S}'} \cdot v = 0$, for all $v \in \mathcal{V}_l$, if and only if each $\mathcal{U}_{p_1,p_2}$ meets $\mathcal{S}'$ in 0 or 2 elements. □

Theorem 5.13 and the following proposition provide an additional way to characterise pseudo-conics.

Proposition 5.14. *Let $\mathcal{S}$ be a pseudo-oval of $Q^-(5,q)$ and $l \in \mathcal{S}$. Let A be the average number of $\mathcal{U}_{p_1,p_2}$ over all flags (B,l) containing two distinct elements of $\mathcal{S} \setminus \{l\}$. Then, $A = q+1$ if and only if each $\mathcal{U}_{p_1,p_2}$ meets $\mathcal{S}\setminus\{l\}$ in 0 or 2 elements.*

Proof. Let μ_i be the number of lines of $\mathcal{S}' = \mathcal{S} \setminus \{l\}$ contained in the i−th set of type $\mathcal{U}_{p_1,p_2}$. We count in two ways the number of pairs $(m, \mathcal{U}_{p_1,p_2})$ such that $m \in \mathcal{U}_{p_1,p_2} \cap \mathcal{S}'$. From Lemma 5.10(b), this number is

$$\sum_i \mu_i = |\mathcal{S}'|(q+1)(q^2-1) = q^2(q+1)(q^2-1).$$

If we double count triples $(m, n, \mathcal{U}_{p_1,p_2})$ where m and n are distinct elements of $\mathcal{S}'$ lying in $\mathcal{U}_{p_1,p_2}$, we see that

$$\sum_i \mu_i(\mu_i - 1) = \sum_{m\in\mathcal{S}'} \sum_{n\in\mathcal{S}'\setminus\{m\}} |\{\mathcal{U}_{p_1,p_2} : m, n \in \mathcal{U}_{p_1,p_2}\}| = q^2(q^2-1)A.$$

Hence,

$$\sum_i \mu_i(2-\mu_i) = \sum_i \mu_i - \sum_i \mu_i(\mu_i-1) = q^2(q^2-1)(q+1-A).$$

Therefore, $A = q+1$ precisely when each μ_i is 0 or 2. □

6 Concluding remarks

In [1], Theorem 5.1 shows that any special set of $H(3,9)$ is of CP-type, or dually, any pseudo-oval in $Q^-(5,3)$ is a pseudo-conic. By using GAP and the mixed integer linear programming software Gurobi [10], we explored the case $q = 5$ and $q = 7$. Indeed, the theory developed in this paper aided in the design of the computation.

We look at a given pseudo-oval as a set $\mathcal{S}$ of lines of $Q^-(5,q)$ such that every non-degenerate hyperplane contains 0 or 2 elements of $\mathcal{S}$ by Proposition 5.4. As we have done throughout this paper, we let l be a fixed line of $Q^-(5,q)$. Let M be the incidence matrix with rows indexed by lines of $Q^-(5,q)$ disjoint from l, and columns indexed by the non-degenerate hyperplanes not containing l. Then, we are seeking a solution to

$$\boldsymbol{x}M = 2\boldsymbol{y}, \tag{17}$$

where $\boldsymbol{x} = (x_1, \dots, x_{q^5})$ and $\boldsymbol{y} = (y_1, \dots, y_{q^5-q^3})$, with $x_i, y_i \in \{0,1\}$, and $\sum x_i = q^2$. In fact, $\boldsymbol{x}$ will be the characteristic vector for $\mathcal{S} \setminus \{l\}$, with $\mathcal{S}$ a pseudo-oval in $Q^-(5,q)$, and $\boldsymbol{y}$ will be the characteristic vector for the set of non-degenerate hyperplanes not containing l, sharing two elements with $\mathcal{S} \setminus \{l\}$.

There are a variety of approaches to solving equations such as (17). In particular, the system of equations can be viewed either as an *integer linear program* or as a *constraint satisfaction problem*. We used the software Gurobi for this problem.

A linear program attempts to find values for variables $x_1, x_2, \dots, x_n$ that maximise (or minimise) a linear objective function subject to linear constraints. An *integer linear program*, for short *integer program*, is a linear program with the additional restriction that the variables must take integral values. Solving (17) does not involve any maximising or minimising, so the objective function can be taken to be a constant, say 0. Then, any feasible solution to the following integer program yields a set of lines with the property:

$$\begin{array}{lrcl} \text{Maximise:} & & 0 & \\ \text{subject to:} & \boldsymbol{x}B - 2\boldsymbol{y} & = & 0 \\ & \sum_i x_i & = & q^2 \\ & x_i, y_j & \in & \{0,1\}. \end{array} \tag{18}$$

There is one more ingredient we need to take into account. For a fixed set $U = \mathcal{U}_{p_1,p_2}$, let $\boldsymbol{u}$ be the characteristic vector for it. We assume that the set U meets $\mathcal{S} \setminus \{l\}$ in precisely 1 element. This adds the linear constraint $\sum u_i x_i = 1$. For $q = 3, 5, 7$, we found that the linear program (18) is infeasible[3] for each $\mathcal{U}_{p_1,p_2}$. Therefore, in these cases, every set $\mathcal{S} \setminus \{l\}$ is forced to meet the sets $\mathcal{U}_{p_1,p_2}$ in 0 or 2 elements, and so every $\mathcal{S}$ is a pseudo-conic by Theorem 5.13.

The above computational results suggest the following conjecture:

Conjecture 1. *Every pseudo-oval in $Q^-(5, q)$ is a pseudo-conic, for any q (odd).*

Results 5.4 and 5.13 allow us to state the above conjecture as follows:

Conjecture 2. *Let $\mathcal{S}$ be a set of $q^2 + 1$ lines of $Q^-(5, q)$, q odd, such that every non-degenerate hyperplane contains 0 or 2 elements of $\mathcal{S}$. Then, each $\mathcal{U}_{p_1,p_2}$ meets $\mathcal{S}\setminus\{l\}$ in 0 or 2 elements, for every $l \in \mathcal{S}$.*

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