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Completeness Theorems on the Boundary for a Parabolic Equation

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Abstract. Let $\{v_\alpha\}$ be a system of polynomial solutions of the parabolic equation $a_{hk}\partial_{x_h x_k} u - \partial_t u = 0$ in a bounded C^1-cylinder Ω_T contained in $\mathbb{R}^{n+1}$. Here, $a_{hk}\partial_{x_h x_k}$ is an elliptic operator with real constant coefficients. We prove that $\{v_\alpha\}$ is complete in $L^p(\Sigma')$, where Σ' is the parabolic boundary of Ω_T. Similar results are proved for the adjoint equation $a_{hk}\partial_{x_h x_k} u + \partial_t u = 0$.

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1. Introduction

The problem of the completeness of certain systems of solutions—in particular polynomial solutions—of a PDE on the boundary has a long history. The prototype of these results is the completeness of harmonic polynomials on the boundary of a bounded domain. The very first results proved in this field are due to Fichera [7], who proved completeness theorems for harmonic polynomials related to Dirichlet, Neumann, Robin and the mixed BVP for Laplace equation. For a full description of the problem and a review of several known results, we refer to Section 2 of [2]. That paper contains a fairly complete list of references, to which we add the more recent [3,4].

Very few results have been obtained for parabolic equations in this field. Completeness theorems for the heat equation in a C^2-cylinder were proved by Magenes in his pioneeristic papers [9,10] (see also Pagni [14]).

Here, we consider the more general parabolic operators

$$H = E - \frac{\partial}{\partial t} \tag{1}$$

and its adjoint

$$H^* = E + \frac{\partial}{\partial t}, \tag{2}$$

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where E is a second order elliptic operator with real constant coefficients and no lower order terms.

The aim of the present paper is to prove completeness theorems related to the Dirichlet problem for the equation $Hu = 0$ ($H^*u = 0$) in a C^1-cylinder $\Omega_T = \Omega \times (0, T)$. More precisely, we shall prove that the system of parabolic polynomials—i.e. polynomials satisfying the equation $Hu = 0$ ($H^*u = 0$) - is complete in $L^p(\Sigma')$ ($1 \le p < \infty$), where Σ' is the parabolic boundary of Ω_T related to the operator H (H^*).

The completeness results hinge on the study of certain classes of solutions, which we denote by $\mathscr{A}^p(\Omega_T)$ and $\mathscr{A}^p_*(\Omega_T)$. A key ingredient of this study is potential theory on C^1-cylinder. The relevant fundamental results have been obtained by Fabes and Rivière [6]. They considered potentials for the heat equation, but their results can be extended to the more general parabolic potentials here considered.

Another important ingredient is an approximation result due to Malgrange [11]. This will enable us to deduce the completeness property of parabolic polynomials from a completeness property we obtain for certain potentials.

The paper is organized as follows. After some preliminaries given in Sect. 2, several properties of parabolic potentials are proved in Sect. 3.

Section 4 is devoted to the study of the classes $\mathscr{A}^p(\Omega_T)$ and $\mathscr{A}^p_*(\Omega_T)$. In particular, some uniqueness theorems are obtained.

In Sect. 5 we construct a system of parabolic polynomials $\{v_\alpha\}$ ($\{w_\alpha\}$), such that any polynomial satisfying the equation $Hu = 0$ ($H^*u = 0$) can be written as a finite linear combination of $\{v_\alpha\}$ ($\{w_\alpha\}$).

Completeness theorems are then proved in Sect. 6.

2. Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^n$ ($n \ge 3$). We assume always that $\mathbb{R}^n \setminus \overline{\Omega}$ is connected and that $\partial\Omega$ is a C^1 boundary.

Given a positive T, we consider the C^1-cylinder $\Omega_T = \Omega \times (0, T)$ in $\mathbb{R}^{n+1}$. Points in $\mathbb{R}^{n+1}$ will be denoted by (x, t), where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, and $\langle \cdot, \cdot \rangle$ stands for the inner product in $\mathbb{R}^n$.

Let H be the parabolic operator (1) and H^* be its adjoint (2), where

$$Eu = \sum_{h,k=1}^{n} a_{hk} \frac{\partial^2 u}{\partial x_h \, \partial x_k},$$

$A = \{a_{hk}\}$ being a real symmetric and positive definite matrix. $|A|$ denotes the determinant of A and by A^{-1} we mean the inverse matrix of A.

If $u, v \in C^2(\overline{\Omega}_T)$ the following Green's formula for the operator H holds

$$\int_{\Omega_T} [vH(u) - uH^*(v)] \mathrm{d}x \, \mathrm{d}t = -\int_{\Sigma_3} \left(v \frac{\partial u}{\partial \overline{\nu}} - u \frac{\partial v}{\partial \overline{\nu}} \right) \mathrm{d}\sigma - \int_{\Sigma_1} uv \, \mathrm{d}\sigma + \int_{\Sigma_2} uv \, \mathrm{d}\sigma \tag{3}$$

where

$$\Sigma_1 = \{(x,T) \mid x \in \Omega\},\ \Sigma_2 = \{(x,0) \mid x \in \Omega\},\ \Sigma_3 = \partial\Omega \times (0,T),$$

ν is the interior normal vector to $\partial\Omega$ and $\overline{\nu}$ is the conormal vector defined by $(A\nu, 0)$ on Σ_3 (see, e.g., [13, p. 20]).

The fundamental solution for the parabolic equation $Hu = 0$ is given by

$$G(x,t) = \begin{cases} \frac{1}{(4\pi t)^{\frac{n}{2}}\,|A|^{\frac{1}{2}}} \exp[-\frac{\langle A^{-1}x,x\rangle}{4t}], & \text{if } t > 0, \\ 0, & \text{if } t \le 0. \end{cases}$$

More precisely, $G(x-y,t-s)$ as a function of (x,t) is of class $C^\infty(\mathbb{R}^{n+1} \setminus \{(y,s)\})$ and is a fundamental solution for the operator $-H$ (see, e.g., [8, p. 146], where the result is given for the heat equation, but it can be easily adapted to $-H$), while as a function of (y,s) is of class $C^\infty(\mathbb{R}^{n+1} \setminus \{(x,t)\})$ and is a fundamental solution for the operator $-H^*$.

We observe that, using the fundamental solution G, we have a “Stokes representation formula” for smooth enough functions:

$$\begin{aligned} &-\int_{\Omega_T} G(x-y,t-s)\, Hu(y,s)\, \mathrm{d}y\mathrm{d}s \\ &\quad + \int_{\Sigma_3} \left(u(y,s)\, \frac{\partial}{\partial \overline{\nu}_{y,s}} G(x-y,t-s) - \frac{\partial u(y,s)}{\partial \overline{\nu}_{y,s}}\, G(x-y,t-s) \right) \mathrm{d}\sigma_{y,s} \\ &\quad - \int_{\Sigma_1} u(y,T)\, G(x-y,t-T)\, \mathrm{d}y + \int_{\Sigma_2} u(y,0)\, G(x-y,t)\, \mathrm{d}y \\ &\quad = \begin{cases} u(x,t) & \text{if } (x,t) \in \Omega_T, \\ 0 & \text{if } (x,t) \notin \overline{\Omega}_T. \end{cases} \end{aligned} \tag{4}$$

Note that, if $t < T$, in particular when $(x,t) \in \Omega_T$,

$$\int_{\Sigma_1} u(y,T)\, G(x-y,t-T)\, \mathrm{d}y = 0.$$

We have also

$$\begin{aligned} &\int_{\Omega_T} G(x-y,t-s)\, H^*u(x,t)\, \mathrm{d}x\mathrm{d}t \\ &\quad + \int_{\Sigma_3} \left(u(x,t)\, \frac{\partial}{\partial \overline{\nu}_{x,t}} G(x-y,t-s) - \frac{\partial u(x,t)}{\partial \overline{\nu}_{x,t}}\, G(x-y,t-s) \right) \mathrm{d}\sigma_{x,t} \\ &\quad + \int_{\Sigma_1} u(x,T)\, G(x-y,T-s)\, \mathrm{d}x - \int_{\Sigma_2} u(x,0)\, G(x-y,-s)\, \mathrm{d}x \\ &\quad = \begin{cases} u(y,s) & \text{if } (y,s) \in \Omega_T, \\ 0 & \text{if } (y,s) \notin \overline{\Omega}_T, \end{cases} \end{aligned} \tag{5}$$

and

$$\int_{\Sigma_2} u(x,0)\, G(x-y,-s)\, \mathrm{d}x = 0$$

if $s > 0$, in particular when $(y,s) \in \Omega_T$.

As a consequence of these representation formulas, we have that, if A is a domain in $\mathbb{R}^{n+1}$, then

$$\begin{aligned} v(x,t) &= -\int_A G(x-y,t-s)\, Hv(y,s)\, \mathrm{d}y\mathrm{d}s, \quad \forall\ (x,t) \in \mathbb{R}^{n+1}, \\ v(y,s) &= -\int_A G(x-y,t-s)\, H^*v(x,t)\, \mathrm{d}x\mathrm{d}t, \quad \forall\ (y,s) \in \mathbb{R}^{n+1} \end{aligned} \tag{6}$$

for any $v \in \mathring{C}^\infty(A)$, where $\mathring{C}^\infty(A)$ is the space of C^∞ functions with compact support contained in A.

As usual, we denote by $L^p(\Sigma)$ $(1 \le p < \infty)$ the vector space of all measurable real valued functions f such that $|f|^p$ is integrable over Σ.

3. The Jump Formulas for Parabolic Potentials

Let $\varphi \in L^p(\Sigma_3)$ $(1 \le p < \infty)$. The double layer parabolic potential with density φ is defined as

$$\int_{\Sigma_3} \varphi(y,s)\, \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{y,s}}\, \mathrm{d}\sigma_{y,s}.$$

The next results show that the jump relations holding on the lateral boundary Σ_3 for caloric potentials also hold for this more general type of potentials.

Theorem 1. *Let $\varphi \in L^p(\Sigma_3)$ $(1 \le p < \infty)$. Then*

$$\begin{aligned} \lim_{(x,t)\to(x_0,t)^\pm} \int_{\Sigma_3} \varphi(y,s)\, \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{y,s}}\, \mathrm{d}\sigma_{y,s} &= \pm\frac{1}{2}\varphi(x_0,t) \\ &+ \int_{\Sigma_3} \varphi(y,s)\, \frac{\partial G(x_0-y,t-s)}{\partial \overline{\nu}_{y,s}}\, \mathrm{d}\sigma_{y,s} \end{aligned} \tag{7}$$

for almost every $(x_0,t) \in \Sigma_3$. Here, the limit $(x,t) \to (x_0,t)^+$ $((x,t) \to (x_0,t)^-)$ has to be understood as an internal (external) angular boundary value and the integral on the right-hand side exists as a singular integral. Moreover, for any $1 < p < \infty$ and for any real $c \neq 0$, the operator

$$c\,\varphi(x,t) + \int_{\Sigma_3} \varphi(y,s)\, \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{y,s}}\, \mathrm{d}\sigma_{y,s} \tag{8}$$

is invertible on $L^p(\Sigma_3)$.

Proof. We begin by considering the potential

$$\int_{\Sigma_3^\infty} \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{y,s}}\, \mathrm{d}\sigma_{y,s},$$

where $\Sigma_3^\infty = \partial\Omega \times (-\infty, +\infty)$. This is equal to

$$\frac{1}{2(4\pi)^{n/2}|A|^{1/2}} \int_{-\infty}^t \frac{\mathrm{d}s}{(t-s)^{1+n/2}} \int_{\partial\Omega} \langle x-y, \nu(y)\rangle \mathrm{e}^{-\frac{\langle A^{-1}(x-y),(x-y)\rangle}{4(t-s)}}\, \mathrm{d}\sigma_y. \tag{9}$$

Setting $u=\langle A^{-1}(x-y),(x-y)\rangle/(4(t-s))$, we find

$$\begin{aligned}&\frac{1}{2\pi^{n/2}|A|^{1/2}}\int_0^{+\infty}u^{\frac{n}{2}-1}\mathrm{e}^{-u}\mathrm{d}u\int_{\partial\Omega}\frac{\langle x-y,\nu(y)\rangle}{\langle A^{-1}(x-y),(x-y)\rangle^{n/2}}\mathrm{d}\sigma_y\\&\quad=\frac{\Gamma(n/2)}{2\pi^{n/2}|A|^{1/2}}\int_{\partial\Omega}\frac{\langle x-y,\nu(y)\rangle}{\langle A^{-1}(x-y),(x-y)\rangle^{n/2}}\mathrm{d}\sigma_y\\&\quad=\frac{1}{\omega_n|A|^{1/2}}\int_{\partial\Omega}\frac{\langle x-y,\nu(y)\rangle}{\langle A^{-1}(x-y),(x-y)\rangle^{n/2}}\mathrm{d}\sigma_y.\end{aligned}$$

This shows that, denoting by $s(x,y)$ the fundamental solution of the elliptic equation $Eu=0$, i.e.

$$s(x,y)=\frac{1}{(2-n)\omega_n|A|^{1/2}}\langle A^{-1}(x-y),(x-y)\rangle^{(2-n)/2},$$

we have

$$\int_{\Sigma_3^\infty}\frac{\partial G(x-y,t-s)}{\partial\overline{\nu}_{y,s}}\,\mathrm{d}\sigma_{y,s}=-\int_{\partial\Omega}\frac{\partial}{\partial\overline{\nu}_y}s(x,y)\,\mathrm{d}\sigma_y$$

and then

$$\int_{\Sigma_3^\infty}\frac{\partial G(x-y,t-s)}{\partial\overline{\nu}_{y,s}}\,\mathrm{d}\sigma_{y,s}=\begin{cases}1 & (x,t)\in\Omega\times(-\infty,+\infty)\\ \frac{1}{2} & (x,t)\in\partial\Omega\times(-\infty,+\infty)\\ 0 & (x,t)\in(\mathbb{R}^n\setminus\overline{\Omega})\times(-\infty,+\infty).\end{cases}\tag{10}$$

Suppose now $p>1$ and let $\varphi\in L^p(\Sigma_3)$. By means of the arguments used in [6, Th. 1.1] one can show that the integral

$$\int_{\Sigma_3}\varphi(y,s)\frac{\partial G(x-y,t-s)}{\partial\overline{\nu}_{y,s}}\,\mathrm{d}\sigma_{y,s}$$

does exist as a singular integral for almost every $(x,t)\in\Sigma_3$.

To prove (7), consider first a function $\varphi\in\mathring{C}^1(\Sigma_3)$. This means that $\varphi\in C^1(\Sigma_3^\infty)$ and its support is contained in $\partial\Omega\times(0,T)$. Let (x_0,t) be in Σ_3; keeping in mind (10), we have

$$\begin{aligned}&\int_{\Sigma_3}\varphi(y,s)\frac{\partial G(x-y,t-s)}{\partial\overline{\nu}_{y,s}}\mathrm{d}\,\sigma_{y,s}\\&\quad=\int_{\Sigma_3^\infty}[\varphi(y,s)-\varphi(x_0,t)]\frac{\partial G(x-y,t-s)}{\partial\overline{\nu}_{y,s}}\mathrm{d}\,\sigma_{y,s}+\varphi(x_0,t)\end{aligned}$$

for any $(x,t)\in\Omega_T$. As far as the integral on the right hand side is concerned, we can write it as

$$C\int_{-\infty}^{t}\frac{\mathrm{d}s}{(t-s)^{1+n/2}}\int_{\partial\Omega}\langle x-y,\nu(y)\rangle[\varphi(y,s)-\varphi(x_0,t)]\mathrm{e}^{-\frac{\langle A^{-1}(x-y),(x-y)\rangle}{4(t-s)}}\,\mathrm{d}\sigma_y.$$

Similarly to what we did for (9), this integral is equal to

$$C\int_0^{+\infty}u^{\frac{n}{2}-1}\mathrm{e}^{-u}\,\mathrm{d}u\int_{\partial\Omega}\frac{\langle x-y,\nu(y)\rangle}{\langle A^{-1}(x-y),(x-y)\rangle^{\frac{n}{2}}}\Lambda(x_0,x,y,t,u)\,\mathrm{d}\sigma_y\tag{11}$$

where

$$\Lambda(x_0,x,y,t,u)=\varphi\left(x,t-\langle A^{-1}(x-y),(x-y)\rangle/4u\right)-\varphi(x_0,t).$$

Since there exists a constant K such that the modulus of the integrand in (11) can be majorized by

$$K\,\frac{u^{\frac{n}{2}-1}\mathrm{e}^{-u}}{|y-x_0|^{n-2}},$$

we may apply Lebesgue's dominated convergence theorem and, recalling (10), conclude that

$$\begin{aligned}
&\lim_{(x,t)\to(x_0,t)^+}\int_{\Sigma_3}\varphi(y,s)\,\frac{\partial G(x-y,t-s)}{\partial\overline{\nu}_{y,s}}\,\mathrm{d}\sigma_{y,s}\\
&\quad=\int_{\Sigma_3^\infty}[\varphi(y,s)-\varphi(x_0,t)]\frac{\partial G(x_0-y,t-s)}{\partial\overline{\nu}_{y,s}}\mathrm{d}\,\sigma_{y,s}+\varphi(x_0,t)\\
&=\int_{\Sigma_3}\varphi(y,s)\frac{\partial G(x_0-y,t-s)}{\partial\overline{\nu}_{y,s}}\mathrm{d}\sigma_{y,s}-\varphi(x_0,t)\int_{\Sigma_3^\infty}\frac{\partial G(x_0-y,t-s)}{\partial\overline{\nu}_{y,s}}\mathrm{d}\sigma_{y,s}\\
&\quad+\varphi(x_0,t)=\int_{\Sigma_3}\varphi(y,s)\frac{\partial G(x_0-y,t-s)}{\partial\overline{\nu}_{y,s}}\mathrm{d}\sigma_{y,s}+\frac{1}{2}\varphi(x_0,t).
\end{aligned}$$

The same proof works for $(x,t)\to(x_0,t)^-$. Formula (7) is then proved if $\varphi\in\mathring{C}^1(\Sigma_3)$.

Let now $\varphi\in L^p(\Sigma_3)$. Given α, $0<\alpha<1$, consider the non tangential maximal function

$$U^*(x,t)=\sup\{|U(\xi,t)|\ :\ \xi\in\Omega,\ |\xi-x|<\delta,\ \langle\xi-x,\nu(x)\rangle\ >\alpha|\xi-x|\},$$

where

$$U(x,t)=\int_{\Sigma_3}\varphi(y,s)\frac{\partial G(x-y,t-s)}{\partial\overline{\nu}_{y,s}}\,\mathrm{d}\sigma_{y,s}.$$

Reasoning as in [6, Th. 2.1], there exists a constant $\delta=\delta_{\Omega,\alpha}$ such that U^* belongs to $L^p(\Sigma_3)$ and

$$\|U^*\|_{L^p(\Sigma_3)}\le C\|\varphi\|_{L^p(\Sigma_3)}$$

with C independent of φ. By standard arguments, the pointwise limit (7) follows from what we have already proved when $\varphi\in\mathring{C}^1(\Sigma_3)$ (see [5, p. 172–173]).

The case $p=1$ is slightly different, because the double layer potential is bounded from $L^1(\Sigma_3)$ into weak-$L^1(\Sigma_3)$. Nevertheless the point-wise limit (7) is still valid, see the Remark in [5, p. 173].

The invertibility of operator (8) can be proved as in [6, Th. 1.3]. $\square$

A similar result holds for the conormal derivative of the single layer parabolic potential

$$\int_{\Sigma_3}\varphi(y,s)G(x-y,t-s)\,d\sigma_{y,s}.$$

Theorem 2. *Let $\varphi\in L^p(\Sigma_3)$ $(1\le p<\infty)$. Then*

$$\begin{aligned}
\lim_{(x,t)\to(x_0,t)^\pm}\int_{\Sigma_3}\varphi(y,s)\,\frac{\partial G(x-y,t-s)}{\partial\overline{\nu}_{x_0,t}}\,\mathrm{d}\sigma_{y,s}&=\mp\frac{1}{2}\varphi(x_0,t)+\\
+\int_{\Sigma_3}\varphi(y,s)\,\frac{\partial G(x_0-y,t-s)}{\partial\overline{\nu}_{x_0,t}}\,\mathrm{d}\sigma_{y,s}&
\end{aligned}\tag{12}$$

for almost every $(x_0, t) \in \Sigma_3$. As in Theorem 1, the limit $(x,t) \to (x_0,t)^+$ ($(x,t) \to (x_0,t)^-$) has to be understood as an internal (external) angular boundary value and the integral on the right hand side exists as a singular integral. Moreover, for any $1 < p < \infty$ and for any real $c \neq 0$, the operator

$$c\,\varphi(x,t) + \int_{\Sigma_3} \varphi(y,s)\,\frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x,t}}\,\mathrm{d}\sigma_{y,s} \tag{13}$$

is invertible on $L^p(\Sigma_3)$.

Proof. The proof is similar to that of Theorem 1. The only difference concerns the proof of the jump relation when $\varphi \in \mathring{C}^1(\Sigma_3)$. Now we have

$$\int_{\Sigma_3^\infty} \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x,t}}\,\mathrm{d}\sigma_{y,s} = -\int_{\partial\Omega} \frac{\partial}{\partial \overline{\nu}_x} s(x,y)\,\mathrm{d}\sigma_y$$

and then a formula like (10) does not hold. However, it is known that

$$\lim_{x \to x_0^\pm} \int_{\partial\Omega} \frac{\partial}{\partial \overline{\nu}_{x_0}} s(x,y)\,\mathrm{d}\sigma_y = \pm\frac{1}{2} + \int_{\partial\Omega} \frac{\partial}{\partial \overline{\nu}_{x_0}} s(x_0,y)\,\mathrm{d}\sigma_y$$

(see e.g. [12, p. 35]) and then

$$\begin{aligned}
&\lim_{(x,t)\to(x_0,t)^+} \int_{\Sigma_3} \varphi(y,s)\,\frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x_0,t}}\,\mathrm{d}\sigma_{y,s}\\
&= \lim_{(x,t)\to(x_0,t)^+} \left(\int_{\Sigma_3^\infty} [\varphi(y,s)-\varphi(x_0,t)]\frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x_0,t}}\mathrm{d}\,\sigma_{y,s} \right.\\
&\qquad \left. +\varphi(x_0,t) \int_{\Sigma_3^\infty} \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x_0,t}}\mathrm{d}\sigma_{y,s} \right)\\
&= \int_{\Sigma_3^\infty} [\varphi(y,s)-\varphi(x_0,t)]\frac{\partial G(x_0-y,t-s)}{\partial \overline{\nu}_{x_0,t}}\mathrm{d}\sigma_{y,s}\\
&\qquad +\varphi(x_0,t)\left(-\frac{1}{2} + \int_{\Sigma_3^\infty} \frac{\partial G(x_0-y,t-s)}{\partial \overline{\nu}_{x_0,t}}\mathrm{d}\sigma_{y,s} \right)\\
&= \int_{\Sigma_3} \varphi(y,s)\frac{\partial G(x_0-y,t-s)}{\partial \overline{\nu}_{x_0,t}}\mathrm{d}\sigma_{y,s} - \frac{1}{2}\varphi(x_0,t).
\end{aligned}$$

As far as the invertibility of (13) is concerned, this can be proved as in [6, Th. 1.4, p. 185]. □

Theorem 3. *Let $\varphi \in L^p(\Sigma_2)$ $(1 \le p < \infty)$. Then*

$$\lim_{t\to 0^+} \int_{\Sigma_2} \varphi(y,0)\,G(x-y,t)\,\mathrm{d}y = \varphi(x,0) \tag{14}$$

for almost every $(x,0) \in \Sigma_2$.

Proof. Putting $u \equiv 1$ in formula (4), we find

$$\int_{\Sigma_3} \frac{\partial}{\partial \overline{\nu}_{y,s}} G(x-y,t-s)\,\mathrm{d}\sigma_{y,s} + \int_{\Sigma_2} G(x-y,t)\,\mathrm{d}y = 1$$

for any $(x,t) \in \Omega_T$. Letting $t \to 0^+$, we obtain (14) when $\varphi \equiv 1$. Then the thesis will follow if we show that

$$\lim_{t\to 0^+} \int_{\Sigma_2} [\varphi(y,0) - \varphi(x,0)])\, G(x-y,t)\, \mathrm{d}y = 0$$

for almost every $(x,0) \in \Sigma_2$. This can be proved in Lebesgue points by standard arguments (see, e.g., [1, p. 91–92]). □

The same results hold for the potentials related to the operator H^*.

Theorem 4. *Let $\varphi \in L^p(\Sigma_3)$ $(1 \le p < \infty)$. Then*

$$\begin{aligned} &\lim_{(y,s)\to(y_0,s)^\pm} \int_{\Sigma_3} \varphi(x,t)\, \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x,t}}\, \mathrm{d}\sigma_{x,t} \\ &\quad = \pm\frac{1}{2}\varphi(y_0,s) + \int_{\Sigma_3} \varphi(x,t)\, \frac{\partial G(x-y_0,t-s)}{\partial \overline{\nu}_{x,t}}\, \mathrm{d}\sigma_{x,t} \end{aligned} \tag{15}$$

$$\begin{aligned} &\lim_{(y,s)\to(y_0,s)^\pm} \int_{\Sigma_3} \varphi(x,t)\, \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{y_0,s}}\, \mathrm{d}\sigma_{x,t} \\ &\quad = \mp\frac{1}{2}\varphi(y_0,s) + \int_{\Sigma_3} \varphi(x,t)\, \frac{\partial G(x-y_0,t-s)}{\partial \overline{\nu}_{y_0,s}}\, \mathrm{d}\sigma_{x,t} \end{aligned} \tag{16}$$

for almost every $(y_0,s) \in \Sigma_3$. As in Theorem 1, the limits $(y,s) \to (y_0,s)^+$ $((y,s) \to (y_0,s)^-)$ have to be understood as internal (external) angular boundary values and the integrals on the right-hand side exist as singular integrals. Moreover, for any $1 < p < \infty$ and for any real $c \neq 0$, the operators

$$\begin{aligned} &c\,\varphi(y,s) + \int_{\Sigma_3} \varphi(x,t)\, \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x,t}}\, \mathrm{d}\sigma_{x,t}, \\ &c\,\varphi(y,s) + \int_{\Sigma_3} \varphi(x,t)\, \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{y,s}}\, \mathrm{d}\sigma_{x,t} \end{aligned}$$

are invertible on $L^p(\Sigma_3)$.

Proof. We start by observing that, reasoning as in the proof of Theorem 1, we have

$$\begin{aligned} \int_{\Sigma_3^\infty} \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x,t}}\, \mathrm{d}\sigma_{x,t} &= \int_{\partial\Omega} \mathrm{d}\sigma_x \int_s^{+\infty} \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x,t}}\, \mathrm{d}t \\ &= -\int_{\partial\Omega} \frac{\partial}{\partial \overline{\nu}_x} s(x,y)\, \mathrm{d}\sigma_x. \end{aligned}$$

Then we can repeat that proof to obtain (15). In a similar way, we get (16) as in Theorem 2. □

Theorem 5. *Let $\varphi \in L^p(\Sigma_1)$ $(1 \le p < \infty)$. Then*

$$\lim_{s\to T^-} \int_{\Sigma_1} \varphi(x,T)\, G(x-y,T-s)\, \mathrm{d}x = \varphi(y,T)$$

for almost every $(y,T) \in \Sigma_1$.

Proof. The proof is the same as in Theorem 3. □

4. The Classes $\mathscr{A}^p$ and $\mathscr{A}_*^p$

Let p be a real number such that $1 \le p < \infty$. We give the following

Definition 1. Let $u \in L^p(\Omega_T)$. We say that u belongs to the class $\mathscr{A}^p(\Omega_T)$ if there exist two functions $A \in L^p(\Sigma)$ and $B \in L^p(\Sigma_3)$ such that

$$\int_{\Omega_T} u\, H^* v\, \mathrm{d}x\mathrm{d}t = \int_{\Sigma_3} \left(B\, v - A\, \frac{\partial v}{\partial \overline{\nu}} \right) \mathrm{d}\sigma + \int_{\Sigma_1} A\, v\, \mathrm{d}\sigma - \int_{\Sigma_2} A\, v\, \mathrm{d}\sigma \tag{17}$$

for all v in $C^\infty(\mathbb{R}^{n+1})$.

Similarly, we define

Definition 2. Let $u \in L^p(\Omega_T)$. We say that u belongs to the $\mathscr{A}_*{}^p(\Omega_T)$ if there exist two functions $A \in L^p(\Sigma)$ and $B \in L^p(\Sigma_3)$ such that

$$\int_{\Omega_T} u\, H v\, \mathrm{d}x\mathrm{d}t = \int_{\Sigma_3} \left(B\, v - A\, \frac{\partial v}{\partial \overline{\nu}} \right) \mathrm{d}\sigma - \int_{\Sigma_1} A\, v\, \mathrm{d}\sigma + \int_{\Sigma_2} A\, v\, \mathrm{d}\sigma \tag{18}$$

for all v in $C^\infty(\mathbb{R}^{n+1})$.

Remark 1. If $u \in \mathscr{A}^p(\Omega_T)$ ($u \in \mathscr{A}_*{}^p(\Omega_T)$) then u is a smooth solution in Ω_T of the equation $Hu = 0$ ($H^*u = 0$). This follows from the hypoellipticity of the operator H (H^*) (see, e.g., [8, p. 146], where the arguments used for the heat operator can be easily adapted to H and H^*).

Remark 2. In conditions (17) and (18) it is sufficient to consider functions v which are C^∞ in a neighborhood of $\overline{\Omega}_T$.

It will be useful to introduce other two classes of functions

Definition 3. We say that u belongs to the class $\mathscr{B}^p(\Omega_T)$ if there exist two functions $A \in L^p(\Sigma)$ and $B \in L^p(\Sigma_3)$ such that

$$\begin{aligned} &\int_{\Sigma_3} \left(A(y,s)\, \frac{\partial}{\partial \overline{\nu}_{y,s}} G(x-y,t-s) - B(y,s)\, G(x-y,t-s) \right) \mathrm{d}\sigma_{y,s} \\ &\quad - \int_{\Sigma_1} A(y,T)\, G(x-y,t-T)\, \mathrm{d}y + \int_{\Sigma_2} A(y,0)\, G(x-y,t)\, \mathrm{d}y \\ &\quad = \begin{cases} u(x,t) & \text{if } (x,t) \in \Omega_T, \\ 0 & \text{if } (x,t) \notin \overline{\Omega}_T. \end{cases} \end{aligned} \tag{19}$$

Definition 4. We say that u belongs to the class $\mathscr{B}_*^p(\Omega_T)$ if there exist two functions $A \in L^p(\Sigma)$ and $B \in L^p(\Sigma_3)$ such that

$$\begin{aligned} &\int_{\Sigma_3} \left(A(x,t)\, \frac{\partial}{\partial \overline{\nu}_{x,t}} G(x-y,t-s) - B(x,t)\, G(x-y,t-s) \right) \mathrm{d}\sigma_{x,t} \\ &\quad + \int_{\Sigma_1} A(x,T)\, G(x-y,T-s)\, \mathrm{d}x - \int_{\Sigma_2} A(x,0)\, G(x-y,-s)\, \mathrm{d}x \\ &\quad = \begin{cases} u(y,s) & \text{if } (y,s) \in \Omega_T, \\ 0 & \text{if } (y,s) \notin \overline{\Omega}_T. \end{cases} \end{aligned}$$

Roughly speaking $\mathscr{A}^p(\Omega_T)$ $({\mathscr{A}_*}^p(\Omega_T))$ is the class of solutions of the equation $Hu=0$ $(H^*u=0)$ such that u and $\partial u/\partial\overline{\nu}$ do exist on the boundary in some weak sense, while $\mathscr{B}^p(\Omega_T)$ $({\mathscr{B}_*}^p(\Omega_T))$ is the class of solutions of the same equation for which a representation Stokes formula holds.

Actually these two spaces coincide.

Theorem 6. *$\mathscr{A}^p(\Omega_T)=\mathscr{B}^p(\Omega_T)$, ${\mathscr{A}_*}^p(\Omega_T)={\mathscr{B}_*}^p(\Omega_T)$.*

Proof. Suppose that u belongs to $\mathscr{A}^p(\Omega_T)$. If $(x,t)\notin\overline{\Omega}_T$, the function $G(x-\cdot,t-\cdot)$ is a solution of the equation $Hu=0$ in Ω_T and C^∞ in a neighborhood of $\overline{\Omega}_T$. Therefore, formula (19) for $(x,t)\notin\overline{\Omega}_T$ follows immediately from (17) and Remark 2.

Let now (x,t) be fixed in Ω_T and consider a scalar function $\psi\in C^\infty(\mathbb{R}^{n+1})$ such that $\psi(y,s)=0$ if $|y|\le 1/2$, $|s|\le 1/2$, and $\psi(y,s)=1$ if $|y|\ge 1$, $|s|\ge 1$. Take $\varepsilon>0$ such that $\overline{B_\varepsilon(x)}\times[t-\varepsilon,t+\varepsilon]\subset\Omega_T$ and define

$$v(y,s)=\psi\left(\frac{y-x}{\varepsilon},\frac{s-t}{\varepsilon}\right)G(x-y,t-s).$$

Since v is a C^∞ function and coincide with $G(x-y,t-s)$ for (y,s) in a neighborhood of $\partial\Omega_T$, by (17) we get

$$\begin{aligned}
&\int_{\Sigma_3}\left(A(y,s)\,\frac{\partial}{\partial\overline{\nu}_{y,s}}G(x-y,t-s)-B(y,s)\,G(x-y,t-s)\right)\mathrm{d}\sigma_{y,s}\\
&\quad-\int_{\Sigma_1}A(y,T)\,G(x-y,t-T)\,\mathrm{d}y+\int_{\Sigma_2}A(y,0)\,G(x-y,t)\,\mathrm{d}y\\
&\quad=\int_{\Sigma_3}\left(A\,\frac{\partial v}{\partial\overline{\nu}}-B\,v\right)\mathrm{d}\sigma-\int_{\Sigma_1}A\,v\,\mathrm{d}\sigma+\int_{\Sigma_2}A\,v\,\mathrm{d}\sigma\\
&\quad=-\int_{\Omega_T}u\,H^*v\,\mathrm{d}y\mathrm{d}s=-\int_{B_\varepsilon(x)\times(t-\varepsilon,t+\varepsilon)}u\,H^*v\,\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

The last equality holds because $H^*v(y,s)=H^*_{(y,s)}G(x-y,t-s)=0$ for $|y-x|>\varepsilon$, $|s-t|>\varepsilon$.

With obvious notations, by Green's formula (3) we can write the last integral as

$$-\int_{\Sigma_3^\varepsilon}\left(v\frac{\partial u}{\partial\overline{\nu}}-u\frac{\partial v}{\partial\overline{\nu}}\right)\mathrm{d}\sigma-\int_{\Sigma_1^\varepsilon}uv\,\mathrm{d}\sigma+\int_{\Sigma_2^\varepsilon}uv\,\mathrm{d}\sigma.$$

Keeping in mind (4), this is equal to

$$\begin{aligned}
&-\int_{\Sigma_3^\varepsilon}\left(\frac{\partial u(y,s)}{\partial\overline{\nu}_{y,s}}\,G(x-y,t-s)-u(y,s)\,\frac{\partial}{\partial\overline{\nu}_{y,s}}G(x-y,t-s)\right)\mathrm{d}\sigma_{y,s}\\
&\qquad+\int_{\Sigma_2^\varepsilon}u(y,t-\varepsilon)\,G(x-y,\varepsilon)\,\mathrm{d}y=u(x,t)
\end{aligned}$$

Note that we can apply Green's formula and (4), because the function u is C^∞ in a neighborhood of $B_\varepsilon(x)\times(t-\varepsilon,t+\varepsilon)$ and it satisfies the equation $Hu=0$ (see Remark 1). This establishes formula (19).

Conversely, if $u\in\mathscr{B}^p(\Omega_T)$, known results of potential theory show that u belongs to $L^p(\Omega_T)$. Thanks to Remark 2, it is enough to prove that (17) holds for any $v \in \mathring{C}^\infty(A)$, A being a domain such that $\overline{\Omega}_T \subset A$. From (19) we get

$$\int_{\Omega_T} u\, H^* v \,\mathrm{d}x\mathrm{d}t = \int_{\Omega_T} H^* v(x,t)$$

$$\times\Bigg(\int_{\Sigma_3} \left(A(y,s) \frac{\partial}{\partial \overline{\nu}_{y,s}} G(x-y,t-s) - B(y,s)\, G(x-y,t-s) \right) \mathrm{d}\sigma_{y,s}$$

$$- \int_{\Sigma_1} A(y,T)\, G(x-y,t-T)\,\mathrm{d}y + \int_{\Sigma_2} A(y,0)\, G(x-y,t)\,\mathrm{d}y \Bigg) \mathrm{d}x\mathrm{d}t$$

$$= \int_A H^* v(x,t)$$

$$\times\Bigg(\int_{\Sigma_3} \left(A(y,s) \frac{\partial}{\partial \overline{\nu}_{y,s}} G(x-y,t-s) - B(y,s)\, G(x-y,t-s) \right) \mathrm{d}\sigma_{y,s}$$

$$- \int_{\Sigma_1} A(y,T)\, G(x-y,t-T)\,\mathrm{d}y + \int_{\Sigma_2} A(y,0)\, G(x-y,t)\,\mathrm{d}y \Bigg) \mathrm{d}x\mathrm{d}t$$

Applying Fubini's Theorem and keeping in mind (6), we get (17).

The proof that $\mathscr{A}_*{}^p(\Omega_T) = \mathscr{B}_*{}^p(\Omega_T)$ is similar. □

The next Theorems show that usual uniqueness results hold in the class $\mathscr{A}^p(\Omega_T)$ ($\mathscr{A}_*{}^p(\Omega_T)$) for the Dirichlet problem and for the mixed problem for the equation $Hu = 0$ ($H^* u = 0$).

Theorem 7. *Let $1 < p < \infty$. If u is solution of the Dirichlet problem*

$$\begin{cases} u \in \mathscr{A}^p(\Omega_T) & \\ Hu = 0 & \text{in } \Omega_T \\ A = 0 & \text{a.e. on } \Sigma_2 \cup \Sigma_3, \end{cases}$$

then $u = 0$ in Ω_T.

Proof. Thanks to Theorem 6, u belongs to $\mathscr{B}^p(\Omega_T)$ and then

$$- \int_{\Sigma_3} B(y,s)\, G(x-y,t-s)\mathrm{d}\sigma_{y,s} - \int_{\Sigma_1} A(y,T)\, G(x-y,t-T)\,\mathrm{d}y$$

$$= \begin{cases} u(x,t) & \text{if } (x,t) \in \Omega_T, \\ 0 & \text{if } (x,t) \notin \overline{\Omega}_T. \end{cases}$$

In particular,

$$u(x,t) = - \int_{\Sigma_3} B(y,s)\, G(x-y,t-s)\mathrm{d}\sigma_{y,s}, \quad (x,t) \in \Omega_T. \tag{20}$$

We have also

$$\int_{\Sigma_3} B(y,s)\, G(x-y,t-s)\mathrm{d}\sigma_{y,s} = 0, \quad (x,t) \notin \overline{\Omega}_T,\ t < T.$$

Therefore, for any fixed $(x_0, t) \in \Sigma_3$, we get

$$\int_{\Sigma_3} B(y,s) \frac{\partial G(x-y,t-s)}{\partial \overline{\nu}_{x_0,t}} \, d\sigma_{y,s} = 0$$

for any $x \notin \overline{\Omega}$. By the jump formula (12), we obtain

$$\frac{1}{2} B(x_0,t) + \int_{\Sigma_3} B(y,s) \frac{\partial G(x_0-y,t-s)}{\partial \overline{\nu}_{x_0,t}} \, d\sigma_{y,s} = 0$$

for almost any $(x_0, t) \in \Sigma_3$. The invertibility of the operator on the left hand side (see Theorem 1) implies $B = 0$ a.e.. The result follows from (20). □

Theorem 8. *Let $1 < p < \infty$. If u is solution of the mixed problem*

$$\begin{cases} u \in \mathscr{A}^p(\Omega_T) & \\ Hu = 0 & \text{in } \Omega_T \\ A = 0 & \text{a.e. on } \Sigma_2, \\ B = 0 & \text{a.e. on } \Sigma_3, \end{cases}$$

then $u = 0$ in Ω_T.

Proof. Theorem 6 implies $u \in \mathscr{B}^p(\Omega_T)$ and then

$$\int_{\Sigma_3} A(y,s) \frac{\partial}{\partial \overline{\nu}_{y,s}} G(x-y,t-s) \, d\sigma_{y,s} - \int_{\Sigma_1} A(y,T) \, G(x-y,t-T) \, dy = \begin{cases} u(x,t) & \text{if } (x,t) \in \Omega_T, \\ 0 & \text{if } (x,t) \notin \overline{\Omega}_T. \end{cases}$$

In particular

$$u(x,t) = \int_{\Sigma_3} A(y,s) \frac{\partial}{\partial \overline{\nu}_{y,s}} G(x-y,t-s) \, d\sigma_{y,s}, \quad (x,t) \in \Omega_T \tag{21}$$

and

$$\int_{\Sigma_3} A(y,s) \frac{\partial}{\partial \overline{\nu}_{y,s}} G(x-y,t-s) \, d\sigma_{y,s} = 0, \quad (x,t) \notin \overline{\Omega}_T, \ t < T.$$

In view of (7), the last condition implies

$$-\frac{1}{2} A(x,t) + \int_{\Sigma_3} A(y,s) \frac{\partial}{\partial \overline{\nu}_{y,s}} G(x-y,t-s) \, d\sigma_{y,s} = 0$$

for almost any $(x,t) \in \Sigma_3$. The invertibility of the operator on the left hand side (see Theorem 2) implies $A = 0$ a.e.. The result follows from (21). □

By similar proofs we find

Theorem 9. *Let $1 < p < \infty$. If u is solution of the Dirichlet problem*

$$\begin{cases} u \in \mathscr{A}^p_*(\Omega_T) & \\ H^* u = 0 & \text{in } \Omega_T \\ A = 0 & \text{a.e. on } \Sigma_1 \cup \Sigma_3, \end{cases}$$

then $u = 0$ in Ω_T.

Theorem 10. *Let $1 < p < \infty$. If u is solution of the mixed problem*

$$\begin{cases} u \in \mathscr{A}^p_*(\Omega_T) & \\ H^*u = 0 & \text{in } \Omega_T \\ A = 0 & \text{a.e. on } \Sigma_1, \\ B = 0 & \text{a.e. on } \Sigma_3, \end{cases}$$

then $u = 0$ in Ω_T.

The next result provides a representation formula for functions belonging to $\mathscr{A}^p(\Omega_T)$.

Theorem 11. *Let $1 < p < \infty$. The function u belongs to $\mathscr{A}^p(\Omega_T)$ if and only if there exists $\varphi \in L^p(\Sigma_2 \cup \Sigma_3)$ such that*

$$u(x,t) = \int_{\Sigma_2 \cup \Sigma_3} \varphi(y,s)\, G(x-y, t-s)\, \mathrm{d}\sigma_{y,s}, \quad (x,t) \in \Omega_T.$$

Proof. Sufficiency. Suppose

$$u(x,t) = \int_{\Sigma_3} \varphi(y,s)\, G(x-y, t-s)\, \mathrm{d}\sigma_{y,s} \tag{22}$$

with $\varphi \in L^p(\Sigma_3)$. For any $v \in C^\infty(\mathbb{R}^{n+1})$ we have

$$\int_{\Omega_T} u\, H^*v\, \mathrm{d}x\mathrm{d}t = \int_{\Sigma_3} \varphi(y,s)\, \mathrm{d}\sigma_{y,s} \int_{\Omega_T} G(x-y, t-s)\, H^*v(x,t)\, \mathrm{d}x\mathrm{d}t.$$

Keeping in mind Theorem 4, representation formula (5) leads to

$$\int_{\Omega_T} G(x-y,t-s)\, H^*v(x,t)\, \mathrm{d}x\mathrm{d}t = -\frac{1}{2}\, v(y,s)$$
$$+ \int_{\Sigma_3} \left(v(x,t)\, \frac{\partial}{\partial \overline{\nu}_{x,t}} G(x-y,t-s) - \frac{\partial v(x,t)}{\partial \overline{\nu}_{x,t}}\, G(x-y,t-s) \right) \mathrm{d}\sigma_{x,t}$$
$$+ \int_{\Sigma_1} v(x,T)\, G(x-y, T-s)\, \mathrm{d}x$$

for almost every $(y,s) \in \Sigma_3$. Then

$$\int_{\Omega_T} u\, H^*v\, \mathrm{d}x\mathrm{d}t = \int_{\Sigma_3} \varphi(y,s) \Big(-\frac{1}{2}\, v(y,s)$$
$$+ \int_{\Sigma_3} \left(v(x,t)\, \frac{\partial}{\partial \overline{\nu}_{x,t}} G(x-y,t-s) - \frac{\partial v(x,t)}{\partial \overline{\nu}_{x,t}}\, G(x-y,t-s) \right) \mathrm{d}\sigma_{x,t}$$
$$+ \int_{\Sigma_1} v(x,T)\, G(x-y,T-s)\, \mathrm{d}x \Big) \mathrm{d}\sigma_{y,s}$$
$$= \int_{\Sigma_3} \left(B\, v - A\, \frac{\partial v}{\partial \overline{\nu}} \right) \mathrm{d}\sigma + \int_{\Sigma_1} A\, v\, \mathrm{d}\sigma$$

where

$$B(x,t) = -\frac{1}{2}\, \varphi(x,t) + \int_{\Sigma_3} \varphi(y,s)\, \frac{\partial}{\partial \overline{\nu}_{x,t}} G(x-y,t-s)\, \mathrm{d}\sigma_{y,s}, \ (x,t) \in \Sigma_3$$

$$A(x,t) = \int_{\Sigma_3} \varphi(y,s)\, G(x-y,t-s)\, \mathrm{d}\sigma_{y,s}, \ (x,t) \in \ \Sigma_1 \cup \Sigma_3.$$

This proves that the potential (22) belongs to $\mathscr{A}^p(\Omega_T)$.

Likewise, if

$$u(x,t) = \int_{\Sigma_2} \varphi(y,0)\, G(x-y,t)\, \mathrm{d}y,$$

keeping in mind also Theorem 3, we find that (17) holds with

$$B(x,t) = \int_{\Sigma_2} \varphi(y,0)\, \frac{\partial}{\partial \overline{\nu}_{x,t}} G(x-y,t)\, \mathrm{d}y, \;\; (x,t) \in \Sigma_3$$
$$A(x,t) = \int_{\Sigma_2} \varphi(y,0)\, G(x-y,t)\, \mathrm{d}y, \;\; (x,t) \in \; \Sigma_1 \cup \Sigma_3,$$
$$A(x,0) = \varphi(x,0), \;\; (x,0) \in \Sigma_2.$$

This completes the proof of the sufficiency.

Necessity. Let $u \in \mathscr{A}^p(\Omega_T)$. Let us consider the parabolic potential

$$v(x,t) = \int_{\Sigma_2 \cup \Sigma_3} \varphi(y,s)\, G(x-y,t-s)\, \mathrm{d}\sigma_{y,s}$$

where

$$\varphi(x,0) = A(x,0), \quad x \in \Omega \tag{23}$$

and φ on Σ_3 is the solution of the integral equation

$$\begin{aligned} -\frac{1}{2}\varphi(x,t) &+ \int_{\Sigma_3} \varphi(y,s)\, \frac{\partial}{\partial \overline{\nu}_{x,t}} G(x-y,t-s)\, \mathrm{d}\sigma_{y,s} \\ &= -\int_{\Sigma_2} A(y,0)\, \frac{\partial}{\partial \overline{\nu}_{x,t}} G(x-y,t)\, \mathrm{d}y + B(x,t), \quad (x,t) \in \Sigma_3. \end{aligned} \tag{24}$$

Here A and B are the ones in (17) (or equivalently, in (19)). Note that the integral equation (24) is solvable, because of the invertibility of operator (13).

By the Sufficiency part of the proof, we know that $v \in \mathscr{A}^p(\Omega_T)$. Then, the function $w = u - v$ belongs to $\mathscr{A}^p(\Omega_T)$. Conditions (23) and (24) show that $w = 0$ a.e. on Σ_2 and $\partial w / \partial \overline{\nu} = 0$ a.e. on Σ_3. The result follows from the uniqueness theorem 8. □

By a similar proof, we get

Theorem 12. *Let $1 < p < \infty$. The function u belongs to $\mathscr{A}^p_*(\Omega_T)$ if and only if there exists $\varphi \in L^p(\Sigma_1 \cup \Sigma_3)$ such that*

$$u(y,s) = \int_{\Sigma_1 \cup \Sigma_3} \varphi(x,t)\, G(x-y,t-s)\, \mathrm{d}\sigma_{x,t}, \quad (y,s) \in \Omega_T.$$

5. Parabolic Polynomials

We define the parabolic polynomials $v_\alpha(x,t)$ as the coefficient of $\xi^\alpha/\alpha!$ in the following power series expansion

$$\exp[\langle x, \xi \rangle + t\, a_{hk} \xi_h \xi_k] = \sum_{|\alpha|=0}^{\infty} v_\alpha(x,t) \frac{\xi^\alpha}{\alpha!}.$$

Here we adopt the usual conventions for a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \ \alpha! = \alpha_1! \cdots \alpha_n!, \ \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

In other words,

$$v_\alpha(x,t) = D_\xi^\alpha \left(\exp[\langle x, \xi\rangle + t\, a_{hk}\xi_h\xi_k]\right)\Big|_{\xi=0}.$$

The polynomial $v_\alpha(x,t)$ satisfies the equation $Hv_\alpha = 0$ and it can be represented as

$$v_\alpha(x,t) = \int_{\mathbb{R}^n} G(x-y,t)\, y^\alpha \mathrm{d}y \tag{25}$$

for any $x \in \mathbb{R}^n$ and $t > 0$. This is a consequence of the fact that, for $t > 0$, we have

$$\sum_{|\alpha|=0}^{\infty} \frac{\xi^\alpha}{\alpha!} \int_{\mathbb{R}^n} G(x-y,t)\, y^\alpha \mathrm{d}y = \int_{\mathbb{R}^n} G(x-y,t) \sum_{|\alpha|=0}^{\infty} \frac{y^\alpha \xi^\alpha}{\alpha!} \mathrm{d}y$$
$$= \int_{\mathbb{R}^n} G(x-y,t) \mathrm{e}^{\langle y,\xi\rangle} \mathrm{d}y = \exp[\langle x,\xi\rangle + t\, a_{hk}\xi_h\xi_k].$$

From (25) we see that v_α is of degree α_j in x_j and

$$v_\alpha(x,0) = x^\alpha, \qquad \forall\, x \in \mathbb{R}^n. \tag{26}$$

The next lemma shows that the linear span of the system $\{v_\alpha\}$ is given by all the polynomials p satisfying the equation $Hp = 0$.

Lemma 1. *Let $p(x,t)$ a polynomial such that $Hp = 0$. There exist a non negative integer m and real constants c_α ($|\alpha| \leq m$) such that*

$$p(x,t) = \sum_{|\alpha| \leq m} c_\alpha v_\alpha(x,t), \quad \forall\, (x,t) \in \mathbb{R}^{n+1}. \tag{27}$$

Proof. Since $p(x,0)$ is a polynomial of a certain degree m, there exist real constants c_α such that

$$p(x,0) = \sum_{|\alpha| \leq m} c_\alpha x^\alpha, \quad \forall\, x \in \mathbb{R}^n.$$

Let q be the polynomial

$$q(x,t) = p(x,t) - \sum_{|\alpha| \leq m} c_\alpha v_\alpha(x,t).$$

We have $Hq = 0$ and then

$$q(x,t) = \int_{\mathbb{R}^n} G(x-y,t)\, q(y,0)\, \mathrm{d}y, \quad x \in \mathbb{R}^n,\, t > 0.$$

On the other hand, (26) implies $q(y,0) = 0$ for any $y \in \mathbb{R}^n$. Therefore, $q(x,t) = 0$ for any $x \in \mathbb{R}^n$, $t > 0$, and (27) follows by analyticity. □

In a similar way, we may define the polynomials

$$w_\alpha(x,t) = D_\xi^\alpha \left(\exp[\langle x, \xi\rangle - t\, a_{hk}\xi_h\xi_k]\right)\Big|_{\xi=0}.$$

They satisfy the equation $H^* w_\alpha = 0$ and, as in Lemma 1, we have

Lemma 2. *Let $q(x,t)$ a polynomial such that $H^*q = 0$. There exist a non negative integer m and real constants c_α ($|\alpha| \le m$) such that*

$$q(x,t) = \sum_{|\alpha|\le m} c_\alpha w_\alpha(x,t), \quad \forall\ (x,t) \in \mathbb{R}^{n+1}.$$

6. Completeness Theorems

Before proving our main result we need a completeness result for a particular class of potentials.

Let B_r be the open ball defined by $|x| < r$. Let us consider the domain $B_r \times (-T^*, T^*)$, where $T^* > T$ and r is such that $\overline{\Omega} \subset B_r$. Denote by $\mathscr{S}$ the class of potentials

$$\Phi(x,t) = \int_{\Sigma_2^* \cup \Sigma_3^*} \varphi(y,s) G(x-y, t-s)\, \mathrm{d}\sigma_{y,s}$$

with φ varying in $C^0(\Sigma_2^* \cup \Sigma_3^*)$, where $\Sigma_2^* = \{(x, -T^*) \mid x \in B_r\}$, $\Sigma_3^* = \partial B_r \times (-T^*, T^*)$.

Theorem 13. *Let $1 \le p < \infty$. The system $\left\{\Phi\big|_{\Sigma_2\cup\Sigma_3} \,\Big|\, \Phi \in \mathscr{S}\right\}$ is complete in $L^p(\Sigma_2 \cup \Sigma_3)$.*

Proof. Thanks to the Hann–Banach theorem, we have to show that, if $\beta \in L^q(\Sigma_2 \cup \Sigma_3)$ $(q = p/(p-1))$ is such that

$$\int_{\Sigma_2\cup\Sigma_3} \beta\, \Phi\, \mathrm{d}\sigma = 0, \quad \forall\, \Phi \in \mathscr{S}, \tag{28}$$

then $\beta = 0$ a.e. on $\Sigma_2 \cup \Sigma_3$. Conditions (28) mean that

$$\int_{\Sigma_2^*\cup\Sigma_3^*} \varphi(y,s)\mathrm{d}\sigma_{y,s} \int_{\Sigma_2\cup\Sigma_3} \beta(x,t)\, G(x-y, t-s)\mathrm{d}\sigma_{x,t} = 0$$

for any $\varphi \in C^0(\Sigma_2^* \cup \Sigma_3^*)$. This implies that the parabolic potential

$$u(y,s) = \int_{\Sigma_2\cup\Sigma_3} \beta(x,t)\, G(x-y, t-s)\, \mathrm{d}\sigma_{x,t}$$

vanishes on $\Sigma_2^* \cup \Sigma_3^*$. We observe that u satisfies the following exterior BVP

$$\begin{cases} u \in C^\infty\left(\mathbb{R}^{n+1} \setminus [B_r \times (-T^*, T^*)]\right) & \\ H^*u = 0, & \text{in } \mathbb{R}^{n+1} \setminus [B_r \times (-T^*, T^*)]\,, \\ u = 0, & \text{on } \Sigma_2^* \cup \Sigma_3^*. \end{cases}$$

The function u vanishing at infinity, we have $u = 0$ in $\mathbb{R}^{n+1} \setminus [B_r \times (-T^*, T^*)]$. This can be proved by means of very classical methods (see [15, Sec. 2]). We repeat these here for the sake of completeness.

Given $\varepsilon > 0$, we may find R large enough so that $|u| < \varepsilon$ on the boundary of $B_R \times (-R, R)$. Since $u = 0$ on $\Sigma_2^* \cup \Sigma_3^*$, we have that $|u| < \varepsilon$ on the parabolic boundary (for the operator H^*) of the set $[B_R \times (-R, R)] \setminus [B_r \times (-T^*, T^*)]$, which is given by

$$\widetilde{\Sigma}_1 \cup \widetilde{\Sigma}_3 \cup \Sigma_2^* \cup \Sigma_3^*,$$

where $\widetilde{\Sigma}_1 = \{(x, R) \mid x \in B_R\}$, $\widetilde{\Sigma}_3 = \partial B_R \times (-R, R)$. By the maximum principle, we get $|u| < \varepsilon$ on $[B_R \times (-R, R)] \setminus [B_r \times (-T^*, T^*)]$, from which easily follows $u = 0$ in $\mathbb{R}^{n+1} \setminus [B_r \times (-T^*, T^*)]$.

Let $0 \le s \le T$. Since $\mathbb{R}^n \setminus \overline{\Omega}$ is connected, $u(\cdot, s)$ is there analytic and $u(y, s) = 0$ for $|y| > r$, we get $u(y, s) = 0$ for any $y \notin \overline{\Omega}$, $0 \le s \le T$. If $s \notin [0, T]$, for similar reasons, we find $u(y, s) = 0$ for any $y \in \mathbb{R}^n$. In other words we have $u(y, s) = 0$ for any $(y, s) \notin \overline{\Omega}_T$.

By the properties of the parabolic potential u on Σ_3, we deduce $u = 0$ on Σ_3. We have also

$$u(y, T) = \int_{\Sigma_2 \cup \Sigma_3} \beta(x, t)\, G(x - y, t - T)\, \mathrm{d}\sigma_{x,t} = 0$$

and then u is solution of the following Dirichlet boundary value problem

$$\begin{cases} H^* u = 0 \quad \text{in} \quad \Omega_T \\ u = 0 \quad \text{on} \quad \Sigma_1 \cup \Sigma_3. \end{cases}$$

Since

$$u(y, s) = \int_{\Sigma_3} \beta(x, t)\, G(x - y, t - s)\, \mathrm{d}\sigma_{x,t}, \quad (y, s) \in \Omega_T,$$

Theorem 12 shows that u belongs to $\mathscr{A}_*{}^p(\Omega_T)$. Then, by Theorem 9, we get $u = 0$ in Ω_T. Jump formulas (16) imply

$$\beta = \left(\frac{\partial u}{\partial \overline{\nu}}\right)^- - \left(\frac{\partial u}{\partial \overline{\nu}}\right)^+ = 0$$

a.e. on Σ_3. Therefore we can write

$$u(y, s) = \int_{\Sigma_2} \beta(x, 0)\, G(x - y, -s)\, dx = 0, \quad (y, s) \notin \Sigma_2.$$

By Theorem 3, we deduce

$$\beta(y, 0) = \lim_{s \to 0^-} \int_{\Sigma_2} \beta(x, 0)\, G(x - y, -s)\, dx = 0$$

a.e. on Σ_2 and this completes the proof for $1 < p < \infty$. This implies also the completeness for $p = 1$. $\square$

Theorem 14. *Let $1 \le p < \infty$. The system of parabolic polynomials $\{v_\alpha\}$ introduced in Sect. 5 is complete in $L^p(\Sigma_2 \cup \Sigma_3)$.*

Proof. Let $1 < p < \infty$ and take $f \in L^p(\Sigma_2 \cup \Sigma_3)$. By Theorem 13, given $\varepsilon > 0$, there exists $\Phi \in \mathscr{S}$ such that

$$\|f - \Phi\|_{L^p(\Sigma_2 \cup \Sigma_3)} < \varepsilon.$$

The cylinder $B_r \times (-T^*, T^*)$ being convex and then H-convex (see [16, Cor. 3, p. 351], by a result of Malgrange [11], we can find a sequence $\{\omega_k\}$ of polynomial solutions of the equation $Hu = 0$ such that $\omega_k \to \Phi$ in $C^\infty(B_r \times (-T^*, T^*))$ (see [16, Th. 4.4, p. 282, and Remark 4.1, p. 284]). This implies that we can find a polynomial ω solution of the equation $Hu = 0$ such that

$$\|\Phi - \omega\|_{L^p(\Sigma_2 \cup \Sigma_3)} < \varepsilon$$

and then

$$\|f-\omega\|_{L^p(\Sigma_2\cup\Sigma_3)} < 2\,\varepsilon.$$

Since ω can be written as a finite linear combinations of v_α (Lemma1), we have the completeness of the system $\{v_\alpha\}$ in $L^p(\Sigma_2\cup\Sigma_3)$ $(1<p<\infty)$. Completeness when $p=1$ follows easily. $\square$

Defining $\mathscr{S}^*$ the class of potentials

$$\Psi(y,s)=\int_{\Sigma_1^*\cup\Sigma_3^*}\psi(x,t)G(x-y,t-s)\,\mathrm{d}\sigma_{x,t}$$

with ψ varying in $C^0(\Sigma_1^*\cup\Sigma_3^*)$, where $\Sigma_1^*=\{(x,T^*)\mid x\in B_r\}$, we have also

Theorem 15. *Let $1<p<\infty$. The system $\left\{\Psi\big|_{\Sigma_1\cup\Sigma_3}\mid\Psi\in\mathscr{S}^*\right\}$ is complete in $L^p(\Sigma_1\cup\Sigma_3)$.*

The proof is similar to that of Theorem 13 and we omit the details. As a consequence we find

Theorem 16. *Let $1\leq p<\infty$. The system of parabolic polynomials $\{w_\alpha\}$ introduced in Sect. 5 is complete in $L^p(\Sigma_1\cup\Sigma_3)$.*

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