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NONLINEAR ANALYSIS OF BRITTLE MATERIALS

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Abstract. *The problem of the evaluation of the nonlinear response of damaged structures is considered. In order to circumvent the numerical difficulties which can affect standard numerical strategies because of the high sensitivity of the stress response to small changes of displacements, a specific implementation of the standard path–following procedure is presented. The approach is based on a parametrization of the equilibrium curve which takes into account also the variables used in the modelling of the damage process. Two damage models, an isotropic one and an orthotropic one, are implemented in order to verify the numerical procedure in the analysis of 2D continua subjected to plane stress condition.*

1 INTRODUCTION

The interest in using brittle material models in structural numerical analyses stems from several engineering applications which includes geomechanics problems, industrial applications and civil structures. Issues which have traditionally been addressed using fracture mechanics: starting from the assumption of an idealized, dominant flaw, fracture mechanics theory provides conditions for the growth of a crack from this flaw. However the approach, while is effective to describe the propagation of a single crack, is not suitable to implement general purpose computational tools to be used in the analysis of brittle structures [1]. In this regard, the notion of a continuous representation of, intrinsically discontinuous, material damage is more promising. Stemming from the work of Kachanov [2] and Rabotnov [3], and further developed in the work of Chaboche[4, 5], the damage approach is based on the introduction of a set of field variables, damage variables, which describe the local loss in material integrity.

The damage mechanics approach is now widely accepted and several numerical models have been proposed for the analysis of brittle materials. In particular in the analysis of masonry structures, Massart explored [6] the use of a multi-scale approach on the basis of micro-scale damage describing the behaviour of mortar joints and bricks, where all the material nonlinearities are localized, and of a macro scale constituted by the FEM description of the masonry panels. Berto et alia, [7, 8], proposed a macro-model, i.e. a synthetic model of the global behaviour of the masonry to be tuned up through specific experimental tests, based on an orthotropic model referring to horizontal and vertical orthotropy planes as expected for regular brickwork textures. Formica, [9], used a multilevel strategy for the nonlinear analysis of brick masonry walls based on the computational transfer of information between a macroscopic level (structural level) and a microscopic one (masonry texture level).

In the context of the previously quoted papers and of a research project *Definition of integrated methods for the analysis of masonry buildings*, supported by the Italian Ministry of University Scientific and Technology Research (MIUR), the present work aims to investigate the computational strategy to be used in the analysis of regular masonry structures or, generally, of 2D brittle materials. In particular, as also proposed in [10] and [9], the problem of the evaluation of the nonlinear response of damaged structures has been reformulated in order to circumvent the converge difficulties experimented by standard approaches and related to the high sensitivity of the stress response to small changes of displacements. This result is obtained by using as description parameters also the damage variables used in the modelling of the damage process. In order to verify the method, two kind of damage models has been considered: an isotropic model, [11], suitable for the analysis of concrete structures, and an orthotropic model, [7], proposed for the analysis of periodic brickwork masonries. The two models have been implemented inside the proposed strategy in the context of 2D continua subjected to plane stress condition and discretized through compatible finite elements.

The paper is organized as follows. Section 2 reports two damage models describing the nonlinear behaviour of quasi-brittle materials. In section 3, after a short review of the standard numerical strategy used in the nonlinear analysis of structures, a particular implementation of the path-following approach more suitable to analyze brittle material is presented. Section 4 reports some numerical results and section 5 some closing comments.

2 DAMAGE MODELS

Some damage models used to describe the nonlinear behavior of quasi-brittle materials are presented. These models are based on continuum damage mechanics and are based on the hypothesis of strain equivalence [12] and on the concept of effective stress tensor. In this work the rate-independent treatment will be considered and the model will be applied to the study of small displacements and small strain problems.

Firstly an isotropic damage model is considered. Generally, an isotropic damage model represents the simplest approach to the non-linear analysis of quasi-brittle continuum where progressive mechanical degradation due to the applied action has to be considered. The model adopted is the damage model proposed by Faria and Oliver [13] for the analysis of reinforced concrete structures, and afterwards enriched by Scotta [11]. Secondly a damage model for the non linear analysis of orthotropic quasi-brittle material in plane stress condition is considered [7]. In particular, the model has been developed for application to regular masonries considered as an orthotropic material. The main assumption of this model is that the evolution of damage can modify the intensity of the anisotropy of the material but does not alter its initial symmetries. This means that the initially orthotropic material may not evolve into more general anisotropy after the onset of damage. This hypothesis is reasonable for those composite materials in which the orientation and the characteristics of the constituents lead to microcracks which are generally parallel or perpendicular to the fibres [8].

Both the damage models are macro-models and as such have some computational advantages. For example, in a FE approach for masonry analysis, the mesh discretization does not have to accurately describe the internal structure of masonry and macro-elements can be of dimensions significantly greater than that of the single brick units.

2.1 Isotropic damage model

The isotropic model proposed in [11] is based on the introduction of an equivalent effective stress which is used to describe the evolution of the unique damage parameter of the model, d. The effective stress tensor $\bar{\boldsymbol{\sigma}} = \{\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}\}$ is defined as the elastic stress tensor

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{E}\boldsymbol{\varepsilon} \tag{1}$$

where $\boldsymbol{E}$ is the fourth-rank elasticity tensor of the material and $\boldsymbol{\varepsilon}$ is the total strain tensor, in our treatment no permanent strains are considered. The equivalent effective stress is defined as follows:

$$\bar{k}[\bar{\boldsymbol{\sigma}}] := (1 + r_\sigma(n-1)) \sqrt{\sum (\bar{\sigma}_i)^2} \tag{2}$$

where $n = f_c/f_t$, being f_c and f_t the compressive and tensile strength of the material and

$$r_\sigma := \frac{\sum \langle \bar{\sigma}_i \rangle}{\sum |\bar{\sigma}_i|} \qquad \text{with} \qquad \langle \pm x \rangle = 1/2(|x| \pm x) \tag{3}$$

$\bar{\sigma}_i$, for $i = 1, 2$, are the principal stress of the effective stress tensor. A failure criterion and an evolution law for the damage parameter are also needed. The failure criterion is defined as

$$\bar{k}[\bar{\boldsymbol{\sigma}}] - r \leq 0 \tag{4}$$

where the current value of the damage threshold, r, controls the size of the expanding damage surfaces. The initial value of r coincides with the initial strength of the material, but it evolves keeping memory of the strain history of the material. The evolution law of the damage parameter is given by the expression proposed in [13]:

$$d = 1 - \frac{f_c}{\bar{k}} \exp A(1 - \frac{\bar{k}}{f_c}) \tag{5}$$

The expression assumed for parameter A [13] is:

$$A := (\frac{n^2 g_f E}{f_c^2} - \frac{1}{2})^{-1} \tag{6}$$

E is the usual Young modulus and g_f is the specific fracture energy defined as $g_f = G_f/l_{ch}$. G_f is the fracture energy and the characteristic length l_{ch}, see [14], is introduced to avoid mesh-dependence and is defined as

$$l_{ch} = \sqrt{\Omega_e} \tag{7}$$

where Ω_e is the area of the finite element. With such an assumption, while G_f still remains a characteristic of the material, the specific fracture energy g_f becomes dependent on the mesh discretization.

2.2 Orthotropic damage model

The model [7] has been specifically developed for masonry. The basic assumption is the acceptance of the natural axes of the masonry $x - y$ (that is the bed joints and head joints directions) also as the principal axes of the damage. In each direction, two independent damage parameters are assumed, one for compression and one for traction, allowing the description of the crack closure effect. In such a way four damage variables (d_x^+, d_x^-, d_y^+, d_y^-) are introduced and, analogously to the previous isotropic model, four equivalent effective stresses are defined. An equivalent effective tensile stress and an equivalent effective compressive stress in x-direction:

$$\bar{k}_x^+ := \bar{\sigma}_x + |\bar{\tau}_{xy}|/\tan\phi_x \qquad \bar{k}_x^- := \bar{\sigma}_x - |\bar{\tau}_{xy}|/\tan\phi_x \tag{8}$$

and the same in y-direction:

$$\bar{k}_y^+ := \bar{\sigma}_y + |\bar{\tau}_{xy}|/\tan\phi_y \qquad \bar{k}_y^- := \bar{\sigma}_y - |\bar{\tau}_{xy}|/\tan\phi_y \tag{9}$$

where the internal friction angles are defined as

$$\tan\phi_x := \frac{f_\tau}{f_{tx}} \qquad \tan\phi_y := \frac{f_\tau}{f_{ty}} \tag{10}$$

on the basis of shear strength f_τ and of the tensile strengths f_{tx} and f_{ty}.

The failure criteria are simply defined as

$$\begin{array}{ll} \bar{k}_x^+ - r_x^+ \leq 0 & \bar{k}_x^- - r_x^- \geq 0 \\ \bar{k}_y^+ - r_y^+ \leq 0 & \bar{k}_y^- - r_y^- \geq 0 \end{array} \tag{11}$$

where the four damage threshold r_x^+, r_y^+, r_x^- and r_y^- define the size of the damage surfaces and evolve independently with the loading history. The damage laws which describe the evolution of the damage of the four parameters assume a different form for tensile condition and compressive condition [15], i.e.

$$\begin{array}{l} d_x^+ = 1 - \dfrac{r_{0x}^+}{\bar{k}_x^+} \exp A_x^+ (1 - \dfrac{\bar{k}_x^+}{r_{0x}^+}) \quad \text{if} \quad \bar{k}_x^+ > r_x^+ \\ d_y^+ = 1 - \dfrac{r_{0y}^y}{\bar{k}_y^+} \exp A_y^+ (1 - \dfrac{\bar{k}_y^+}{r_{0y}^+}) \quad \text{if} \quad \bar{k}_y^+ > r_y^+ \end{array} \tag{12}$$

and

$$d_x^- = 1 - \frac{r_{0x}^-}{\bar{k}_x^-}(1 - B_x^-) - B_x^- \exp A_x^-(1 - \frac{\bar{k}_x^-}{r_{0x}^-}) \quad \text{if} \quad \bar{k}_x^- < r_x^- \qquad (13)$$
$$d_y^- = 1 - \frac{r_{0y}^-}{\bar{k}_y^-}(1 - B_y^-) - B_y^- \exp A_y^-(1 - \frac{\bar{k}_y^-}{r_{0y}^-}) \quad \text{if} \quad \bar{k}_y^- < r_y^-$$

Parameters A_x^+ and A_y^+ are defined in the same way as the isotropic model

$$A_x^+ = (\frac{G_f E}{l_{ch} {f_{tx}}^2} - \frac{1}{2})^{-1} \qquad A_y^+ = (\frac{G_f E}{l_{ch} {f_{ty}}^2} - \frac{1}{2})^{-1} \qquad (14)$$

Parameters A^- and B^- can be defined by imposing the $\sigma - \varepsilon$ $1D$ numerical curve to convey two selected points on a curve extracted from a $1D$ compressive test [7].

3 NONLINEAR ANALYSIS

The objective of a nonlinear FE analysis is to trace the response of the structural model subject to a given load history. This is usually done using an *incremental-iterative procedure*. The load is applied in a number of steps, and the structural response after each step is computed from the equilibrium equations. The nonlinearity of the equations is induced by the damaging process, accordingly they have to be solved by an iterative method. In this way the equilibrium path of the structure is described by a finite number of distinct points.

A generic point of the equilibrium path is obtained by an iterative process which repeats the following steps in each iteration:

1. *Evaluation of the damaged response of the structure*. On the basis of the current value of the displacements and of the assumed damage law, the corresponding structural response is calculated by assembling the contributions of the elements of the discretization.

2. *Correction of the residuum of the equilibrium condition*. The structural response, evaluated in the previous step, is used to calculate the residual to the equilibrium condition and then to update the current estimate of the displacement parameters.

3.1 Evaluation of the damaged response of the structure

The structural response is evaluated in a standard way by assembling the contribution of each element of the mesh, i.e.

$$\boldsymbol{s}[\boldsymbol{u}] := \mathcal{A} \boldsymbol{s}_e \qquad (15)$$

where $\boldsymbol{u}$ is the current estimate of the displacement parameters, $\mathcal{A}$ represents the standard assembling operator and $\boldsymbol{s}_e$ is the vector of the element response which is evaluated by integration over the domain of the finite element Ω_e through the formula

$$\boldsymbol{s}_e := \int_{\Omega_e} \boldsymbol{B}^T \boldsymbol{\sigma} \qquad (16)$$

Referring to a standard compatible finite element whose relative quantities are evaluated by numerical integration over the Gauss points, $\boldsymbol{B}$ is the discrete compatibility operator and $\boldsymbol{\sigma}$ is damaged stress tensor evaluated on the basis of the effective stress tensor $\bar{\boldsymbol{\sigma}}$, see section 2, and of the assigned damage law, whichever it is.

3.2 Correction of the residuum of the equilibrium condition

The equilibrium of a structure subjected to proportional loadings $\mathbf{p}[\lambda] := \lambda\hat{\mathbf{p}}$, where $\hat{\mathbf{p}}$ is the nominal load vector and λ the current load multiplier, can be expressed by the nonlinear equation:

$$\boldsymbol{r}[\boldsymbol{u},\lambda] := \boldsymbol{s}[\boldsymbol{u}] - \lambda\hat{\boldsymbol{p}} = \boldsymbol{0} \tag{17}$$

Equation (17) represents a system of N equations with $N+1$ unknowns and describes a curve in $\Re^{N+1}$. We aim to recover this curve by determining a sequence of sufficiently close points $\{\boldsymbol{u}^{(k)},\lambda^{(k)}\}$ through a Riks path-following approach [16]: increments of the load parameter λ are not independent on the increments of the variables $\mathbf{u}$ but are constrained by controlling the step-size of the analysis. In other terms, if the first k solutions have already been obtained, the new $(k+1)$-th solution is achieved using equilibrium equation (17) and a further constrain condition fixing the step amplitude in the $\{\boldsymbol{u},\lambda\}$ space:

$$g^{(k+1)}[\boldsymbol{u},\lambda] = 0 \tag{18}$$

The non linear system, made by (17-18), can be solved iteratively using a Newton-Raphson. Following a standard predictor-corrector scheme, a first estimate $\{\boldsymbol{u}_1,\lambda_1\}$ is obtained by extrapolating the previous k-th solution point of the analysis

$$\begin{aligned}\boldsymbol{u}_1 &:= \boldsymbol{u}^k + \beta^{(k+1)}(\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(k-1)}) \\ \lambda_1 &:= \lambda^k + \beta^{(k+1)}(\lambda^{(k)} - \lambda^{(k-1)})\end{aligned} \tag{19}$$

$\beta^{(k+1)}$ is an appropriate step-size amplification factor which can be used for conveniently reducing or extending the step-size during the analysis. This predictor estimate is iteratively updated in the corrector phase, i.e.

$$\begin{aligned}\boldsymbol{u}_{j+1} &:= \boldsymbol{u}_j + \dot{\boldsymbol{u}}_j \\ \lambda_{j+1} &:= \lambda_j + \dot{\lambda}_j\end{aligned} \tag{20}$$

The iterative correction $\{\dot{\boldsymbol{u}}_j,\dot{\lambda}_j\}$ is obtained by solving the linear system

$$\begin{aligned}\boldsymbol{J}_j\dot{\boldsymbol{u}}_j - \hat{\mathbf{p}}\dot{\lambda}_j &= -\mathbf{r}_j \\ \boldsymbol{g}_u^T\dot{\boldsymbol{u}}_j + g_\lambda\dot{\lambda}_j &= 0\end{aligned} \tag{21}$$

where

$$\boldsymbol{J}_j := \left.\frac{d\boldsymbol{s}[\boldsymbol{u}]}{d\boldsymbol{u}}\right|_{\boldsymbol{u}=\boldsymbol{u}_j} \tag{22}$$

represents the Jacobian matrix associated to equation (17), i.e. the tangent stiffness matrix, and equation $(21)_2$ is obtained by substituting equation (18) with the simplified form

$$g^{(k+1)}[\boldsymbol{u},\lambda] := \boldsymbol{g}_u^T(\boldsymbol{u} - \boldsymbol{u}_j) + g_\lambda(\lambda - \lambda_j) = 0 \tag{23}$$

The expression (23), see [16], defines a hyperplane passing through the current estimate $\{\boldsymbol{u}_j,\lambda_j\}$ of the solution point $\{\boldsymbol{u}^{k+1},\lambda^{k+1}\}$ and normal to the local tangent of the equilibrium path, on the basis of a suitable metric.

The linear system (21) can be solved by partitioning, i.e.

$$\begin{aligned} \dot{\boldsymbol{u}}_j &= \boldsymbol{J}_j^{-1}(-\boldsymbol{r}_j + \dot{\lambda}_j \hat{\boldsymbol{p}}) \\ \dot{\lambda}_j &= \frac{\boldsymbol{g}_u^T \boldsymbol{J}_j^{-1} \boldsymbol{r}_j}{\boldsymbol{g}_u^T \boldsymbol{J}_j^{-1} \hat{\boldsymbol{p}} + g_\lambda} \end{aligned} \tag{24}$$

Regarding equation (23) a convenient choice, see [17], is to define a constrain function weighted in a homogeneous way by means of the energy metric

$$g[\boldsymbol{u}, \lambda] := \Delta \boldsymbol{u}^T \tilde{\boldsymbol{K}} \dot{\boldsymbol{u}}_j + \mu \Delta\lambda \dot{\lambda}_j = 0 \tag{25}$$

where $\tilde{\boldsymbol{K}}$ is the initial stiffness matrix. Assuming also $\mu = 0$ and

$$\Delta \boldsymbol{u} = \hat{\boldsymbol{u}} := \tilde{\boldsymbol{K}}^{-1} \hat{\boldsymbol{p}} \tag{26}$$

the expression for $\dot{\lambda}_j$ assumes the following form

$$\dot{\lambda}_j = -\frac{\hat{\boldsymbol{u}}^T \tilde{\boldsymbol{K}} \boldsymbol{J}_j^{-1} \boldsymbol{r}_j}{\hat{\boldsymbol{u}}^T \tilde{\boldsymbol{K}} \boldsymbol{J}_j^{-1} \hat{\boldsymbol{p}}} \tag{27}$$

3.3 An alternative implementation

As shown in [18], the convergence behaviour of the iterative process previously described is highly sensitive to the formulation of the problem and mainly depends on which variables are used for its description. This is an important consideration in the present context because the damage growth introduces an unstable behaviour which tends to slow the iterative process because of the high sensitivity of the stress response to small changes of displacements. A simple and effective way to improve the convergence behaviour of the iterative process is the use of alternative formulations able to produce a better control of the nonlinearities of the problem, by means of an appropriate choice of the description variables, see also [9]. This result can be obtained by using as description variables also the damage parameters of the structure and by treating them as independent variables. To this end the equilibrium problem (17) can be augmented with an independent description of the evolution of the damage parameters of the structures, i.e.

$$\begin{aligned} \boldsymbol{r}_d[\boldsymbol{u}, \boldsymbol{d}] := \boldsymbol{d} - \boldsymbol{I} + \boldsymbol{f}[\boldsymbol{u}] &= \boldsymbol{0} \\ \boldsymbol{r}_u[\boldsymbol{u}, \boldsymbol{d}, \lambda] := \boldsymbol{s}[\boldsymbol{u}, \boldsymbol{d}] - \lambda \hat{\boldsymbol{p}} &= \boldsymbol{0} \end{aligned} \tag{28}$$

The first equation has a local nature being related to each Gauss point of the discretization where are usually defined the damage parameters. $\boldsymbol{d}$ is a vector collecting all the damage parameters, $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{f}$ is also a vector containing for each damage parameter the function used to represent the evolution of the damage parameter, e.g. the exponential functions assumed for the damage models described in section 2.

Accordingly to the variable choice represented by the equation set (28), the constraint condition (23) can be reformulated as follows

$$g^{(k+1)}[\boldsymbol{u}, \boldsymbol{d}, \lambda] := \boldsymbol{g}_u^T(\boldsymbol{u} - \boldsymbol{u}_j) + \boldsymbol{g}_d^T(\boldsymbol{d} - \boldsymbol{d}_j) + g_\lambda(\lambda - \lambda_j) = 0 \tag{29}$$

As in the previous case, the nonlinear system (28-29) can be solved iteratively on the basis of a Newton-Raphson predictor-corrector scheme. During the predictor phase, a first estimate $\{\boldsymbol{u}_1, \lambda_1\}$ is obtained by extrapolating the previous k-th solution point of the analysis:

$$\boldsymbol{u}_1 := \boldsymbol{u}^k + \beta^{(k+1)}(\boldsymbol{u}^{(k)} - \boldsymbol{u}^{(k-1)}) \quad (30)$$

$$\boldsymbol{d}_1 := \boldsymbol{d}^k + \beta^{(k+1)}(\boldsymbol{d}^{(k)} - \boldsymbol{d}^{(k-1)}) \quad (31)$$

$$\lambda_1 := \lambda^k + \beta^{(k+1)}(\lambda^{(k)} - \lambda^{(k-1)})$$

The predictor is then iteratively updated by the following corrector procedure

$$\begin{aligned} \boldsymbol{u}_{j+1} &:= \boldsymbol{u}_j + \dot{\boldsymbol{u}}_j \\ \boldsymbol{d}_{j+1} &:= \boldsymbol{d}_j + \dot{\boldsymbol{d}}_j \\ \lambda_{j+1} &:= \lambda_j + \dot{\lambda}_j \end{aligned} \quad (32)$$

where the j-th correction $\{\dot{\boldsymbol{u}}_j, \dot{\boldsymbol{d}}_j, \dot{\lambda}_j\}$ is obtained by solving the linear system

$$\begin{aligned} \boldsymbol{f}_{uj}\dot{\boldsymbol{u}}_j + \dot{\boldsymbol{d}}_j &= -\boldsymbol{r}_{dj} \\ \boldsymbol{J}_j\dot{\boldsymbol{u}}_j + \boldsymbol{s}_{dj}\dot{\boldsymbol{d}}_j - \hat{\boldsymbol{p}}\dot{\lambda}_j &= -\boldsymbol{r}_{uj} \\ \boldsymbol{g}_u^T\dot{\boldsymbol{u}}_j + \boldsymbol{g}_d^T\dot{\boldsymbol{d}}_j + g_\lambda\dot{\lambda}_j &= \boldsymbol{0} \end{aligned} \quad (33)$$

Where the symbols introduced have the following meanings

$$\boldsymbol{s}_{dj} := \left.\frac{\partial \boldsymbol{s}[\boldsymbol{u},\boldsymbol{d}]}{\partial \boldsymbol{d}}\right|_{\boldsymbol{u}=\boldsymbol{u}_j,\boldsymbol{d}=\boldsymbol{d}_j} \qquad \boldsymbol{J}_j := \left.\frac{\partial \boldsymbol{s}[\boldsymbol{u},\boldsymbol{d}]}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}=\boldsymbol{u}_j,\boldsymbol{d}=\boldsymbol{d}_j} \qquad \boldsymbol{f}_{uj} := \left.\frac{\partial \boldsymbol{f}[\boldsymbol{u}]}{\partial \boldsymbol{u}}\right|_{\boldsymbol{u}=\boldsymbol{u}_j} \quad (34)$$

Also in this case the linear system (33) can be solved by partitioning as follows

$$\begin{aligned} \dot{\boldsymbol{u}}_j &= -\boldsymbol{J}_j^{-1}\boldsymbol{r}_{uj} - \boldsymbol{J}_j^{-1}\boldsymbol{s}_{dj}\dot{\boldsymbol{d}}_j + \dot{\lambda}_j\boldsymbol{J}_j^{-1}\hat{\boldsymbol{p}} \\ \dot{\boldsymbol{d}}_j &= -\boldsymbol{r}_{dj} \\ \dot{\lambda}_j &= \frac{\boldsymbol{g}_u^T\boldsymbol{J}_j^{-1}\boldsymbol{r}_{uj} + \boldsymbol{g}_u^T\boldsymbol{J}_j^{-1}\boldsymbol{s}_{dj}\dot{\boldsymbol{d}}_j - \boldsymbol{g}_d^T\boldsymbol{f}_{uj}\boldsymbol{J}_j^{-1}\boldsymbol{r}_{uj} - \boldsymbol{g}_d^T\boldsymbol{f}_{uj}\boldsymbol{J}_j^{-1}\boldsymbol{s}_{dj}\boldsymbol{r}_{uj} + \boldsymbol{g}_d^T\boldsymbol{r}_{dj}}{\boldsymbol{g}_u^T\boldsymbol{J}_j^{-1}\hat{\boldsymbol{p}} - \boldsymbol{g}_d^T\boldsymbol{f}_{uj}\boldsymbol{J}_j^{-1}\hat{\boldsymbol{p}} + g_\lambda} \end{aligned} \quad (35)$$

It worth noting to mention that such a solution is obtained by neglecting the term $\boldsymbol{f}_{uj}$ only in the evaluation of the correction $\dot{\boldsymbol{d}}_j$.

A further comment about the constraint equation (29) is at order. A suitable choice is to take into account all the quantities of the problem, $\boldsymbol{u}, \boldsymbol{d}, \lambda$ weighted in a homogeneous way by means of an energy metric:

$$\boldsymbol{g}[\boldsymbol{u}, \boldsymbol{d}, \lambda] := \Delta\boldsymbol{u}^T\tilde{\boldsymbol{K}}\dot{\boldsymbol{u}}_j + \boldsymbol{Y}^T\dot{\boldsymbol{d}}_j + \mu\Delta\lambda\dot{\lambda}_j = 0 \quad (36)$$

where $\{\Delta\boldsymbol{u}, \Delta\lambda\}$ defines the tangent to the solution curve. Vector $\boldsymbol{Y}$, in the case of the isotropic damage model, is the associated variable to the damage parameters $\boldsymbol{d}$ and this is defined as the elastic strain energy, [19],

$$\boldsymbol{Y} = \frac{1}{2}\boldsymbol{\varepsilon}^T\boldsymbol{E}\boldsymbol{\varepsilon} \quad (37)$$

associated to the current value $\boldsymbol{u}_j$ of displacement vector. Making use also in the present case of the assumption (26) and imposing $\mu = 0$, $\dot{\lambda}$ becomes

$$\dot{\lambda}_j = \frac{\hat{\boldsymbol{u}}^T\tilde{\boldsymbol{K}}\boldsymbol{J}_j^{-1}\boldsymbol{r}_{uj} + \hat{\boldsymbol{u}}^T\tilde{\boldsymbol{K}}\boldsymbol{J}_j^{-1}\boldsymbol{s}_{dj}\dot{\boldsymbol{d}}_j - \boldsymbol{Y}^T\boldsymbol{f}_{uj}\boldsymbol{J}_j^{-1}\boldsymbol{r}_{uj} - \boldsymbol{Y}^T\boldsymbol{f}_{uj}\boldsymbol{J}_j^{-1}\boldsymbol{s}_{dj}\boldsymbol{r}_{uj} + \boldsymbol{Y}^T\boldsymbol{r}_{dj}}{\hat{\boldsymbol{u}}^T\tilde{\boldsymbol{K}}\boldsymbol{J}_j^{-1}\hat{\boldsymbol{p}} - \boldsymbol{Y}^T\boldsymbol{f}_{uj}\boldsymbol{J}_j^{-1}\hat{\boldsymbol{p}}} \quad (38)$$

4 NUMERICAL RESULTS

The nonlinear procedure described in the previous section has been used to analyze some simple tests subjected to monotonic increasing loads. Two tests, a traction test and a bending test, were performed on the basis of the isotropic damage model. In the third test, the orthotropic damage model was used to analyze a masonry wall. The finite elements used are 2D compatible 4-node elements subjected to plane-stress condition.

4.1 Traction test, isotropic model

The Figure 1 shows the geometry and the applied loads for the traction test. It is a simple 10 mm thick rectangular plate, with homogeneous characteristics except for a small zone, shown in bold in the previous figure, for which a reduction of 5% in the elastic modulus is assumed. The mechanical characteristics assumed for the material are reported in Table 1. The analysis

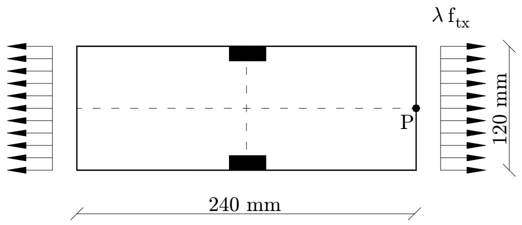

Figure 1: Traction test: geometry and applied loads.

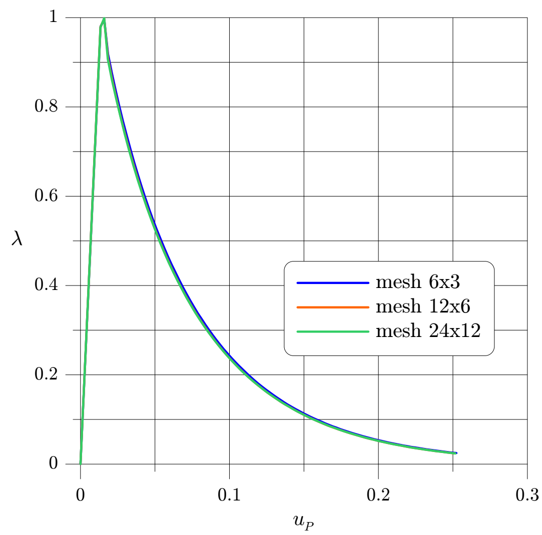

Figure 2: Traction test: equilibrium paths.

	E	ν	f_t	f_c	G_f
mat. 1	2460	0.18	0.28	0.56	0.02
mat. 2	2337	0.18	0.28	0.56	0.02

Table 1: Traction test: isotropic model parameters.

was carried out taking only a quarter of the plate and considering three different meshes. The Figure 2 shows the equilibrium paths obtained. No particular problem has been noticed in the calculation of the equilibrium points along the curve and no mesh dependence can be observed along the softening branch.

4.2 Bending test, isotropic model

The rectangular plate shown in Figure 3 was subjected to a distributed load that leads to a bending stress in the medium plane of the plate. The plate is 10 mm thick and its mechanical characteristics are reported in Table 2. The results of the analysis are shown in Figure 4 for the two meshes considered.

	E	ν	f_t	f_c	G_f
mat. 1	2460	0.18	0.28	0.56	0.02

Table 2: Bending test: isotropic model parameters.

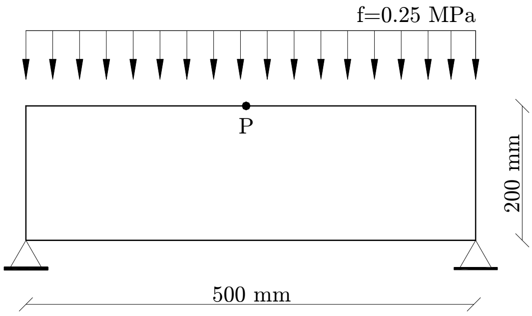

Figure 3: Bending test: geometry and applied loads.

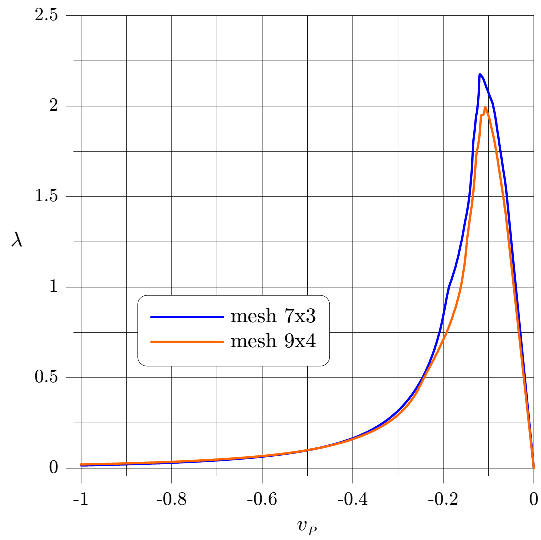

Figure 4: Bending test: equilibrium paths.

4.3 Analysis of a wall, orthotropic model

The last test is relative to the masonry wall shown in Figure 5, where the three meshes used in the analyzes are also reported. The mechanical parameters of the orthotropic damage model used in the analysis are reported in table 3 and are typical of a masonry structures [8]. Figure

E_x	1000 Mpa	f_{tx}	0.5 MPa	f_{ty}	0.18 MPa
E_y	1450 Mpa	f_{cx0}	1.8 MPa	f_{cy0}	3.15 MPa
ν_{xy}	0.1	f_{cx}	3 MPa	f_{cy}	6.3 MPa
G_{xy}	545 Mpa	f_τ	0.4 MPa		
		G_{fx}	0.2 N/mm	G_{fy}	0.1 N/mm
		A_x^-	0.3	A_y^-	0.34
		B_x^-	1	B_y^-	1.3

Table 3: Analysis of a wall: parameters of the orthotropic model.

6 reports the equilibrium paths obtained where, also in this case, no particular problems have observed in the calculation of the equilibrium curves. Figure 7 shows the damage contour plots relative to last step of the analysis.

5 CONCLUSIONS

The problem of the nonlinear analysis of brittle materials on the basis of damage models has been considered. A specific implementation of the path-following strategy is adopted in order to avoid the loss of convergency which can affect the standard procedures in the analysis of damaged structures. The proposed procedure uses a parametrization of the equilibrium curve

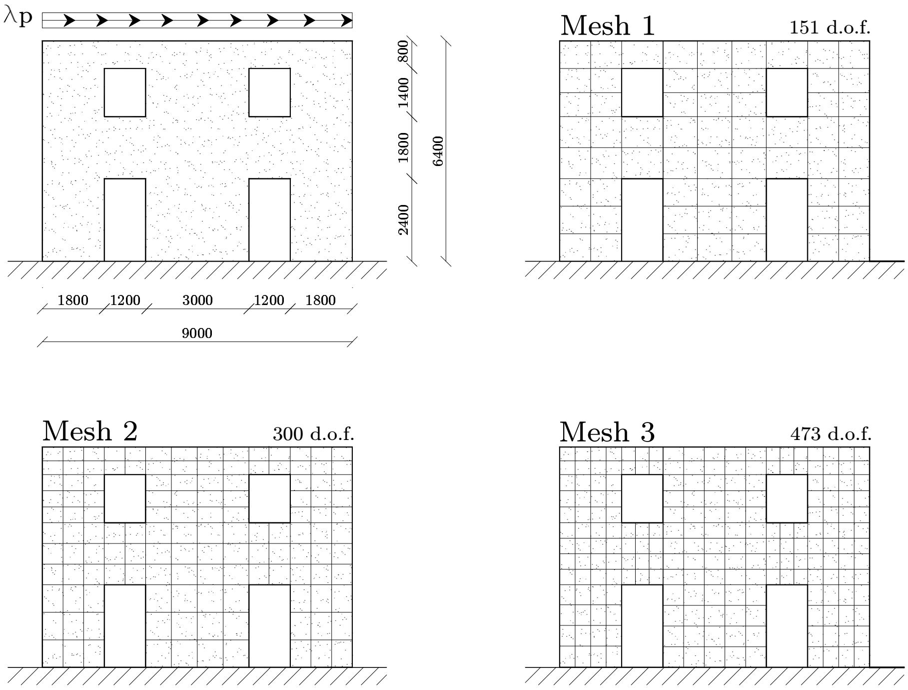

Figure 5: Analysis of a wall : geometry and applied loads, meshes utilized.

based on the variables usually adopted in the description of the damage process, variable which are treated as independent parameters of the analysis. An implementation of the procedure has been tested by performing some numerical simulation of standard 2D problems on the basis of two different kinds of damage models.

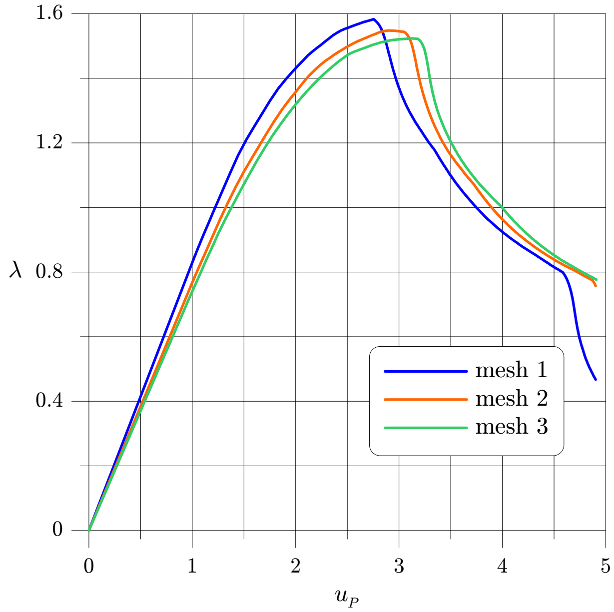

Figure 6: Analysis of a wall-orthotropic model: equilibrium paths.

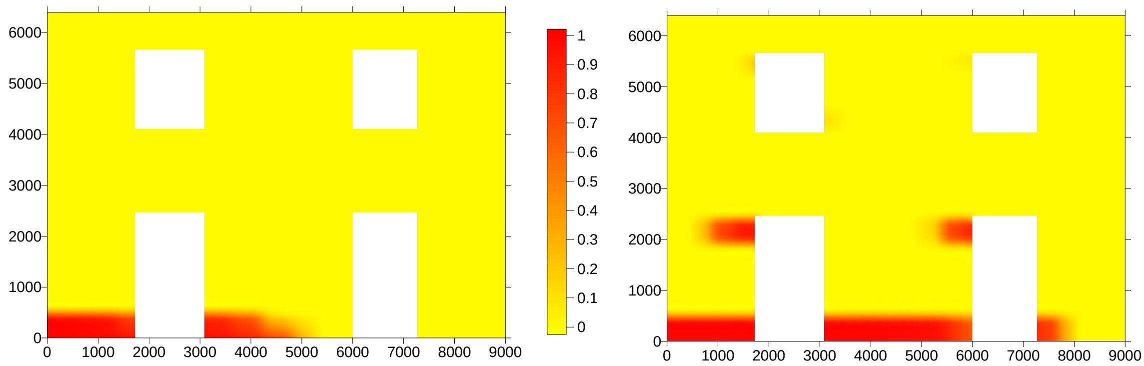

Figure 7: Analysis of a wall-orthotropic model (Mesh 3): damage map d_x^+ on the left and d_y^+ on the right.

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