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Embedding theorems with an exponential weight on the real semiaxis

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Abstract

We state embedding theorems between spaces of functions defined on the real semi-axis, which can grow exponentially both at 0 and at $+\infty$.

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1 Introduction

The aim of this paper is to state some embedding theorems between function spaces related to the weight

(1) $$u(x) = x^\gamma \mathrm{e}^{-x^{-\alpha}-x^\beta}, \qquad \alpha > 0,\ \beta > 1,\ \gamma \geq 0, \qquad x \in (0,+\infty),$$

i.e. spaces of functions defined on the real semiaxis, which can grow exponentially at 0 and $+\infty$.

2 Preliminary results

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$.

Moreover, we denote by $\|\cdot\|_p$ the L^p-norm on $(0,+\infty)$ for $1 \leq p \leq \infty$ and, by a slight abuse of notation, the quasinorm of the L^p-spaces for $0 < p < 1$, defined in the usual way. Finally, $\mathbb{P}_m$ will be the set of all algebraic polynomials of degree at most m.

2.1 Polynomial inequalities

First of all we observe that the exponential part of the weight u, i.e. $w(x) = \mathrm{e}^{-x^{-\alpha}-x^\beta}$ can be reduced to a weight belonging to the class $\mathcal{F}(C^2+)$ defined by Levin and Lubinsky in [1]. We denote by $\varepsilon_\tau = \varepsilon_\tau(w)$ and $a_\tau = a_\tau(w)$ the Mhaskar–Rakhmanov–Saff numbers related to w, with

$$\lim_{\tau\to+\infty} \varepsilon_\tau = 0\,, \qquad \lim_{\tau\to+\infty} a_\tau = +\infty\,.$$

From the results in [1], we deduce

(2) $$\varepsilon_\tau \sim \left(\frac{\sqrt{a_\tau}}{\tau}\right)^{\frac{1}{\alpha+1/2}}$$

and

(3) $$a_\tau \sim \tau^{1/\beta}\,,$$

where the constants in "$\sim$" are independent of τ.

Hence we easily get the following restricted range inequality. For any $P_m \in \mathbb{P}_m$, $0 < p \leq \infty$, setting $n = m + \lceil\gamma\rceil$, we have

$$\|P_m u\|_p \leq \mathcal{C}\, \|P_m u\|_{L^p[\varepsilon_n, a_n]}\,,$$

where $\mathcal{C} \neq \mathcal{C}(m, P_m)$, $\varepsilon_n = \varepsilon_n(w)$ and $a_n = a_n(w)$.

The following Bernstein and Markov inequalities have been proved in [3].

Theorem 2.1 *Let $0 < p \leq \infty$. For any $P_m \in \mathbb{P}_m$, we have*

$$\|P'_m \varphi u\|_p \leq \mathcal{C} \frac{m}{\sqrt{a_m}} \|P_m u\|_p \tag{4}$$

$$\|P'_m u\|_p \leq \mathcal{C} \frac{m}{\sqrt{\varepsilon_m a_m}} \|P_m u\|_p \,, \tag{5}$$

where $\varphi(x) = \sqrt{x}$ and $\mathcal{C} \neq \mathcal{C}(m, P_m)$.

In analogy with the Bernstein and Markov inequalities we have two versions of the Nikolskii inequalities (see [3]).

Theorem 2.2 *Let $1 \leq p < q \leq \infty$. Then, for any $P_m \in \mathbb{P}_m$, we get*

$$\left\| P_m \varphi^{\frac{1}{p} - \frac{1}{q}} u \right\|_q \leq \mathcal{C} \left(\frac{m}{\sqrt{a_m}} \right)^{\frac{1}{p} - \frac{1}{q}} \|P_m u\|_p \tag{6}$$

and

$$\|P_m u\|_q \leq \mathcal{C} \left(\frac{m}{\sqrt{\varepsilon_m a_m}} \right)^{\frac{1}{p} - \frac{1}{q}} \|P_m u\|_p \tag{7}$$

where $\varphi(x) = \sqrt{x}$ and $\mathcal{C} \neq \mathcal{C}(m, P_m)$.

2.2 Function spaces and polynomial approximation

Let us now define some function spaces related to the weight u (see [2]). By L^p_u, $1 \leq p < \infty$, we denote the set of all measurable functions f such that

$$\|f\|_{L^p_u} := \|fu\|_p = \left(\int_0^{+\infty} |fu|^p (x) \, \mathrm{d}x \right)^{1/p} < \infty \,,$$

while, for $p = \infty$, by a slight abuse of notation, we set

$$L^\infty_u = C_u = \left\{ f \in C^0(0, +\infty) : \lim_{x \to 0^+} f(x)u(x) = 0 = \lim_{x \to +\infty} f(x)u(x) \right\}$$

with the norm

$$\|f\|_{L^\infty_u} := \|fu\|_\infty = \sup_{x \in (0,+\infty)} |f(x)u(x)| \,.$$

To characterize functions in these spaces, we introduce the following moduli of smoothness. For any $f \in L^p_u$, $1 \leq p \leq \infty$, $r \geq 1$ and $0 < t < t_0$, we set

$$\Omega^r_\varphi(c, f, t)_{u,p} = \sup_{0 < h \leq t} \left\| \Delta^r_{h\varphi} (f) \, u \right\|_{L^p(\mathcal{I}_h(c))} \,,$$

where $\mathcal{I}_h(c) = \left[h^{1/(\alpha+1/2)}, c\, h^{-1/(\beta-1/2)}\right]$, $c > 1$ fixed, and

$$\Delta^r_{h\varphi} f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f\left(x + (r-i)h\varphi(x)\right), \quad \varphi(x) = \sqrt{x}\,.$$

Then we define the complete rth modulus of smoothness by

$$\begin{aligned} \omega^r_\varphi(f,t)_{u,p} = \Omega^r_\varphi(f,t)_{u,p} &+ \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,u\|_{L^p\left(0,t^{1/\left(\alpha+\frac{1}{2}\right)}\right]} \\ &+ \inf_{q\in\mathbb{P}_{r-1}} \|(f-q)\,u\|_{L^p\left[c\,t^{-1/\left(\beta-\frac{1}{2}\right)},+\infty\right)} \end{aligned} \tag{8}$$

with $c > 1$ a fixed constant. Let $r \geq 1$ and $0 < t < t_0$ for some

By means of the main part of the modulus of smoothness, for $1 \leq p \leq \infty$, we can define the Zygmund-type spaces

$$Z^p_s(u) = \left\{ f \in L^p_u : \sup_{t>0} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^s} < \infty,\ r > s \right\},$$

$s \in \mathbb{R}^+$, with the norm

$$\|f\|_{Z^p_s(u)} = \|f\|_{L^p_u} + \sup_{t>0} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^s}\,.$$

We remark that, in the definition of $Z^p_s(u)$, the main part of the rth modulus of smoothness $\Omega^r_\varphi(f,t)_{u,p}$ can be replaced by the complete modulus $\omega^r_\varphi(f,t)_{u,p}$, as can be deduced from next theorem.

Let us denote by $E_m(f)_{u,p} = \inf_{P\in\mathbb{P}_m} \|(f-P)\,u\|_p$ the error of best polynomial approximation of a function $f \in L^p_u$, $1 \leq p \leq \infty$. The following Jackson, weak Jackson and Stechkin inequalities have been proved in [2].

Theorem 2.3 *For any $f \in L^p_u$, $1 \leq p \leq \infty$, and $m > r \geq 1$, we have*

$$E_m(f)_{u,p} \leq \mathcal{C}\,\omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p}, \tag{9}$$

and, assuming $\Omega^r_\varphi(f,t)_{u,p}\, t^{-1} \in L^1[0,1]$,

$$E_m(f)_{u,p} \leq \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t}\,\mathrm{d}t\,, \qquad r < m\,. \tag{10}$$

Finally for any $f \in L^p_u$, $1 \leq p \leq \infty$, we get

$$\omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p} \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^r \sum_{i=0}^{m} \left(\frac{i}{\sqrt{a_i}}\right)^r \frac{E_i(f)_{u,p}}{i}\,. \tag{11}$$

In any case $\mathcal{C}$ is independent of m and f.

3 Embedding theorems

Now, using the Nikolskii inequalities (6) and (7), by arguments analogous to [4,5], we can prove some embedding theorems, connecting function spaces related to the weight u defined in the previous Section.

Theorem 3.1 *For any $f \in L^p_u$, $1 \le p < \infty$, such that*

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+\eta/p}}\,\mathrm{d}t < \infty\,, \tag{12}$$

where $\eta = (2\alpha+2)/(2\alpha+1)$, we have

$$E_m(f)_{u,\infty} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+\eta/p}}\,\mathrm{d}t\,, \tag{13}$$

$$\Omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,\infty} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+\eta/p}}\,\mathrm{d}t \tag{14}$$

and

$$\|fu\|_\infty \le \mathcal{C}\left\{\|fu\|_p + \int_0^1 \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+\eta/p}}\,\mathrm{d}t\right\}, \tag{15}$$

where $\mathcal{C}$ depends only on r.

Theorem 3.2 *For any $f \in L^p_u$, $1 \le p < \infty$ such that*

$$\int_0^1 \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t < \infty\,, \tag{16}$$

we have

$$E_m(f)_{\varphi^{1/p}u,\infty} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\,, \tag{17}$$

$$\Omega^r_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{\varphi^{1/p}u,\infty} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t \tag{18}$$

and

$$\|f\varphi^{1/p}u\|_\infty \le \mathcal{C}\left\{\|fu\|_p + \int_0^1 \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t\right\}, \tag{19}$$

where $\mathcal{C}$ *depends only on* r.

From Theorem 3.2 we can easily deduce the following corollary, useful in several contexts.

Corollary 3.3 *If* $f \in L_u^p$, $1 \le p < \infty$, *is such that*

$$\int_0^1 \frac{\Omega_\varphi^r(f,t)_{u,p}}{t^{1+1/p}}\,\mathrm{d}t < \infty\,, \tag{20}$$

then f *is continuous on* $(0,+\infty)$.

References

[1] A. L. Levin and D. S. Lubinsky, *Orthogonal polynomials for exponential weights*, CSM Books in Mathematics/Ouvrages de Mathématiques de la SMC, 4. Springer-Verlag, New York, 2001.

[2] G. Mastroianni and I. Notarangelo, *Polynomial approximation with an exponential weight on the real semiaxis*, to appear in Acta Mathematica Hungarica.

[3] G. Mastroianni, I. Notarangelo and J. Szabados, *Polynomial inequalities with an exponential weight on* $(0,+\infty)$, Mediterranean Journal of Mathematics **10** (2013), no. 2, 807–821.

[4] G. Mastroianni and J. Szabados, *Polynomial approximation on the real semiaxis with generalized Laguerre weights*, Studia Univ. "Babes-Bolyai" Studia Europa, Vol. LII, Number 4, (2007), 89-103.

[5] I. Notarangelo, *Polynomial inequalities and embedding theorems with exponential weights in* $(-1,1)$, Acta Mathematica Hungarica **134** (2012), no. 3, 286–306.