

# A REMARK ON MULTIVARIATE PROJECTION OPERATORS

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**Abstract.** This paper is a recapitulation of the work of L. Szili and P. Vértesi [4] on multivariate Fourier series with triangular type partial sums. Namely, we give another proof for the corresponding lower estimation, which, in a way, is more direct than the previous one in [4].

## 1. Introduction

**1.1.** Multivariate Fourier series has been the object of an intensive study (see, e.g., A. Zygmund [7, Ch. XVII] and E. M. Stein, and G. Weiss [3, Ch. VII]). First we introduce some notations.

Let  $\mathbb{R}^d$  (direct product) be the Euclidean  $d$ -dimensional space ( $d \geq 1$ , fixed) and let  $\mathbb{T}^d = \mathbb{R}^d \pmod{2\pi\mathbb{Z}^d}$  denote the  $d$ -dimensional torus, where  $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ .

Further, let  $C(\mathbb{T}^d)$  denote the space of (complex valued) continuous functions on  $\mathbb{T}^d$ . By definition they are  $2\pi$ -periodic in each variable.

For  $g \in C(\mathbb{T}^d)$  we define its Fourier series by

$$(1.1) \quad g(\vartheta) \sim \sum_{\mathbf{k}} \hat{g}(\mathbf{k}) e^{i\mathbf{k} \cdot \vartheta}, \quad \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(\mathbf{t}) e^{-i\mathbf{k} \cdot \mathbf{t}} d\mathbf{t},$$

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where in the above vector notation  $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \dots, \vartheta_d) \in \mathbb{T}^d$ ,  $\mathbf{k} = (k_1, k_1, \dots, k_d) \in \mathbb{Z}^d$  and  $\mathbf{k} \cdot \boldsymbol{\vartheta} = \sum_{l=1}^d k_l \vartheta_l$  (scalar product).

The *rectangular*  $n$ -th partial sum of the Fourier series is defined by

$$(1.2) \quad S_{nd}^{[r]}(g, \boldsymbol{\vartheta}) := \sum_{|\mathbf{k}|_\infty \leq n} \hat{g}(\mathbf{k}) e^{i\mathbf{k} \cdot \boldsymbol{\vartheta}} \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \dots\});$$

the *triangular* one is

$$(1.3) \quad S_{nd}^{[t]}(g, \boldsymbol{\vartheta}) := \sum_{|\mathbf{k}|_1 \leq n} \hat{g}(\mathbf{k}) e^{i\mathbf{k} \cdot \boldsymbol{\vartheta}} \quad (n \in \mathbb{N}_0).$$

Above,  $|\mathbf{k}|_\infty = \max_{1 \leq l \leq d} |k_l|$  and  $|\mathbf{k}|_1 = \sum_{l=1}^d |k_l|$  (these are the  $l_\infty$  and  $l_1$  norms of the multiindex  $\mathbf{k}$ , resp.). The names “rectangular” and “triangular” refer to the shape of the corresponding indices of terms when  $d = 2$  and  $0 \leq k_1, k_2$ ,  $|\mathbf{k}|_\infty \leq n$ ,  $|\mathbf{k}|_1 \leq n$ , respectively.

In a way the investigation of  $S_{nd}^{[r]}$  is apparent: in many cases in essence it is a one variable problem (see [3] and [7]).

However there were only relatively few works dealing with the triangular (or  $l_1$ ) summability: see J. G. Herriot [2], Y. Xu [6], H. Berens and Y. Xu [1], L. Szili and P. Vértesi [4] and F. Weisz [5].

## 2. Results

**2.1.** Let us consider

$$(2.1) \quad D_{nd}(\boldsymbol{\vartheta}) = \sum_{|\mathbf{k}|_1 \leq n} e^{i\mathbf{k} \cdot \boldsymbol{\vartheta}}, \quad (n \geq 1), \quad \text{where } \mathbf{k} \in \mathbb{Z}^d.$$

One can see that

$$(2.2) \quad S_{nd}(g, \boldsymbol{\vartheta}) = (g * D_{nd})(\boldsymbol{\vartheta}) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(\boldsymbol{\vartheta} - \mathbf{t}) D_{nd}(\mathbf{t}) d\mathbf{t} \\ = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(\boldsymbol{\vartheta} + \mathbf{t}) D_{nd}(\mathbf{t}) d\mathbf{t},$$

where as before,  $g \in C(\mathbb{T}^d)$ ,  $\boldsymbol{\vartheta}, \mathbf{t} \in \mathbb{T}^d$  (cf. (1.3) or [1], [3, Chs. I, VII]).

Let  $\|g\| := \max_{\boldsymbol{\vartheta} \in \mathbb{T}^d} |g(\boldsymbol{\vartheta})|$ ,

$$\|S_{nd}\| := \max_{\substack{g \in C(\mathbb{T}^d) \\ \|g\| \leq 1}} \|S_{nd}(g, \boldsymbol{\vartheta})\| \quad (n \geq 1).$$

Let us denote by  $L^p$ ,  $1 \leq p < \infty$ , the set of all measurable  $2\pi$ -periodic (in each variable) functions  $g(\boldsymbol{\vartheta})$  on  $\mathbb{T}^d$ , for which

$$\|g\|_p := \left( \int_{\mathbb{T}^d} |g(\boldsymbol{\vartheta})|^p d\boldsymbol{\vartheta} \right)^{1/p}$$

is bounded.

In the paper L. Szili and P. Vértesi [4] (using many ideas of Gábor Halász) the following result has been proved.

**THEOREM A.** *For any fixed  $d \geq 1$ ,*

$$(2.3) \quad \|D_{nd}\|_1 = \|S_{nd}\| \sim (\log n)^d \quad (n \geq 2).^1$$

**REMARK.** The proof of Theorem A consists of two parts; first we prove that the left hand side is  $\leq C_1(\log n)^d$ , then we get that the left hand side is  $\geq C_2(\log n)^d$ .

The aim of this paper is to get a more direct proof for the second part, that means we give another proof of

**THEOREM 2.1.** *We have for any fixed  $d \geq 1$*

$$\|D_{nd}\|_1 \geq C(\log n)^d \quad (n \geq 2).$$

### 3. Proof

**3.1.** As it was proved by Xu (cf. [6, (4.2.5)]),

$$(3.1) \quad D_{nd}(\boldsymbol{\vartheta}) = (-1)^{\lfloor \frac{d-1}{2} \rfloor} \sum_{l=1}^d \frac{2 \cos \frac{\vartheta_l}{2} (\sin^{d-2} \vartheta_l) \operatorname{soc} \frac{2n+1}{2} \vartheta_l}{\prod_{\substack{j=1 \\ j \neq l}}^d (\cos \vartheta_l - \cos \vartheta_j)} \\ =: (-1)^{\lfloor \frac{d-1}{2} \rfloor} \sum_{l=1}^d s_{ln}(\boldsymbol{\vartheta}),$$

where the function  $\operatorname{soc}$  (sin or cos) is defined by

$$(3.2) \quad \operatorname{soc} \vartheta := \begin{cases} \sin \vartheta & \text{if } d \text{ is odd,} \\ \cos \vartheta & \text{if } d \text{ is even.} \end{cases}$$

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<sup>1</sup> Here and later  $a_n \sim b_n$  means that  $0 < C_1 \leq a_n b_n^{-1} \leq C_2$  where  $C, C_1, C_2, \dots$  are positive constants, not depending on  $n$ ; they may denote different values in different formulae.

We suppose  $d = \text{odd}$  (the case of “ $d = \text{even}$ ” is similar). For a fixed  $0 < \varepsilon < 1$  we define

$$(3.3) \quad \begin{cases} g_{1kn}(\varepsilon) = \left\{ \vartheta_1, \vartheta_1 \in \left[ \frac{(2k+1)\pi - \varepsilon\pi/2}{2n+1}, \frac{(2k+1)\pi + \varepsilon\pi/2}{2n+1} \right] \cap \left[ 0, \frac{\pi}{2} \right] \right\}, \\ \mathcal{G}_{1n}(\varepsilon) = \bigcup_{k=1,2,3,\dots} g_{1kn}(\varepsilon). \end{cases}$$

Then we introduce for  $\ell = 2, 3, \dots, d$

$$(3.4) \quad \begin{cases} g_{\ell kn}(\varepsilon) = \left\{ \vartheta_\ell, \vartheta_\ell \in \left[ \frac{2k\pi - \varepsilon\pi/2}{2n+1}, \frac{2k\pi + \varepsilon\pi/2}{2n+1} \right] \cap \left[ 0, \frac{\pi}{2} \right] \right\}, \\ \mathcal{G}_{\ell n}(\varepsilon) = \bigcup_{k=1,2,3,\dots} g_{\ell kn}(\varepsilon). \end{cases}$$

REMARKS. 1. Notice the difference between the definition of the ‘small intervals’ in  $\mathcal{G}_{1n}(\varepsilon)$  and  $\mathcal{G}_{\ell n}(\varepsilon)$  ( $\ell = 2, 3, \dots, d$ ).

2. By definition, the measure of  $\mathcal{G}_{ln}(\varepsilon)$ ,  $1 \leq l \leq d$ , is of order  $\varepsilon$ .

**3.2.** Obviously (sometimes omitting the superfluous indices)

$$|D_{nd}(\boldsymbol{\vartheta})| \geq |s_1(\boldsymbol{\vartheta})| - \sum_{\ell=2}^d |s_\ell(\boldsymbol{\vartheta})|.$$

Now we estimate  $s_1(\boldsymbol{\vartheta})$  for  $\boldsymbol{\vartheta} \in \mathcal{G}_{1n}(\varepsilon) \times [0, \frac{\pi}{2}]^{d-1}$ . By definition (cf. [4, (3.7)])

$$\pm s_1(\boldsymbol{\vartheta}) = 2 \frac{\cos \frac{\vartheta_1}{2} \sin^{d-1} \vartheta_1 \sin \frac{(2n+1)\vartheta_1}{2}}{\sin \vartheta_1 \prod_{\ell=2}^d \left( 2 \sin \frac{\vartheta_1 - \vartheta_\ell}{2} \sin \frac{\vartheta_1 + \vartheta_\ell}{2} \right)}.$$

Since  $\vartheta_1 \in \mathcal{G}_{1n}(\varepsilon)$ , then with a proper  $0 < \varepsilon < 1$ ,

$$\left| \sin \frac{2n+1}{2} \vartheta_1 \right| \geq \frac{1}{2}; \quad \text{moreover by } 0 \leq \vartheta_1 \leq \frac{\pi}{2}, \quad \cos \frac{\vartheta_1}{2} \geq \frac{\sqrt{2}}{2}.$$

From now on we suppose that  $\vartheta_1 \geq \vartheta_\ell$ ,  $2 \leq \ell \leq d$  (it defines a cone), whence we get

$$\sin \vartheta_1 = \sin \left( \frac{\vartheta_1}{2} + \frac{\vartheta_1}{2} \right) \geq \sin \left( \frac{\vartheta_1}{2} + \frac{\vartheta_\ell}{2} \right), \quad 2 \leq \ell \leq d.$$

Therefore

$$(3.5) \quad |s_1(\boldsymbol{\vartheta})| \geq \frac{C}{\sin \vartheta_1 \prod_{\ell=2}^d \sin \frac{\vartheta_1 - \vartheta_\ell}{2}}, \quad \boldsymbol{\vartheta} \in \mathcal{G}_{1n}(\varepsilon) \times \left[0, \frac{\pi}{2}\right]^{d-1}.$$

**3.3.** Let

$$H_{dn}(\varepsilon) = \{\boldsymbol{\vartheta}, \vartheta_\ell \in \mathcal{G}_{\ell n}(\varepsilon), \vartheta_1 \geq \vartheta_\ell, \ell = 1, 2, \dots, d\}$$

and

$$A_{dn} = \left[\frac{1}{n}, \frac{\pi}{2}\right]^d \cap \left\{\boldsymbol{\vartheta}, |\vartheta_i - \vartheta_j| \geq \frac{1}{n}, 1 \leq i \neq j \leq d\right\}.$$

Integrating we have

$$\int_{A_{dn} \cap H_{dn}(\varepsilon)} \cdots \int |s_1(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \geq C \int_{A_{dn} \cap H_{dn}(\varepsilon)} \cdots \int \frac{1}{\sin \vartheta_1 \prod_{l=2}^d \sin \frac{\vartheta_1 - \vartheta_l}{2}} d\boldsymbol{\vartheta} =: I.$$

Now, using symmetry, the property of the sine function, the definition of the Riemann integral we write as follows.

$$\begin{aligned} I &\geq C \int_{A_{dn} \cap H_{dn}(\varepsilon)} \cdots \int \frac{1}{\vartheta_1 \prod_{l=2}^d (\vartheta_1 - \vartheta_l)} d\boldsymbol{\vartheta} \\ &\geq C \sum_{k_1=1}^n \left( \frac{\varepsilon}{n} \left( \sum_{k_2=1}^n \prime \frac{\varepsilon}{n} \cdots \left( \sum_{k_d=1}^n \prime \frac{\varepsilon}{n} \cdot \frac{1}{\frac{k_1}{n} \prod_{l=2}^d \frac{k_1 - k_l}{n}} \right) \right) \right) \\ &\geq C \varepsilon^d \sum_{k_1=1}^n \left( \frac{1}{n} \left( \sum_{k_2=1}^n \prime \frac{1}{n} \cdots \left( \sum_{k_d=1}^n \prime \frac{1}{n} \cdot \frac{1}{\frac{k_1}{n} \prod_{l=2}^d \frac{k_1 - k_l}{n}} \right) \right) \right) \\ &\geq C \varepsilon^d \int_{A_{dn}} \cdots \int \frac{1}{\vartheta_1 \prod_{l=2}^d (\vartheta_1 - \vartheta_l)} d\boldsymbol{\vartheta} =: J. \end{aligned}$$

(Above,  $\sum'$  means that we omit those terms when  $k_1 = k_l$ ,  $l = 2, 3, \dots, d$ .)

Using similar substitutions as in [4, p. 159], let  $u_1 = \vartheta_1$ ,  $u_j = \vartheta_1 - \vartheta_j$ , if  $2 \leq j \leq d$ , so  $\vartheta_1 = u_1$ ,  $\vartheta_j = \vartheta_1 - u_j$  ( $2 \leq j \leq d$ ); i.e.  $\frac{1}{n} \leq u_j \leq \vartheta_1 = u_1$ . We get

$$\begin{aligned} & C|V_d|\varepsilon^d \int_{U_1} \int_{U_2} \cdots \int_{U_d} \frac{1}{\prod_{j=1}^d u_j} d\mathbf{u} \\ & \geq C|V_d|\varepsilon^d \int_{A/n}^{\pi/2} \int_{c/n}^{u_1-1/n} \cdots \int_{c/n}^{u_d-1/n} \frac{1}{\prod_{h=1}^d u_h} d\mathbf{u} \\ & \geq C|V_d|\varepsilon^d \left( \int_{1/\sqrt{n}+1/n}^{\pi/2} \frac{1}{u_1} \left( \int_{c/n}^{1/\sqrt{n}} \frac{1}{u_2} \cdots \left( \int_{c/n}^{1/\sqrt{n}} \frac{1}{u_d} du_d \right) du_{d-1} \right) \cdots du_1 \right) \\ & \geq C|V_d|\varepsilon^d \log^d n, \end{aligned}$$

where

$$\mathcal{U}_d = \begin{pmatrix} \frac{\partial \vartheta_1}{\partial u_1} & \frac{\partial \vartheta_1}{\partial u_2} & \cdots & \frac{\partial \vartheta_1}{\partial u_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \vartheta_d}{\partial u_1} & \frac{\partial \vartheta_d}{\partial u_2} & \cdots & \frac{\partial \vartheta_d}{\partial u_d} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

is the matrix of transformation, with  $V_d = |\det \mathcal{U}_d| = 1$ .

**3.3.1.** Let us remark that an analogous argument results in the estimation  $I \leq C(\varepsilon \log n)^d$  even if we do not suppose  $\vartheta_1 - \vartheta_l \geq 0$  anymore (for more details you may consult [4, Part 3.1.3]).

**3.4.** Moreover if  $\vartheta_2 \in \mathcal{G}_{2n}(\varepsilon)$ , then, by definition  $|\sin \frac{2n+1}{2} \vartheta_2| \leq C\varepsilon$ , i.e.

$$(3.6) \quad |s_2(\vartheta)| \leq C\varepsilon \frac{\cos \frac{\vartheta_2}{2} \sin^{d-1} \vartheta_2}{\sin \vartheta_2 \left| \prod_{\ell \neq 2} \sin \frac{\vartheta_2 - \vartheta_\ell}{2} \sin \frac{\vartheta_2 + \vartheta_\ell}{2} \right|}.$$

Since (cf. [4, (3.6)])

$$\sin \vartheta_2 \leq 2 \sin \frac{\vartheta_2 + \vartheta_\ell}{2},$$

if  $1 \leq \ell \leq d$  it follows from (3.6)

$$|s_2(\boldsymbol{\vartheta})| \leq \frac{C\varepsilon}{\sin \vartheta_2 \prod_{\ell \neq 2} \sin \frac{|\vartheta_2 - \vartheta_\ell|}{2}} =: C\varepsilon |s_2^*(\boldsymbol{\vartheta})|.$$

Integrating as before we get

$$\int \cdots \int_{A_{dn} \cap H_{dn}} |s_2^*(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq C\varepsilon^d (\log n)^d$$

(consider 3.3.1 and use symmetry). Hence

$$\int \cdots \int_{A_{dn} \cap H_{dn}} |s_2(\boldsymbol{\vartheta})| d\boldsymbol{\vartheta} \leq C\varepsilon^{d+1} (\log n)^d.$$

**3.5.** If  $\ell > 2$ , we get similar estimations. That means

$$\begin{aligned} \int \cdots \int_{A_{dn} \cap H_{dn}} \left[ |s_1(\boldsymbol{\vartheta})| - \sum_{\ell=2}^d |s_\ell(\boldsymbol{\vartheta})| \right] d\boldsymbol{\vartheta} &\geq C\varepsilon^d (\log n)^d (1 - (d-1)\varepsilon) \\ &\geq C\varepsilon^d (\log n)^d, \end{aligned}$$

if  $0 < \varepsilon < 1$  is small enough.  $\square$

## References

- [1] H. Berens and Y. Xu, Fejér means for multivariate Fourier series, *Mat. Z.*, **221** (1996), 449–465.
- [2] J. G. Herriot, Nörlund summability of multiple Fourier series, *Duke Math. J.*, **11** (1944), 735–754.
- [3] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press (Princeton, 1971).
- [4] L. Szili and P. Vértési, On multivariate projection operators, *J. Approx. Theory*, **159** (2009), 154–164.
- [5] F. Weisz,  $l_1$ -summability of higher-dimensional Fourier series, *J. Approx. Theory*, **163** (2011), 99–116.
- [6] Y. Xu, Christoffel functions and Fourier series for multivariate orthogonal polynomials, *J. Approx. Theory*, **82** (1995), 205–239.
- [7] A. Zygmund, *Trigonometric Series, Vol. I and Vol. II*, Cambridge University Press (Cambridge, 1959).