# Dynamic analysis of immersed tapered column carrying an eccentric mass at the free.

N.M. Auciello
School of Engineering - University of Basilicata,
Via dell'Ateneo Lucano, 10 - 85100
Potenza, Italy
e-mail: nicola.auciello@unibas.it

*Abstract*

**The dynamic behavior of beams with varying cross section and partially immersed in liquid is examined by means of a direct integration approach of the equation of motion. The proposed structural model is quite useful as a simplified model of structures with complex geometries and unusual load conditions. The Euler-Bernoulli hypothesis is assumed to be valid, henceforth the solutions can be expressed in term of Bessel functions, while the press are frequency-dependent. The problem is solved by employing an ad hoc transformation of the equation of motion, and the orthogonality conditions of the eigenfunctions are computed. Numerous numerical examples are proposed for the calculation of the free frequencies of vibration to vary some parameter of taper of the beams.**

***Keywords: Free vibrations, immersed beam, exact analysis.***

## I. INTRODUCTION

The dynamic behavior of structures submerged in liquid been discussed by various authors, mainly adopting the simplified model of a cantilever beam with constant cross section and tip mass [1]. As suggested in [2], the influence of the liquid is taken into account in the equation of motion by augmenting the mass density, so influencing the beam flexibility. With this device, the usual general solutions can be retained. On the other hand, from a more practical point of view, the largest part of engineering structures has a more complex geometry, and they can be more appropriately studied as beams with varying cross section [3,4].

In offshore engineering, since the dynamic behaviors of the structures such as piles and towers, surrounded by water, can be predicted from a cantilever beam carrying a tip mass with reasonable accuracy, the literature concerned is plenty. For example, Uscilowska et al. [5] and Oz [7] have calculated the values of the natural vibration frequencies for a uniform tower offshore, partially immersed in a fluid. The column under consideration has been modeled as a distributed parameter cantilever with the lumped mass at the top and in the analysis the rotary inertia of the lumped mass has been taken into account. In order to investigate the effect of liquid, Auciello [6], used exact formulation to obtain the free frequencies for tapered beam carrying an eccentric tip mass with mass moment of inertia. Following this approach, Wu and al [8,9], extend the result for tapered beam with complex geometry, instead , De Rosa et al. in [10] studied the tapered beam supported by a translational and rotational springs.

In this paper, the exact deduction of the free vibration frequencies is given for beams with slightly varying cross section. In fact, the Euler-Bernoulli hypotheses are retained, and consequently the exact solutions can be expressed in terms of Bessel functions whereas the presence of the liquid leads to an increment of the mass density of the submerged part of the beam. Due to particular location of the applied mass, the boundary conditions are frequency-dependent. The frequency equations and mode shapes are obtained by formulating equations of motion for each of two span beams of the column. Both the rotary inertia and eccentric of the tip mass are taken into account, so that the boundary condition become frequency dependent.

The roots of the transcendental frequency equations have been obtained by means of the improved conventional analytical (closed-form) solution; in the analysis the influence of the various parameters is examined and some of the results are presented in tabular and graphical form.

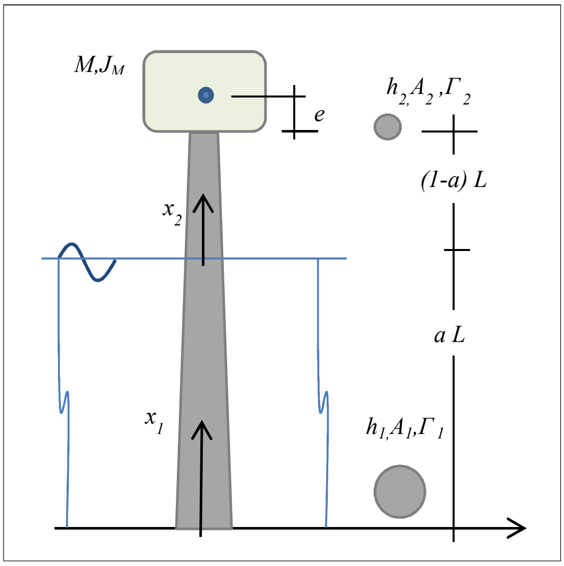


Figure 1. Configuration of a rotating, non-uniform, cantilever beam.

## II. MATHEMATICAL FORMULATIONS

The slope of the hanging in the submerged beam is supposed to be negligible, so that the resulting structural model is given by a "cone" partially immersed, in which the cross section is a function of a tapered parameter, $\varepsilon = 1 - \frac{h_1}{h_2}$ .

The equations of the problem can be simplified if two different reference systems are employed. The first one has origin at the base of the beam, and the second one has origin at the section of the free liquid surface (see Fig. 1). Consequently, area and moment of inertia of the cross section can be written as:

$$A(x_1) = A_1 u_1^2, \qquad \Gamma(x_1) = \Gamma_1 u_1^4 \ \rightarrow \ 0 \le x_1 \le a\,L \tag{1}$$

$$A(x_2) = A_1 u_2^2, \qquad \Gamma(x_2) = \Gamma_1 u_2^4 \ \rightarrow \ 0 \le x_2 \le (1-a)L \tag{2}$$

where the following quantities have been introduced:

$$u_1(x_1) = 1 + \varepsilon\left(\frac{x_1}{L} - 1\right), \qquad u_2(x_2) = 1 + \varepsilon\left(a + \frac{x_2}{L} - 1\right). \tag{3,4}$$

The variables can be separated, and the transverse displacements can be written as:

$$w_1(x_1,t) = W_1(x_1)e^{j\omega t}, \qquad w_2(x_2,t) = W_2(x_2)e^{j\omega t}, \tag{5,6}$$

where $\omega$ is the natural frequency and $j=(-1)^{0.5}$.

By neglecting the effects of shear deformation and rotary inertia using the Euler-Bernoulli hypotheses and the above expression, the equation of the motions are given by:

$$\left[E\Gamma(x_1)W_1(x_1)''\right]'' - \rho_1 A(x_1)\omega^2 W_1(x_1) = 0, \quad 0 \le x_1 \le aL$$

$$\left[E\Gamma(x_2)W_2(x_2)''\right]'' - \rho_2 A(x_2)\omega^2 W_2(x_2) = 0, \quad 0 \le x_2 \le (1-a)L \tag{7,8}$$

where E is Young's modulus, $\rho_1$ and $\rho_2$ are the masses density of the two parts of the beam; i.e. $\rho_2 = \nu\ \rho_1$. The derivatives with respect to the coordinates are defined with the related superscript (').

Eqns (1-4) can be conveniently introduced into the equations of motion, and taken into account that

$$\frac{du_1}{dx_1} = \frac{du_2}{dx_2} = \frac{\varepsilon}{L}, \tag{9}$$

we can write:

$$u_i^2\,\frac{d^4W_i}{du_i^4} + 8u_i\,\frac{d^3W_i}{du_i^3} + 12\frac{d^2W_i}{du_i^2} - q_i\,W_i = 0, \quad (i = 1,2) \tag{10}$$

where the following non-dimensional quantities have been defined:

$$p = \frac{\rho_1 A_1}{E\Gamma_1}L^4\omega^2, \qquad q_1 = -\frac{p}{\varepsilon}, \qquad q_2 = -\sqrt[4]{\nu}\,\frac{p}{\varepsilon}. \tag{11}$$

Therefore, according to Watson [12], the solution of the Eqns. (10) takes the forms; (Auciello et al. [3-4])

$$W_i = \frac{1}{u_i}\left[C_{i1}J_2(a_i) + C_{i2}Y_2(a_i) + C_{i3}I_2(a_i) + C_{i4}K_2(a_i)\right], \quad (i=1,2) \tag{12}$$

where

$$a_1 = 2q_1\,u_1^{0.5}, \qquad a_2 = 2q_2\,u_2^{0.5}. \tag{13}$$

In Eqns (12), $J_2$ and $Y_2$ denote the 2nd order Bessel function of first and second kind, respectively, $I_2$ and $K_2$ denote the 2nd order modified Bessel function of first and second kind, respectively, while $C_{ij}$ *(i=1,2; j=1,..4),* are the integration constants determined by the following boundary conditions.

To this end, it is convenient to express the geometrical parameters in anon-dimensional form:

$$d = \frac{e}{L}, \qquad \mu = \frac{M}{m_t}, \qquad k^2 = \frac{J_M}{M\,L^2}, \tag{14}$$

where $M$ is the applied tip mass and $J_M$ is its rotary inertia. The whole mass of the beam $m_t$, is written as:

$$m_t = \rho_1 A_1 \int_0^L u_1^2\,dx_1, \tag{15}$$

or else, put $Z = \frac{\varepsilon^2 - 3\varepsilon + 3}{3}$:

$$m_t = \rho_1 A_1 L\,\frac{\varepsilon^2 - 3\varepsilon + 3}{3}. \tag{16}$$

## III. THE BOUNDARY CONDITION

The frequency equation is deduced from the boundary conditions at the beam ends and at the section of the free liquid surface. As a function of $u_1$ and $u_2$ they can be imposed as:

$$x_1 = 0 \quad \rightarrow \quad u_1 = 1 - \varepsilon,$$

$$W_1 = 0, \qquad \frac{dW_1}{du_1} = 0 \tag{17,18}$$

at

$$x_1 = a\,L,\ x_0 = a\,L, \ \rightarrow \quad u_1 = u_2 = 1 + \varepsilon(a-1),$$

$$W_1 = W_2, \qquad \frac{dW_1}{du_1} = \frac{dW_2}{du_2} \tag{19,20}$$

$$4\frac{d^2W_1}{du_1^2} + u_1\,\frac{d^3W_1}{du_1^3} = 4\frac{d^2W_2}{du_2^2} - u_2\,\frac{d^3W_2}{du_2^3} \tag{21}$$

$$\frac{d^2W_1}{du_1^2} = \frac{d^2W_2}{du_2^2}. \tag{22}$$

At the abscissa of the applied tip mass, at

$$x_2 = (1-a)L \quad \rightarrow \quad u_2 = 1,$$

$$\frac{d^2W_2}{du_2^2}-(k^2+d^2)\,\mu_1\frac{dW_2}{du_2}-\mu_1\frac{dW_2}{du_2}-\frac{d}{\varepsilon}\mu_1 W_2=0 \quad (23)$$

$$\frac{d^3W_2}{du_2^3}+4\frac{d^2W_2}{du_2^2}+\frac{d}{\varepsilon}\mu_1\frac{dW_2}{du_2}+\frac{\mu_1}{\varepsilon^2}W_2=0 \quad (24)$$

where $\mu_1=\nu\frac{Z}{\varepsilon}p^4\mu$.

The following recurrent formulae for the functions $J_n$ , $Y_n$ , $K_n$ [12];

$$\frac{d}{du}J_n(u)=\frac{n}{u}J_n-J_{n+1} \quad (25)$$

and $I_n$:

$$\frac{d}{du}I_n(u)=\frac{n}{u}I_n+I_{n+1}\,, \quad (26)$$

can be inserted, along with the eqns. (12) into the boundary conditions, and a homogeneous system for the constants $C_{ij}$ is deduced. The matrix **A** of the coefficients, look like:

$$\mathbf{A}=\begin{bmatrix} J_2(b_1) & Y_2(b_1) & I_2(b_1) & K_2(b_1) & 0 & 0 & 0 & 0 \\ J_3(b_1) & Y_3(b_1) & -I_3(b_1) & K_3(b_1) & 0 & 0 & 0 & 0 \\ J_2(b_2) & Y_2(b_2) & I_2(b_2) & K_2(b_2) & -J_2(c_2) & -Y_2(c_2) & -I_2(c_2) & -K_2(c_2) \\ J_3(b_2) & Y_3(b_2) & I_3(b_2) & K_3(b_2) & a_{45} & a_{46} & a_{47} & a_{48} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ J_4(b_2) & Y_4(b_2) & I_4(b_2) & K_4(b_2) & a_{65} & a_{66} & a_{67} & a_{68} \\ 0 & 0 & 0 & 0 & a_{75} & a_{76} & a_{77} & a_{78} \\ 0 & 0 & 0 & 0 & a_{85} & a_{86} & a_{87} & a_{88} \end{bmatrix}$$

with the terms given in Appendix.

The condition

$$\det\mathbf{A}=0\,, \quad (27)$$

given the frequency characteristic equation in $p$.

The root of eqns. (27) can be calculated, in principle, with various iterative numerical procedures, even if the classical bisection approach and even the related methods as for example the "*False position method*" can have serious instability problems. In order to avoid such drawbacks, the *Aiken* approach [13] has been employed, so that even the convergence rate is speeded up. The terms of the series $\{\hat{p}_n\}_{n=0}^{\infty}$ are replaced by the sequence

$$\hat{p}_n=p_n-\frac{(p_{n+1}-p_n)^2}{p_{n+2}-2p_{n+1}+p_n} \quad (28)$$

wich converge more rapidly to the root $p$.

## IV. THE EINGENFUNCTIONS

As already said, the presence of the tip mass leads to a non-adjoint problem, where the $\omega^2$ parameter is necessarily included into the boundary conditions. This difficulty can be overcome by properly manipulating the equations of motions. First of all, let us define:

$$L_1=aL,\ L_2=L(1-a),\ m_1=\rho_1\,A(x_1),\ m_2=\rho_2\,A(x_2) \quad (29)$$

$$r_1=E\Gamma_1(x_1),\qquad r_2=E\Gamma_1(x_2)\,. \quad (30)$$

The properties of the Dirac function $\delta(x_2-L_2^-)$ can be used in order to write:

$$\int_{\infty}^{\infty}W(x_2)\,\delta(x_2-L_2^-)dx_2=W(x_2) \quad (31)$$

$$\int_{\infty}^{\infty}W(x_2)\,\delta'(x_2-L_2^-)dx_2=W'(x_2)\,. \quad (32)$$

Finally, eqns. (5-6) become:

$$\left(r_1W_1''\right)''-\omega^2\,m_1W_1=0, \quad (33)$$

$$\left(r_2W_2''\right)''=\omega^2\left\{\begin{matrix} m_2W_2+M\left[W_2+W_2'd\right]\delta(x_2-L_2^-)+ \\ -M\left[W_2d+\left(J_M+d^2\right)\right]\delta'(x_2-L_2^-)\end{matrix}\right\}=0, \quad (34)$$

and the corresponding boundary conditions of a Clamped-Free beam:

$$W_1(x_1)=0,\qquad W_1'(x_2)=0,\qquad x_1=0 \quad (35)$$

$$r_2W_2''(x_2)=0,\qquad \left(r_2W_2''(x_2)\right)'=0,\qquad x_2=L_2 \quad (36)$$

together, with usual continuity conditions at $x_1=a\,L$.

In this way, a classical self-adjoint problem is recovered, with a different orthogonality condition for eigensolutions. In fact, from eqns. (33,34), taken (31,32) into account, for $\omega_i^2\neq\omega_j^2$ we have:

$$\begin{aligned}&\int_0^{L_1}\left\{W_{1j}\left(r_1W_{1i}''\right)''-W_{1i}\left(r_1W_1''\right)''\right\}dx_1+\\&\int_0^{L_2}\left\{W_{2j}\left(r_2W_{2i}''\right)''-W_{2i}\left(r_2W_{2j}''\right)''\right\}dx_2=\\&\left(\omega_i^2-\omega_j^2\right)\left\{\begin{matrix}\int_0^{L_1}m_1W_{1j}W_{1i}\,dx_1+\int_0^{L_2}m_2W_{2j}W_{2i}\,dx_1+\\+M\left(\left[W_{2i}W_{2j}\right]_{L_2}+d\left[W_{2i}'W_{2j}\right]_{L_2}+d\left[W_{2i}W_{2j}'\right]_{L_2}\right)+\\+\left(J_M+M\,d^2\right)\left[W_{2i}'W_{2j}'\right]_{L_2}\end{matrix}\right\}\end{aligned}$$

(37)

The terms on the left hand side can be integrated by parts:

$$\int_0^{L_1} \left\{ W_{1j}\left(r_1 W_{1i}^{''}\right)^{''} - W_{1i}\left(r_1 W_{1j}^{''}\right)^{''} \right\} dx_1 =$$
$$\left[ W_{1j}\left(r_1 W_{1i}^{''}\right)^{'} - W_{1i}\left(r_1 W_{1j}^{''}\right)^{'} - W_{1j}^{'}\left(r_1 W_{1i}^{''}\right)^{'} + W_{1i}^{'}\left(r_1 W_{1j}^{''}\right)^{''} \right]_0^{L_1} \tag{38}$$

and

$$\int_0^{L_2} \left\{ W_{2j}\left(r_2 W_{1j}^{''}\right)^{''} - W_{1i}\left(r_1 W_{1j}^{''}\right)^{''} \right\} dx_1 =$$
$$\left[ W_{2j}\left(r_2 W_{2i}^{''}\right)^{'} - W_{2i}\left(r_2 W_{2j}^{''}\right)^{'} - W_{2j}^{'}\left(r_2 W_{2i}^{''}\right)^{'} + W_{1i}^{'}\left(r_2 W_{2j}^{''}\right)^{''} \right]_0^{L_1}. \tag{39}$$

Finally, explaining the terms of equation (37) we obtain the orthogonality condition:

$$\int_0^{L_1} m_1 W_{1j} W_{1i}\, dx_1 + \int_0^{L_2} m_2 W_{2j} W_{2i}\, dx_1 +$$
$$+M\left(\left[W_{2i}W_{2j}\right]_{L_2} + d\left[W_{2i}^{'}W_{2j}\right]_{L_2} + d\left[W_{2i}W_{2j}^{'}\right]_{L_2}\right) + \tag{40}$$
$$+\left(J_M + M\, d^2\right)\left[W_{2i}^{'}W_{2j}^{'}\right]_{L_2} = 0.$$

If $d=0$, $a=0$ and $a=L$ this condition coincide with the Morgan condition in [9]. Side by side with the well-known Gram–Schmidt procedure, eqn. (40) allows the deduction of an orthonormal bases for the configuration space [14].

## V. Results and Discussion

As a first numerical example, the first three frequencies have been calculated for a partial immersed beam with constant cross section for various values of the non-dimensional parameters. The results are given in Table I, where the corresponding values are obtained employing the method of the transfer matrix approach by Chang et al. [1] and Uściłowska et al. [5]; reported in italics.

TABLE I. Comparison of first three natural frequencies; d=0, ν=0.887.

| ε | k | μ | a | $p_1$ | $p_2$ | $p_3$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | *1.28589* | *4.15381* | *7.35122* |
| | 0 | 2 | 0.5 | *1.10878* | *4.05978* | *7.20006* |
| | $\sqrt{1/2}$ | 2 | 0.5 | *0.91261* | *1.74004* | *4.89985* |
| | | | 1 | *0.91147* | *1.73393* | *4.84047* |
| 1 | $\sqrt{1/2}$ | 2 | 0.25 | 1.49465 | 2.32176 | 5.98984 |
| | | | 0.5 | 1.40460 | 2.32130 | 5.92173 |
| | | | 0.75 | 1.40427 | 2.31948 | 5.83678 |
| | | | 1 | 1.40324 | 2.31730 | 5.82744 |

The results are in excellent agreement. It is worth noting that as the parameter a increases, the free frequencies diminish, and a takes into account the submerged part of the structure. This is evident for the fundamental frequencies, whereas the effect becomes less noticeable for higher vibration modes. From a practical point of view, it is readily seen that the influence of the applied mass is predominant on the other parameters and it completely, and it completely characterize the structure behavior.

In Table II, the first five free frequencies are given for a tapered beam and for various taper ratio ε. The particular case ε=0 given the values for constant cross section and cross sectional area *A*, for increasing ε values we have a reduction of the beam flexibility.

TABLE II. Comparison of first five natural frequencies; d=0, ν=0.887.

| ε | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ |
|---|---|---|---|---|---|
| 0 | 1.12086 | 2.14826 | 5.05136 | 8.13794 | 11.2748 |
| 0.1 | 1.15866 | 2.19817 | 5.16323 | 8.32617 | 11.5382 |
| 0.2 | 1.19044 | 2.24609 | 5.27026 | 8.50649 | 11.7904 |
| 0.3 | 1.21706 | 2.29234 | 5.37307 | 8.67994 | 12.0328 |
| 0.4 | 1.23928 | 2.33708 | 5.47221 | 8.84735 | 12.2665 |
| 0.5 | 1.25774 | 2.38045 | 5.56810 | 9.00941 | 12.4925 |
| 1 | 1.31062 | 2.57877 | 6.00991 | 9.75602 | 13.5314 |

However, the whole method holds for beams with small taper ratio ε. If ε becomes too large, it is necessary to take into account the shear deformability and the rotary inertia of the cross section, and the Timoshenko model should be used.

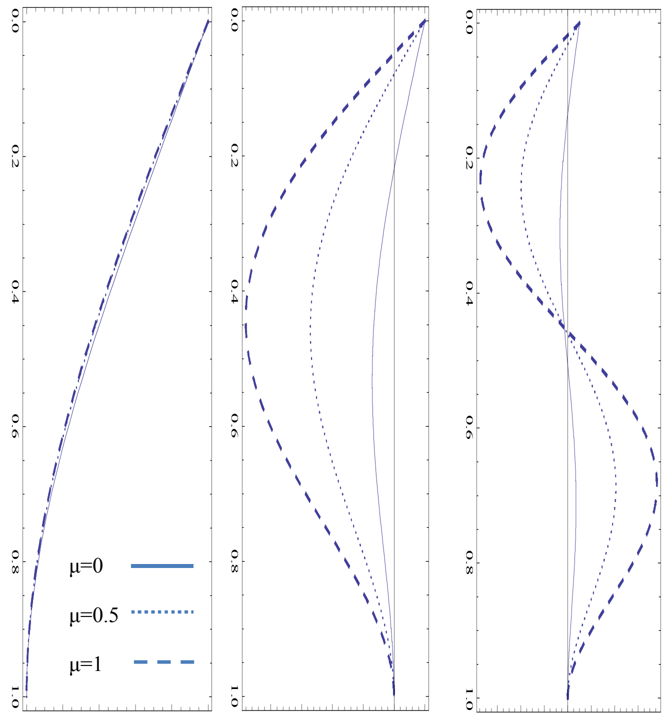


Figure 2. First three shape modes of an offshore tower with fixed base; d=0, km=0 for various parameters μ.

In Figure 2 the first three vibration modes are illustrated for $\varepsilon$=0, $\mu$=0, $\mu$=0.5 and $\mu$=1. It is interesting to note that, as the applied mass increases, the tip displacements become smaller, while the intermediate displacements increase noticeably. Therefore, the presence of the mass is particularly important the higher modes.

In Figure 3, setting $\varepsilon$=0, a=1/2, d=0 and $k_m$=0, the free frequencies values of tapered beam are obtained by varying the parameter $\mu$. As shown, when the parameter $\mu$ increases, the frequencies tends a asymptotic way.

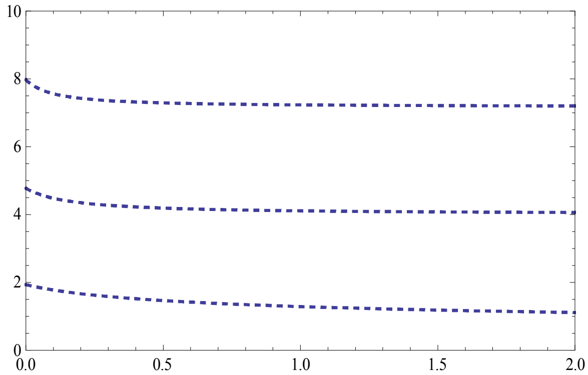


Figure 3. First three frquencies for various parameters $\mu$; $\varepsilon$=0, a=0,5, d=0 and $k_m$=0.

## VI. CONCLUSIONS

An analytical approach is proposed for the free vibration analysis of a partially immersed beam with variable cross section and tip mass. The influence of the liquid is taken into account by increasing the mass density of the immerged part of the beam, whereas the presence of the tip mass leads to a non self-adjoint problem, which must be properly addressed by defining a different orthogonality condition. The numerical example show an excellent agreement with some previously deduced results. The present paper represents an useful tool of investigation in order to study the dynamic behaviour of the immersed tapered beam. Moreover, it can be used to control and optimize the tapered beams.

## REFERENCES


[1] J.Y. Chang and L.H. Liu, “Some studies on the natural frequencies of immersed restrained column”, J Sound Vib., **130**, 516-524, 1989.

[2] Y. Narita, “Vibration of a non-uniform cantilever beam partially immersed in an ideal liquid. J Sound Vib., **81**, 583-586, 1982.

[3] N.M. Auciello and A. Ercolano, “Exact solution for the transverse vibration of a beam a part of which is a wedge and other part is a uniform beam”, J Sound Vib., **34**, 2115-2129, 1997.

[4] N.M. Auciello and G. Nolè, “Vibrations of a cantilever tapered beam with varying section properties and carrying a mass at the free end”, J Sound Vib., **214**, 105-119, 1998.

[5] A. Uściłowska and J.A. Kołodziej, “Free vibration of immersed column carrying a tip mass”, J Sound Vib., **216**, 147-157, 1998.

[6] N.M. Auciello, “Analisi dinamica di colonne immerse con massa all’estremo”, X Congresso Nazionale, Ingegneria Sismica in Italia, Potenza-Matera 9-13 Settembre 2001.

[7] H.R. Oz, “Natural frequencies of an immersed beam carrying a tip mass with rotary inertia”, J. Sound Vib., **266**, 1099-1108, 2003.

[8] J.S. Wu and C.T. Chen, “An exact solution for the natural frequencies and mode shapes of an immersed elastically restrained wedge beam carrying an eccentric tip mass with mass moment of inertia”, J. of Sound and Vib., **286**, 549-568, 2005.

[9] J.S. Wu and S.H. Hsu, “A unified approach for the free vibration analysis of an elastically supported immersed uniform beam carrying an eccentric tip mass with rotary inertia. J. of Sound and Vib., **291**, 1122-1147, 2006.

[10] M.A. De Rosa, M. Lippiello, F. Vairo and M.J. Maurizi, “Free vibration analysis of a variable cross-section column partially immersed in a liquid”, Ocean Engineering (submitted) 2013.

[11] G.W. Morgan, “Some remars on a class of eingenvalue problems with special boundary conditions”. Quaterly Appl. Math., **11**, 157-165, 1953.

[12] G.N. Watson, “A treatise on the theory on the theory of Bessel function”, Cambrige, University Press, 1952.

[13] R.L. Burden and J.D. Faires, Numerical Analysis, Boston, PWS-Kent, 1993.

[14] K. Rektorys, Variational Methods in Mathematics, Science and Engineering. D. Reidel Publishing Comp., Dordrecht-Holland, 1977.


## APPENDIX

$$a_{45} = -\nu^{0.25} J_3(c_2), \quad a_{46} = -\nu^{0.25} Y_3(c_2), \quad a_{47} = \nu^{0.25} I_3(c_2), \quad a_{48} = -\nu^{0.25} K_3(c_2)$$

$$a_{51} = 4\varepsilon J_4(b_2) + p b^{0.5} J_5(b_2), \quad a_{52} = 4\varepsilon Y_4(b_2) + p b^{0.5} Y_5(b_2),$$
$$a_{53} = 4\varepsilon I_4(b_2) + p b^{0.5} I_5(b_2), \quad a_{54} = 4\varepsilon K_4(b_2) + p b^{0.5} K_5(b_2)$$

$$a_{55} = -4\varepsilon\, \nu^{0.5} J_4(c_2) - p b^{0.5} \nu^{0.75} J_5\left(c_2\right), \quad a_{56} = -4\varepsilon\, \nu^{0.5} Y_4(c_2) - p b^{0.5} \nu^{0.75} Y_5\left(c_2\right),$$
$$a_{57} = -4\varepsilon\, \nu^{0.5} I_4(c_2) + p b^{0.5} \nu^{0.75} I_5\left(c_2\right), \quad a_{58} = -4\varepsilon\, \nu^{0.5} K_4(c_2) - p b^{0.5} \nu^{0.75} K_5\left(c_2\right),$$

$$a_{65} = -\,\nu^{0.5} J_4(c_2), \quad a_{66} = -\,\nu^{0.5} Y_4(c_2), \quad a_{67} = -\nu^{0.5} I_4(c_2), \quad a_{68} = -\,\nu^{0.5} K_4(c_2),$$

$$a_{75} = p\ \nu^{0.5} J_4(c_3) - p\,(k^2 + d^2)\,\mu_1\,\varepsilon\,\nu^{0.25}\,J_3(c_3) - d\,\varepsilon\,\mu_1 J_2(c_3),$$
$$a_{76} = p\ \nu^{0.5} Y_4(c_3) - p\,(k^2 + d^2)\,\mu_1\,\varepsilon\,\nu^{0.25}\,Y_3(c_3) - d\,\varepsilon\,\mu_1 Y_2(c_3),$$
$$a_{77} = p\ \nu^{0.5} I_4(c_3) + p\,(k^2 + d^2)\,\mu_1\,\varepsilon\,\nu^{0.25}\,I_3(c_3) - d\,\varepsilon\,\mu_1 I_2(c_3),$$
$$a_{78} = p\ \nu^{0.5} K_4(c_3) - p\,(k^2 + d^2)\,\mu_1\,\varepsilon\,\nu^{0.25}\,K_3(c_3) - d\,\varepsilon\,\mu_1 K_2(c_3),$$

$$a_{85} = p^3\ \nu^{0.75} J_5(c_3) + 4\,\varepsilon\,p^2\ \nu^{0.5} J_4(c_3) + p\,d\,\varepsilon\,\mu_1\,\nu^{0.25} J_3(c_3) + \mu_1\,\varepsilon\,J_2(c_3),$$
$$a_{86} = p^3\ \nu^{0.75} Y_5(c_3) + 4\,\varepsilon\,p^2\ \nu^{0.5} Y_4(c_3) + p\,d\,\varepsilon\,\mu_1\,\nu^{0.25} Y_3(c_3) + \mu_1\,\varepsilon\,Y_2(c_3),$$
$$a_{87} = -p^3\ \nu^{0.75} I_5(c_3) + 4\,\varepsilon\,p^2\ \nu^{0.5} I_4(c_3) - p\,d\,\varepsilon\,\mu_1\,\nu^{0.25} I_3(c_3) + \mu_1\,\varepsilon\,I_2(c_3),$$
$$a_{88} = p^3\ \nu^{0.75} K_5(c_3) + 4\,\varepsilon\,p^2\ \nu^{0.5} K_4(c_3) + p\,d\,\varepsilon\,\mu_1\,\nu^{0.25} K_3(c_3) + \mu_1\,\varepsilon\,K_2(c_3),$$

where

$$b = 1 + \varepsilon(a-1), \quad b_1 = 2p_1\left(1-\varepsilon\right)^{0.5}, \quad b_2 = 2p_1\left(1+\varepsilon(a-1)\right)^{0.5}, \quad c_2 = 2p_2\left(1+\varepsilon(a-1)\right)^{0.5}, \quad c_3 = 2p_2.$$