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Vibration analysis of rotating non-uniform Rayleigh beams using "CDM" method

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Abstract— **In this paper the vibration analysis of rotating non-uniform tapered beams is performed. The structure is discretized by means of the so-called "Cell Discretization Method" (CDM), a numerical procedure already employed for the vibration analysis of arches and beams, [1-4]. Based on a dynamic variational approach, the equation of motion, for the multiple degree of freedom systems (MDOF), is derived from Lagrange formulation. The effect of rotational speed parameter, hub radius, taper ratio and slenderness ratio on natural frequencies are also investigated. Some numerical examples are presented and the results are validated by making comparisons with the results in literature and reported in bibliography. It is demonstrated that the proposed algorithm provides a simple and powerful tool in dealing with the parametric analysis of the free vibrations of rotating Rayleigh beams.**

Keywords- CDM method, Rayleigh beams, flapwise bending.

I. INTRODUCTION

In recent years, an upsurge of interest in the vibrational analysis of elastic rotating structures has been developed. There are many engineering examples which can be idealized as rotating non-uniform beams: helicopter rotor blades, wind mill turbines, satellites and aircraft propellers are some of structural configurations which fall into this category. Due to the effect of the rotational speed, the dynamic behaviour of these structures is strongly influenced by geometrical and inertial parameters. For example, in the helicopter rotor blades it is well-known that the deformability of the beam can be influenced by the centrifugal force due to rotational motion, which tends to vary the free vibrations values

Several authors have been carried out a considerable research on the free vibrations of rotating non-uniform beams and different types of solution procedures may be found in the literature. A large number of papers based on numerical approaches have been published and standard approximate methods for the modal analysis of rotating beams were employed. In general, the governing equations can be solved analytically in closed form assuming Euler-Bernoulli hypotheses and considering non-rotating tapered beams, subjected to the geometrical conditions depending on specific tapering ratios. In this case, the frequencies are derived in terms of Bessel functions [5-6]. Many approximated solutions have been developed by numerous authors for Euler-Bernoulli and Timoshenko tapered beams rotating around to axis. The equations of motion are derived from Lagrange equation and the free vibration frequencies are given by a weak solution of the dynamic problem in integral terms [7-10]. Among different numerical techniques, Finite Element Method (FEM) has acquired a dominant position due to its simplicity and generality, mostly for the vibration analysis of uniform cross-section beams: in this case, the beam is modelled as assemblage consisting of uniform elements [11]. There are, also, several numerical procedures, based on differential approaches, which lead to semi-analytical solutions of the rotating Euler-Bernoulli and Timoshenko beams. Amongst them is the Differential Transformation Method (DTM): a well-known semi-analytical method that depends on the Taylor Series Expansion. Zhou [19] has introduced this method in order to calculate the free frequencies of uniform longitudinal beams. By employing DTM, Mei [14] has developed a method for solving free vibrations problems of rotating beams taking into account the bending behaviour. In recent years, Banerjee et al. [13] has developed the Dynamic Stiffness Method (DSM) for a rotating Timoshenko beams based on the Frobenius Series Expansion [12].

Recently Jackson et al. [16] have applied the DTM method to study the dynamic problem of rotating Rayleigh beams. The theory takes into account both the tapering of the cross section, and the rotary inertia effect, and the accuracy of the Rayleigh beam theory with respect to Euler-Bernoulli hypotheses is clearly illustrated. The importance of the rotational inertia parameter in solving the dynamic problem of uniform beams has been emphasized in [10]. Finally, a couple of papers based on the well-known Rayleigh-Ritz approach [7-8] should be mentioned. A common peculiarity of these papers is that they give approximate solutions, which represent upper bounds to the true values. [17]. Instead, if the beam is discretized reducing it to a set of rigid bars linked together by elastic sections (elastic cells), the resulting approximate free frequencies values give a lower bound to the exact values. Consequently, it is possible to obtain a useful lower-upper approximation of free frequencies of vibration, [1-4].

In the present paper, the CDM method is extended to cover the dynamic analysis of rotating Rayleigh beams.

II. THEORY OF DISCRETE ELEMENT MODEL FOR BEAMS

Consider a tapered beam of length L, rotating about the z_1 axis, fixed at a distance r from the left side of the beam, as shown in Fig.1.

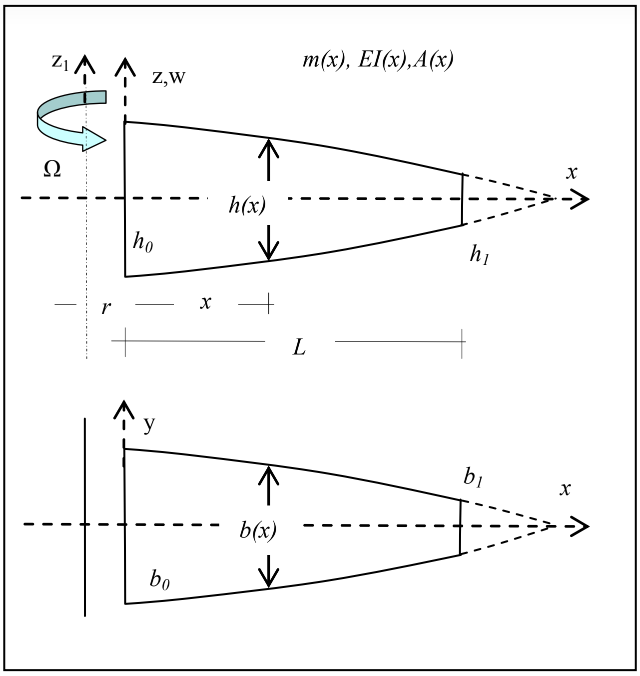

Figure 1. Rotating non-uniform beam element geometry.

The beam, under consideration, is discretized reducing it to a set of rigid bars, linked together by elastic sections (elastic cells). In this way, as shown in detail in [1-5], the structure is reduced to a system with finite number of degrees of freedom (MDOF). The Lagrangian parameters can be assumed to be the *t* rotations of the rigid bars, i.e. the generalized coordinates of the rigid-elastic system. All the possible configurations are functions of the following vector:

$$\mathbf{c} = [\varphi_1, \varphi_2, \ldots \ \varphi_t]^T \qquad (1)$$

and the vertical components of the nodal displacements are given by the following expressions:

$$w_1 = 0, \quad w_2 = -\varphi_1 l_1, \quad w_i = -\varphi_i l_i, \quad w_{t+i} = -\varphi_t l_t.$$

In matrix form, being **A** the displacements matrix, it is possible to write:

$$\mathbf{w} = \mathbf{A}\,\mathbf{c}\,. \qquad (2)$$

Similarly, the relative rotations between the two faces of the elastic cells are given by:

$$\Delta\varphi_1 = \varphi_1, \quad \Delta\varphi_2 = \varphi_2 - \varphi_1, \quad \Delta\varphi_i = \varphi_i - \varphi_{i-1}, \quad \Delta\varphi_{t+1} = 0$$

and in matrix form:

$$\boldsymbol{\Delta\varphi} = \mathbf{B}\,\mathbf{c}. \qquad (3)$$

The rectangular matrices **A** and **B** have t+1 rows and t columns, and each term can be calculated according to Fig.2.

Quite often it is possible to neglect both the axial and the shear deformation effects, limiting oneself to the bending deformations. In such hypothesis, at each "cell", the following

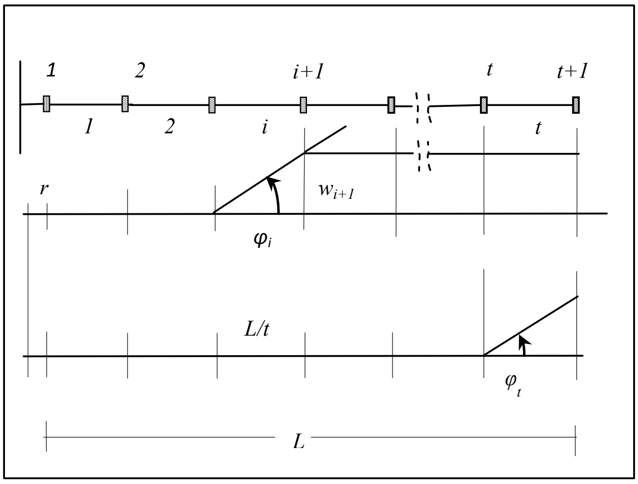

Figure 2. A discrete element beam.

relation between the relative rotation $\Delta\varphi_i$ and the moment M_i can be written, as follows:

$$M_i = k_i\,\Delta\varphi_i \qquad (4)$$

where the cells stiffness k_i, according to the present discretization, can be written as:

$$k_1 = 2\frac{t}{L}EI_1,$$
$$k_i = 2\frac{t}{L}E\frac{I_i\,I_{i+1}}{I_i + I_{i+1}} \quad (i = 2,\ldots t), \qquad (5)$$
$$k_{t+1} = 2\frac{t}{L}EI_{t+1},$$

III. Equation of Motion

In the case of multiple degree of freedom system (MDOF), the equation of motion is derived from Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}_i}\right) - \frac{\partial T}{\partial \varphi_i} + \frac{\partial U}{\partial \varphi_i} = 0, \qquad i = 1,2\ldots t \qquad (6)$$

where *U* is the total potential energy of the system. It is given by the sum of two terms: one due to bending deformation and the other due to centrifugal force deformation.

Similarly, *T* is the kinetic energy of the system, and if the Rayleigh beam model is assumed, the effects of the rotary inertia must be taken into account.

A. Elastic energy

In the hypothesis of Rayleigh beam model, the potential energy *U* of the system is defined by:

$$U = \frac{1}{2}\int_0^L E(I_y\,w'')^2\,dx + \frac{1}{2}\int_0^L F_x\,(w')^2\,dx \qquad (7)$$

where the apex sign represents the derivative respect to abscissa *x* and F_x is the centrifugal force acting on the beam at a distance x from the axis of the rotation motion.

In the rigid-elastic formulation, relations (3-4) must be expressed as functions of the rotations of the rigid bars, and the following relationship is easily obtained:

$$U = \frac{1}{2}\mathbf{c}^T\,\mathbf{B}^T\,\mathbf{k_f}\,\mathbf{B}\,\mathbf{c} + \frac{1}{2}\mathbf{c}^T\,\mathbf{F}_x\,\mathbf{c} \qquad (8)$$

where

$$\mathbf{F}_x = \frac{L}{t}\int_x^L \rho\,A_x\,\Omega^2\,(r+x)\,dx + F_0 \ , \qquad (9)$$

is the centrifugal force acting on the beam at a distance from the origin and is due to the effect of the spin around the axis z_1. The term F_0 is constant and related to the static inertia, in dynamic conditions its contribution is zero.

Thus potential energy is made of two different terms: one due to bending deformation and the other due to centrifugal force deformation.

B. Kinetic energy

The kinetic energy of the beam is defined as:

$$T = \frac{1}{2}\int_0^L \rho\,A_x\,(\dot{\imath}\,)^2\,dx + \frac{1}{2}\int_0^L \rho I_y\left[\left(\dot{\imath}\,'\right)^2 + \Omega^2\left(w'\right)^2\right]dx, \qquad (10)$$

the first term, T_m, is the classical kinetic energy, the second one, T_R, is due to the rotary inertia, according to the Rayleigh hypothesis.

In the aproximate form, the nodal rotations can be written as follows:

$$w_1^{'} = \varphi_1, \quad w_2^{'} = \varphi_2\ , \quad w_i^{'} = \varphi_i, \quad w_{t+1}^{'} = \varphi_t, \qquad (11)$$

or:

$$\mathbf{w}^{'} = \mathbf{R}\,\mathbf{c}\,, \qquad (12)$$

where **R** is the matrix of the rotations with dimension [t+1, t].

With the proposed method, the mass of the beam is properly concentrated at the middle point of the rigid segments. Therefore, the mass distribution becomes:

$$\begin{aligned} m_1 &= \rho\frac{L}{2t}A_1, \\ m_i &= \rho\frac{L}{t}\frac{A_i + A_{i+1}}{2} \qquad i = 2\ldots,t \qquad (13) \\ m_{t+1} &= \rho\frac{L}{2t}A_{t+1}, \end{aligned}$$

and the kinetic energy relative to the mass, taking into account the relations (2), has the following form:

$$T_m = \frac{1}{2}\sum_1^{t+1} m_i\,\dot{\imath}_{\imath}^{\,2} = \frac{1}{2}\,\dot{}\ \mathbf{A}^T\,\mathbf{m}\,\mathbf{A}\ \dot{}\ , \qquad (14)$$

where **m** is the mass matrix with dimension [t+1, t+1].

In the Rayleigh hypothesis it is necessary to consider the inertia effect of the section respect to axis y. The inertia of generic element is applied at nodes. Being I_y, the moment of inertia with respect to the axis y, it can be conveniently written:

$$\begin{aligned} \tilde{\imath}_{1} &= \rho\frac{L}{2t}I_{y1}, \\ \tilde{\imath}_{i} &= \rho\frac{L}{t}\frac{I_{yi} + I_{y(i+1)}}{2} \qquad i = 2\ldots,t \qquad (15) \\ \tilde{\imath}_{t+1} &= \rho\frac{L}{2t}I_{y\,(t+1)}, \end{aligned}$$

where $\tilde{\mathbf{\imath}}$ is the inertia matrix with dimension [t+1, t+1].

The kinetic energy due to the rotary inertia [7-8] holds:

$$T_R = \frac{1}{2}\ \dot{}\ \ \tilde{}\ \ \dot{}\ \ \ \tilde{}\ \ . \qquad (16)$$

Finally, taking into account the relations (14) and (16), the kinetic energy of the system can be written as:

$$T = \frac{1}{2}\ \dot{}\ \ \mathbf{M}\ \dot{} \qquad (17)$$

where **M**, the so-called generalized matrix of the mass is a square matrix of dimension t, and it is given by:

$$\mathbf{M} = \mathbf{A}^T\,\mathbf{m}\,\mathbf{A} + \mathbf{R}^T\,\tilde{\mathbf{\imath}}\ \ . \qquad (18)$$

The term due to the angular velocity Ω^2 appears both in the stiffness matrix and in the rotary inertia kinetic energy. Applying the relation (6), the equation of motion becomes:

$$(\mathbf{B}^T\,\mathbf{k_f}\,\mathbf{B} + \mathbf{F}_x - \Omega^2\,\mathbf{R}^T\,\tilde{\mathbf{\imath}}\ \mathbf{R}\ )\,\mathbf{c} + \omega^2\,\mathbf{M}\ \ddot{}\ = 0\ . \qquad (19)$$

The free frequencies have calculated by solving the eigenvalues problem given by the following algebraic system:

$$(\mathbf{K} - \omega^2\,\mathbf{M})\,\mathbf{c} = 0 \qquad (20)$$

where **K** is the square stiffness matrix of dimension t, and it is given by:

$$\mathbf{K} = \mathbf{B}^T\,\mathbf{k_f}\,\mathbf{B} + \mathbf{F}_x - \Omega^2\,\mathbf{R}^T\,\tilde{\mathbf{\imath}}\ \mathbf{R}\ \ . \qquad (21)$$

IV. Numerical Example and Discussion

In order to develop some numerical examples and to compare the relative results with those of the literature reported in bibliography, some non-dimensional parameters have been introduced, as follows:

$$\begin{aligned} &\xi = \frac{x}{L}, \quad \delta = \frac{r}{L}, \quad \gamma^2 = \rho\,\Omega\frac{A_1\,L^4}{E\,I_{y1}}, \\ &r_H = \sqrt{\frac{A_1\,L^2}{EI_{y1}}}, \qquad \lambda_i = \omega_i\sqrt{\rho\frac{A_1\,L^4}{E\,I_{y1}}}, \end{aligned} \qquad (22)$$

In order to study several typologies of tapered beams and define the sectional area and the moment of inertia of the beam, two functions, which define the geometric characteristics of the structure, can be conveniently introduced as follows:

$$A(\xi) = A_1\, G(\xi), \qquad I_y(\xi) = I_{y1}\, H(\xi)\,. \qquad (23)$$

- *Uniform beam*

The first numerical example deals with the dynamic analysis of uniform beams. Putting; $G(\xi)=1, H(\xi)=1$, the free frequencies are calculated for various values of the radius of inertia (slenderness ratio r_H). In literature the results for Rayleigh beams model are reader scarce and mainly obtained by employing the Finite Element Method (FEM). For example, in the paper [9], Rossi et al. calculate the first five non dimensional frequencies for different values of r_H. In the absence of rotation - γ=0 - and employing a rigid-elastic model with *t=100* rigid bars, the "CDM" method gives free frequencies values which numerically coincide with the results obtained in [9]. Assuming the Euler-Bernoulli hypothesis, in Table I the natural frequencies are also given. As mentioned before, the frequencies values are always lower bounds to the corresponding values obtained by using the finite element method (FEM). In this way, it is possible to obtain a lower-upper bound for the frequencies exact values.

TABLE I. COMPARISON OF NATURAL FREQUENCIES; γ=0, UNIFORM BEAMS

γ=0	C.D.M.			
	r_H=1/20	r_H=1/13.33	r_H=1/10	***E-B***
λ_1	3.4954	3.4706	3.4365	3.5157
λ_2	21.1821	20.2503	19.1298	22.0250
λ_3	56.4416	51.5192	46.4658	61.6414
λ_4	103.6940	90.1918	78.1436	120.7140
λ_5	159.4150	132.8860	111.6750	199.3861

γ=0	Rossi, [9]			***E-B [10]***
	r_H=1/20	r_H=1/13.33	r_H=1/10	r_H--->0
λ_1	3.4957	3.4709	3.4363	3.5160
λ_2	21.1907	20.2579	19.1364	22.0345
λ_3	56.4863	51.5548	46.4936	61.6972
λ_4	103.8238	90.2861	78.2131	120.9020
λ_5	159.6954	133.0771	111.6750	199.8600

One can see that the natural frequencies decrease for increasing values of the slenderness radius r_H .

In Figure 3, the first three shape modes of Rayleigh beam (r_H=1/10) are compared with the corresponding modes for the Euler-Bernoulli beam. As shown and taking into account the rotary inertia effect, the Rayleigh beam theory gives a model closer to the real structure behavior. Moreover, for the higher modes, the discrepancies between the Rayleigh theory and the simpler Euler-Bernoulli theory appear to be relevant. An accurate knowledge of the nodal point location can be useful for analysing and controlling the behaviour of rotating beam.

The effect of the rotary parameter has a deep impact even on the free frequencies results. For values of γ≠0, the centrifugal force influences the dynamic behavior of beam: the free frequencies increase to increase, for increasing values of the rotation, γ. From a practical point of view, greater values of the angular velocity lead to greater centrifugal forces, and in turn to stiffener beams. By increasing the extensional deformation, one gets increasing values of the natural frequencies of vibration.

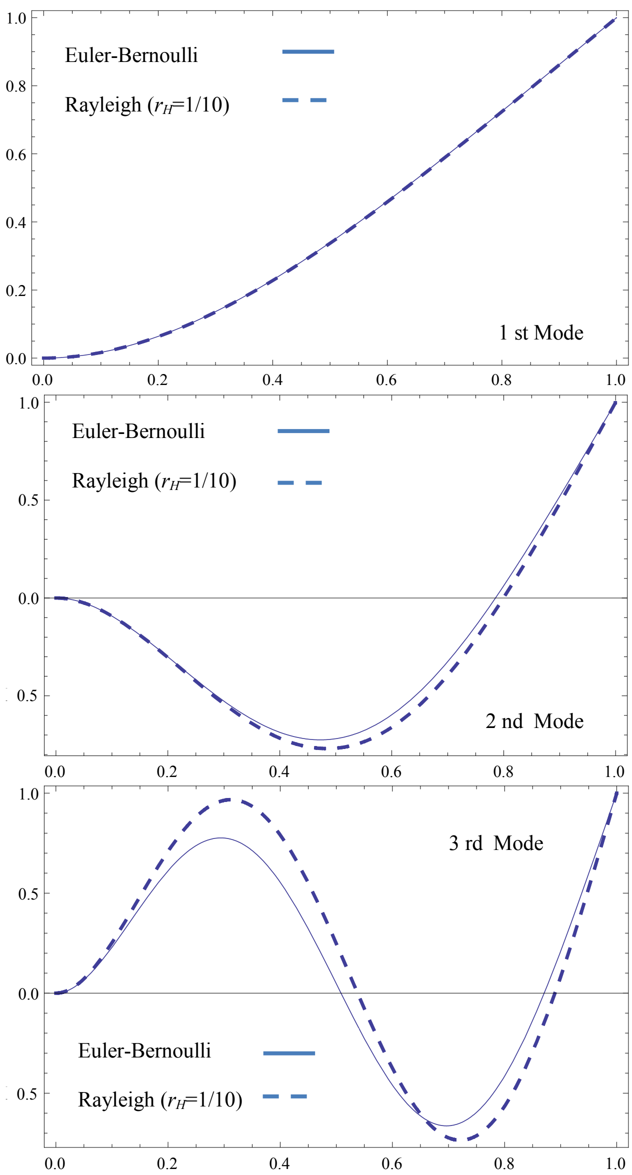

Figure 3. Bending mode shape for uniform beam; γ=0.

When the rotational speed increases, the fundamental frequency of the beam can assume the same value of the speed: this condition is known as resonance phenomenon and the corresponding angular speed is named tuned angular speed. For r_H=1/10 and r_H=1/30, varying the γ and δ parameters, the values of λ_I (i=1…5) have been calculated and the relative results are reported in Table II.

TABLE II. NATURAL FREQUENCIES; γ=0, UNIFORM RAYLEIGH BEAMS,

	r_H=1/10			r_H=1/30		
δ=0	γ=5	γ=10	γ=50	γ=5	γ=10	γ=50
λ_1	6.2296	10.8148	49.4097	6.4244	11.1584	50.8854
λ_2	22.8322	28.3933	95.8741	24.9572	32.9432	123.1830
λ_3	48.7246	55.2136	136.3991	62.4456	71.3887	197.6440
λ_4	80.0209	86.4222	188.0070	115.3910	124.8530	280.9950
λ_5	113.1210	119.1211	231.6666	181.1661	190.7340	370.9030
δ=0.5						
λ_1	7.5657	13.7589	64.8856	7.7713	14.1248	66.5609
λ_2	23.5418	33.1757	118.3671	26.9432	38.5822	156.3140
λ_3	50.4130	60.6491	166.5590	64.6587	78.6804	246.0710
λ_4	81.6709	92.2256	228.3361	117.7315	133.105	344.2930
λ_5	114.6591	124.803	279.0991	183.5310	199.436	449.0700

In the proposed Rayleigh beam, with $r_H=1/10$ and δ=0, the resonance phenomenon occurs in the range of $\gamma\in[10,50]$. In Figure 4, the phenomenon is sketched. The curve of the fundamental frequencies intersects the line of the speed (ω=γ), at the point γ=32.82, and this intersection gives the tuned angular speed. Increasing the δ parameter (named hub radius ratio) the free frequencies increase.

By observing the Figure 5, one gets that for δ=0.5, the resonance phenomenon cannot occur. The curves of frequencies, for increasing γ values, are anywhere above the line of the speeds (ω=γ), so that a tuned angular speed do not exist.

➢ *Tapered beam*

Let us consider that the variation of the cross section of the beam is given by the equations (23). In the case of tapered beams, the cross section area and moment of inertia are represented by the following expressions:

$$A(\xi)=A_1\left(1-\alpha\xi\right)\left(1-\beta\xi\right)$$
$$I_y(\xi)=I_{y1}\left(1-\alpha\xi\right)^3\left(1-\beta\xi\right). \quad (23)$$

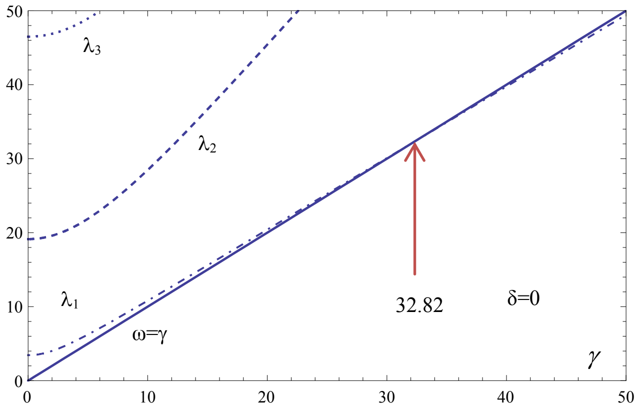

Figure 4. Tuned angular speed in flapwise bending vibration; rH=1/10

where the α and β parameters define the variation of height and base of the beam cross section along its span. As already said, in the literature few papers exist which deal with the tapered Rayleigh beam theory, so that the Authors – for the sake of comparisons - will refer to the natural frequencies calculated by Jackson et al., as reported in [16]. In the Table III, for α=0.5 and r_H=1/30 data, the obtained values are reported. As shown and by applying the CDM method, the natural frequencies values are always lower bounds to the values determined by the DTM.

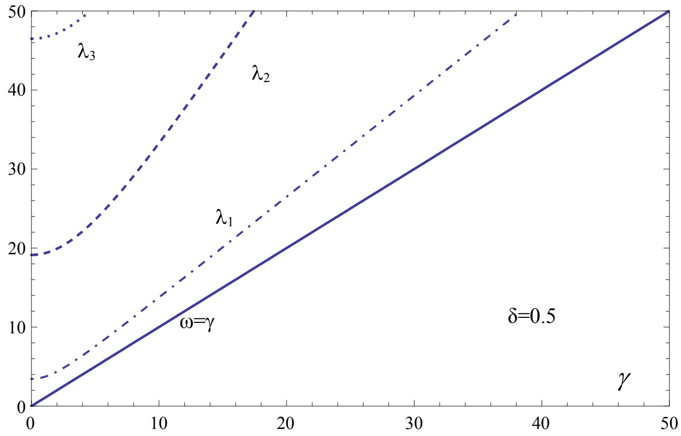

Figure 5. Angular speed and flapwise bending natural frequencies; r_H=1/30.

where the α and β parameters define the variation of height and base of the beam cross section along its span. As already said, in the literature few papers exist which deal with the tapered Rayleigh beam theory, so that the Authors – for the sake of comparisons - will refer to the natural frequencies calculated by Jackson et al., as reported in [16]. In the Table III, for α=0.5 and r_H=1/30 data, the values obtained are reported. As shown, the natural frequencies obtained by applying the CDM method are always lower bounds to the values determined by the DTM method.

TABLE III. NATURAL FREQUENCIES, TAPERED BEAMS; δ=0, r_H=1/30.

β=0	C.D.M.			Jackson [16]		
α=0,5	γ=0	γ=5	γ=10	γ=0	γ=5	γ=10
λ_1	3.8177	6.7332	11.4838	3.82109	6.7356	11.4856
λ_2	18.1601	21.7123	29.8974	18.2245	21.7911	30.0232
λ_3	46.2759	49.8385	59.1543	46.5757	50.1876	59.6737
λ_4	86.9716	90.4926	100.1831	87.7974	91.4413	101.5422
λ_5	139.0892	142.4541	151.9492	140.8192	144.4462	154.7865

In Figure 6, setting r_H=1/30, δ=β=0 and α=0,5, the free frequencies values of tapered beam are obtained by varying the angular speed γ. As shown, when the speed increases, the curve of fundamental frequencies tends, in asymptotic way, to the straight line of the angular speeds.

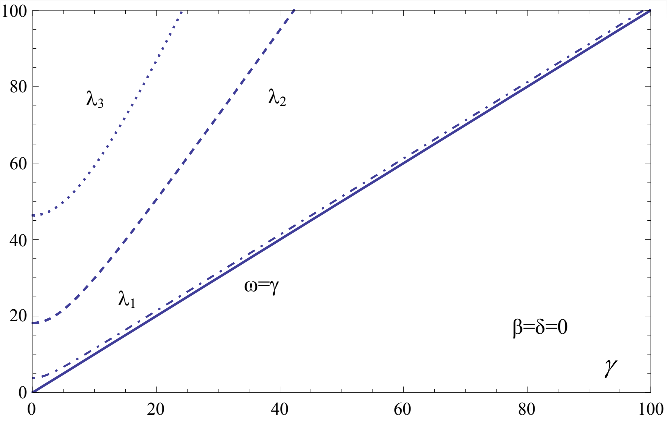

Figure 6. Natural frequencies for tapered Rayleigh beam α=0,5, and r_H=1/30.

In the Rayleigh beam case, for the higher modes of vibrations, the effect of taper has a relevant impact. For the higher modes, the shape of cross-section beam has a relevant influence on the free frequencies results.

In Fig. 7, for the different values of taper parameter, the curve of the first five fundamental frequencies is reported. If the dimensional parameter, α increases, the free frequencies of vibration decrease and their effect is relevant on the higher modes.

In the case r_H=1/10 (slenderness ratio) and considering the same Rayleigh beam, a different behaviour can be observed. The resonance phenomenon occurs for γ = 62.4 (tuned angular speed), which represents the intersection between the curve relative to the fundamental frequencies, and the line of the angular speeds γ=ω, see Figure 7. If the δ parameter (named hub radius) is allowed to increase, the centrifugal force leads to an increase of the extension deformation of the beam, so that the fundamental frequencies migrate away from the line of speeds and the resonance phenomenon does not occur.

This conclusion has a practical usefulness in analyzing the rotors, where the control devices are of paramount importance.

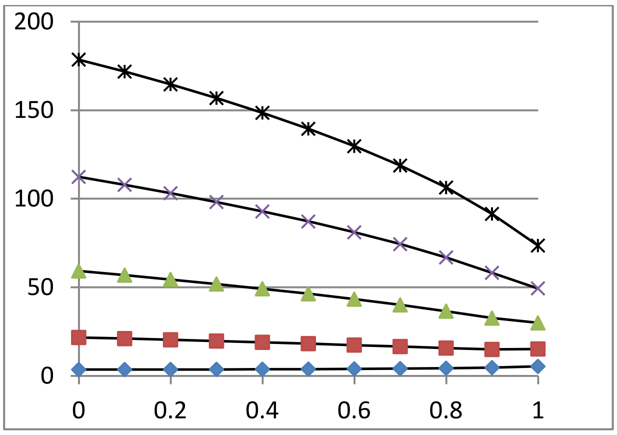

Figure 7. Variation of the nondimensional frequencies with respect to the taper ratio, α, for the five lowest modes of the beam with γ=0, β=0 and δ=0.

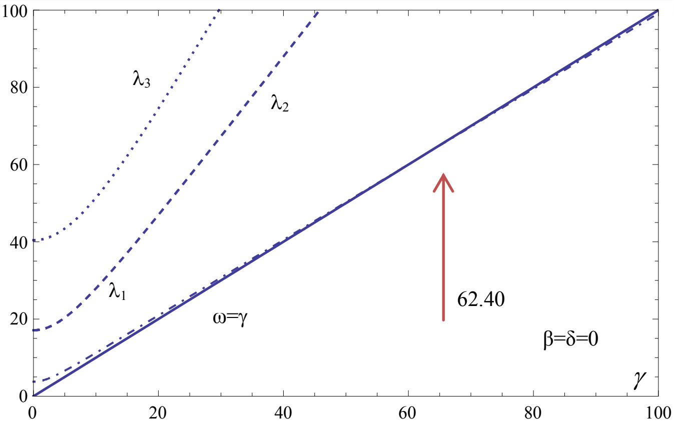

Figure 8. Natural frequencies for tapered Rayleigh beam α=0,5, and r_H=1/10.

➢ *Beam with parabolic thickness variation*

This section is concerned with the transverse vibration of the non-uniform beam shown in Fig. 9, a beam of constant breadth and depth proportional to the square of the axial co-ordinate.

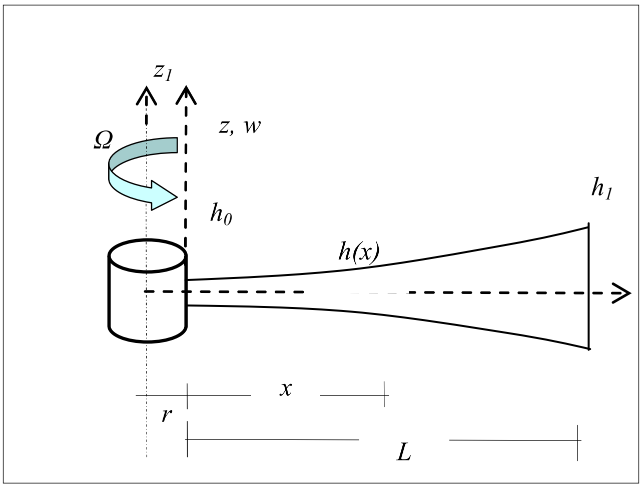

Figure 9. Non-uniform beam; parabolic thickness variation.

In particular, the geometry of the structure is given:

$$h(\xi)=h_0\left(1+(\alpha-1)\xi^2\right), \qquad b(\xi)=b_0, \qquad (25)$$

the area and the inertia assume the form

$$\begin{aligned} A(\xi)&=A_0\left(1+(\alpha-1)\xi^2\right) \\ I(\xi)&=I_0\left(1+\xi^2(\alpha-1)\right)^3. \end{aligned} \qquad (26)$$

In Tab IV, the first five natural frequencies for rotating parabolic non-uniform Euler beam (r_H=1/1000), *α=5*, are presented for various angular speed, γ and hub ratios, δ.

TABLE IV. COMPARISON OF FIRST FIVE NATURAL FREQUENCIES; PARABOLIC THICKESS VARIATION (EULER BEAM).

α=5		δ=0	δ=5
γ=0	λ_1	2.2608	
	λ_2	30.0559	
	λ_3	110.7650	
	λ_4	230.4959	
	λ_5	395.3289	
γ=12	λ_1	12.5806	33.4471
	λ_2	50.0796	106.4972
	λ_3	137.4859	234.4189
	λ_4	260.9709	397.4638
	λ_5	428.5189	605.8781
γ=100	λ_1	100.9850	276.5721
	λ_2	301.1600	769.3038
	λ_3	550.2041	1320.2100
	λ_4	854.6310	1968.2940
	λ_5	1223.8710	2735.9000

If δ = 0 as γ increases the first natural frequency of the beam tends to the angular velocity (ω ≡ γ), causing the phenomenon of resonance, which is usually referred to as "angular speed turner". An increase of parameter δ results in an increase in the fundamental frequency of the beam for which the phenomenon of "turner angular speed" doesn't occur. Assuming the previous hypothesis, in the simple Euler-Bernoulli beam model the resonance phenomenon doesn't occur, even for the higher values of parameter γ, as shown in Fig. 10.

From a practical point of view, greater values of the angular velocity lead to greater centrifugal forces, and in turn to stiffener beams. By increasing the extensional deformation, one gets increasing values of the natural frequencies of vibration. The effect of the parameter α, has a deep impact even on the free frequencies results. For values of γ≠0, the centrifugal force influences the dynamic behavior of beam: the

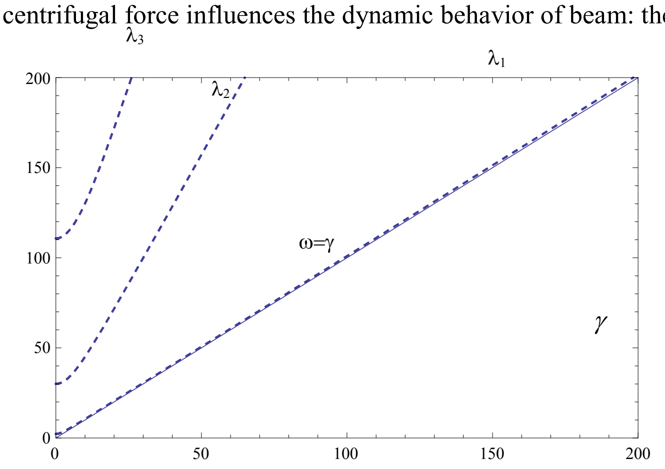

Figure 10. Angular speed and flapwise bending; α=5, r=0, rH=1/1000 (Euler Beam).

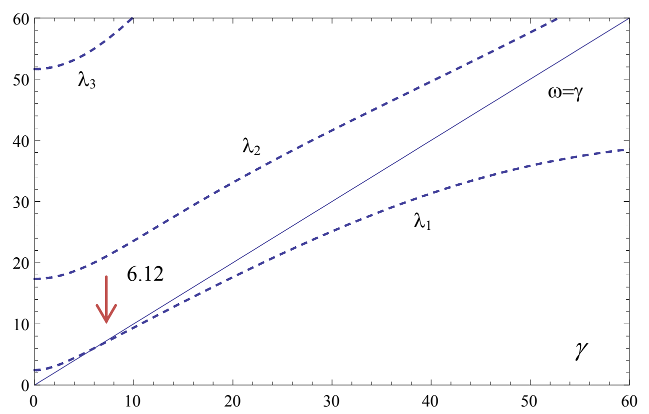

Figure 11. Angular speed and flapwise bending; α=3, r=0, rH=1/1000 (Euler Beam).

free frequencies increase to increase, for increasing values of the rotation, γ.

If the non-dimensional parameter r_H is allowed to increase, the phenomenon of resonance occurs. For α=3 and r_H=1/10 (Rayleigh beam), the values of the first three frequencies are determined and reported in Fig. 11.

V. CONCLUSIONS

In the present paper, the equation of motion for a rotating Rayleigh beam has been derived using the Lagrange's equation. The structure is reduced to a system with finite number of degrees of freedom (MDOF) by employing the CDM method.

The effects of slenderness ratio r_H, hub radius, δ, and γ dynamic parameters are discussed in detail and the following considerations can be made:

- The natural frequencies increase for increasing angular speed. That is due to the increase of the beam stiffness, and it is due to the increase of centrifugal force. In particular, the effect is evident on higher mode shapes.
- The rotary inertia effect, which must be taken into account in the Rayleigh beam theory, influences the beam deformation. When the parameter r_H (slenderness ratio) increases, the natural frequencies decrease; for $r_H=0$ the classical Euler-Bernoulli values are recovered.
- For values of $r_H \neq 0$ (Rayleigh beam model) a tuned angular speed exists, for which the resonance phenomenon is observed.

The present paper represents a useful tool of investigation in order to study the dynamic behaviour of the rotating beam. Moreover, it can be used to control and optimize the rotating beams.

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