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Sorin Dragomir & André Minor

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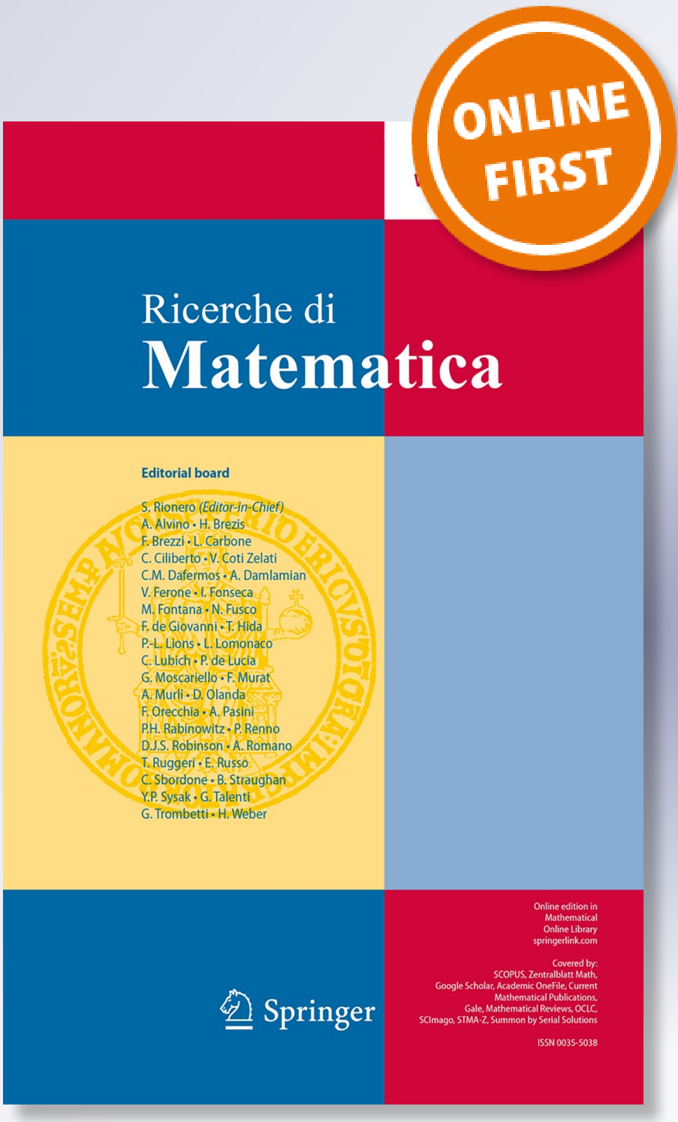

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CR immersions and Lorentzian geometry

Part II: A Takahashi type theorem

Sorin Dragomir · André Minor

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Abstract Using tools from Lorentzian geometry (arising from the presence of the Fefferman metric) we prove a Takahashi type theorem (for a class of pseudohermitian immersions covered by connection-preserving equivariant immersions among the total spaces of the canonical circle bundles) thus relating the geometry of a pseudohermitian immersion from a strictly pseudoconvex CR manifold M into an odd dimensional sphere, to the spectrum of the sublaplacian on M.

Keywords Fefferman's metric · Pseudohermitian immersion · Sublaplacian

Mathematics Subject Classification 32V20 · 53C50

5 Introduction and abridged statement of results

This is the second part of the paper [4]. Section 6 is devoted to recalling the essentials on the Fefferman metric (cf. also [3]). In Sect. 7 we study the geometry of the second fundamental form of a pseudohermitian immersion $\phi : M \to A$ covered by

Dedicated to Carlo Miranda, founder of *Ricerche di Matematica*, in celebration of a century from his birth.

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S. Dragomir (✉)
Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata,
Viale dell'Ateneo Lucano 10, Campus Macchia Romana, 85100 Potenza, Italy
e-mail: sorin.dragomir@unibas.it

A. Minor
Department of Mathematics, University of California, 9500 Gilman Drive 0112,
La Jolla, San Diego, CA 92093-0112, USA
e-mail: aminor@math.ucsd.edu

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a connection-preserving equivariant immersion $\Phi : C(M) \to C(A)$. The main result in Sect. 7 is Theorem 2 admitting the following

Corollary 5 *Let $\phi : M \to S^{2N+1}$ be a pseudohermitian immersion of (M, θ) into (S^{2N+1}, Θ). Let $\hat{\Delta}_b$ be the sublaplacian of $(M, \hat{\theta} = \lambda\theta)$ where $\lambda = \Lambda \circ \phi$ and $(i/2)\, j^*_{N+1}(\overline{\partial} - \partial)|Z|^2 = \Lambda\Theta$. If there is a connection-preserving equivariant immersion $\Phi : C(M) \to C(S^{2N+1})$ covering ϕ then $u^j = Z^j \circ j_{N+1} \circ \phi \in \mathrm{Eigen}(\hat{\Delta}_b\, , \, 2n)$ i.e. each u^j is an eigenfunction of the sublaplacian $\hat{\Delta}_b$ corresponding to the eigenvalue $2n$. Conversely let M be a strictly pseudoconvex CR manifold of CR dimension n and $\phi : M \to \mathbb{C}^{N+1}$ a smooth map covered by an isometric immersion $\Phi : C(M) \to \mathbb{C}^{N+1} \times \mathbb{C}^*$ of $(C(M), F_{\hat{\theta}})$ into $(\mathbb{C}^{n+1} \times \mathbb{C}^*, G)$ for some contact form $\hat{\theta}$ on M with $G_{\hat{\theta}}$ positive definite. Let $u^j = Z^j \circ \phi$ and $f = \zeta \circ \Phi$. If*

$$\hat{\Delta}_b u^j = \mu\, u^j, \qquad \hat{\Box} f = -2f, \qquad 0 \le j \le N,$$

for some $\mu \in \mathbb{R}$ and $|f| = 1$ everywhere on $C(M)$ then

$$\mu > 0, \quad \phi(M) \subset S^{2N+1}(\sqrt{2n/\mu}). \tag{124}$$

Corollary 5 bears a close analogy to a result by Takahashi [14], relating the geometry of minimal immersions among Riemannian manifolds to the spectrum of the Laplace–Beltrami operator of the given submanifold. In the context of our Theorem 2 and Corollary 5 the role of the Laplacian is played by a second order subelliptic operator appearing naturally on a strictly pseudoconvex CR manifold, the sublaplacian Δ_b.

6 Fefferman's metric

A complex valued p-form ω on M is a $(p, 0)$-*form* if $T_{0,1}(M) \rfloor \omega = 0$. Let $\Lambda^{p,0}(M) \subset \Lambda^p T^*(M) \otimes \mathbb{C}$ be the relevant subbundle. If M has CR dimension n then the top degree $(p, 0)$-forms are the sections of $\Lambda^{n+1,0}(M)$ (a complex line bundle over M, the *canonical bundle* of $(M, T_{1,0}(M))$). There is a canonical action of the multiplicative positive reals $\mathrm{GL}^+(1, \mathbb{R}) = (0, +\infty)$ on $\Lambda^{n+1,0}(M)\setminus(0)$. Let $C(M) = \left[\Lambda^{n+1,0}(M)\setminus(0)\right]/\mathrm{GL}^+(1, \mathbb{R})$ be the quotient space and $\pi : C(M) \to M$ the projection, so that $C(M)$ is the total space of a principal S^1-bundle over M (the *canonical circle bundle*). Let $S^2[C(M)]$ and $\mathrm{Lor}[C(M)]$ denote, respectively, the space of all symmetric $(0, 2)$-tensor fields and the set of all Lorentzian metrics on $C(M)$. We endow $S^2[C(M)]$ with the distance function

$$d^\infty_{g_M}(h, h') = \sup_{c \in C(M)} \left[\mathrm{trace}(\tilde{h}_c - \tilde{h}'_c)^2\right]^{1/2} \tag{125}$$

where g_M is a fixed Riemannian metric on $C(M)$ while $\tilde{h}$, $\tilde{h}'$ are the $(1, 1)$-tensor fields determined by $h,\ h' \in S^2[C(M)]$ with respect to g_M e.g.

$$g_M(\tilde{h}(X), Y) = h(X, Y), \quad X, Y \in \mathfrak{X}(C(M)).$$

Then $\mathrm{Lor}[C(M)]$ is an open set of the metric space $(S^2[C(M)], d^\infty_{g_M})$ (cf., e.g. Mounoud [11], p. 49). When M is strictly pseudoconvex for each contact form θ (with G_θ positive definite) there is a Lorentzian metric $L_\theta \in \mathrm{Lor}[C(M)]$ (the *Fefferman metric* of (M, θ)) given by

$$L_\theta = \pi^* \tilde{G}_\theta + 2(\pi^*\theta) \odot \sigma \tag{126}$$

where $\tilde{G}_\theta$ is the extension of G_θ to a symmetric degenerate $(0, 2)$-tensor field on M given by $\tilde{G}_\theta = G_\theta$ on $H(M) \otimes H(M)$ and $\tilde{G}_\theta(X, T) = 0$ for any $X \in \mathfrak{X}(M)$ while $\odot$ denotes the symmetric tensor product e.g. $\alpha \odot \beta = (1/2)\{\alpha \otimes \beta + \beta \otimes \alpha\}$ for any 1-forms α, β. Also $\sigma \in C^\infty(T^*(C(M)))$ is (cf. [7]) a canonical connection 1-form in the principal bundle $S^1 \to C(M) \to M$ given (cf. [10]) by

$$\sigma = \frac{1}{n+2}\left\{d\gamma + \pi^*\left(i\,\omega_\alpha{}^\alpha - \frac{i}{2}\, g^{\alpha\overline{\beta}}\, dg_{\alpha\overline{\beta}} - \frac{\rho}{4(n+1)}\,\theta\right)\right\}. \tag{127}$$

Here γ is a local fibre coordinate on $C(M)$

$$\gamma : \pi^{-1}(U) \to \mathbb{R}, \quad \gamma(c) = \arg\left(\frac{\lambda}{|\lambda|}\right), \quad c \in \pi^{-1}(U),$$

with respect to a local frame $\{T_\alpha : 1 \le \alpha \le n\}$ of $T_{1,0}(M)$ defined on the open set $U \subset M$, i.e. c is represented as

$$c = \left[\lambda\left(\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n\right)_x\right], \quad \lambda \in \mathbb{C}\backslash\{0\}, \quad x \in U.$$

Here $\{\theta^\alpha : 1 \le \alpha \le n\}$ is an *adapted* local coframe (i.e. frame of $T_{1,0}(M)^*$) determined by

$$\theta^\alpha(T_\beta) = \delta^\alpha_\beta, \quad \theta^\alpha(T_{\overline{\beta}}) = 0, \quad \theta^\alpha(T) = 0,$$

and $\arg : S^1 \to [0, 2\pi)$. For each $\omega \in \Lambda^{n+1,0}(M)\backslash(0)$ we denote by $[\omega] \in C(M)$ the class of ω (mod $\mathrm{GL}^+(1, \mathbb{R})$). Also ρ is the *pseudohemitian scalar curvature* of (M, θ). With respect to $\{T_\alpha : 1 \le \alpha \le n\}$ we set

$$g_{\alpha\overline{\beta}} = L_\theta(T_\alpha, T_{\overline{\beta}}), \quad [g^{\alpha\overline{\beta}}] = [g_{\alpha\overline{\beta}}]^{-1},$$
$$\nabla T_\beta = \omega_\beta{}^\alpha \otimes T_\alpha, \quad \omega_\beta{}^\alpha \in \Omega^1(U),$$
$$R_{\alpha\overline{\beta}} = R_\alpha{}^\gamma{}_{\gamma\overline{\beta}}, \quad R_\alpha{}^\delta{}_{\lambda\overline{\sigma}} T_\delta = R^\nabla(T_\lambda, T_{\overline{\sigma}})T_\alpha,$$
$$\rho = R_\alpha{}^\alpha, \quad R_\alpha{}^\gamma = g^{\gamma\overline{\beta}}\, R_{\alpha\overline{\beta}}.$$

Here R^∇ is the curvature tensor field of the Tanaka–Webster connection ∇ of (M, θ). The *restricted conformal class*

$$[F_\theta] = \{e^{u\circ\pi}\, F_\theta : u \in C^\infty(M, \mathbb{R})\}$$

is a CR invariant (by a result of Lee [10], or Theorem 2.3 in [3], p. 128).

Let $X^{\uparrow} \in \mathfrak{X}(C(M))$ denote the horizontal lift of $X \in \mathfrak{X}(M)$ with respect to the connection 1-form σ, i.e. $X^{\uparrow} \in \mathrm{Ker}(\sigma)$ and $(d_c \pi) X^{\uparrow}_c = X_{\pi(c)}$ for any $c \in C(M)$. Let S be the tangent to the S^1-action i.e. the tangent vector field $S \in \mathfrak{X}(C(M))$ locally given by $S = [(n+2)/2]\, \partial/\partial\gamma$. Then $T^{\uparrow} - S$ is timelike i.e. $(C(M), F_\theta)$ is time oriented by $T^{\uparrow} - S$. Hence $(C(M), F_\theta)$ is a space-time. However when M is compact $(C(M), F_\theta)$ is not chronological (cf. Proposition 2.6 in [2], p. 23). Note that S is null i.e. $F_\theta(S, S) = 0$. Hence $\pi : C(M) \to M$ is not a semi-Riemannian submersion (its fibres are degenerate) (cf. also [13], p. 212). Nevertheless we may (in the spirit of [12]) relate the Levi–Civita connection ∇^{F_θ} of $(C(M), F_\theta)$ to the Tanaka–Webster connection ∇ of (M, θ).

Lemma 7 *For any* $X, Y \in C^\infty(H(M))$

$$F_\theta(X^{\uparrow}, Y^{\uparrow}) = g_\theta(X, Y) \circ \pi, \quad F_\theta(X^{\uparrow}, T^{\uparrow}) = 0, \quad F_\theta(T^{\uparrow}, T^{\uparrow}) = 0. \tag{128}$$

Moreover

$$\begin{aligned} \nabla^{F_\theta}_{X^{\uparrow}} Y^{\uparrow} = (\nabla_X Y)^{\uparrow} - [(d\theta)(X, Y) \circ \pi]\, T^{\uparrow} \\ + \left[\sigma([X^{\uparrow}, Y^{\uparrow}]) - 2\, A(X, Y) \circ \pi\right] S, \end{aligned} \tag{129}$$

$$\nabla^{F_\theta}_{X^{\uparrow}} T^{\uparrow} = (\tau(X) + \mathfrak{M}(X))^{\uparrow}, \tag{130}$$

$$\nabla^{F_\theta}_{T^{\uparrow}} X^{\uparrow} = (\nabla_T X + \mathfrak{M}(X))^{\uparrow} + 4\,(d\sigma)(X^{\uparrow}, T^{\uparrow})\, S, \tag{131}$$

$$\nabla^{F_\theta}_{X^{\uparrow}} S = \nabla^{F_\theta}_S X^{\uparrow} = \frac{1}{2}\, (JX)^{\uparrow}, \tag{132}$$

$$\nabla^{F_\theta}_{T^{\uparrow}} T^{\uparrow} = 2\, V^{\uparrow}, \quad \nabla^{F_\theta}_S S = \nabla^{F_\theta}_S T^{\uparrow} = \nabla^{F_\theta}_{T^{\uparrow}} S = 0, \tag{133}$$

where $\mathfrak{M} : H(M) \to H(M)$ *and* $V \in H(M)$ *are, respectively, the bundle morphism and the tangent vector field determined by*

$$G_\theta(\mathfrak{M}(X), Y) \circ \pi = (d\sigma)(X^{\uparrow}, Y^{\uparrow}), \quad G_\theta(V, X) \circ \pi = (d\sigma)(T^{\uparrow}, X^{\uparrow}), \tag{134}$$

for any $X, Y \in H(M)$. *Locally* $\mathfrak{M}$ *and* V *are given by*

$$\mathfrak{M}_\alpha{}^\beta = \frac{i}{2(n+2)} \left\{ R_\alpha{}^\beta - \frac{\rho}{2(n+1)}\, \delta^\beta_\alpha \right\}, \quad \mathfrak{M}_\alpha{}^{\overline{\beta}} = 0, \quad \mathfrak{M}_\alpha{}^0 = 0, \tag{135}$$

$$V^{\alpha} = g^{\alpha\overline{\beta}}\, V_{\overline{\beta}}, \quad V_{\overline{\beta}} = \frac{1}{2(n+2)} \left\{ \frac{1}{4(n+1)}\, \rho_{\overline{\beta}} + i\, W^{\alpha}_{\alpha\overline{\beta}} \right\}. \tag{136}$$

Proof The identities (128) follow from (126). Next for any tangent vector fields $\tilde{X},\ \tilde{Y},\ \tilde{Z} \in \mathfrak{X}(C(M))$

$$\begin{aligned} 2F_{\theta}\left(\nabla^{F_{\theta}}_{\tilde{X}}\tilde{Y}, \tilde{Z}\right) &= \tilde{X}\left(F_{\theta}(\tilde{Y}, \tilde{Z})\right) + \tilde{Y}\left(F_{\theta}(\tilde{Z}, \tilde{X})\right) - \tilde{Z}\left(F_{\theta}(\tilde{X}, \tilde{Y})\right) \\ &\quad + F_{\theta}\left(\left[\tilde{X}, \tilde{Y}\right], \tilde{Z}\right) + F_{\theta}\left(\tilde{Y}, \left[\tilde{Z}, \tilde{X}\right]\right) - F_{\theta}\left(\left[\tilde{Y}, \tilde{Z}\right], \tilde{X}\right). \end{aligned} \tag{137}$$

Let us set $\tilde{X} = X^{\uparrow}$, $\tilde{Y} = Y^{\uparrow}$ and $\tilde{Z} = Z^{\uparrow}$ in (137) for any $X, Y, Z \in H(M)$. The vertical distribution $\mathrm{Ker}(d\pi)$ is spanned by S. Thus [by (126), (127)] $\mathrm{Ker}(d\pi)$ and $H(M)^{\uparrow} \subset \mathrm{Ker}(\sigma)$ are orthogonal (with respect to F_{θ}). On the other hand (by [8,9], vol. I, p. 65) $[X, Y]^{\uparrow}$ is the horizontal component of $\left[X^{\uparrow}, Y^{\uparrow}\right]$ hence

$$\begin{aligned} F_{\theta}\left(\left[X^{\uparrow}, Y^{\uparrow}\right], Z^{\uparrow}\right) &= F_{\theta}\left([X, Y]^{\uparrow}, Z^{\uparrow}\right) = (\pi^{*}\tilde{G}_{\theta})\left([X, Y]^{\uparrow}, Z^{\uparrow}\right) \\ &= \tilde{G}_{\theta}([X, Y], Z) \circ \pi = G_{\theta}\left(\Pi_{H}[X, Y], Z\right) \circ \pi = g_{\theta}([X, Y], Z) \circ \pi. \end{aligned}$$

Here $\Pi_{H} : T(M) \to H(M)$ is the projection (associated to the decomposition (3) in Part I of this paper). Then (137) yields

$$\begin{aligned} 2F_{\theta}\left(\nabla^{F_{\theta}}_{X^{\uparrow}} Y^{\uparrow}, Z^{\uparrow}\right) &= X^{\uparrow}\left(g_{\theta}(Y, Z) \circ \pi\right) + Y^{\uparrow}\left(g_{\theta}(Z, X) \circ \pi\right) - Z^{\uparrow}\left(g_{\theta}(X, Y) \circ \pi\right) \\ &\quad + g_{\theta}\left([X, Y], Z\right) \circ \pi + g_{\theta}\left(Y, [Z, X]\right) \circ \pi \\ &\quad - g_{\theta}\left([Y, Z], X\right) \circ \pi \\ &= 2\, g_{\theta}\left(\nabla^{g_{\theta}}_{X} Y, Z\right) \circ \pi \end{aligned}$$

where $\nabla^{g_{\theta}}$ is the Levi–Civita connection of the Riemannian manifold. We recall [cf. (1.61) in [3], p. 37]

$$\nabla^{g_{\theta}} = \nabla - (d\theta + A) \otimes T + \tau \otimes \theta + 2(\theta \odot \varphi). \tag{138}$$

Thus $\Pi_{H}\nabla^{g_{\theta}}_{X} Y = \nabla_{X} Y$ for any $X, Y \in H(M)$. Also note that

$$T(C(M)) = \mathrm{Ker}(\sigma) \oplus \mathrm{Ker}(d\pi) = H(M)^{\uparrow} \oplus (\mathbb{R}T^{\uparrow}) \oplus (\mathbb{R}S). \tag{139}$$

Consequently

$$F_{\theta}\left(\nabla^{F_{\theta}}_{X^{\uparrow}} Y^{\uparrow}, Z^{\uparrow}\right) = G_{\theta}(\nabla_{X} Y, Z) \circ \pi = F_{\theta}((\nabla_{X} Y)^{\uparrow}, Z^{\uparrow})$$

yields

$$\nabla^{F_{\theta}}_{X^{\uparrow}} Y^{\uparrow} = (\nabla_{X} Y)^{\uparrow} + a\, T^{\uparrow} + b\, S \tag{140}$$

for some $a, b \in C^\infty(C(M))$ (depending on X and Y). Note that (by $\pi_*(S) = 0$)

$$F_\theta(T^\uparrow,\ S) = 2\left((\pi^*\theta)\odot\sigma\right)(T^\uparrow,\ S) = \sigma(S) = 1/2.$$

Then [by taking the inner product of (140) with S]

$$a = 2F_\theta\left(\nabla^{F_\theta}_{X^\uparrow} Y^\uparrow,\ S\right) =$$

[by (137) for $\tilde X = X^\uparrow$, $\tilde Y = Y^\uparrow$ and $\tilde Z = S$]

$$\begin{aligned} &= X^\uparrow\left(F_\theta(Y^\uparrow, S)\right) + Y^\uparrow\left(F_\theta(S,\ X^\uparrow)\right) - S\left(F_\theta(X^\uparrow,\ Y^\uparrow)\right) \\ &\quad + F_\theta\left(\left[X^\uparrow, Y^\uparrow\right],\ S\right) + F_\theta\left(Y^\uparrow,\ \left[S, X^\uparrow\right]\right) - F_\theta\left(\left[Y^\uparrow, S\right],\ X^\uparrow\right) = \end{aligned}$$

[by (128) and $[X^\uparrow,\ S] = 0$ (cf. [8,9], vol. I, p. 79)]

$$\begin{aligned} &= -S(g_\theta(X, Y)\circ\pi) + F_\theta\left(\left[X^\uparrow, Y^\uparrow\right],\ S\right) = \quad (\text{by}\, \pi_*(S) = 0) \\ &= \left(\pi^*\theta\right)([X^\uparrow, Y^\uparrow])\sigma(S) = (1/2)\,\theta([X, Y])\circ\pi \end{aligned}$$

that is

$$a = -(d\theta)(X, Y)\circ\pi. \tag{141}$$

Similarly (by taking the inner product of (140) with $T^\uparrow$)

$$b = 2F_\theta(\nabla^{F_\theta}_{X^\uparrow} Y^\uparrow,\ T^\uparrow) =$$

(by (137) with $\tilde X = X^\uparrow$, $\tilde Y = Y^\uparrow$ and $\tilde Z = T^\uparrow$)

$$\begin{aligned} &= X^\uparrow\left(F_\theta(Y^\uparrow,\ T^\uparrow)\right) + Y^\uparrow\left(F_\theta(T^\uparrow,\ X^\uparrow)\right) - T^\uparrow\left(F_\theta(X^\uparrow,\ Y^\uparrow)\right) \\ &\quad + F_\theta\left(\left[X^\uparrow, Y^\uparrow\right],\ T^\uparrow\right) + F_\theta\left(Y^\uparrow,\ \left[T^\uparrow, X^\uparrow\right]\right) - F_\theta\left(\left[Y^\uparrow, T^\uparrow\right],\ X^\uparrow\right). \end{aligned}$$

On the other hand (as $\theta(X) = \theta(Y) = 0$)

$$\begin{gathered} F_\theta(X^\uparrow,\ T^\uparrow) = (\pi^*\tilde G_\theta)(X^\uparrow,\ T^\uparrow) = \tilde G_\theta(X, T)\circ\pi = 0, \\ F_\theta([X^\uparrow, Y^\uparrow],\ T^\uparrow) = \sigma([X^\uparrow, Y^\uparrow]), \\ F_\theta([X^\uparrow, T^\uparrow],\ Y^\uparrow) = \tilde G_\theta([X, T], Y)\circ\pi, \end{gathered}$$

hence

$$b = \sigma([X^{\uparrow}, Y^{\uparrow}]) + \{-T(g_\theta(X, Y)) + g_\theta([T, X], Y) + g_\theta([T, Y], X)\} \circ \pi. \tag{142}$$

Also

$$\begin{aligned} 2\, g_\theta\left(\nabla^{g_\theta}_X Y, T\right) &= X(g_\theta(Y, T)) + Y(g_\theta(T, X)) - T(g_\theta(X, Y)) + g_\theta([X, Y], T) \\ &\quad + g_\theta(Y, [T, X]) - g_\theta([Y, T], X) \\ &= -T(g_\theta(X, Y)) + g_\theta([T, X], Y) + g_\theta([T, Y], X) + \theta([X, Y]) \end{aligned}$$

leads to the identity

$$\begin{aligned} &- T(g_\theta(X, Y)) + g_\theta([T, X], Y) + g_\theta([T, Y], X) \\ &\quad = 2\,\theta\left(\nabla^{g_\theta}_X Y\right) - \theta([X, Y]). \end{aligned} \tag{143}$$

Let us substitute from (143) into (142) so that to yield

$$b = 2\,\theta\left(\nabla^{g_\theta}_X Y\right) - \theta([X, Y]) + \sigma([X^{\uparrow}, Y^{\uparrow}])$$

or (by $\theta\left(\nabla^{g_\theta}_X Y\right) = -(d\theta)(X, Y) - A(X, Y)$)

$$b = \sigma([X^{\uparrow}, Y^{\uparrow}]) - 2\, A(X, Y). \tag{144}$$

Finally we may substitute from (141) and (144) into (140) so that to yield

$$\nabla^{F_\theta}_{X^{\uparrow}} Y^{\uparrow} = (\nabla_X Y)^{\uparrow} - [(d\theta)(X, Y) \circ \pi]\; T^{\uparrow} + \left[\sigma([X^{\uparrow}, Y^{\uparrow}]) - 2\, A(X, Y) \circ \pi\right] S$$

for any $X, Y \in H(M)$. This proves (129). To prove (130) let us set $\tilde{X} = X^{\uparrow}$, $\tilde{Y} = T^{\uparrow}$ and $\tilde{Z} = Z^{\uparrow}$ in (137) with $X, Z \in H(M)$

$$\begin{aligned} 2F_\theta\left(\nabla^{F_\theta}_{X^{\uparrow}} T^{\uparrow}, Z^{\uparrow}\right) &= X^{\uparrow}\left(F_\theta(T^{\uparrow}, Z^{\uparrow})\right) + T^{\uparrow}\left(F_\theta(X^{\uparrow}, Z^{\uparrow})\right) \\ &\quad - Z^{\uparrow}\left(F_\theta(X^{\uparrow}, T^{\uparrow})\right) + F_\theta\left(\left[X^{\uparrow}, T^{\uparrow}\right], Z^{\uparrow}\right) \\ &\quad + F_\theta\left(T^{\uparrow}, \left[Z^{\uparrow}, X^{\uparrow}\right]\right) - F_\theta\left(\left[T^{\uparrow}, Z^{\uparrow}\right], X^{\uparrow}\right) \end{aligned}$$

or

$$\begin{aligned} 2F_\theta\left(\nabla^{F_\theta}_{X^{\uparrow}} T^{\uparrow}, Z^{\uparrow}\right) &= -\sigma([X^{\uparrow}, Z^{\uparrow}]) + \{T(g_\theta(X, Z)) \\ &\quad + g_\theta([X, T], Z) + g_\theta([Z, T], X)\} \circ \pi. \end{aligned} \tag{145}$$

On the other hand

$$2\,g_\theta\left(\nabla^{g_\theta}_X T, Z\right) = X(g_\theta(T,Z)) + T(g_\theta(X,Z)) - Z(g_\theta(X,T)) + g_\theta([X,T],Z) + g_\theta(T,[Z,X]) - g_\theta([T,Z],X) = T(g_\theta(X,Z)) + g_\theta([X,T],Z) + g_\theta([Z,T],X) - \theta([X,Z])$$

yields the identity

$$T(g_\theta(X,Z)) + g_\theta([X,T],Z) + g_\theta([Z,T],X) = 2\,g_\theta\left(\nabla^{g_\theta}_X T, Z\right) + \theta([X,Z]). \tag{146}$$

Substitution from (146) into (145) gives

$$2F_\theta\left(\nabla^{F_\theta}_{X^\uparrow} T^\uparrow\,,\ Z^\uparrow\right) = \left\{2\,g_\theta\left(\nabla^{g_\theta}_X T, Z\right) + \theta([X,Z])\right\}\circ\pi - \sigma([X^\uparrow, Z^\uparrow])$$

hence [by $\nabla^{g_\theta}_X T = \tau(X) + J(X)$]

$$2F_\theta\left(\nabla^{F_\theta}_{X^\uparrow} T^\uparrow\,,\ Z^\uparrow\right) = 2\,A(X,Z)\circ\pi - \sigma([X^\uparrow, Z^\uparrow]). \tag{147}$$

Next one has

$$\nabla^{F_\theta}_{X^\uparrow} T^\uparrow = W^\uparrow + \lambda\,T^\uparrow + \mu\,S \tag{148}$$

for some $W \in H(M)$ and $\lambda, \mu \in C^\infty(C(M))$ (depending on X). Taking the inner product of (148) with $Z^\uparrow$ leads [by (147)] to

$$W = \tau(X) + \mathfrak{M}(X) \tag{149}$$

where $\mathfrak{M} : H(M) \to H(M)$ is given by (134). Taking the inner product of (148) with S leads [by (128) and by (137) for $\tilde{X} = X^\uparrow$, $\tilde{Y} = T^\uparrow$ and $\tilde{Z} = S$] to

$$\lambda = 2F_\theta\left(\nabla_{X^\uparrow} T^\uparrow\,,\ S\right) = F_\theta([X^\uparrow, T^\uparrow], S) = \frac{1}{2}\,\theta([X,T])$$

i.e. $\lambda = 0$ (as $[X,T] \in H(M)$). Similarly

$$\mu = 2F_\theta(\nabla_{X^\uparrow} T^\uparrow, T^\uparrow) = 0$$

and (148), (149) yield (130). To prove (131) let us set $\tilde{X} = T^{\uparrow}$, $\tilde{Y} = Y^{\uparrow}$ and $\tilde{Z} = Z^{\uparrow}$ in (137)

$$
\begin{aligned}
2F_\theta\left(\nabla^{F_\theta}_{T^\uparrow} Y^\uparrow,\ Z^\uparrow\right) &= T^\uparrow\left(F_\theta(Y^\uparrow,\ Z^\uparrow)\right) + F_\theta\left(\left[T^\uparrow, Y^\uparrow\right],\ Z^\uparrow\right) \\
&\quad + F_\theta\left(Y^\uparrow,\ \left[Z^\uparrow, T^\uparrow\right]\right) - F_\theta\left(\left[Y^\uparrow, Z^\uparrow\right],\ T^\uparrow\right) \\
&= \{T(g_\theta(Y, Z)) + g_\theta([T, Y], Z) + g_\theta([Z, T], Y)\} \circ \pi \\
&\quad - \sigma([Y^\uparrow, Z^\uparrow])
\end{aligned}
$$

and substitution from

$$
2\, g_\theta(\nabla^{g_\theta}_T Y\,,\ Z) + \theta([Y, Z]) = T(g_\theta(Y, Z)) + g_\theta([T, Y], Z) + g_\theta(Y, [Z, T])
$$

furnishes

$$
2F_\theta\left(\nabla^{F_\theta}_{T^\uparrow} Y^\uparrow\,,\ Z^\uparrow\right) = 2\, g_\theta(\nabla^{g_\theta}_T Y\,,\ Z) + \theta([Y, Z]) - \sigma([Y^\uparrow, Z^\uparrow])
$$

or (by $\nabla^{g_\theta}_T Y = \nabla_T Y + J(Y)$)

$$
F_\theta\left(\nabla^{F_\theta}_{T^\uparrow} Y^\uparrow\,,\ Z^\uparrow\right) = g_\theta(\nabla_T Y\,,\ Z) + (d\sigma)(Y^\uparrow\,,\ Z^\uparrow). \tag{150}
$$

Consequently

$$
\nabla^{F_\theta}_{T^\uparrow} Y^\uparrow = (\nabla_T Y + \mathfrak{M}(Y))^\uparrow + \alpha\, T^\uparrow + \beta\, S
$$

where [by (137) for $\tilde{X} = T^{\uparrow}$, $\tilde{Y} = Y^{\uparrow}$ and $\tilde{Z} = S$, respectively for $\tilde{Z} = T^{\uparrow}$]

$$
\begin{aligned}
\alpha &= 2F_\theta\left(\nabla^{F_\theta}_{T^\uparrow} Y^\uparrow\,,\ S\right) = F_\theta\left(\left[T^\uparrow, Y^\uparrow\right]\,,\ S\right) = \frac{1}{2}\,\theta([T, Y]) = 0, \\
\beta &= 2F_\theta\left(\nabla^{F_\theta}_{T^\uparrow} Y^\uparrow\,,\ T^\uparrow\right) = 2F_\theta\left(\left[T^\uparrow, Y^\uparrow\right]\,,\ T^\uparrow\right) \\
&= 2\,\sigma([T^\uparrow, Y^\uparrow]) = -4\,(d\sigma)(T^\uparrow, Y^\uparrow),
\end{aligned}
$$

thus leading to (131). To prove (132) we set $\tilde{X} = X^{\uparrow}$, $\tilde{Y} = S$ and $\tilde{Z} = Z^{\uparrow}$ in (137)

$$
\begin{aligned}
2\, F_\theta\left(\nabla^{F_\theta}_{X^\uparrow} S\,,\ Z^\uparrow\right) &= S\left(F_\theta(X^\uparrow\,,\ Z^\uparrow)\right) + F_\theta\left(S\,,\ \left[Z^\uparrow, X^\uparrow\right]\right) \\
&= S\,(g_\theta(X, Z) \circ \pi) + \frac{1}{2}\,\theta([Z, X]) \circ \pi
\end{aligned}
$$

and obtain

$$
2\, F_\theta\left(\nabla^{F_\theta}_{X^\uparrow} S\,,\ Z^\uparrow\right) = (d\theta)(X, Z) \tag{151}
$$

so that

$$\nabla^{F_\theta}_{X^\uparrow} S = W^\uparrow + \lambda T^\uparrow + \mu S$$

for some $W \in H(M)$ and $\lambda, \mu \in C^\infty(C(M))$. Taking the inner product with $Z^\uparrow$ leads to [by (151)] $W = (1/2)\, JX$. Also [by $\nabla^{F_\theta} F_\theta = 0$ and (130)]

$$\begin{aligned}
\lambda &= 2\, F_\theta\left(\nabla^{F_\theta}_{X^\uparrow} S,\ S\right) = 0,\\
\mu &= 2\, F_\theta\left(\nabla^{F_\theta}_{X^\uparrow} S,\ T^\uparrow\right) = 2\left\{X^\uparrow\left(F_\theta(S, T^\uparrow)\right) - F_\theta\left(S,\ \nabla^{F_\theta}_{X^\uparrow} T^\uparrow\right)\right\}\\
&= -2\, F_\theta\left(S,\ (\tau(X) + \mathfrak{M}(X))^\uparrow\right) = 0,
\end{aligned}$$

and (132) is proved. Finally let us prove (133). To this end we set $\tilde{X} = \tilde{Y} = T^\uparrow$ and $\tilde{Z} = Z^\uparrow$ in (137)

$$F_\theta\left(\nabla^{F_\theta}_{T^\uparrow} T^\uparrow,\ Z^\uparrow\right) = F_\theta\left(T^\uparrow,\ \left[Z^\uparrow, T^\uparrow\right]\right) = \sigma([Z^\uparrow,\ T^\uparrow])$$

or

$$F_\theta\left(\nabla^{F_\theta}_{T^\uparrow} T^\uparrow,\ Z^\uparrow\right) = 2\,(d\sigma)(T^\uparrow, Z^\uparrow) \tag{152}$$

so that

$$\nabla^{F_\theta}_{T^\uparrow} T^\uparrow = 2\, V^\uparrow + \lambda\, T^\uparrow + \mu\, S$$

where $V \in H(M)$ is given by the second of the identities (134) and [by (137)]

$$\lambda = 2\, F_\theta\left(\nabla^{F_\theta}_{T^\uparrow} T^\uparrow,\ S\right) = 0, \quad \mu = 2\, F_\theta\left(\nabla^{F_\theta}_{T^\uparrow} T^\uparrow,\ T^\uparrow\right) = 0,$$

and the first of the formulae (133) is proved. The second identity in (133) is an immediate consequence of (137), $\nabla^{F_\theta} F_\theta = 0$, and (132). Moreover we set

$$\sigma_0 = i\, \omega_\alpha{}^\alpha - \frac{i}{2}\, g^{\alpha\overline{\beta}}\, dg_{\alpha\overline{\beta}} - \frac{\rho}{4(n+1)}\, \theta$$

so that $\sigma = [1/(n+2)]\,\{d\gamma + \pi^*\sigma_0\}$. Note that

$$dg^{\alpha\overline{\beta}} \wedge dg_{\alpha\overline{\beta}} = 0$$

as a consequence of $\nabla g_\theta = 0$. Then

$$d\sigma_0 = i\, d\omega_\alpha{}^\alpha - \frac{1}{4(n+1)}\, d(\rho\theta).$$

At this point we need to recall (1.90) in [3], p. 55

$$\Omega_\alpha{}^\beta = R_\alpha{}^\beta{}_{\lambda\overline{\mu}}\, \theta^\lambda \wedge \theta^{\overline{\mu}} + W^\beta_{\alpha\lambda}\, \theta^\lambda \wedge \theta - W^\beta_{\alpha\overline{\lambda}}\, \theta^{\overline{\lambda}} \wedge \theta \tag{153}$$

where

$$\Omega_\alpha{}^\beta = d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta - 2i\, \theta_\alpha \wedge \tau^\beta + 2i\, \tau_\alpha \wedge \theta^\beta$$

while $W^\beta_{\alpha\lambda}$ and $W^\beta_{\alpha\overline{\lambda}}$ are certain contractions of covariant derivatives of $A^\alpha_{\overline{\beta}}$. Here we set $\tau(T_{\overline{\beta}}) = A^\alpha_{\overline{\beta}} T_\alpha$. Let us contract α and β in (153). As A is symmetric

$$\theta_\alpha \wedge \tau^\alpha = A_{\overline{\alpha}\overline{\beta}}\, \theta^{\overline{\alpha}} \wedge \theta^{\overline{\beta}} = 0, \quad \theta_\alpha \wedge \theta^\alpha = A_{\alpha\beta}\, \theta^\alpha \wedge \theta^\beta = 0.$$

Also $\omega_\alpha{}^\beta \wedge \omega_\beta{}^\alpha = 0$ hence $\Omega_\alpha{}^\alpha = d\omega_\alpha{}^\alpha$. Next [by (1.99) in [3], p. 56] $R_{\lambda\overline{\mu}} = R_\alpha{}^\alpha{}_{\lambda\overline{\mu}}$ hence

$$d\omega_\alpha{}^\alpha = R_{\lambda\overline{\mu}}\, \theta^\lambda \wedge \theta^{\overline{\mu}} + \left(W^\alpha_{\alpha\lambda} \theta^\lambda - W^\alpha_{\alpha\overline{\lambda}} \theta^{\overline{\lambda}} \right) \wedge \theta. \tag{154}$$

Then by (134)

$$G_\theta(\mathfrak{M}(X), Y) = \frac{1}{n+2} \left\{ i\, R_{\alpha\overline{\beta}}\, \theta^\alpha \wedge \theta^{\overline{\beta}} - \frac{\rho}{4(n+1)}\, d\theta \right\} (X, Y), \tag{155}$$

$$G_\theta(V, X) = \frac{1}{2(n+2)} \left\{ \frac{d\rho}{4(n+1)} - i \left(W^\alpha_{\alpha\lambda}\, \theta^\lambda - W^\alpha_{\alpha\overline{\lambda}}\, \theta^{\overline{\lambda}} \right) \right\} (X), \tag{156}$$

for any $X, Y \in H(M)$. Finally (155), (156) and $d\theta = 2i\, g_{\alpha\overline{\beta}}\, \theta^\alpha \wedge \theta^{\overline{\beta}}$ yield (135), (136). The proof of Lemma 7 is complete. □

7 CR immersions covered by equivariant connection-preserving maps

Let M and A be strictly pseudoconvex CR manifolds of CR dimensions n and $N = n + k$. Let θ and Θ be contact forms on M and A such that the Levi forms G_θ and G_Θ are positive definite. Let $\phi : M \to A$ be a pseudohermitian immersion of (M, θ) into (A, Θ). Let $C(M)$ and $C(A)$ be the canonical circle bundles. Let $\sigma \in C^\infty(T^*(C(M)))$ and $\sigma_A \in C^\infty(T^*(C(A)))$ be the connection 1-forms associated to θ and Θ. Precisely σ is given by (127) and

$$\sigma_A = \frac{1}{N+2} \left\{ d\Gamma + \pi_A^* \left(i\Omega_j{}^j - \frac{i}{2}\, G^{j\overline{k}}\, dG_{j\overline{k}} - \frac{\rho_A}{4(N+1)}\, \Theta \right) \right\}.$$

Here Γ is a local fibre coordinate on $C(A)$. Also $G_{j\overline{k}}$ and $\Omega_j{}^k$ are the local coefficients of the Levi form G_Θ and the connection 1-forms of the Tanaka–Webster connection of (A, Θ) i.e.

$$G_{j\bar{k}} = G_\Theta(W_j\,,\ W_{\bar{k}}), \quad \nabla^A W_j = \Omega_j{}^k W_k\,,$$

with respect to a local frame $\{W_j : 1 \leq j \leq N\}$ of $T_{1,0}(A)$. A smooth map $\Phi : C(M) \to C(A)$ is *equivariant* if $\Phi(a\, c) = a\, \Phi(c)$ for any $a \in S^1$ and $c \in C(M)$. Also Φ is *connection-preserving* if

$$\Phi^* \sigma_A = \sigma. \tag{157}$$

Through Sect. 5 we assume that there is a connection-preserving equivariant C^∞ immersion $\Phi : C(M) \to C(A)$ covering $\phi : M \to A$ i.e. such that

$$\begin{array}{ccccc} S^1 \to & C(M) & \xrightarrow{\Phi} & C(A) & \leftarrow S^1 \\ & \downarrow \pi & & \downarrow \pi_A & \\ & M & \xrightarrow{\phi} & A & \end{array} \tag{158}$$

is a commutative diagram. Then the pair (Φ, ϕ) is a morphism of principal circle bundles.

Lemma 8 *Let $S \in \mathfrak{X}(C(M))$ and $S_A \in \mathfrak{X}(C(A))$ be the tangents to the S^1-actions. Then*

- i) $\Phi_* S = S_A$ *and* $\Phi_* \Gamma \subset \Gamma_A$,
- ii) $\Phi_* X^\uparrow = (\phi_* X)^\uparrow$ *for any* $X \in \mathfrak{X}(M)$,
- iii) $\nu(\Phi) = \nu(\phi)^\uparrow$,
- iv) Φ *is an isometric immersion among the Lorentzian manifolds* $(C(M), F_\theta)$ *and* $(C(A), F_\Theta)$.

Here $\nu(\Phi) \to C(M)$ is the normal bundle of the immersion Φ. Also $\Gamma \subset T(C(M))$ and $\Gamma_A \subset T(C(A))$ are the horizontal distributions associated to the connection 1-forms σ and σ_A.

Proof If $c \in C(M)$ let $a_c : \mathbb{R} \to C(M)$ be the curve given by $a_c(t) = e^{it} c$ for any $t \in \mathbb{R}$. Then $S_c = (da_c/dt)_{t=0}$. As Φ is equivariant Φ maps a_c into $a_{\Phi(c)}$ so that $(d_c \Phi) S_c = S_{A,\Phi(c)}$ for any $c \in C(M)$. Next

$$\Gamma = \mathrm{Ker}(\sigma), \quad \Gamma_A = \mathrm{Ker}(\sigma_A),$$

hence [by (157)] Φ_* maps Γ into Γ_A. To prove (ii) let $X \in \mathfrak{X}(M)$. Then [by the commutativity of the diagram (158)]

$$\Phi_* X^\uparrow - (\phi_* X)^\uparrow \in \Gamma_A \cap \mathrm{Ker}(d\pi_A) = (0).$$

To prove (iii) we consider the decompositions

$$T_{\Phi(c)}(C(A)) = [(d_c\Phi)\, T_c(C(M))] \oplus \nu(\Phi)_c\,, \quad c \in C(M), \tag{159}$$

$$T(C(M)) = H(M)^{\uparrow} \oplus \mathbb{R}T^{\uparrow} \oplus \mathbb{R}S. \tag{160}$$

Let $\xi \in \nu(\phi)$. Then for any $X \in H(M)$

$$\begin{aligned}
F_\Theta(\Phi_* X^{\uparrow}\,,\,\xi^{\uparrow}) &= F_\Theta((\phi_* X)^{\uparrow}\,,\,\xi^{\uparrow}) = g_\Theta(\phi_* X\,,\,\xi)\circ\pi_A = 0,\\
F_\Theta(\Phi_* T^{\uparrow}\,,\,\xi^{\uparrow}) &= F_\Theta((\phi_* T)^{\uparrow}\,,\,\xi^{\uparrow}) = F_\Theta(T_A^{\uparrow}\,,\,\xi^{\uparrow}) = \tilde{G}_\theta(T_A\,,\,\xi) = 0,\\
F_\Theta(\Phi_* S\,,\,\xi^{\uparrow}) &= F_\Theta(S_A\,,\,\xi^{\uparrow}) = 2\left[(\pi_A^*\Theta)\odot\sigma_A\right](S_A\,,\,\xi^{\uparrow})\\
&= \Theta(\xi)\sigma_A(S_A) = 2\, g_\Theta(T_A\,,\,\xi) = 0.
\end{aligned}$$

Therefore [by (159), (160)] $\xi^{\uparrow} \in \nu(\Phi)$, i.e. $\nu(\phi)^{\uparrow} \subseteq \nu(\Phi)$. Equality holds because both $\nu(\phi)$ and $\nu(\Phi)$ have rank $2k$. Finally let us note that $\phi^*\tilde{G}_\Theta = \tilde{G}_\theta$. This follows from (14) in Part I of this paper and

$$(\phi^*\tilde{G}_\Theta)(T, X) = \tilde{G}_\Theta(T_A\,,\,\phi_* X) = 0 = \tilde{G}_\theta(T, X)$$

for any $X \in \mathfrak{X}(M)$. Hence [by (126), (157), and (158)]

$$\Phi^* F_\Theta = \Phi^*\left\{\pi_A^*\tilde{G}_\Theta + 2(\pi_A^*\Theta)\odot\sigma_A\right\} = F_\theta\,.$$

Lemma 8 is proved. □

Let g_M and g_A be fixed Riemannian metrics on $C(M)$ and $C(A)$ respectively. Let us endow Lor$[C(M)]$ and Lor$[C(A)]$ with the distance functions $d^\infty_{g_M}$ and $d^\infty_{g_A}$ [given by (125)]. Then

Proposition 6 *Let M and A be two compact strictly pseudoconvex CR manifolds. Then for any connection-preserving equivariant immersion $\Phi : C(M) \to C(A)$ covering the pseudohermitian immersion $\phi : M \to A$ the map $\Phi^* : S^2[C(A)] \to S^2[C(M)]$ is a continuous surjection of $[F_\Theta]$ onto $[F_\theta]$.*

Proof For each Fefferman metric $H = e^{V\circ\pi_A} F_\Theta$ on $C(A)$ (with $V \in C^\infty(A, \mathbb{R})$) one has $\Phi^* H = e^{v\circ\pi} F_\theta$ where $v = V\circ\phi \in C^\infty(M, \mathbb{R})$. Hence $\Phi^*[F_\Theta] \subseteq [F_\theta]$. Equality holds because each C^∞ function on $\phi(M)$ extends smoothly to a function on A. Let $\{X_j : 1 \le j \le 2N+2\}$ be a local g_A-orthonormal (i.e. $g_A(X_j\,,\,X_k) = \delta_{jk}$) frame of $T(C(A))$ defined on the open subset $\mathcal{U} \subset C(A)$. Then

$$\tilde{H}(X_j) = \sum_{k=1}^{2N+2} e^{V\circ\pi_A} F_\Theta(X_j\,,\,X_k) X_k$$

on $\mathcal{U}$. In particular

$$\text{trace}\,[(\tilde{H})^2] = e^{2(V\circ\pi_A)}\,\|F_\Theta\|^2 \tag{161}$$

where the (pointwise) norm of F_Θ is taken with respect to g_A. Consequently by (161)

$$d^\infty_{g_A}(H_1\,,\ H_2) = \sup_{c\in C(A)} \left| e^{V_1(\pi_A(c))} - e^{V_2(\pi_A(c))} \right| \,\|F_\Theta\|_c$$

for any $H_i = e^{V_i\circ\pi_A}F_\Theta$ and $i \in \{1, 2\}$. Let $\{H_\nu\}_{\nu\geq 1} \subset [F_\Theta]$ be a sequence of Fefferman metrics such that $H_\nu \to H$ as $\nu \to \infty$ for some $H \in [F_\Theta]$. Then for any $\epsilon > 0$ there is $\nu_\epsilon \geq 1$ such that

$$\left| e^{V_\nu\circ\pi_A} - e^{V\circ\pi_A} \right| \,\|F_\Theta\| < \epsilon$$

for any $\nu \geq \nu_\epsilon$ everywhere on $C(A)$. Here $H_\nu = e^{V_\nu\circ\pi_A}F_\Theta$ for some $V_\nu \in C^\infty(A, \mathbb{R})$. We claim that $\|F_\Theta\|$ is bounded away from zero. Indeed as A is compact $C(A)$ is compact as well hence $\inf_{c\in C(A)}\|F_\Theta\|_c = 0$ yields (by the Weierstrass theorem) $F_{\Theta,c} = 0$ at some $c \in C(A)$, a contradiction (as $F_{\Theta,c}$ is nondegenerate on $T_c(C(A))$). Let $a = \inf_{c\in C(A)}\|F_\Theta\|_c > 0$. Then $e^{V_\nu} - e^{V} < \epsilon/a$ for any $\nu \geq \nu_\epsilon$ i.e. $\{e^{V_\nu}\}_{\nu\geq 1}$ converges to e^V uniformly on A and in particular on $\phi(M)$. Let $v_\nu = V_\nu \circ \phi$ and $v = V \circ \phi$. Then $\{e^{v_\nu}\}_{\nu\geq 1}$ converges to e^v uniformly on M. Finally as M is compact $\|F_\theta\|$ (computed with respect to g_M) is bounded from above and

$$d^\infty_{g_M}\left(\Phi^* H_\nu\,,\ \Phi^* H\right) = \sup_{c\in C(M)} \left| e^{v_\nu(\pi(c))} - e^{v(\pi(c))} \right| \,\|F_\theta\| \to 0, \quad \nu \to \infty.$$

□

Theorem 2 *Let M and A be strictly pseudoconvex CR manifolds. Let θ and Θ be contact forms on M and A such that the Levi forms G_θ and G_Θ are positive definite. Let $\phi : M \to A$ be a pseudohermitian immersion of (M, θ) into (A, Θ). Then* (i) *any connection-preserving equivariant immersion $\Phi : C(M) \to C(A)$ covering $\phi : M \to A$ is a minimal isometric immersion of $(C(M), F_\theta)$ into $(C(A), F_\Theta)$. In particular* (ii) *if $A = S^{2N+1}$ then*

$$\alpha(\Phi)(V, W) = \tan_{C(S^{2N+1})}\,[\alpha(\iota_0 \circ \Phi)(V, W)] \tag{162}$$

for any $V, W \in \mathfrak{X}(C(M))$. Here

$$\iota_0 = p^{-1} \circ (j_{N+1} \times j_1) \circ \Psi : C(S^{2N+1}) \to V_{N+2}$$

while $\Psi : C(S^{2N+1}) \to S^{2N+1} \times S^1$ and $p : V_{N+2} \to \mathbb{C}^{N+1} \times (\mathbb{C}\setminus\{0\})$ are, respectively, the the natural diffeomorphism induced by the $(N+1, 0)$-form

$\eta = j^*_{N+1}(dZ_0 \wedge dZ_1 \wedge \cdots \wedge dZ_N)$ on S^{2N+1} and the biholomorphism given by $p([Z,\zeta]) = (Z/\zeta\,,\,\zeta^{N+2})$ for any $[Z,\zeta] \in V_{N+2}$. Also

$$V_{N+2} = \left[\mathbb{C}^{N+1} \times (\mathbb{C}\setminus\{0\})\right]/I_{N+2}\,, \quad I_{N+2} = \{\zeta \in \mathbb{C} : \zeta^{N+2} = 1\},$$

and $\tan_{C(S^{2N+1})} : \iota_0^{-1} T(V_{N+2}) \to T(C(S^{2N+1}))$ *is the tangential projection associated to the decomposition*

$$T_c(V_{N+2}) = \left[(d_c\iota_0)\, T_c(C(S^{2N+1}))\right] \oplus E(\nu(\iota_0))_c\,, \quad c \in C(S^{2N+1}).$$

Finally (iii) *if $f^j = Z^j \circ \iota_0 \circ \Phi$ $(0 \leq j \leq N)$ and $f = \zeta \circ \iota_0 \circ \Phi$ (with respect to a local coordinate system $(Z^j\,,\,\zeta)$ on V_{N+2}) then*

$$\hat{\Box} f^j = 2n f^j\,, \quad \hat{\Box} f = -2f. \tag{163}$$

Here $\hat{\Box}$ is the Laplace-Beltrami operator of $(C(M), F_{\hat{\theta}})$ and $\hat{\theta} = \lambda\,\theta$. Also $\lambda = \Lambda \circ \phi$ where $\Lambda \in C^\infty(S^{2N+1})$ is given by

$$\hat{\Theta} = \Lambda\,\Theta, \quad \hat{\Theta} = (i/2)\, j^*_{N+1}\left(\overline{\partial} - \partial\right)|Z|^2\,.$$

To prove Theorem 2 we first establish

Lemma 9 *Let $\alpha(\Phi)$ be the second fundamental form of the isometric immersion $\Phi : C(M) \to C(A)$.*

$$\alpha(\Phi)(X^\uparrow\,,\,Y^\uparrow) = [\alpha(\phi)(X,Y)]^\uparrow\,, \tag{164}$$

$$\alpha(\Phi)(X^\uparrow\,,\,T^\uparrow) = \mathrm{nor}\,[\mathfrak{M}_A(\phi_* X)]^\uparrow\,, \tag{165}$$

$$\alpha(\Phi)(X^\uparrow\,,\,S) = 0, \tag{166}$$

$$\alpha(\Phi)(T^\uparrow\,,\,T^\uparrow) = 2\,\mathrm{nor}(V_A^\uparrow), \tag{167}$$

$$\alpha(\Phi)(S\,,\,S) = \alpha(\Phi)(S\,,\,T^\uparrow) = 0, \tag{168}$$

for any $X, Y \in H(M)$. Here $\mathfrak{M}_A : H(A) \to H(A)$ and $V_A \in H(A)$ are the bundle morphism and the vector field determined by

$$G_\Theta(\mathfrak{M}_A(\mathfrak{X}), \mathfrak{X}') \circ \pi_A = (d\sigma_A)(\mathfrak{X}^\uparrow\,,\,\mathfrak{X}'^\uparrow),$$
$$G_\Theta(V_A\,,\,\mathfrak{X}^\uparrow) \circ \pi_A = (d\sigma_A)(T_A^\uparrow\,,\,\mathfrak{X}^\uparrow),$$

for any $\mathfrak{X}, \mathfrak{X}' \in H(A)$.

Lemma 9 follows from Lemma 7 and the Gauss formula

$$\nabla^{F_\Theta}_{\Phi_* \tilde{X}} \Phi_* \tilde{Y} = \Phi_* \nabla^{F_\theta}_{\tilde{X}} \tilde{Y} + \alpha(\Phi)(\tilde{X}, \tilde{Y}), \quad \tilde{X}, \tilde{Y} \in \mathfrak{X}(C(M)). \tag{169}$$

Indeed if $X, Y \in H(M)$ then

$$\begin{aligned} \nabla^{F_\Theta}_{\Phi_* X^\uparrow} \Phi_* Y^\uparrow &= \nabla^{F_\Theta}_{(\phi_* X)^\uparrow} (\phi_* Y)^\uparrow = \qquad \text{(by (129))} \\ &= \left(\nabla^A_{\phi_* X} \phi_* Y\right)^\uparrow - [(d\Theta)(\phi_* X, \phi_* Y) \circ \pi_A] \, T_A^\uparrow \\ &\quad -2 \left\{(d\sigma_A)\left((\phi_* X)^\uparrow, (\phi_* Y)^\uparrow\right) + g_\Theta(\tau_A(\phi_* X), \phi_* Y) \circ \pi_A\right\} S_A = \end{aligned}$$

(by the pseudohermitian Gauss formula (24) and identity (38) in Part I of this paper)

$$\begin{aligned} &= \Phi_* (\nabla_X Y)^\uparrow + [\alpha(\phi)(X, Y)]^\uparrow - [(d\theta)(X, Y) \circ \pi] \, \Phi_* T^\uparrow \\ &\quad -2 \left\{(d\sigma)(X^\uparrow, Y^\uparrow) + A(X, Y) \circ \pi\right\} \Phi_* S \end{aligned}$$

and a comparison of the normal components yields (164). Next

$$\begin{aligned} \nabla^{F_\Theta}_{\Phi_* X^\uparrow} \Phi_* T^\uparrow &= \nabla^{F_\Theta}_{(\phi_* X)^\uparrow} (\phi_* T)^\uparrow = \nabla^{F_\Theta}_{(\phi_* X)^\uparrow} T_A^\uparrow = \quad [\text{by}(7)] \\ &= [\tau_A(\phi_* X) + \mathfrak{M}_A(\phi_* X)]^\uparrow = (\phi_* \tau X)^\uparrow + \mathfrak{M}_A(\phi_* X)^\uparrow \end{aligned}$$

while a calculation based on the very definitions shows that

$$\tan\{\mathfrak{M}_A(\phi_* X)\} = \mathfrak{M}(X). \tag{170}$$

The explicit calculation of the normal component is more tedious (and is not required by the proof at hand). A comparison to (169) (for $\tilde{X} = X^\uparrow$ and $\tilde{Y} = T^\uparrow$) leads to (165). Similarly the Gauss formula (169) together with (132), (133) yields (166)–(168). Lemma 9 is proved. To prove statement (i) in Theorem 2 let $H(\Phi) = [1/(2n+2)] \, \mathrm{trace}_{F_\theta} \alpha(\Phi)$ be the mean curvature vector of $\Phi : C(M) \to C(A)$. Let $\{E_a : 1 \leq a \leq 2n\}$ be a local orthonormal (i.e. $g_\theta(E_a, E_b) = \delta_{ab}$) frame of $H(M)$. Then $\{E_a^\uparrow, T^\uparrow \pm S : 1 \leq a \leq 2n\}$ is a local orthonormal frame of $T(C(M))$ (with respect to the Lorentzian metric F_θ). Consequently [by (164), (166)–(168), (iii) in Proposition 3 in Part I of this paper, and $T \rfloor \alpha(\phi) = 0$]

$$
\begin{aligned}
2(n+1)\, H(\Phi) &= \sum_{a=1}^{2n} \alpha(\Phi)(E_a^{\uparrow}\,,\, E_a^{\uparrow}) \\
&\quad + \alpha(\Phi)(T^{\uparrow}+S\,,\, T^{\uparrow}+S) - \alpha(\Phi)(T^{\uparrow}-S\,,\, T^{\uparrow}-S) \\
&= (2n+1)\, H(\phi)^{\uparrow} = 0.
\end{aligned}
$$

The proof of statement (ii) in Theorem 2 requires some preparation. For every real hypersurface $A \subset \mathbb{C}^{N+1}$ the canonical circle bundle is trivial i.e. $C(A) \approx A \times S^1$ (a principal bundle isomorphism). Indeed the $(N+1, 0)$-form $\eta = j^*(dZ_0 \wedge dZ_1 \wedge \cdots \wedge dZ_N)$ determines a global section in $C(A)$. Here $j : A \to \mathbb{C}^{N+1}$ is the inclusion. Let $\Omega \subset \mathbb{C}^{N+1}$ be a smoothly bounded strictly pseudoconvex domain. By work of Fefferman [5,6], there is a smooth defining function u of Ω satisfying the complex Monge–Ampère equation

$$
J(u) \equiv \det \begin{pmatrix} u & \partial u/\partial \overline{Z}_k \\ \partial u/\partial Z_j & \partial^2 u/\partial Z_j\, \partial \overline{Z}_k \end{pmatrix} = 1 \tag{171}
$$

to second order along $A = \partial\Omega$ and such that

$$
\Psi^* h = F_{\hat{\Theta}} \tag{172}
$$

i.e. $\Psi^* h$ is the Fefferman metric corresponding to the choice of contact form $\hat{\Theta} = (i/2) j^* \left(\overline{\partial} - \partial\right) u(Z)$. Also $\Psi : C(\partial\Omega) \to \partial\Omega \times S^1$ is the diffeomorphism induced by η while h is the Lorentzian metric on $\partial\Omega \times S^1$ whose construction we briefly recall below. First one sets (cf. [5,6] or [3], p. 150)

$$
H(Z, \zeta) = |\zeta|^{2/(N+2)} u(Z), \quad Z \in \Omega, \quad \zeta \in \mathbb{C}\backslash\{0\},
$$

and considers the $(0, 2)$-tensor field G on $\Omega \times (\mathbb{C}\backslash\{0\})$ given by

$$
G = \sum_{A,B=0}^{N+1} \frac{\partial^2 H}{\partial Z_A\, \partial \overline{Z}_B}\, dZ_A \odot d\overline{Z}_B .
$$

Here $Z_{N+1} = \zeta$. By a result in [5,6] G is a semi-Riemannian metric. It may be written explicitly

$$
\begin{aligned}
G &= \frac{u(Z)}{(N+2)^2}\, |\zeta|^{2/(N+2)-2}\, d\zeta \odot d\overline{\zeta} + \frac{|\zeta|^{2/(N+2)}}{N+2}\, (\partial u) \odot \left(\frac{1}{\overline{\zeta}}\, d\overline{\zeta} \right) \\
&\quad + \frac{|\zeta|^{2/(N+2)}}{N+2} \left(\frac{1}{\zeta}\, d\zeta \right) \odot (\overline{\partial} u) + |\zeta|^{2/(N+2)} \sum_{j,k=0}^{N} \frac{\partial^2 u}{\partial Z_j \partial \overline{Z}_k}\, dZ_j \odot d\overline{Z}_k .
\end{aligned} \tag{173}
$$

Then h may be found by taking the pullback of G to $\Omega \times S^1$ and passing to the limit with $Z \to \partial\Omega$. From now on let $\Omega = B_{N+1}$ be the unit ball in $\mathbb{C}^{N+1}$ so that $A = S^{2N+1}$.

Then $u(Z) = |Z|^2 - 1$ is an exact solution to (171) i.e. $J(u) = 1$ everywhere in $\mathbb{C}^{N+1}$ and (173) becomes

$$G = |\zeta|^{2/(N+2)} \left\{ dZ^j \odot d\overline{Z}_j + \frac{1}{(N+2)^2} \frac{|Z|^2 - 1}{|\zeta|^2} d\zeta \odot d\overline{\zeta} \right. \\ \left. + \frac{1}{N+2} \left[\left(\overline{Z}_j \, dZ^j \right) \odot \frac{d\overline{\zeta}}{\overline{\zeta}} + \frac{d\zeta}{\zeta} \odot \left(Z^j \, d\overline{Z}_j \right) \right] \right\}$$

where $Z^j = Z_j$. The group $I_{N+2} = \{\zeta \in \mathbb{C} : \zeta^{N+2} = 1\}$ of complex roots of unity of order $N+2$ acts freely [by setting $a \cdot (Z, \zeta) = (aZ, a\zeta)$ for any $a \in I_{N+2}$ and $Z \in \mathbb{C}^{N+1}$, $\zeta \in \mathbb{C}$, $\zeta \neq 0$] on $\mathbb{C}^{N+1} \times (\mathbb{C} \backslash \{0\})$ as a properly discontinuous group of holomorphic transformations hence the quotient space

$$V_{N+2} = \left(\mathbb{C}^{N+1} \times (\mathbb{C} \backslash \{0\}) \right) / I_{N+2}$$

is a complex $(N+2)$-dimensional manifold (cf. [1]). Also the map

$$p : V_{N+2} \to \mathbb{C}^{N+1} \times (\mathbb{C} \backslash \{0\}), \\ p([Z, \zeta]) = \left(\frac{Z}{\zeta}, \, \zeta^{N+2} \right), \quad [Z, \zeta] \in V_{N+2},$$

is a biholomorphism. Let us set

$$G_0 = dZ^j \odot d\overline{Z}_j - d\zeta \odot d\overline{\zeta}. \tag{174}$$

The right hand side of (174) is I_{N+2}-invariant hence gives rise to a globally defined semi-Riemannian metric G_0 of index 2 on V_{N+2}. In other words $\mathbb{R}_2^{2(N+2)}$ is the universal semi-Riemannian covering space of $(V_{N+2}, \, G_0)$. A calculation shows that

$$p^* G = G_0. \tag{175}$$

Let $\phi : M \to S^{2N+1}$ be a CR immersion from the strictly pseudoconvex CR manifold M and let θ and Θ be contact forms on M and S^{2N+1} such that ϕ is a pseudohermitian immersion of (M, θ) into (S^{2N+1}, Θ). There is a C^∞ function $\Lambda : S^{2N+1} \to (0, +\infty)$ such that $\hat{\Theta} = \Lambda \Theta$. Let $\Phi : C(M) \to C(S^{2N+1})$ be a connection-preserving bundle map with base map ϕ. Let us consider the immersion

$$\iota : C(S^{2n+1}) \to \mathbb{C}^{N+1} \times (\mathbb{C} \backslash \{0\}), \quad \iota = (j_{N+1} \times j_1) \circ \Psi,$$

and set $\iota_0 = p^{-1} \circ \iota$. If $\Phi_0 = \iota_0 \circ \Phi$ and $\lambda = \Lambda \circ \phi \in C^\infty(M)$ then [by (172)]

$$\Phi^* F_{\hat{\Theta}} = F_{\hat{\theta}}, \quad \iota^* G = F_{\hat{\Theta}}, \quad \Phi_0^* G_0 = F_{\hat{\theta}}, \tag{176}$$

where $\hat{\theta} = \lambda\,\theta$. The various metrics and isometries introduced so far are summarized in the diagram below

$$\begin{array}{ccccc}
(M, g_{\hat\theta}) & \overset{\pi}{\longleftarrow} & (C(M), F_{\hat\theta}) & & \\
\phi \downarrow & & \downarrow \Phi & & \\
(S^{2N+1}, g_{\hat\Theta}) & \overset{\pi_A}{\longleftarrow} & (C(S^{2N+1}), F_{\hat\Theta}) & \overset{\Psi}{\longrightarrow} & (S^{2N+1} \times S^1, h) \\
 & & \iota_0 \downarrow & & \downarrow j_{N+1} \times j_1 \\
 & & (V_{N+2}, G_0) & \overset{p}{\longrightarrow} & (\mathbb{C}^{N+1} \times (\mathbb{C}\setminus\{0\}), G)
\end{array}$$

By the Gauss formula for the immersions Φ_0, ι_0 and Φ

$$\tan_{C(S^{2N+1})}[\alpha(\Phi_0)(V, W)] = \tan_{C(S^{2N+1})}\left[\nabla^G_{(d\Phi_0)V}(d\Phi_0)W\right] - \Phi_* \nabla^{F_{\hat\theta}}_V W = \nabla^{F_{\hat\Theta}}_{\Phi_* V}\Phi_* W - \Phi_* \nabla^{F_{\hat\theta}}_V W = \alpha(\Phi)(V, W)$$

for any $V, W \in \mathfrak{X}(C(M))$. The identity (162) is proved. For each $B \in \mathfrak{X}(V_{N+2})$ we denote by $B^p \in \mathfrak{X}(\mathbb{C}^{N+1} \times (\mathbb{C}\setminus\{0\})$ the tangent vector field given by

$$B^p_y = (d_{p^{-1}(y)}p)B_{p^{-1}(y)}, \quad y \in \mathbb{C}^{N+1} \times (\mathbb{C}\setminus\{0\}).$$

One has

$$\left(\frac{\partial}{\partial Z_j}\right)^p = \zeta^{-1/(N+2)}\,\frac{\partial}{\partial Z_j}, \tag{177}$$

$$\left(\frac{\partial}{\partial \zeta}\right)^p = \zeta^{-1/(N+2)}\left\{-Z^j\,\frac{\partial}{\partial Z^j} + (N+2)\,\zeta\,\frac{\partial}{\partial \zeta}\right\}. \tag{178}$$

To prove statement (iii) in Theorem 2 we first establish

Lemma 10 *The mean curvature vector of the isometric immersion $\Phi_0 : (C(M), F_{\hat\theta}) \to (V_{N+2}, G_0)$ is given by*

$$2(n+1)\,H(\Phi_0) = -\left\{\left(\hat{\Box} f^j\right)\frac{\partial}{\partial Z^j} + \left(\hat{\Box} f\right)\frac{\partial}{\partial \zeta}\right\} + \text{ complex conjugates} \tag{179}$$

where $\hat{\Box}$ is the wave operator of $(C(M), F_{\hat\theta})$. Also $f^j = Z^j \circ \Phi_0$ and $f = \zeta \circ \Phi_0$.

Proof For any $X \in \mathfrak{X}(C(M))$

$$(d\Phi_0)X = X(f^j)\frac{\partial}{\partial Z^j} + X(\overline{f}_j)\frac{\partial}{\partial \overline{Z}_j} + X(f)\frac{\partial}{\partial \zeta} + X(\overline{f})\frac{\partial}{\partial \overline{\zeta}}$$

with respect to the local coordinate system (Z^j, ζ) on V_{N+2}. Hence

$$\nabla^{G_0}_{(d\Phi_0)X}(d\Phi_0)X = X^2(f^j)\frac{\partial}{\partial Z^j} + X^2(f)\frac{\partial}{\partial \zeta} + \text{complex conjugates.} \quad (180)$$

Let $\{X_a : 1 \le a \le 2n+2\}$ be a local orthonormal (i.e. $F_{\hat{\theta}}(X_a, X_b) = \epsilon_a \delta_{ab}$ where $\epsilon_1 = \cdots = \epsilon_{2n+1} = 1$ and $\epsilon_{2n+2} = -1$) frame of $T(C(M))$. Then (180) and the Gauss formula for Φ_0 together with

$$(2n+2)\, H(\Phi_0) = \sum_{a=1}^{2n+2} \epsilon_a\, \alpha(\Phi_0)(X_a, X_a)$$

yield (179) as $\hat{\Box}$ is locally given by

$$\hat{\Box} u = -\sum_{a=1}^{2n+2}\left\{X_a^2(u) - (\nabla^{F_{\hat{\theta}}}_{X_a} X_a)(u)\right\}, \quad u \in C^2(C(M)).$$

□

Lemma 11 *The tangent vector fields* $\xi_1, \xi_2 \in \mathfrak{X}(\mathbb{C}^{N+2}\setminus\{\zeta = 0\})$ *given by*

$$\xi_1 = \frac{1}{C_N}\left\{Z^j \frac{\partial}{\partial Z^j} + \overline{Z}_j \frac{\partial}{\partial \overline{Z}_j} + \zeta \frac{\partial}{\partial \zeta} + \overline{\zeta}\frac{\partial}{\partial \overline{\zeta}}\right\},$$
$$\xi_2 = \frac{1}{C_N}\left\{Z^j \frac{\partial}{\partial Z^j} + \overline{Z}_j \frac{\partial}{\partial \overline{Z}_j} - (N+3)\left[\zeta \frac{\partial}{\partial \zeta} + \overline{\zeta}\frac{\partial}{\partial \overline{\zeta}}\right]\right\},$$

form an orthonormal frame of the normal bundle $\nu(j_{N+1} \times j_1) \to S^{2N+1} \times S^1$ *i.e.*

$$G(\xi_1, \xi_1) = 1, \quad G(\xi_2, \xi_2) = -1, \quad G(\xi_1, \xi_2) = 0. \quad (181)$$

Here $C_N = \sqrt{(N+4)/(N+2)}$.

Proof Note that

$$G(\partial/\partial Z^j, \partial/\partial \overline{Z}_k) = \frac{1}{2}\delta^j_k, \quad G(\partial/\partial \zeta, \partial/\partial \overline{\zeta}) = 0,$$
$$G(\partial/\partial Z^j, \partial/\partial \overline{\zeta}) = \frac{1}{2(N+2)}\overline{Z}_j/\overline{\zeta},$$

everywhere on $S^{2N+1} \times S^1$. For each $n \geq 1$ let $\mathbf{n}_n = \sum_{j=1}^{n} (z^j\, \partial/\partial z^j + \overline{z}_j\, \partial/\partial \overline{z}_j)$ (the unit normal to $S^{2n-1} \subset \mathbb{C}^n$). If $N_a = \mathbf{n}_{N+1} + a\, \mathbf{n}_1$ with $a \in \mathbb{R}$ then $G(N_a\,,\, N_b) = 1 + (a+b)/(N+2)$. Then $\xi_1 = (1/C_N)\, N_1$ and $\xi_2 = (1/C_N)\, N_{-(N+3)}$ satisfy the requirement (181). □

Lemma 12 *Let* $L_N = \sqrt{(N+2)(N+4)}$ *and* $\eta_1\,,\, \eta_2 \in \mathfrak{X}(V_{N+2})$ *given by* $(\eta_a)^p = \xi_a$ *for* $a \in \{1, 2\}$. *Then*

$$\eta_1 = \frac{1}{L_N}\{(N+3)\, \mathbf{n}_{N+1} + \mathbf{n}_1\}, \quad \eta_2 = -\frac{1}{L_N}\{\mathbf{n}_{N+1} + (N+3)\mathbf{n}_1\},$$

is a local frame of $\nu(\iota_0) \to C(S^{2N+1})$.

The proof follows from (177), (178) and Lemma 11. At this point we may take traces in (162) to get

$$\tan_{C(S^{2N+1})}[H(\Phi_0)] = 0.$$

Therefore [by (179)]

$$\left(\hat{\Box} f^j\right) \frac{\partial}{\partial Z^j} + \left(\hat{\Box} f\right) \frac{\partial}{\partial \zeta} + \text{complex conjugates} = \lambda_1\, \eta_1 + \lambda_2 \eta_2$$

for some $\lambda_a \in C^\infty(C(S^{2N+1}))$ so that (by Lemma 12)

$$\hat{\Box} f^j = \left(\frac{N+3}{L_N}\lambda_1 - \frac{1}{L_N}\lambda_2\right) f^j\,, \quad 0 \leq j \leq N, \tag{182}$$

$$\hat{\Box} f = \left(\frac{1}{L_N}\lambda_1 - \frac{N+3}{L_N}\lambda_2\right) f. \tag{183}$$

Let us contract (182), (183) with $\overline{f}_j$ and $\overline{f}$ respectively and use

$$\hat{\Box}(uv) = (\hat{\Box} u)\, v + u\, (\hat{\Box} v) - 2\, F_{\hat{\theta}}(\hat{D}u, \hat{D}v), \quad u, v \in C^2(C(M)),$$

where $\hat{D}u$ is the gradient of u i.e. $F_{\hat{\theta}}(\hat{D}u\,,\, X) = X(u)$ for any $X \in \mathfrak{X}(C(M))$. We obtain

$$(N+3)\lambda_1 - \lambda_2 = L_N\, F_{\hat{\theta}}(\hat{D} f^j\,,\, \hat{D}\overline{f}_j), \tag{184}$$

$$\lambda_1 - (N+3)\lambda_2 = L_N\, F_{\hat{\theta}}(\hat{D} f\,,\, \hat{D}\overline{f}). \tag{185}$$

On the other hand $\Phi_0^* G_0 = F_{\hat\theta}$ yields

$$2\epsilon_a \delta_{ab} = X_a(f^j) X_b(\overline{f}_j) - X_a(f) X_b(\overline{f}) + \text{complex conjugate}$$

hence (multiplying by ϵ_a and contracting a and b)

$$2n + 2 = F_{\hat\theta}(\hat D f^j \,,\, \hat D \overline{f}_j) - F_{\hat\theta}(\hat D f \,,\, \hat D \overline{f}). \tag{186}$$

Therefore (184)–(186) imply $\lambda_1 + \lambda_2 = 2(n+1) C_N$. If $\lambda_1 = \mu$ then

$$\hat\Box f^j = \left[C_N \mu - \frac{2(n+1)}{N+2} \right] f^j, \quad 0 \le j \le N, \tag{187}$$

$$\hat\Box f = \left[C_N \mu - \frac{2(n+1)(N+3)}{N+2} \right] f. \tag{188}$$

Next [by (184)–(186)]

$$\mu = \frac{1}{L_N} \left\{ (N+2)\, F_{\hat\theta}(\hat D f^j \,,\, \hat D \overline{f}_j) + 2n + 2 \right\}. \tag{189}$$

Lemma 13 *Let $\phi : M \to S^{2N+1}$ be a pseudohermitian immersion and $\Phi : C(M) \to C(S^{2N+1})$ a connection-preserving equivariant map covering ϕ. Let us consider*

$$u^j = Z^j \circ j_{N+1} \circ \phi \in C^\infty(M, \mathbb{C}), \quad 0 \le j \le N.$$

Then f^j is the vertical lift of u^j i.e. $f^j = u^j \circ \pi$. In particular

$$\hat\Box f^j = (\hat\Delta_b u^j) \circ \pi \tag{190}$$

where $\hat\Delta_b$ is the sublaplacian of $(M, \hat\theta)$.

The verification of the first statement requires a rather pedantic notational distinction among the (local) complex coordinates $(\tilde Z^j)$, (W^j, η) and (Z^j, ζ) on $\mathbb{C}^{N+1}$, $\mathbb{C}^{N+1} \times (\mathbb{C} \setminus \{0\})$ and V_{N+2} respectively. Here $Z^j = W^j \circ p$ for any $0 \le j \le N$. Let

$$\Pi_{N+1} : \mathbb{C}^{N+1} \times \mathbb{C} \setminus \{0\} \to \mathbb{C}^{N+1}, \quad \pi_{N+1} : S^{2n+1} \times S^1 \to S^{2N+1},$$

be the natural projections so that

$$j_{N+1} \circ \pi_{N+1} = \Pi_{N+1} \circ (j_{N+1} \times j_1), \quad W^j = \tilde Z^j \circ \Pi_{N+1}.$$

Also the diagram

$$\begin{array}{ccc} C(S^{2N+1}) & \stackrel{\Psi}{\longrightarrow} & S^{2N+1} \times S^1 \\ \pi_A \downarrow & & \downarrow \pi_{N+1} \\ S^{2N+1} & = & S^{2N+1} \end{array}$$

is commutative. Then

$$\begin{aligned} u^j \circ \pi &= \tilde{Z}^j \circ j_{N+1} \circ \phi \circ \pi = \tilde{Z}^j \circ j_{N+1} \circ (\pi_A \circ \Phi) \\ &= \tilde{Z}^j \circ j_{N+1} \circ (\pi_{N+1} \circ \Psi) \circ \Phi = \tilde{Z}^j \circ \left[\Pi_{N+1} \circ (j_{N+1} \times j_1)\right] \circ \Psi \circ \Phi \\ &= W^j \circ \iota \circ \Phi = \left(W^j \circ p\right) \circ \left(p^{-1} \circ \iota\right) \circ \Phi = Z^j \circ \iota_0 \circ \Phi = f_j. \end{aligned}$$ □

We recall that the *sublaplacian* of (M, θ) is the second order differential operator

$$\Delta_b u = -\mathrm{div}\left(\nabla^H u\right), \quad u \in C^2(M),$$

where $\mathcal{L}_X\,(\theta \wedge (d\theta)^n) = \mathrm{div}(X)\,\theta \wedge (d\theta)^n$ for any $X \in \mathfrak{X}(M)$. Also $\mathcal{L}_X$ denotes the Lie derivative. As $\Box$ (the wave operator of $(C(M), F_\theta)$) is S^1-invariant it admits a natural push-forward $\pi_*\Box$ (defined on $C^2(M)$). By a result of Lee [10], $\pi_*\Box = \Delta_b$ (cf. also Proposition 2.8 in [3], p. 140) thus yielding (190). □

Let $\{E_a : 1 \le a \le 2n\}$ be a local G_θ-orthonormal frame of $H(M)$. Then $\left\{(\lambda \circ \pi)^{-1/2} E_a^\uparrow,\ (\lambda \circ \pi)^{-1/2}(T^\uparrow \pm S) : 1 \le a \le 2n\right\}$ is a local $F_{\hat\theta}$-orthonormal frame of $T(C(M))$. Horizontal lifting is meant with respect to the connection 1-form σ. Then (by Lemma 13)

$$\begin{aligned} F_{\hat\theta}(\hat{D} f^j, \hat{D}\overline{f}_j) &= \frac{1}{\lambda \circ \pi}\left\{\sum_{a=1}^{2n} E_a^\uparrow(f^j) E_a^\uparrow(\overline{f}_j)\right. \\ &\qquad \left. + (T^\uparrow + S)(f^j)\,(T^\uparrow + S)(\overline{f}_j) - (T^\uparrow - S)(f^j)\,(T^\uparrow - S)(\overline{f}_j)\right\} \\ &= \frac{1}{\lambda \circ \pi} \sum_{a=1}^{2n} \left[E_a(u^j) \circ \pi\right] \left[E_a(\overline{u}_j) \circ \pi\right]. \end{aligned}$$

Let $G_{N+1} = dZ^j \odot d\overline{Z}_j$ be the canonical flat metric on $\mathbb{C}^{N+1}$ and $\hat{E}_a = \lambda^{-1/2} E_a$. The Webster metric $g_{\hat\Theta}$ and the metric induced on S^{2N+1} by G_{N+1} actually coincide. Then (by Lemma 4 in Part I of this paper) $g_{\hat\theta} = (j_{N+1} \circ \phi)^* G_{N+1}$ so that

$$\delta_{ab} = g_{\hat\theta}(\hat{E}_a\,,\ \hat{E}_b) = \frac{1}{2\lambda}\left\{E_a(u^j)\,E_b(\overline{u}_j) + E_a(\overline{u}_j)\,E_b(u^j)\right\}$$

or (by contracting a and b)

$$2n = \frac{1}{\lambda} \sum_{a=1}^{2n} E_a(u^j)\, E_a(\overline{u}_j).$$

We may conclude that

$$F_{\hat{\theta}}(\hat{D} f^j, \hat{D} \overline{f}_j) = 2n \tag{191}$$

so that [by (189) and (191)]

$$\mu = \frac{1}{L_N} \left[2n(N+2) + 2n + 2 \right].$$

This yields the multipliers of f^j and f in (187), (188) thus leading to (163). Theorem 2 is proved. □

At this point we may prove Corollary 5 (as stated in Sect. 5). The direct statement there follows from (163) in Theorem 2 and (190) in Lemma 13. As to the converse let us set $\Phi_0 = p^{-1} \circ \Phi$ so that $\Phi_0^* G_0 = F_{\hat{\theta}}$. Since Φ covers ϕ (i.e. $\Pi_{N+1} \circ \Phi = \phi \circ \pi$) it follows that $u^j \circ \pi = f^j$ where $f^j = Z^j \circ \Phi$. Therefore $\hat{\Box} f^j = \mu\, f^j$ so that (by observing that Lemma 10 holds for *any* isometric immersion $\Phi_0 : C(M) \to V_{N+2}$ of $(C(M), F_{\hat{\theta}})$ into (V_{N+2}, G_0))

$$\begin{aligned} 2(n+1) H(\Phi_0) &= -\left(\hat{\Box} f^j \right) \frac{\partial}{\partial Z^j} - \left(\hat{\Box} f \right) \frac{\partial}{\partial \zeta} + \text{complex conjugate} \\ &= -\mu\, f^j \frac{\partial}{\partial Z^j} + 2\, f \frac{\partial}{\partial \zeta} + \text{complex conjugate}. \end{aligned}$$

On the other hand for any $X \in \mathfrak{X}(C(M))$ the vector fields $(d\Phi_0)X$ and $H(\Phi_0)$ are respectively tangent and normal to $\Phi_0(C(M))$ hence

$$\begin{aligned} 0 &= 2(n+1)\, G_0((d\Phi_0)X\,,\ H(\Phi_0)) \\ &= -\frac{\mu}{2}\, X(f^j)\, \overline{f}_j + X(f)\, \overline{f} + \text{complex conjugate} \\ &= -\frac{\mu}{2}\, X(f^j \overline{f}_j) + X(|f|^2) \end{aligned}$$

so that $f^j \overline{f}_j = R^2$ for some constant $R > 0$. In particular

$$\begin{aligned} 0 = \hat{\Box} \left(f^j \overline{f}_j \right) &= \left(\hat{\Box} f^j \right) \overline{f}_j + f^j\, \hat{\Box} \overline{f}_j - 2\, F_{\hat{\theta}}(\hat{D} f^j\,,\ \hat{D} \overline{f}_j) \\ &= 2\mu\, f^j \overline{f}_j - 2\, F_{\hat{\theta}}(\hat{D} f^j\,,\ \hat{D} \overline{f}^j), \end{aligned}$$

$$0 = \hat{\Box} \left(|f|^2 \right) = \left(\hat{\Box} f \right) \overline{f} + f\, \hat{\Box} \overline{f} - 2\, F_{\hat{\theta}}(\hat{D} f\,,\ \hat{D} \overline{f}) = -4\, |f|^2 - 2\, F_{\hat{\theta}}(\hat{D} f\,,\ \hat{D} \overline{f}).$$

Let us subtract the two previous equations and use (186) (a consequence of $\Phi_0^* G_0 = F_{\hat{\theta}}$ alone). We obtain $\mu f^j \overline{f}_j + 2|f|^2 = 2(n+1)$ i.e. $\mu R^2 = 2n$. □

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