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CONSTRUCTIVE APPROXIMATION

The Solution of a Problem of Coifman, Meyer, and Wickerhauser on Wavelet Packets

Sandra Saliani

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Abstract Wavelet packets provide an algorithm with many applications in signal processing together with a large class of orthonormal bases of $L^2(\mathbb{R})$, each one corresponding to a different splitting of $L^2(\mathbb{R})$ into a direct sum of its closed subspaces. The definition of wavelet packets is due to the work of Coifman, Meyer, and Wickerhauser, as a generalization of the Walsh system. A question has been posed since then: one asks if a (general) wavelet packet system can be an orthonormal basis for $L^2(\mathbb{R})$ whenever a certain set linked to the system, called the "exceptional set" has zero Lebesgue measure. This answer to this question affects the quality of wavelet packet approximation. In this paper we show that the answer to this question is negative by providing an explicit example. In the proof we make use of the "local trace function" by Dutkay and the generalized shift-invariant system machinery developed by Ron and Shen.

Keywords Wavelet packets · Local trace function · Tight frame · Generalized shift-invariant systems

Mathematics Subject Classification (2000) 42C40

1 Introduction

Wavelet packets are employed in many applications in signal processing. They give rise to a large class of orthonormal bases of $L^2(\mathbb{R})$, each one corresponding to a different splitting of $L^2(\mathbb{R})$ into a direct sum of its closed subspaces.

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S. Saliani (✉)
Dipartimento di Matematica e Informatica, Università degli Studi della Basilicata, 85100 Potenza, Italy
e-mail: sandra.saliani@unibas.it

The definition of wavelet packets is due to the work of Coifman, Meyer, and Wickerhauser [3]. It is an attempt to construct an orthonormal basis (ONB) associated with an arbitrary partition of the time-frequency plane by some Heisenberg boxes, together with an algorithm that allows a given signal to be effectively represented by time-frequency atoms. Any wavelet packet $2^{q/2}w_n(2^q x-k)$ is linked to a Heisenberg box $R_{n,q,k}$ which indicates the time and frequency regions where the energy of this wavelet packet is mostly concentrated, namely $R_{n,q,k}=[k2^{-q},(k+1)2^{-q}]\times[n2^q,(n+1)2^q]$. The time interval $k2^{-q}\le x\le (k+1)2^{-q}$ is the time support of the Walsh wavelet packet, while the frequency interval $n2^q\le\xi\le(n+1)2^q$ is the positive frequency support of the Shannon wavelet packet. In this way any wavelet packet basis realizes an exact partition of the time-frequency plane even though general wavelet packets have a time and frequency spread that is wider than the Heisenberg box. A wavelet packet basis divides the frequency axis into intervals of varying sizes, and the pavings of the time frequency plane are provided with horizontal strips. In other words, wavelet packets correspond to adaptive filtering of the frequency axis.

The construction is realized as follows. Let us start with a pair of quadratic mirror filters (QMF) with transfer functions $m_0(\theta)$ and $m_1(\theta)=e^{-i\theta}\overline{m_0(\theta+\pi)}$ associated to a multiresolution analysis (MRA) with wavelet ψ and scaling function φ. Let us define first the basic wavelet packets, recursively, by the formulas (for the Fourier transform):

$$\hat{w}_0(\theta)=\hat{\varphi}(\theta),\qquad \hat{w}_1(\theta)=\hat{\psi}(\theta),$$

$$\hat{w}_{2n}(\theta)=m_0\left(\frac{\theta}{2}\right)\hat{w}_n\left(\frac{\theta}{2}\right),$$

$$\hat{w}_{2n+1}(\theta)=m_1\left(\frac{\theta}{2}\right)\hat{w}_n\left(\frac{\theta}{2}\right).$$

The general wavelet packets are defined by taking some of the dilation and translation of the basic ones, i.e.,

$$2^{q/2}w_n(2^q x-k),\quad k\in\mathbb{Z},\ (n,q)\in E\subset\mathbb{N}\times\mathbb{Z}. \tag{1}$$

For the sake of brevity we shall call (1) wavelet packets again.

It is well known that (1) is an orthonormal basis of $L^2(\mathbb{R})$ provided the set E satisfies the following assumption: the dyadic (frequency) intervals $I_{n,q}=[2^q n,2^q(n+1))$, $(n,q)\in E$, form a disjoint covering of $[0,+\infty)$. In [3], Coifman, Meyer, and Wickerhauser proved that, in the case of the Lemarié–Meyer wavelet, the above result could be extended to disjoint coverings of the form

$$[0,+\infty)=\bigcup_{(n,q)\in E} I_{n,q}\cup A,$$

where the set A is any denumerable set called, hereafter, the "exceptional" set associated with (1).

Each choice of E corresponds to a different splitting of $L^2(\mathbb{R})$ and so to a different orthonormal basis: $E = \{1\} \times \mathbb{Z}$ leads to the wavelet basis, $E = \mathbb{N} \times \{0\}$ to the basis $w_n(x-k)$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$. In the first case $A = \{0\}$, in the second case A is the empty set. For a fixed scale 2^j, $L^2(\mathbb{R})$ is then decomposed, by the wavelet packet procedure, into the orthogonal sum of closed subspaces $W_{n,j} = \overline{\text{span}}\{2^{j/2}w_n(2^j x - k) \mid k \in \mathbb{Z}\}$, $n \in \mathbb{N}$. Note that any $W_{0,j} = V_j$ is a subspace of the underlying MRA.

However, there are choices of E where the intervals $I_{n,q}$ form a disjoint covering of $[0, +\infty)$ and the exceptional set A is not denumerable: think of A as a Cantor like set.

Therefore Coifman, Meyer, and Wickerhauser have posed the question of whether the above result could be generalized to exceptional sets A with zero Lebesgue measure. Even if this situation does not occur in applications, where A is generally empty or at most finite, this is a deep mathematical question whose answer influences the quality of wavelet packet approximations.

In the case of the Lemarié–Meyer wavelet, in [12] we gave a positive answer to this question when the Hausdorff dimension of A is strictly less then $1/2$, while in [13] we provided an orthonormal basis corresponding to A with Hausdorff dimension exactly $1/2$.

In this work we show that the answer to the problem is negative by providing an explicit example of a set A, with Hausdorff dimension exactly $1/2$, corresponding to a wavelet packet system for the Lemarié–Meyer wavelet, which is not an orthonormal basis for $L^2(\mathbb{R})$. As a consequence we obtain that the result in [12] is sharp.

It seems, from the proof, that the value $1/2$ depends on the chosen wavelet and that the more $m_0(\theta)$ is close to the ideal filter, the more orthonormal bases one could get. Here for the ideal filter we intend that of the Shannon system, i.e., a 2π periodic extension of the function $\chi_{[-\frac{\pi}{2},\frac{\pi}{2})}$.

The proof, obtained by reductio ad absurdum, has been made possible by considering two recent tools. The first one is the "local trace function" by Dutkay [4] and its property of being invariant with respect to different choices of normalized tight frames. The second one is the "generalized shift-invariant system" machinery developed by Ron and Shen [11]. It allows us, under the hypothesis that our wavelet packet system is an orthonormal basis of $L^2(\mathbb{R})$, to obtain another $\mathbb{Z}$-shift invariant normalized tight frame for $L^2(\mathbb{R})$. A crucial role is played by our choice of E and thus of A. By previous results in [14] on the continuous measures μ_k, $k \in \mathbb{Z}$ induced by the wavelet packet algorithm on the Borel sets of $[0, 1)$ [3], we are then forced to contradiction.

Indeed the problem can be reformulated by stating that any measure μ_k is absolutely continuous with respect to Lebesgue measure. Actually, it follows by definition that absolute continuity for one μ_k implies the same property for all, so, as a consequence we can say that none of them is absolutely continuous.

On the other hand, in the extreme cases of the Walsh system and the Shannon system, one can easily compute the values of such measures in dyadic intervals by means of (9). In the first case the quadrature mirror filters are $m_0(\theta) = \frac{1}{2}(e^{-i\theta} + 1)$ and $m_1(\theta) = \frac{1}{2}(e^{-i\theta} - 1)$, while in the second case ideal filters are being used. In both cases one easily sees that each measure is exactly Lebesgue measure, thus in these two cases it obviously suffices that A be a set with zero Lebesgue measure.

We have to mention that the above measures are a particular case of measures studied by Jorgensen in [9], and therefore we expect that a generalization of the tools involved could be successfully applied also in the more general setting; this will be the matter for a further study.

The paper is structured as follows. In Sect. 2 we introduce the wavelet packet system which leads to the negative answer. In Sect. 3 we review the local trace function. In Sect. 4 we show how wavelet packet systems fit into the scheme of generalized shift-invariant systems. In Sect. 5 we recall properties of measures μ_k, and finally in Sect. 6 we combine things together to obtain the contradiction.

2 A Counterexample to the Wavelet Packet Conjecture

In this section we introduce the exceptional set which contradicts the conjecture, and we establish in Theorem 2.2 a fundamental property of our wavelet packet system. Let us define

$$E = \bigcup_{M\in\mathbb{N}} E_M,$$

where

$$\begin{aligned} E_0 = \Bigg\{&(n,q)\in\mathbb{N}\times\mathbb{Z} \mid q=-2p, p\in\mathbb{N}^*, n=\sum_{h=1}^{2p}\varepsilon_h 2^{h-1}, \varepsilon_1=0,\\ &\varepsilon_{2i+1}=1, i=1,0,\dots,p-1\Bigg\},\\ E_1 &= \big\{\big(n+2^{-q},q\big)\in\mathbb{N}\times\mathbb{Z} \mid (n,q)\in E_0\big\},\\ E_M &= \big\{(M,0)\big\},\quad M\geq 2. \end{aligned}$$

For any $(n,q)\in E$, let us set $I_{n,q}=[2^q n, 2^q(n+1))$ and the corresponding subspace by $W_{n,q}=\overline{\operatorname{span}}\{2^{q/2}w_n(2^q x-k)\mid k\in\mathbb{Z}\}$. It is easy to see (see Fig. 1) that for $(n,q)\in E_0\cup E_1$, the value of each ε_h in the dyadic expansion of n is related to the "path" in the tree structure from the subspace $W_{n,q}$ up: $\varepsilon_h=0$ if one takes the left "leaf," $\varepsilon_h=1$ otherwise. Obviously,

$$[2,+\infty)=\bigcup_{M\geq 2}[M,M+1)=\bigcup_{M\geq 2}\bigcup_{(n,q)\in E_M} I_{n,q},$$

while

$$[0,2)=\bigcup_{(n,q)\in E_0\cup E_1} I_{n,q}\cup A,$$

where the set A has Lebesgue measure zero and Hausdorff dimension equal to $1/2$. Indeed if we set

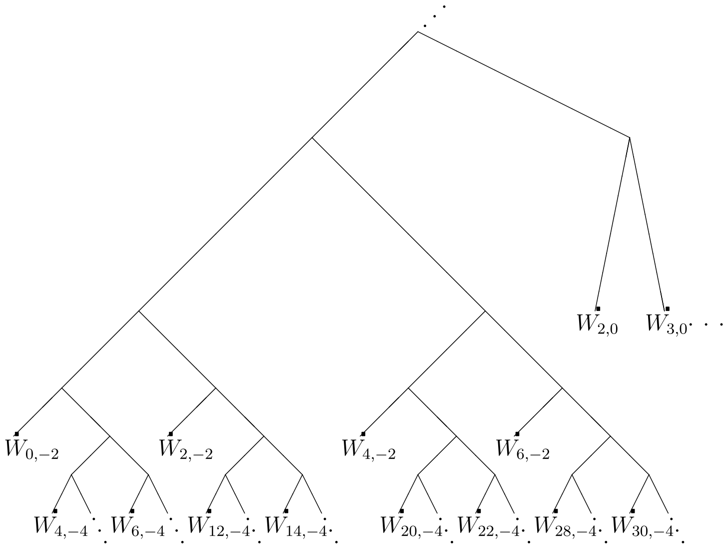

Fig. 1 The closed subspaces of $L^2(\mathbb{R})$ corresponding to our choice of E

$$F_0 = \left\{ (n,q) \in \mathbb{N} \times \mathbb{Z} \mid q = -2p,\, p \in \mathbb{N}^*, n = \sum_{h=1}^{2p} \varepsilon_h 2^{h-1}, \right.$$
$$\left. \varepsilon_{2i+1} = 1, i = 0, \ldots, p-1 \right\},$$
$$F_1 = \left\{ \left(n + 2^{-q}, q\right) \in \mathbb{N} \times \mathbb{Z} \mid (n,q) \in F_0 \right\},$$

we can write A as the disjoint union of two sets:

$$A = \bigcap_{(n,q)\in F_0} I_{n,q} \cup \bigcap_{(n,q)\in F_1} I_{n,q} = A_0 \cup A_1.$$

It is easy to see that A_0, as well as A_1, is constructed like a Cantor set. In particular the compact set A_0 is the invariant set of the iterated function system (f_1, f_2), $f_i : \mathbb{R} \to \mathbb{R}$, $f_1(x) = \frac{x+1}{4}$, $f_2(x) = \frac{x+3}{4}$, which satisfies Moran's open set condition, see [6]. So the Hausdorff dimension of A_0, $\dim_H A_0$ is equal to the similarity dimension $s = \frac{\log 2}{\log 4} = 1/2$. One gets the same conclusion for A_1 and finally $\dim_H A = \max(\dim_H A_0, \dim_H A_1) = 1/2$.

Therefore

$$[0, +\infty) = \bigcup_{M\in\mathbb{N}} \bigcup_{(n,q)\in E_M} I_{n,q} \cup A,$$

and A is our exceptional set.

Consider the Lemarié–Meyer wavelet and the wavelet packet system corresponding to our choice of E:

$$\mathbf{X} = \left\{ 2^{q/2} w_n\left(2^q x - k\right) \mid k \in \mathbb{Z},\ (n,q) \in E \right\}. \tag{2}$$

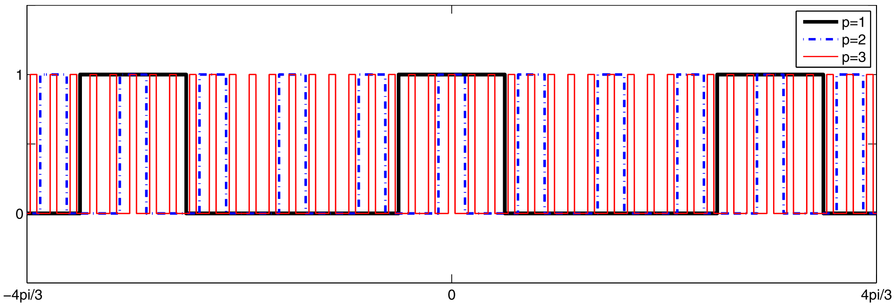

Fig. 2 Intervals where $m_0(2^{2p-1}\xi)=1$, for $p=1,2,3$

We shall assume that it is an orthonormal basis in order to get a contradiction. Before that, however, we shall show an inequality which does not rely on orthonormality. We need a preliminary lemma.

Lemma 2.1 *Let m_0 and φ be, respectively, the transfer and the scaling function of the Lemarié–Meyer wavelet. Then for almost all $\xi \in [-4\pi/3, 4\pi/3] = \operatorname{supp}\hat{\varphi}$, there exists $N \geq 1$ such that $m_0(2^{2N-1}\xi)=1$.*

Proof It is sufficient to prove that ($|\cdot|$ denotes the Lebesgue measure)

$$\begin{aligned}
&\left|\left\{\xi \in \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right] \mid \exists N \geq 1, \text{such that } m_0\big(2^{2N-1}\xi\big)=1\right\}\right| \\
&\quad = \sum_{N=1}^{+\infty}\left|\left\{\xi \in \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right] \mid m_0\big(2^{2N-1}\xi\big)=1\right\}\right. \\
&\qquad \left.\Big\backslash \left\{\xi \in \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right] \mid m_0\big(2^{2N-3}\xi\big)=1\right\}\right| \\
&\quad = \frac{8\pi}{3},
\end{aligned}$$

and this will be done by looking at the support of each dilation of m_0. First of all recall that in $[-\pi,\pi]$, $m_0(\xi)=1$ in $[-\pi/3,\pi/3]$ and $m_0(\xi)=0$ in $[-\pi,-2\pi/3]\cup[2\pi/3,\pi]$. Now

$$\operatorname{supp} m_0\big(2^{2N-1}\cdot\big) = \left[-\frac{2\pi}{3\cdot 2^{2N-1}}, \frac{2\pi}{3\cdot 2^{2N-1}}\right] + \frac{2\pi\mathbb{Z}}{2^{2N-1}},$$

and if we look for the number of subintervals in $[-\frac{4\pi}{3}, \frac{4\pi}{3}]$ where $m_0(2^{2N-1}\xi)$ equals 1, a rapid inspection and the symmetry of m_0 show that this number is equal to

$$\alpha_N = 2\cdot\frac{2^{2N}-1}{3}+1 = \frac{2^{2N+1}+1}{3}.$$

Indeed, by symmetry, it is sufficient to examine $[0, \frac{4\pi}{3})$ and to look for the biggest $\beta_N \in \mathbb{N}$ such that

$$\frac{2\pi}{3 \cdot 2^{2N-1}} + \frac{2\pi\beta_N}{2^{2N-1}} \leq \frac{4\pi}{3}.$$

We get

$$\beta_N = \frac{2^{2N}-1}{3} = 4\beta_{N-1} + 1,$$

and so

$$\alpha_N = 2\beta_N + 1 = 2 \cdot \frac{2^{2N}-1}{3} + 1 = \frac{2^{2N+1}+1}{3}.$$

Now let γ_N be the number of subintervals in $[-\frac{4\pi}{3}, \frac{4\pi}{3}]$ where $m_0(2^{2N-1}\xi) = 1$, out of all those (bigger ones) where also $m_0(2^{2h-1}\xi) = 1$, $h = 1, \ldots, N-1$, $N \geq 2$. Set $\gamma_1 = \alpha_1 = 3$. The measure we look for is equal to

$$\sum_{N=1}^{+\infty} \gamma_N \frac{2\pi}{3 \cdot 2^{2N-1}}.$$

We have the relation

$$\gamma_N + \sum_{p=1}^{N-1} \beta_{N-p}\gamma_p = \alpha_N.$$

Indeed, by symmetry and the structure of m_0, it is easy to see that, for any $h = 1, \ldots, N-1$, the number of subintervals where $m_0(2^{2N-1}\xi) = 1$ which fall into each one of those where $m_0(2^{2p-1}\xi) = 1$, $p = 1, \ldots, N-1$, decreases with p and it is exactly $\beta_{N-p} = \frac{2^{2(N-p)}-1}{3}$. This is shown in the following calculation (see Fig. 2). If we examine any interval where $m_0(2\xi) = 1$, for example the one which contains the origin, we notice that the right endpoint coincides with the right endpoint of an interval where $m_0(2^3\xi) = 0$. Indeed, $\frac{\pi}{3 \cdot 2} = \frac{4\pi}{3 \cdot 2^3}$. This feature applies, by periodicity and re-scaling, to any couple $m_0(2^{2N-1}\xi)$ and $m_0(2^{2p-1}\xi)$, $p = 1, \ldots, N-1$, so any bigger interval contains, by symmetry, $2k+1$ smaller intervals, where $k \in \mathbb{N}$ is such that

$$\frac{\pi}{3 \cdot 2^{2p-1}} = \frac{4\pi}{3 \cdot 2^{2N-1}} + \frac{2\pi k}{2^{2N-1}}.$$

We get $k = \frac{2^{2(N-p)-1}-2}{3}$, and the total number is equal to

$$2k + 1 = \frac{2^{2(N-p)}-1}{3} = \beta_{N-p},$$

as claimed. Hence

$$\gamma_N = \alpha_N - \sum_{p=1}^{N-1} \beta_{N-p}\gamma_p = \alpha_N - \gamma_{N-1} - \sum_{p=1}^{N-2} (4\beta_{N-p-1} + 1)\gamma_p$$

$$= \alpha_N - \gamma_{N-1} - 4(\alpha_{N-1} - \gamma_{N-1}) - \sum_{p=1}^{N-2} \gamma_p$$

$$= 3\gamma_{N-1} - 1 - \sum_{p=1}^{N-2} \gamma_p.$$

We claim that $\gamma_N = 2\gamma_{N-1} + 2^{N-1}$, which is equivalent to

$$\gamma_{N-1} - 1 - \sum_{h=1}^{N-2} \gamma_h = 2^{N-1},$$

easily proved by induction on $N \geq 3$.

So, by a recurrence argument and since $\gamma_1 = 3$, we find

$$\gamma_N = 2\gamma_{N-1} + 2^{N-1} = 2\big(2\gamma_{N-2} + 2^{N-2}\big) + 2^{N-1}$$
$$= \cdots = 2^{N-1}\gamma_1 + (N-1)2^{N-1} = (N+2)2^{N-1}.$$

Finally the measure we look for is equal to

$$\sum_{N=1}^{+\infty} \gamma_N \frac{2\pi}{3 \cdot 2^{2N-1}} = \sum_{N=1}^{+\infty} (N+2)2^{N-1} \frac{2\pi}{3 \cdot 2^{2N-1}} = \frac{8\pi}{3} \sum_{N=1}^{+\infty} \frac{(N+2)}{2^{N+2}} = \frac{8\pi}{3}. \qquad \square$$

The specific choice of E, and thus of A, is crucial in the following theorem:

Theorem 2.2 *Let us consider the wavelet packet system* $\mathbf{X}$ (2). *Then*, *a.e.*

(1) $$\sum_{(n,q)\in E_0} \big|\hat{w}_n\big(2^{-q}\xi\big)\big|^2 \geq \big|\hat{\varphi}(\xi)\big|^2 \quad \textit{and} \quad \sum_{(n,q)\in E_1} \big|\hat{w}_n\big(2^{-q}\xi\big)\big|^2 \geq \big|\hat{\psi}(\xi)\big|^2;$$

(2) $$\sum_{(n,q)\in E} \big|\hat{w}_n\big(2^{-q}\xi\big)\big|^2 \geq 1.$$

Proof We prove the first inequality in (1); the other one is proved similarly, replacing φ by ψ.

For a fixed $p \in \mathbb{N}^*$ we collect all indexes n such that $(n, -2p) \in E_0$ in the set $E_0^p = \{n \in \mathbb{N} | (n, -2p) \in E_0\}$. Then, by unconditional convergence, we get

$$\sum_{(n,q)\in E_0} \big|\hat{w}_n\big(2^{-q}\xi\big)\big|^2 = \lim_{N\to+\infty} \sum_{p=1}^{N} \sum_{n\in E_0^p} \big|\hat{w}_n\big(2^{2p}\xi\big)\big|^2. \tag{3}$$

We claim that

(i) $$\sum_{n\in E_0^1} \big|\hat{w}_n\big(2^2\xi\big)\big|^2 = \big|m_0(2\xi)\hat{\varphi}(\xi)\big|^2;$$

$$\text{(ii)} \quad \sum_{n \in E_0^p} \left|\hat{w}_n\left(2^{2p}\xi\right)\right|^2 = \left|m_0\left(2^{2p-1}\xi\right)m_1\left(2^{2p-3}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2, \quad p > 1.$$

To show (i) we note first that $E_0^1 = \{0, 2\}$, hence

$$\begin{aligned}\sum_{n \in E_0^1} \left|\hat{w}_n\left(2^2\xi\right)\right|^2 &= \left|\hat{\varphi}\left(2^2\xi\right)\right|^2 + \left|m_0(2\xi)\hat{w}_1(2\xi)\right|^2 \\ &= \left|m_0(2\xi)\right|^2\left[\left|m_0(\xi)\right|^2\left|\hat{\varphi}(\xi)\right|^2 + \left|m_1(\xi)\right|^2\left|\hat{\varphi}(\xi)\right|^2\right] = \left|m_0(2\xi)\hat{\varphi}(\xi)\right|^2.\end{aligned}$$

To show (ii) we observe that, for a fixed $p > 1$, any $n \in E_0^p$ is in the form

$$n = \varepsilon_2 2 + 2^2 + \varepsilon_4 2^3 + 2^4 + \dots + 2^{2p-2} + \varepsilon_{2p} 2^{2p-1}, \quad \varepsilon_{2i} = 0, 1.$$

This fact is reflected in the expression of $\hat{w}_n$ with the occurrence either of m_0 or m_1. Indeed,

$$\hat{w}_n(\xi) = m_0\left(\frac{\xi}{2}\right)m_{\varepsilon_2}\left(\frac{\xi}{2^2}\right)m_1\left(\frac{\xi}{2^3}\right)m_{\varepsilon_4}\left(\frac{\xi}{2^4}\right)\dots m_1\left(\frac{\xi}{2^{2p-1}}\right)m_{\varepsilon_{2p}}\left(\frac{\xi}{2^{2p}}\right)\hat{\varphi}\left(\frac{\xi}{2^{2p}}\right).$$

Therefore, $|m_0(\xi)|^2 + |m_1(\xi)|^2 = 1$ leads to

$$\begin{aligned}\sum_{n \in E_0^p} \left|\hat{w}_n\left(2^{2p}\xi\right)\right|^2 = \sum_{\substack{i=1,\dots,p \\ \varepsilon_{2i}=0,1}} &\left|m_0\left(2^{2p-1}\xi\right)m_{\varepsilon_2}\left(2^{2p-2}\xi\right)m_1\left(2^{2p-3}\xi\right)\right. \\ &\left.\dots m_{\varepsilon_{2p-2}}\left(2^2\xi\right)m_1(2\xi)m_{\varepsilon_{2p}}(\xi)\hat{\varphi}(\xi)\right|^2 \\ = &\left|m_0\left(2^{2p-1}\xi\right)m_1\left(2^{2p-3}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2,\end{aligned}$$

and (ii) is proved. Returning back to (3) we get

$$\begin{aligned}\sum_{(n,q) \in E_0} \left|\hat{w}_n\left(2^{-q}\xi\right)\right|^2 &= \left|m_0(2\xi)\hat{\varphi}(\xi)\right|^2 \\ &\quad + \sum_{p=2}^{+\infty} \left|m_0\left(2^{2p-1}\xi\right)m_1\left(2^{2p-3}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2. \quad (4)\end{aligned}$$

Next we show that, for any $N \geq 2$, we have

$$\sum_{p=2}^{N} \left|m_0\left(2^{2p-1}\xi\right)m_1\left(2^{2p-3}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2 = \left|m_1(2\xi)\hat{\varphi}(\xi)\right|^2 \quad (5)$$

in the set where $m_0(2^{2N-1}\xi) = 1$. Indeed,

$$\sum_{p=2}^{N} \left|m_0\left(2^{2p-1}\xi\right)m_1\left(2^{2p-3}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2$$

$$
\begin{aligned}
&= \left|m_0\left(2^{2N-1}\xi\right)m_1\left(2^{2N-3}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2 \\
&\quad + \left|m_0\left(2^{2(N-1)-1}\xi\right)m_1\left(2^{2(N-1)-3}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2 \\
&\quad + \sum_{p=2}^{N-2}\left|m_0\left(2^{2p-1}\xi\right)m_1\left(2^{2p-3}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2 \\
&= \left|m_1\left(2^{2(N-1)-3}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2 + \sum_{p=2}^{N-2}\left|m_0\left(2^{2p-1}\xi\right)\dots m_1(2\xi)\hat{\varphi}(\xi)\right|^2,
\end{aligned}
$$

since $m_0(2^{2N-1}\xi) = 1$ and $|m_1(2^{2N-3}\xi)|^2 + |m_0(2^{2(N-1)-1}\xi)|^2 = 1$. The same argument, repeated a finite number of times, leads to the result.

Finally, equality (4) implies the desired inequality for all points out of the (compact) support of $\hat{\varphi}$, while Lemma 2.1, (4), and (5) yield the result for a.e. $\xi \in \operatorname{supp}\hat{\varphi}$. Indeed, Lemma 2.1 implies that for a.e. $\xi \in \operatorname{supp}\hat{\varphi}$ there exists N such that $m_0(2^{2N-1}\xi) = 1$. If $N = 1$ we invoke (4), while for $N \geq 2$, (5) implies

$$
\sum_{(n,q)\in E_0}\left|\hat{w}_n\left(2^{-q}\xi\right)\right|^2 \geq \left|m_0(2\xi)\hat{\varphi}(\xi)\right|^2 + \left|m_1(2\xi)\hat{\varphi}(\xi)\right|^2 = \left|\hat{\varphi}(\xi)\right|^2.
$$

This completes the proof of (1).

To show (2) we use the definition of wavelet packets and (1) to estimate

$$
\begin{aligned}
\sum_{(n,q)\in E}\left|\hat{w}_n\left(2^{-q}\xi\right)\right|^2 &= \sum_{(n,q)\in E_0}\left|\hat{w}_n\left(2^{-q}\xi\right)\right|^2 + \sum_{(n,q)\in E_1}\left|\hat{w}_n\left(2^{-q}\xi\right)\right|^2 + \sum_{n=2}^{+\infty}\left|\hat{w}_n(\xi)\right|^2 \\
&\geq \left|\hat{\varphi}(\xi)\right|^2 + \left|\hat{\psi}(\xi)\right|^2 + \sum_{n=2}^{+\infty}\left|\hat{w}_n(\xi)\right|^2 \\
&= \sum_{n=0}^{+\infty}\left|\hat{w}_{2n}(\xi)\right|^2 + \sum_{n=0}^{+\infty}\left|\hat{w}_{2n+1}(\xi)\right|^2 \\
&= \sum_{n=0}^{+\infty}\left|m_0\left(\frac{\xi}{2}\right)\hat{w}_n\left(\frac{\xi}{2}\right)\right|^2 + \sum_{n=0}^{+\infty}\left|m_1\left(\frac{\xi}{2}\right)\hat{w}_n\left(\frac{\xi}{2}\right)\right|^2 \\
&= \sum_{n=0}^{+\infty}\left|\hat{w}_n\left(\frac{\xi}{2}\right)\right|^2 = \dots = \sum_{n=0}^{+\infty}\left|\hat{w}_n\left(\frac{\xi}{2^j}\right)\right|^2,
\end{aligned}
$$

for any $j \in \mathbb{N}$. Now, since the system $\{w_n(\cdot - k), n \in \mathbb{N}, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, Proposition 3.3 of [5] applies to $V_0 = L^2(\mathbb{R})$ and we have

$$
\lim_{j\to+\infty}\sum_{n=0}^{+\infty}\left|\hat{w}_n\left(\frac{\xi}{2^j}\right)\right|^2 = 1, \quad \text{a.e. } \xi \in \mathbb{R},
$$

thus completing the proof. □

3 The Local Trace Function

Both subspaces V_0 and W_0 arising in the wavelet packet decomposition of $L^2(\mathbb{R})$ are $\mathbb{Z}$ shift invariant spaces. In the setting of such spaces Dutkay [4], has defined an invariant, the local trace function. The fact that it can be calculated with any normalized tight frame generator will be used in the proof of our main result: We shall compute the local trace function in two different ways and the resulting quantities will be obliged to be equal. In this section we recall definitions and necessary results to our scope; the interested reader can find more in Dutkay's work [4].

We say that a closed subspace $V \subset L^2(\mathbb{R})$ is shift invariant (SI) if for every $f \in V$ we also have $T_k f \in V$ when $k \in \mathbb{Z}$, where $T_k f(x) = f(x-k)$. For any $\Phi \subset L^2(\mathbb{R})$ let

$$S(\Phi) = \overline{\operatorname{span}}\{T_k\varphi \mid \varphi \in \Phi, k \in \mathbb{Z}\}$$

be the SI space generated by Φ.

A periodic range function is any measurable mapping

$$J_{\mathrm{per}} : \mathbb{R} \to \{\text{closed subspaces of } \ell^2(\mathbb{Z})\}$$

satisfying the periodicity condition

$$J_{\mathrm{per}}(\xi + 2\pi k) = \lambda(k)^*\bigl(J_{\mathrm{per}}(\xi)\bigr) \quad \text{for all } k \in \mathbb{Z}, \xi \in \mathbb{R},$$

where λ denotes the shift operator on $\ell^2(\mathbb{Z})$,

$$\lambda(k)\bigl((a_h)_{h\in\mathbb{Z}}\bigr) = (a_{h-k})_{h\in\mathbb{Z}}, \quad k \in \mathbb{Z}.$$

Measurable means weakly operator measurable, i.e., $\xi \mapsto \langle P_{J_{\mathrm{per}}(\xi)}a, b\rangle$ is measurable for any choice of $a, b \in \ell^2(\mathbb{Z})$.

Note that the periodic range function is uniquely determined by its values on the representatives of the cosets of $\mathbb{R}/2\pi\mathbb{Z}$ identified with $[-\pi,\pi]$. Sometimes we shall indicate with the same letter both the subspace $J_{\mathrm{per}}(\xi)$ and the projection onto $J_{\mathrm{per}}(\xi)$.

Let us now recall some spaces and maps. The first one is the Hilbert space of measurable vector-valued functions

$$L^2\bigl([-\pi,\pi],\ell^2(\mathbb{Z})\bigr) = \left\{F : [-\pi,\pi] \to \ell^2(\mathbb{Z}) \mid \int_{[-\pi,\pi]} \|F(\xi)\|^2_{\ell^2(\mathbb{Z})}\, d\xi < +\infty\right\},$$

where the scalar product is given by

$$\langle F, G\rangle := \int_{[-\pi,\pi]} \bigl\langle F(\xi), G(\xi)\bigr\rangle_{\ell^2(\mathbb{Z})}\, d\xi. \tag{6}$$

The map

$$\tau : L^2(\mathbb{R}) \to L^2\bigl([-\pi,\pi],\ell^2(\mathbb{Z})\bigr),$$

defined for $f \in L^2(\mathbb{R})$ and $\xi \in [-\pi,\pi]$, by $\tau f(\xi) = (\hat{f}(\xi + 2\pi k))_{k\in\mathbb{Z}}$, is an isometric isomorphism up to multiplication by $1/(2\pi)^{1/2}$, where the Fourier Transform is defined for $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by

$$\hat{g}(\xi) = \int_{\mathbb{R}} g(x) e^{-ix\xi}\, dx.$$

Consider also the Hilbert space $L^2_{\text{per}}(\mathbb{R}, \ell^2(\mathbb{Z}))$ of measurable vector-valued functions $F : \mathbb{R} \to \ell^2(\mathbb{Z})$ such that $F|_{[-\pi,\pi]}$ belongs to $L^2([-\pi,\pi], \ell^2(\mathbb{Z}))$ and is periodic in the following sense:

$$F(\xi + 2\pi k) = \lambda(k)^* F(\xi) \quad \text{for all } k \in \mathbb{Z}, \xi \in \mathbb{R}.$$

The scalar product is defined again as in (6). The corresponding map

$$\tau_{\text{per}} : L^2(\mathbb{R}) \to L^2_{\text{per}}\big(\mathbb{R}, \ell^2(\mathbb{Z})\big)$$

is defined, for $f \in L^2(\mathbb{R})$ and $\xi \in \mathbb{R}$, by $\tau_{\text{per}} f(\xi) = (\hat{f}(\xi + 2\pi k))_{k\in\mathbb{Z}}$, and verifies the periodicity condition

$$\tau_{\text{per}} f(\xi + 2\pi h) = \big(\hat{f}(\xi + 2\pi k + 2\pi h)\big)_{k\in\mathbb{Z}} = \lambda(k)^* \tau_{\text{per}} f(\xi).$$

The following theorem, due to Helson [7], characterizes SI space in terms of the periodic range function:

Theorem 3.1 *A closed subspace $V \subset L^2(\mathbb{R})$ is SI if and only if*

$$V = \big\{ f \in L^2(\mathbb{R}) \mid \tau_{\text{per}} f(\xi) \in J_{\text{per}}(\xi) \textit{ for a.e. } \xi \in \mathbb{R} \big\},$$

where J_{per} is a measurable periodic range function.

The correspondence between V and J_{per} is bijective under the convention that range functions are identified if they are equal a.e. Furthermore, if $V = S(\Phi)$ for some countable $\Phi \subset L^2(\mathbb{R})$ then, for a.e. $\xi \in \mathbb{R}$,

$$J_{\text{per}}(\xi) = \overline{\operatorname{span}}\big\{ \tau_{\text{per}} \varphi(\xi) \mid \varphi \in \Phi \big\}.$$

Definition 3.2 If V is an SI subspace of $L^2(\mathbb{R})$, then J_{per} associated with V as in Theorem 3.1 is called the periodic range function of V.

It is necessary now to recall the definition of a frame.

Definition 3.3 A subset $\{e_i | i \in I\}$ of a Hilbert space H is called a frame with constants $b, B > 0$, if

$$b\|f\|^2 \le \sum_{i\in I} \big|\langle e_i, f\rangle\big|^2 \le B\|f\|^2 \quad \text{for all } f \in H.$$

If $b = B = 1$, it is called a normalized tight frame (NTF).

Definition 3.4 Let V be a shift invariant subspace of $L^2(\mathbb{R})$. A subset Φ of V is called a normalized tight frame generator (NTF generator) for V if the set of translates

$$\{T_k\varphi \mid k \in \mathbb{Z}, \varphi \in \Phi\}$$

is an NTF for V.

Theorem 3.5 *Let V be an SI subspace of $L^2(\mathbb{R})$, J_{per} its periodic range function, and Φ a countable subset of V.*

$\{T_k\varphi \mid \varphi \in \Phi, k \in \mathbb{Z}\}$ is a frame with constants b and B for V if and only if $\{\tau_{\mathrm{per}}\varphi(\xi) \mid \varphi \in \Phi\}$ is a frame with constants b and B for J_{per}, for a.e. $\xi \in \mathbb{R}$.

The notion of the local trace function is based on the notion of the trace of a positive operator on a Hilbert space. We refer the reader to the work of Dutkay [4] for a survey on the trace function.

Definition 3.6 Let H be a Hilbert space and T a positive operator on H. The trace of T is the positive number (it could be $+\infty$) defined by

$$\mathrm{Trace}(T) = \sum_{i\in I} \langle Te_i, e_i\rangle,$$

where $\{e_i \mid i \in I\}$ is an orthonormal basis for H.

Definition 3.7 Let V be an SI subspace of $L^2(\mathbb{R})$, T a positive operator on $\ell^2(\mathbb{Z})$, and let $J_{\mathrm{per}}(\xi)$ be the periodic range function of V. We define the local trace function associated with V and T as the map

$$\tau_{V,T} : \mathbb{R} \to [0, +\infty)$$

defined by

$$\tau_{V,T}(\xi) = \mathrm{Trace}\big(T J_{\mathrm{per}}(\xi)\big), \quad \xi \in \mathbb{R}.$$

We define the restricted local trace function associated with V and $f \in \ell^2(\mathbb{Z})$ by

$$\tau_{V,f}(\xi) = \mathrm{Trace}\big(P_f J_{\mathrm{per}}(\xi)\big) = \tau_{V,P_f}(\xi), \quad \xi \in \mathbb{R},$$

where P_f is the operator on $\ell^2(\mathbb{Z})$ defined by $P_f(v) = \langle v, f\rangle f$.

The following theorem says that the local trace function can be calculated with any NTF generator. This fact will be used in the proof of our main result: We shall compute the local trace function in two different ways and the resulting quantities will be obliged to be equal.

Theorem 3.8 *Let V be an SI subspace of $L^2(\mathbb{R})$, and $\Phi \subset V$ an NTF generator for V. Then for every positive operator T on $\ell^2(\mathbb{Z})$ and for every $f \in \ell^2(\mathbb{Z})$, we have*

$$\tau_{V,T}(\xi) = \sum_{\varphi\in\Phi} \big\langle T\tau_{\mathrm{per}}\varphi(\xi), \tau_{\mathrm{per}}\varphi(\xi)\big\rangle, \quad a.e.\ \xi \in \mathbb{R}, \tag{7}$$

$$\tau_{V,f}(\xi) = \sum_{\varphi\in\Phi} \left|\left\langle f, \tau_{\mathrm{per}}\varphi(\xi)\right\rangle\right|^2, \quad a.e.\ \xi \in \mathbb{R}.$$

4 Wavelet Packets as Generalized Shift-Invariant Systems

It is well known that a wavelet system is never $\mathbb{Z}$-SI. Nevertheless, one can enlarge it by adding more elements and construct a $\mathbb{Z}$-SI system, called a *quasi-affine system*, which shares most of the properties of the wavelet system. This object was studied by Ron and Shen in [10] and further by Chui, Shi, and Stöckler in [2]. Generally speaking, wavelet systems, and wavelet packet systems as well (WP for short), are *generalized shift-invariant systems* (GSI for short) in the sense of Ron and Shen work [11], and in this general context the analog of a quasi-affine system is the so-called *oblique oversampling* [11].

In this section we consider our example of WP system $\mathbf{X}$ (2), and we shall construct an oblique oversampling of it. Under the hypothesis that $\mathbf{X}$ is an orthonormal basis of $L^2(\mathbb{R})$, we shall show that the oblique oversampling is a normalized tight frame for $L^2(\mathbb{R})$. A crucial role in this is played by part (2) of Theorem 2.2 based on our choice of E and thus of A. This fact will allow us to supply each V_0 and W_0 with another $\mathbb{Z}$-SI NTF different from the one inherited from the wavelet structure.

For the sake of clarity and simplicity we recall definitions and results of [11] in one dimension and we show how our WP system fits into this scheme, but the same argument can be extended to any WP system. We emphasize that the case of WP systems is not included among the special types of GSI systems discussed in [11]. In what follows, $\mathbf{X}$ will always denote our WP system defined in (2).

Definition 4.1 Let J be a countable index set. For any $j \in J$, let $\Gamma_j = c_j\mathbb{Z}$, where c_j is a nonzero, positive real number. Set $\mathcal{L} = (\Gamma_j)_{j\in J}$. For any $j \in J$ assume there is an associate function $\varphi_j \in L^2(\mathbb{R})$, and consider

$$Y_j = \left\{\varphi_j(\cdot + \gamma) \mid \gamma \in \Gamma_j\right\}.$$

The union

$$\mathbf{Y} = \bigcup_{j\in J} Y_j,$$

is called a generalized shift-invariant (GSI) system. Sometimes we shall write $J(\mathbf{Y})$ for the index set.

Remark 1 WP system $\mathbf{X} = \{2^{q/2}w_n(2^q x - k) \mid k \in \mathbb{Z},\ (n,q) \in E\}$, is a GSI system. Indeed, let us take as index set $J = E$, and for any $(n,q) \in E$, the lattice $\Gamma_{(n,q)} = 2^{-q}\mathbb{Z}$, and associate function $\varphi_{(n,q)} = D_{2^q}w_n$, so

$$Y_{(n,q)} = \left\{\varphi_{(n,q)}\left(\cdot - 2^{-q}k\right) \mid 2^{-q}k \in 2^{-q}\mathbb{Z}\right\} = \left\{2^{q/2}w_n\left(2^q \cdot - k\right) \mid k \in \mathbb{Z}\right\}.$$

$\mathbf{X}$ is a nested GSI system, in the following sense. Let us define a total ordering on $E \subset \mathbb{N}\times\mathbb{Z}$ as $(n,q) \le (n',q')$ if and only if either $q < q'$, or, in the case $q = q'$,

$n \leq n'$. We have

$$(n, q) \leq (n', q') \Longleftrightarrow 2^{-q'} \leq 2^{-q} \Longleftrightarrow 2^{-q}\mathbb{Z} \subset 2^{-q'}\mathbb{Z}.$$

Definition 4.2 Let $\mathbf{Y} = \bigcup_{j \in J} Y_j$ and $\mathbf{Y^0} = \bigcup_{j \in J} Y_j^0$ be two GSI systems with the same index set J. Let φ_j and φ_j^0 be the relative associate functions. We say that $\mathbf{Y^0}$ is an oversampling of $\mathbf{Y}$ if for every $j \in J$ the following holds:

(1) The lattice $c_j^0\mathbb{Z}$ (of the layer Y_j^0) is a superlattice of $c_j\mathbb{Z}$;
(2) The following relation holds:

$$\varphi_j^0 = \left(\frac{c_j^0}{c_j}\right)^{\frac{1}{2}} \varphi_j.$$

If $\mathbf{Y}$ is nested we say that the oversampling is oblique if there exists $\overline{j} \in J$ such that for every $j \in J$,

$$c_j^0 = \begin{cases} c_j, & j > \overline{j}, \\ c_{\overline{j}}, & j \leq \overline{j}. \end{cases}$$

Remark 2 Let us construct an oblique oversampling of $\mathbf{X}$. We define

$$\mathbf{X^0} = \bigcup_{(n,q) \in E} Y_{(n,q)}^0,$$

where, if $q \geq 0$, we take as associated function $\varphi_{(n,q)}^0 = D_{2^q} w_n$, lattice $\Gamma_{(n,q)}^0 = 2^{-q}\mathbb{Z}$, and

$$Y_{(n,q)}^0 = \left\{2^{q/2} w_n\left(2^q x - k\right) \mid (n, q) \in E, q \geq 0, k \in \mathbb{Z}\right\},$$

while, if $q < 0$, we take as associated function $\varphi_{(n,q)}^0 = 2^{q/2} D_{2^q} w_n$, lattice $\Gamma_{(n,q)}^0 = \mathbb{Z}$, and

$$Y_{(n,q)}^0 = \left\{2^q w_n\left(2^q\left(x - k\right)\right) \mid (n, q) \in E, q < 0, k \in \mathbb{Z}\right\}.$$

It is easy to see that, in both cases, $\Gamma_{(n,q)}^0$ is a superlattice of $\Gamma_{(n,q)} = 2^{-q}\mathbb{Z}$. The equality (2) is also satisfied, since, if $q < 0$,

$$\varphi_{(n,q)}^0(x) = 2^q w_n\left(2^q x\right) = \left(\frac{1}{2^{-q}}\right)^{\frac{1}{2}} 2^{q/2} w_n\left(2^q x\right) = \left(\frac{1}{2^{-q}}\right)^{\frac{1}{2}} \varphi_{(n,q)},$$

the other case being trivial.

The oversampling is oblique, with respect to the total ordering of E, since for every $(n, q) \in E$,

$$\Gamma_{(n,q)}^0 = \begin{cases} 2^{-q}\mathbb{Z} = \Gamma_{(n,q)}, & (n, q) > (n, 0), \\ \mathbb{Z} = \Gamma_{(n,0)}, & (n, q) \leq (n, 0). \end{cases}$$

Finally the oversampling is *tailless*, which means (in one dimension) that the set of all different numbers in the set $\{2^{-q}, q \geq 0\}$ is bounded and has no accumulation points other then 0. (See Definition 2.18 in [11].) This fact will be useful later, see Proposition 4.7. Note also that the WP we start with is not tailless.

The main object of the study of GSI systems in [11] is the dual Gramian. In the case of **X** it specializes as $\widetilde{G} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$,

$$\begin{aligned}\widetilde{G}(\xi, \eta) &= \sum_{(n,q)\in\kappa(\xi-\eta)} \frac{D_{2^{-q}}\hat{w}_n(\xi)\overline{D_{2^{-q}}\hat{w}_n(\eta)}}{|\Gamma_{(n,q)}|} \\ &= \sum_{(n,q)\in\kappa(\xi-\eta)} 2^q D_{2^{-q}}\hat{w}_n(\xi)\overline{D_{2^{-q}}\hat{w}_n(\eta)},\end{aligned}$$

where $|\Gamma_{(n,q)}| = 2^{-q}$ and the valuation function κ is defined as

$$\kappa(\xi) := \left\{(n, q) \in E \mid \xi \in 2\pi 2^q \mathbb{Z}\right\}. \tag{8}$$

In other words, for a fixed $\xi \in \mathbb{R}$, $\widetilde{G}(\xi, \eta) = 0$ unless η lies in the countable set $\xi + \bigcup_{(n,q)\in F}(2\pi 2^q \mathbb{Z})$, and if this is the case for a certain $(m, p) \in E$ and $k \in \mathbb{Z}$,

$$\widetilde{G}\big(\xi, \xi + 2\pi 2^p k\big) = \sum_{(n,q)\in E, 2^q|2^p k} \hat{w}_n\big(2^{-q}\xi\big)\overline{\hat{w}_n\big(2^{-q}\xi + 2\pi 2^{p-q}k\big)}.$$

Another important tool is the diagonal function, and for **X** it becomes

$$\widetilde{g}(\xi) = \widetilde{G}(\xi, \xi) = \sum_{(n,q)\in E} \left|\hat{w}_n\big(2^{-q}\xi\big)\right|^2,$$

since $\kappa(0) = \{(n, q) \in E \mid 0 \in 2\pi 2^q \mathbb{Z}\} = E$.

It is needless to say that the above objects could be defined for any wavelet packet system, with the obvious changes, as it is for the following lemma, which says that the diagonal function is a.e. bounded (by Schwarz's inequality the dual Gramian is bounded, too).

Lemma 4.3 *For almost all $\xi \in \mathbb{R}$,*

$$\widetilde{g}(\xi) = \sum_{(n,q)\in E} \left|\hat{w}_n\big(2^{-q}\xi\big)\right|^2 \leq 1.$$

Proof The proof follows from the observation that the diagonal function is related to the spectral function σ_V^Γ defined by Bownik and Rzeszotnik in the following way (see Lemma 2.5 in [1]). If we take $W_{n,q}$ and $\Gamma_q = 2^{-q}\mathbb{Z}$, then

$$\widetilde{g}(\xi) = \sum_{(n,q)\in E} \left|\hat{w}_n\big(2^{-q}\xi\big)\right|^2 = \sum_{(n,q)\in E} \sigma_{W_{n,q}}^{\Gamma_q}(\xi).$$

Now for any finite set G in E let us denote $V = \bigoplus_{(n,q)\in G} W_{n,q}$. By (b) and (d) in Proposition 2.8 [1],

$$\sum_{(n,q)\in G} \sigma_{W_{n,q}}^{\Gamma_q}(\xi) = \sigma_V(\xi) \le \sigma_{L^2(\mathbb{R})}^{\mathbb{Z}}(\xi) = 1,$$

so, by unconditional convergence the result follows. □

The previous lemma and Theorem 2.2 yield, for the specific choice of $\mathbf{X}$, the next crucial identity.

Corollary 4.4 *Let us consider the wavelet packet system* $\mathbf{X}$. *Then for almost all* $\xi \in \mathbb{R}$,

$$\widetilde{g}(\xi) = \sum_{(n,q)\in E} \left|\hat{w}_n\left(2^{-q}\xi\right)\right|^2 = 1.$$

The above equality relates to the so-called *discrete* Calderón condition. For an orthonormal wavelet system (where a finite number of mother wavelets occurs) it is equivalent to completeness in $L^2(\mathbb{R})$, see the work by Hernández, Labate, and Weiss [8] and references therein. What fails in our case is the fact that there is no lattice Γ which makes the direct sum $\bigoplus_{(n,q)\in E} W_{n,q}$ a Γ shift invariant space, nor does the wavelet packet system yet verify the local integrability condition (2.6) in Corollary 4.4 in [8].

Remark 3 The dual Gramian of the GSI $\mathbf{X^0}$ is $\widetilde{G}_{\mathbf{X^0}} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$,

$$\begin{aligned}\widetilde{G}_{\mathbf{X^0}}(\xi,\eta) &= \sum_{(n,q)\in\kappa^0(\xi-\eta)} \frac{\hat{\varphi}^0_{(n,q)}(\xi)\overline{\hat{\varphi}^0_{(n,q)}(\eta)}}{|\Gamma^0_{(n,q)}|} \\ &= \sum_{(n,q)\in\kappa^0(\xi-\eta)} 2^q D_{2^{-q}}\hat{w}_n(\xi)\overline{D_{2^{-q}}\hat{w}_n(\eta)},\end{aligned}$$

since $|\Gamma^0_{(n,q)}|$ is either 2^{-q}, $q \ge 0$, or 1, $q < 0$, and the valuation function κ^0 is

$$\kappa^0(\xi) = \left\{(n,q)\in E \mid q\ge 0, \xi \in 2\pi 2^q\mathbb{Z}\right\} \cup \left\{(n,q)\in E| \mid q<0, \xi\in 2\pi\mathbb{Z}\right\}.$$

Corollary 4.5 (cf. Proposition 3.9 in [11]) *Let us consider the wavelet packet system* $\mathbf{X}$ *and its oblique oversampling* $\mathbf{X^0}$. *Let* κ *and* κ^0 *be the corresponding valuation functions given by* (8). *Then*:

(1) $\widetilde{G}_{\mathbf{X}}(\xi,\eta) = \widetilde{G}_{\mathbf{X^0}}(\xi,\eta)$ *whenever* $\kappa(\xi-\eta) = \kappa^0(\xi-\eta)$;
(2) $\widetilde{g}_{\mathbf{X}}(\xi) = \widetilde{g}_{\mathbf{X^0}}(\xi) = 1$.

We shall now assume that $\mathbf{X}$ is an orthonormal basis for $L^2(\mathbb{R})$, and we shall show that the (tailless) oblique oversampling $\mathbf{X^0}$ is a normalized tight frame (for $L^2(\mathbb{R})$). In order to do so, we need only to prove that $\mathbf{X^0}$ is a Bessel system with Bessel bound

≤ 1. This follows from the above corollary and the following results in [11] applied to $\mathbf{X}$ and $\mathbf{X^0}$.

Proposition 4.6 ([11], Corollary 1.14) *Let* $\mathbf{Y}$ *be a GSI Bessel system with Bessel bound* ≤ 1. *Let* $\widetilde{G}$ *be the associated dual Gramian and* $\widetilde{g}$ *the associated diagonal function*. *Then*:

If $\widetilde{g} \geq 1$ *a.e., then* $\widetilde{G}(\xi, \eta) = 0$, *for a.e.* ξ *and every* η, $\eta \neq \xi$.

Proposition 4.7 (cf. [11], Corollary 3.7) *Let* $\mathbf{Y}$ *be a tailless GSI system. Then* $\mathbf{Y}$ *is a normalized tight frame if and only if* $\mathbf{Y}$ *is scalar, which means there is a null set* $\mathcal{N} \subset \mathbb{R}$ *such that* $\widetilde{G}_{\mathbf{Y}}(\xi, \eta) = \delta_{\xi,\eta}$ *for every* $\xi, \eta \in \mathbb{R} \backslash \mathcal{N}$.

In order to show that $\mathbf{X^0}$ is a Bessel system with Bessel bound ≤ 1 we need first the notion of "dominance," (see Definition 3.14 in [11]) which requires, more or less, that the dual Gramian $\widetilde{G}_{\mathbf{Y}}$ coincides with $\widetilde{G}_{\mathbf{Y^0}}$ at all the entries which are involved in the Bessel and frame properties of $\mathbf{Y^0}$. Since in our case the intersection of all lattices associated to $\mathbf{X^0}$, $\bigcap_{j \in J(\mathbf{X^0})} \Gamma_j^0$ is the 1-dimensional lattice $\mathbb{Z}$, we can slightly simplify the definition as follows:

Definition 4.8 Let $\mathbf{Y}$ and $\mathbf{Y^0}$ be two GSI systems. Let Γ_j, $j \in J(\mathbf{Y^0})$, be the lattices associated to $\mathbf{Y^0}$. Assume that $\Gamma_0 = \bigcap_{j \in J(\mathbf{Y^0})} \Gamma_j$ is a 1-dimensional lattice.

We say that $\mathbf{Y}$ dominates $\mathbf{Y^0}$ if there exists a null set $\mathcal{N} \subset \mathbb{R}$ such that $\widetilde{G}_{\mathbf{Y}}(\xi, \eta) = \widetilde{G}_{\mathbf{Y^0}}(\xi, \eta)$ whenever $(\xi - \eta) \cdot \gamma \in 2\pi\mathbb{Z}$ for any $\gamma \in \Gamma_0$ and $\xi \in \mathbb{R} \backslash \mathcal{N}$.

Lemma 4.9 $\mathbf{X}$ *dominates the oblique oversampling* $\mathbf{X^0}$.

Proof Since $\mathbf{X^0}$ is $\mathbb{Z}$-shift invariant, by Corollary 4.5 it suffices to show the equality $\kappa(\xi - \eta) = \kappa^0(\xi - \eta)$ whenever $(\xi - \eta) \cdot \gamma \in 2\pi\mathbb{Z}$ for any $\gamma \in \mathbb{Z}$, which means $\kappa(2\pi r) = \kappa^0(2\pi r)$ for any $r \in \mathbb{Z}$. But this follows easily from the choice of lattices in $\mathbf{X^0}$. Indeed, since $r \in \mathbb{Z}$,

$$\begin{aligned} \kappa(2\pi r) &= \big\{(n,q) \in E \mid r \in 2^q\mathbb{Z}\big\} \\ &= \big\{(n,q) \in E \mid q \geq 0, r \in 2^q\mathbb{Z}\big\} \cup \big\{(n,q) \in E \mid q < 0\big\}, \end{aligned}$$

while, by definition, $\kappa^0(2\pi r)$ equals to

$$\big\{(n,q) \in E \mid q \geq 0, r \in 2^q\mathbb{Z}\big\} \cup \big\{(n,q) \in E \mid q < 0, r \in \mathbb{Z}\big\}. \qquad \square$$

Since $\mathbf{X^0}$ is tailless and dominated by $\mathbf{X}$, it follows that in order to show that $\mathbf{X^0}$ is Bessel we need to control, in a certain way, the Gramian of $\mathbf{X}$. The following definition presents a tool to do so.

Definition 4.10 Let $\mathbf{Y}$ be a GSI system. Given a finite $P \subset \mathbb{R}$ we denote by $\widetilde{G}(P)$ the submatrix of $\widetilde{G}$ whose rows and columns are indexed by P and whose (p,q)-entry is $\widetilde{G}(p,q)$, $p, q \in P$. We denote by $\mathcal{G}(P)$ the norm of the matrix $\widetilde{G}(P)$ viewed as an endomorphism of $\ell^2(P)$. If any of the entries of $\widetilde{G}(P)$is not well defined or is not finite, we define $\mathcal{G}(P) = +\infty$.

We say that $\mathcal{G}$ is bounded by $A > 0$ if, for every finite $P \subset \mathbb{R}$,

$$\left\|\mathcal{G}(\cdot + P)\right\|_{L^\infty(\mathbb{R})} \leq A.$$

We call $\mathcal{G} = \mathcal{G}_{\mathbf{Y}}$ the norm function of $\mathbf{Y}$.

Theorem 4.11 (cf. Theorem 2.14, and 3.4 [11]) *Let* $\mathbf{Y}$ *be a GSI system associated with a norm function* $\mathcal{G}$. *If* $\mathbf{Y}$ *is Bessel with Bessel bound* A, *then* $\mathcal{G}$ *is bounded by* A. *Moreover, if* $\mathbf{Y}$ *is tailless the converse is also true.*

Corollary 4.12 *Let us assume that* $\mathbf{X}$ *is an orthonormal basis. Then* $\mathcal{G}_{\mathbf{X}}$ *is bounded by* 1.

Proposition 4.13 (cf. Proposition 3.15, [11]) *Let* $\mathbf{Y}$, $\mathbf{Y^0}$ *be GSI systems. Assume that* $\mathbf{Y^0}$ *is tailless and that* $\mathbf{Y}$ *dominates* $\mathbf{Y^0}$. *If* $\mathbf{Y}$ *is Bessel, then* $\mathbf{Y^0}$ *is Bessel too, and its Bessel bound is no larger than that of* $\mathbf{Y}$.

Proof The proof follows the first part of the proof of Proposition 3.15, [11], (at this stage of the proof $\mathbf{Y}$ is not required to have small tails), and we get that, by dominance, $\mathcal{G}_{\mathbf{Y^0}}$ is bounded by the Bessel bound A of $\mathbf{Y}$. By Theorem 4.11 applied to $\mathbf{Y^0}$, we obtain the thesis. □

Corollary 4.14 *Let us assume that* $\mathbf{X}$ *is an orthonormal basis. Then* $\mathbf{X^0}$ *is a normalized tight frame for* $L^2(\mathbb{R})$.

Corollary 4.15 *Let us assume that* $\mathbf{X}$ *is an orthonormal basis. Then, for any fixed* $m \in \mathbb{N}$, *the system*

$$\left\{2^q w_n\left(2^q(x-k)\right) \mid (n,q) \in E, W_{n,q} \subseteq W_{m,0}, k \in \mathbb{Z}\right\}$$

is an NTF for the subspace $W_{m,0}$.

Proof It follows from the previous corollary and the orthonormality of the spaces $W_{n,q}$ and $W_{m,0}$ for $W_{n,q} \nsubseteq W_{m,0}$. □

5 Measures Associated to Wavelet Packets

In this section we summarize the properties of the continuous measures induced by the wavelet packet algorithm on the Borel sets of $[0, 1)$ (see [3] for definition and details) as they appear in [14].

The measure μ_k is first defined in any dyadic interval as follows. Let

$$\mathcal{I}_j^n = \left[\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{4} + \cdots + \frac{\varepsilon_j}{2^j}, \frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{4} + \cdots + \frac{\varepsilon_j}{2^j} + \frac{1}{2^j}\right) \subset [0,1),$$

where $n=\varepsilon_1+2\varepsilon_2+\cdots+2^{j-1}\varepsilon_j$, and $\varepsilon_i=0,1$. Let us define

$$\mu_k(\mathcal{I}_j^n)=2^j\sum_{p\in\mathbb{Z}}\left|\int_0^{2\pi} m_{\varepsilon_1}(\theta)m_{\varepsilon_2}(2\theta)\dots m_{\varepsilon_j}(2^{j-1}\theta)e^{-ik\theta}e^{i2^j p\theta}\frac{d\theta}{2\pi}\right|^2. \tag{9}$$

The occurrence of the symbols m_0 and m_1 reflects the position of the interval.

μ_k extends to a regular continuous Borel measure and its value in dyadic intervals is linked to wavelet packets w_n as shown in the following theorem:

Theorem 5.1 *Let $n=\varepsilon_1+2\varepsilon_2+\cdots+2^{j-1}\varepsilon_j$. Consider*

$$\mathcal{I}_j^n=\left[\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{4}+\cdots+\frac{\varepsilon_j}{2^j},\frac{\varepsilon_1}{2}+\frac{\varepsilon_2}{4}+\cdots+\frac{\varepsilon_j}{2^j}+\frac{1}{2^j}\right)=\left[\frac{\overline{n}}{2^j},\frac{\overline{n}+1}{2^j}\right),$$

where we set $\overline{n}=\varepsilon_j+2\varepsilon_{j-1}+\cdots+2^{j-1}\varepsilon_1$.

Then:

(1) *If n is even* ($\varepsilon_1=0$), *then*

$$\mu_k(\mathcal{I}_j^n)=\frac{1}{2^j}\sum_{p\in\mathbb{Z}}\left|w_{\overline{n}}\left(p-\frac{k}{2^j}\right)\right|^2.$$

(2) *If n is odd* ($\varepsilon_1=1$), *then*

$$\mu_k(\mathcal{I}_j^n)=\frac{1}{2^j}\sum_{p\in\mathbb{Z}}\left|w_{\overline{n}-2^{j-1}}\left(p-\frac{(k-1)}{2^j}\right)\right|^2.$$

(3) *For any measurable set $\Omega\subset[0,\frac{1}{2})$, we have*

$$\mu_{2k}\left(\frac{\Omega}{2}\right)=\frac{1}{2}\mu_k(\Omega).$$

(4) *For any measurable set $\Omega\subset[0,1)$, we have*

$$\mu_k\left(\frac{\Omega}{2}\right)=\mu_{k+1}\left(\frac{\Omega+1}{2}\right).$$

Remark 4 Note that, from the definition of the measures μ_k, the following relations are always true for any dyadic interval of length $1/2^j$:

$$\sum_{k=0}^{2^j-1}\mu_k(\mathcal{I})=1,\quad \mu_{k+2^j}(\mathcal{I})=\mu_k(\mathcal{I}). \tag{10}$$

In particular we can deduce the following values for the Lemarié–Meyer wavelet ψ and the scaling function φ:

$$\sum_{p\in\mathbb{Z}}\left|\psi(p)\right|^2=\sum_{p\in\mathbb{Z}}\left|\varphi(p)\right|^2=1+4a,\qquad \sum_{p\in\mathbb{Z}}\left|\varphi\left(p+\frac{1}{2}\right)\right|^2=1-4a, \tag{11}$$

where

$$a = \frac{1}{2\pi}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} m_0(\theta)m_0(\theta+\pi)\,d\theta > 0.$$

6 Final Calculations

The overlapped region between $m_0(\theta)$ and $m_0(\theta+\pi)$ is responsible for (11). Thus none of the measures μ_k is the Lebesgue measure. As we shall see, however, the existence of another NTF will force the same measures to behave like that, yielding a contradiction.

Theorem 6.1 *Consider the Lemarié–Meyer wavelet and the corresponding wavelet packet system*

$$\mathbf{X} = \bigl\{2^{q/2} w_n\bigl(2^q x - k\bigr),\ k \in \mathbb{Z},\ (n,q) \in E\bigr\},$$

where $E = \bigcup_{M\in\mathbb{N}} E_M$, *and*

$$E_0 = \Biggl\{(n,q) \in \mathbb{N}\times\mathbb{Z} \mid q = -2p,\ p \in \mathbb{N}^*, n = \sum_{h=1}^{2p} \varepsilon_h 2^{h-1}, \varepsilon_1 = 0,$$
$$\varepsilon_{2i+1} = 1, i = 1, \dots, p-1\Biggr\},$$
$$E_1 = \bigl\{\bigl(n + 2^{-q}, q\bigr) \in \mathbb{N}\times\mathbb{Z} \mid (n,q) \in E_0\bigr\},$$
$$E_M = \bigl\{(M,0)\bigr\}, \quad M \ge 2.$$

Then $\mathbf{X}$ *is not an ONB of* $L^2(\mathbb{R})$.

Proof The proof is obtained by reductio ad absurdum.

Let us assume that the wavelet packet $\mathbf{X}$ is an ONB for $L^2(\mathbb{R})$ and consider the corresponding oblique oversampling $\mathbf{X^0}$. Then, by Corollary 4.15, we know that the system

$$\bigl\{2^q w_n\bigl(2^q(x-k)\bigr) \mid (n,q) \in E_0, q < 0, k \in \mathbb{Z}\bigr\}$$

is an NTF for the subspace $W_{0,0} = V_0$ and the system

$$\bigl\{2^q w_n\bigl(2^q(x-k)\bigr) \mid (n,q) \in E_1, q < 0, k \in \mathbb{Z}\bigr\}$$

is an NTF for the subspace $W_{1,0} = W_0$.

In other words, $\{2^{q/2} D_{2^q} w_n \mid (n,q) \in E_0, q < 0\}$ is an NTF generator for V_0, while $\{2^{q/2} D_{2^q} w_n \mid (n,q) \in E_1, q < 0\}$ is an NTF generator for W_0, where the underlying lattice is $\Gamma = \mathbb{Z}$. So, by Theorem 3.8 (7), for any $f \in \ell^2(\mathbb{Z})$,

$$\tau^{\mathbb{Z}}_{V_0,f}(\xi) = \sum_{(n,q)\in E_0} \bigl|\bigl\langle f, \tau_{\mathrm{per}}\bigl(2^{q/2} D_{2^q} w_n\bigr)(\xi)\bigr\rangle\bigr|^2, \quad \text{a.e. } \xi \in \mathbb{R}^n,$$

and

$$\tau^{\mathbb{Z}}_{W_0,f}(\xi)=\sum_{(n,q)\in E_1}\left|\left\langle f,\tau_{\mathrm{per}}\left(2^{q/2}D_{2^q}w_n\right)(\xi)\right\rangle\right|^2,\quad \text{a.e. } \xi\in\mathbb{R}^n.$$

By definition,

$$\tau_{\mathrm{per}}\left(2^{q/2}D_{2^q}w_n\right)(\xi)=\left(\hat{w}_n\left(2^{-q}(\xi+2\pi k)\right)\right)_{k\in\mathbb{Z}}.$$

On the other hand, the integer translates of φ and ψ form, respectively, an ONB for V_0 and W_0, so, since the local trace function does not depend on the choice of the NTF generator, we have the equalities, for any $f\in\ell^2(\mathbb{Z})$,

$$\begin{aligned}\sum_{(n,q)\in E_0}\left|\sum_{l\in\mathbb{Z}}f_l\,\overline{\hat{w}_n\left(2^{-q}(\xi+2\pi l)\right)}\right|^2&=\tau^{\mathbb{Z}}_{V_0,f}(\xi)=\left|\left\langle f,\tau_{\mathrm{per}}(\varphi)(\xi)\right\rangle\right|^2\\&=\left|\sum_{l\in\mathbb{Z}}f_l\,\overline{\hat{\varphi}(\xi+2\pi l)}\right|^2,\end{aligned}$$

and

$$\begin{aligned}\sum_{(n,q)\in E_1}\left|\sum_{l\in\mathbb{Z}}f_l\,\overline{\hat{w}_n\left(2^{-q}(\xi+2\pi l)\right)}\right|^2&=\tau^{\mathbb{Z}}_{W_0,f}(\xi)=\left|\left\langle f,\tau_{\mathrm{per}}(\psi)(\xi)\right\rangle\right|^2\\&=\left|\sum_{l\in\mathbb{Z}}f_l\,\overline{\hat{\psi}(\xi+2\pi l)}\right|^2.\end{aligned}$$

Since the above equalities hold for any $f\in\ell^2(\mathbb{Z})$, they imply

$$\left|\sum_{r\in\mathbb{Z}}\hat{\varphi}(\xi+2\pi r)\right|^2=\sum_{(n,q)\in E_0}\left|\sum_{r\in\mathbb{Z}}\hat{w}_n\left(2^{-q}(\xi+2\pi r)\right)\right|^2$$

and

$$\left|\sum_{r\in\mathbb{Z}}\hat{\psi}(\xi+2\pi r)\right|^2=\sum_{(n,q)\in E_1}\left|\sum_{r\in\mathbb{Z}}\hat{w}_n\left(2^{-q}(\xi+2\pi r)\right)\right|^2.$$

The next step consists of applying the Poisson summation formula to each sum (recall that φ, ψ, w_n are in the Schwartz class), thus obtaining

$$\left|\sum_{r\in\mathbb{Z}}\varphi(r)e^{-ir\xi}\right|^2=\sum_{(n,q)\in E_0}\left|\frac{1}{2^{-q}}\sum_{r\in\mathbb{Z}}w_n\left(\frac{r}{2^{-q}}\right)e^{-ir\xi}\right|^2$$

and

$$\left|\sum_{r\in\mathbb{Z}}\psi(r)e^{-ir\xi}\right|^2=\sum_{(n,q)\in E_1}\left|\frac{1}{2^{-q}}\sum_{r\in\mathbb{Z}}w_n\left(\frac{r}{2^{-q}}\right)e^{-ir\xi}\right|^2.$$

If we integrate between 0 and 2π, we get

$$\sum_{r\in\mathbb{Z}}\left|\varphi(r)\right|^2=\sum_{(n,q)\in E_0}\frac{1}{2^{-2q}}\sum_{r\in\mathbb{Z}}\left|w_n\left(\frac{r}{2^{-q}}\right)\right|^2 \tag{12}$$

and

$$\sum_{r\in\mathbb{Z}}\left|\psi(r)\right|^2=\sum_{(n,q)\in E_1}\frac{1}{2^{-2q}}\sum_{r\in\mathbb{Z}}\left|w_n\left(\frac{r}{2^{-q}}\right)\right|^2. \tag{13}$$

The left-hand sums are given in (11), and they are equal to $1+4a$ each. The right-hand sums are linked to measures μ_s, associated to wavelet packets (see (9)), and their values in dyadic intervals are determined by the pairs $(n,q)\in E_0\cup E_1$. Indeed, by our choice of E_0 and E_1, the right-hand side of (12) and (13) are, respectively, equal to

$$\sum_{(n,q)\in E_0}\frac{1}{2^{-2q}}\sum_{s=0}^{2^{-q}-1}\sum_{r\in\mathbb{Z}}\left|w_n\left(r-\frac{s}{2^{-q}}\right)\right|^2$$

and

$$\sum_{(n,q)\in E_0}\frac{1}{2^{-2q}}\sum_{s=0}^{2^{-q}-1}\sum_{r\in\mathbb{Z}}\left|w_{n+2^{-q}}\left(r-\frac{s}{2^{-q}}\right)\right|^2.$$

We now treat the two sums above singularly. We emphasize that the following calculations heavily depend on the nature of E and thus of A. Let us begin with the first one.

If $(n,q)\in E_0$, then $q=-2p,\ p\in\mathbb{N}$, and

$$n=\varepsilon_1+2\varepsilon_2+\cdots+2^{2p-1}\varepsilon_{2p},\quad \varepsilon_1=0,\ \varepsilon_{2i+1}=1, i=1,\dots,p-1.$$

Set

$$\Omega_{n,p}=\left[\frac{n}{2^{2p}},\frac{n+1}{2^{2p}}\right).$$

If $\varepsilon_{2p}=0$, then $\overline{n}=\varepsilon_{2p}+2\varepsilon_{2p-1}+\cdots+2^{2p-1}\varepsilon_1$ is even. Also we note that, for our choice of E_0, since $\varepsilon_1=0$, $n+1$ cannot be larger then 2^{2p-1}, so $\Omega_{n,p}\subset[0,\frac{1}{2})$, and by Theorem 5.1,

$$\mu_s(\Omega_{n,p})=2\mu_{2s}\left(\frac{\Omega_{n,p}}{2}\right),$$

so we get

$$\begin{aligned}\sum_{r\in\mathbb{Z}}\left|w_n\left(r-\frac{s}{2^{-q}}\right)\right|^2&=\sum_{r\in\mathbb{Z}}\left|w_{\overline{\overline{n}}}\left(r-\frac{s}{2^{-q}}\right)\right|^2=2^{2p}\mu_s\left(\left[\frac{\overline{\overline{n}}}{2^{2p}},\frac{\overline{\overline{n}}+1}{2^{2p}}\right)\right)\\&=2^{2p}\mu_s(\Omega_{n,p})=2^{2p+1}\mu_{2s}\left(\frac{\Omega_{n,p}}{2}\right).\end{aligned}$$

On the other hand, if $\varepsilon_{2p}=1$, then we can write $n=\varepsilon_1+2\varepsilon_2+\cdots+2^{2p-1}\varepsilon_{2p}+2^{2p}\cdot 0$, and so $\overline{n}=0+2\varepsilon_{2p}+2^2\varepsilon_{2p-1}+\cdots+2^{2p-1}\varepsilon_2+2^{2p}\varepsilon_1$ is even and by Theorem 5.1, we have again

$$\sum_{r\in\mathbb{Z}}\left|w_n\left(r-\frac{s}{2^{-q}}\right)\right|^2=\sum_{r\in\mathbb{Z}}\left|w_{\overline{\overline{n}}}\left(r-\frac{2s}{2^{-q+1}}\right)\right|^2=2^{2p+1}\mu_{2s}\left(\left[\frac{\overline{\overline{n}}}{2^{2p+1}},\frac{\overline{\overline{n}}+1}{2^{2p+1}}\right)\right)$$
$$=2^{2p+1}\mu_{2s}\left(\frac{\Omega_{n,p}}{2}\right).$$

Therefore, substituting in equality (12), we obtain

$$1+4a=\sum_{(n,-2p)\in E_0\ p\in\mathbb{N}^*}\frac{1}{2^{2p-1}}\sum_{s=0}^{2^{2p}-1}\mu_{2s}\left(\frac{\Omega_{n,p}}{2}\right).$$

Let us proceed with the second sum.

This time $n+2^{2p}=\varepsilon_1+2\varepsilon_2+\cdots+2^{2p-1}\varepsilon_{2p}+2^{2p}+2^{2p+1}\cdot 0$, and so $\overline{n+2^{2p}}=0+2+2^2\varepsilon_{2p}+\cdots+2^{2p}\varepsilon_2+2^{2p+1}\varepsilon_1$ is even, hence by Theorem 5.1, we have

$$\sum_{r\in\mathbb{Z}}\left|w_{n+2^{2p}}\left(r-\frac{s}{2^{2p}}\right)\right|^2=\sum_{r\in\mathbb{Z}}\left|w_{\overline{\overline{n+2^{2p}}}}\left(r-\frac{2s}{2^{2p+1}}\right)\right|^2$$
$$=2^{2p+1}\mu_{2s}\left(\left[\frac{\overline{\overline{n+2^{2p}}}}{2^{2p+1}},\frac{\overline{\overline{n+2^{2p}}}+1}{2^{2p+1}}\right)\right)$$
$$=2^{2p+1}\mu_{2s}\left(\left[\frac{n+2^{2p}}{2^{2p+1}},\frac{n+2^{2p}+1}{2^{2p+1}}\right)\right).$$

Now we note that the interval

$$I=\left[\frac{n+2^{2p}}{2^{2p+1}},\frac{n+2^{2p}+1}{2^{2p+1}}\right)=\frac{1}{2}\left(\left[\frac{n}{2^{2p}},\frac{n+1}{2^{2p}}\right)+1\right)=\frac{1}{2}(\Omega_{n,p}+1),$$

so by Theorem 5.1,

$$\mu_{2s}\left(\left[\frac{n+2^{2p}}{2^{2p+1}},\frac{n+2^{2p}+1}{2^{2p+1}}\right)\right)=\mu_{2s}\left(\frac{\Omega_{n,p}+1}{2}\right)=\mu_{2s-1}\left(\frac{\Omega_{n,p}}{2}\right).$$

Therefore, substituting in equality (13), we obtain

$$1+4a=\sum_{(n,-2p)\in E_0\ p\in\mathbb{N}^*}\sum_{s=0}^{2^{2p}-1}\frac{1}{2^{2p-1}}\mu_{2s-1}\left(\frac{\Omega_{n,p}}{2}\right).$$

Finally we add term by term the two equalities, thus obtaining, by properties (10) of μ_s, since for any fixed p the number of elements $(n,-2p)\in E_0$ is 2^p,

$$\begin{aligned}
2+8a &= \sum_{(n,-2p)\in E_0,\, p\in\mathbb{N}^*} \frac{1}{2^{2p-1}} \sum_{s=0}^{2^{2p}-1} \left\{\mu_{2s}\left(\frac{\Omega_{n,p}}{2}\right)+\mu_{2s-1}\left(\frac{\Omega_{n,p}}{2}\right)\right\} \\
&= \sum_{(n,-2p)\in E_0,\, p\in\mathbb{N}^*} \frac{1}{2^{2p-1}} \sum_{s=0}^{2^{2p+1}-1} \mu_s\left(\frac{\Omega_{n,p}}{2}\right) \\
&= \sum_{(n,-2p)\in E_0,\, p\in\mathbb{N}^*} \frac{1}{2^{2p-1}}\cdot 1 \\
&= \sum_{p=1}^{+\infty} \frac{1}{2^{2p-1}}\cdot 2^p \\
&= 2,
\end{aligned}$$

and the contradiction $a=0$. □

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