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Polynomial Inequalities with an Exponential Weight on $(0, +\infty)$

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Abstract. We consider the weight $u(x) = x^\gamma e^{-x^{-\alpha}-x^\beta}$ with $x \in (0, +\infty)$, $\alpha > 0$, $\beta > 1$ and $\gamma \geq 0$, and prove Remez-, Bernstein–Markoff-, Schur- and Nikolskii-type inequalities for algebraic polynomials with the weight u on $(0, +\infty)$.

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1. Introduction

The aim of this paper is to extend some classical polynomial inequalities to the case of weighted polynomial inequalities on the real semiaxis, considering weights of the form

$$u(x) = x^\gamma e^{-x^{-\alpha}-x^\beta}, \qquad x \in (0, +\infty),\ \alpha > 0,\ \beta > 1,\ \gamma \geq 0. \tag{1.1}$$

Though the weight u can be seen as a combination of a Pollaczeck-type weight $e^{-x^{-\alpha}}$ and a Laguerre-type weight $x^\gamma e^{-x^\beta}$, as we will see, the polynomial inequalities cannot be deduced from previous results in the literature concerning these two weights. Namely, one cannot investigate the problem reducing it to a combination of a Pollaczeck-type case (on $[0, 1]$, say) and a Laguerre-type case (on $[1, +\infty)$).

Nevertheless, we will see that the exponential part of the weight (1.1), i.e.

$$w(x) = e^{-x^{-\alpha}-x^\beta}, \qquad \alpha > 0, \quad \beta > 1, \qquad x \in (0, +\infty). \tag{1.2}$$

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can be reduced to a weight belonging to the class $\mathcal{F}(C^2+)$ defined by Levin and Lubinsky in [7, p. 7]. From the results in [7], we will deduce some estimates for the Mhaskar–Rakhmanov–Saff numbers related to w. Then, we will prove our weighted polynomial inequalities considering the complete weight u.

To be more precise, after showing the restricted range inequality with the weight u on $(0,+\infty)$, we are going to prove the related Bernstein–Markoff, Schur and Nikolskii inequalities. These are, as is well-known, important tools in order to estimate the degree of convergence of the best weighted polynomial approximation.

The paper is structured as follows. In Section 2 we recall some basic facts concerning the weight w and deduce a Remez-type inequality. In Section 3 we state Bernstein–Markoff-, Schur- and Nikolskii-type inequalities, which will be proved in Section 4. Finally, in the Appendix we show how the weight w can be reduced to a weight belonging to the class $\mathcal{F}(C^2+)$.

2. Basic facts and preliminary results

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore $A \sim B$ will mean that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$. By a slight abuse of notation, we will denote by $\|\cdot\|_p$ the quasinorm of the L^p-spaces for $0 < p < 1$, defined in the usual way. Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m.

Let us consider the weight function (1.2). By using a linear transformation this weight can be reduced to a weight belonging to the class $\mathcal{F}(C^2+)$ defined in [7, p. 7] (see the Appendix for further details). We denote by $\varepsilon_t = \varepsilon_t(w)$ and $a_t = a_t(w)$ the Mhaskar–Rakhmanov–Saff numbers related to w, with

$$\lim_{t\to+\infty} \varepsilon_t = 0\,, \qquad \lim_{t\to+\infty} a_t = +\infty\,.$$

From the results in [7], we deduce

$$\varepsilon_t \sim \left(\frac{\sqrt{a_t}}{t}\right)^{\frac{1}{\alpha+1/2}} \tag{2.1}$$

and

$$a_t \sim t^{1/\beta}\,, \tag{2.2}$$

where the constants in "$\sim$" are independent of t.

The importance of these numbers lies, e.g., in the following restricted range inequalities. Letting $0 < p \leq \infty$, for any $P_m \in \mathbb{P}_m$, we have

$$\|P_m\, w\|_p \leq \mathcal{C}\, \|P_m\, w\|_{L^p[\varepsilon_m, a_m]}\,, \tag{2.3}$$

and, with $s > 1$ fixed,

$$\|P_m w\|_{L^p(\mathbb{R}^+\setminus[\varepsilon_{sm},a_{sm}])} \le \mathcal{C}\mathrm{e}^{-Am^\nu} \|P_m w\|_p \,, \tag{2.4}$$

where

$$\nu = \left(1 - \frac{1}{2\beta}\right) \frac{2\alpha}{2\alpha+1} \,, \tag{2.5}$$

and $\mathcal{C}$ is in both cases independent of m and P_m.

From (2.3) and (2.4) we can easily deduce the analogous inequalities with the weight w replaced by u. The following remark will allow us to avoid considering the Mhaskar-Rahmanov-Saff numbers related to u, but only the ones related to w.

Remark 2.1. Let $\gamma \in \mathbb{R}$ and $n = m + \lceil|\gamma|\rceil$. For any $P_m \in \mathbb{P}_m$, with $0 < p \le \infty$, we have

$$\|P_m u\|_p \le \mathcal{C} \|P_m u\|_{L^p[\varepsilon_n,a_n]} \,, \tag{2.6}$$

where $\mathcal{C} \ne \mathcal{C}(m, P_m)$, and $\varepsilon_n = \varepsilon_n(w)$, $a_n = a_n(w)$ are defined by (2.1) and (2.2).

Analogously, from inequality (2.4) we can deduce

$$\|P_m u\|_{L^p(\mathbb{R}^+\setminus[\varepsilon_{sn},a_{sn}])} \le \mathcal{C}\mathrm{e}^{-Am^\nu} \|P_m u\|_p \,, \qquad s > 1 \,, \tag{2.7}$$

where $n \sim m$, $\mathcal{C} \ne \mathcal{C}(m, P_m)$, $A \ne A(m, P_m)$, and ν is defined by (2.5).

3. Main results

Now we are able to state the polynomial inequalities related to the weight (1.1).

The following lemma will be a crucial tool in order to prove our polynomial inequalities. For the rest of the paper, let

$$\varphi(x) = \sqrt{x} \qquad \text{and} \qquad v_\delta(x) = x^\delta \,.$$

Lemma 3.1. *For a sufficiently large m (say $m \ge m_0$), there exists a polynomial $R_{lm} \in \mathbb{P}_{lm}$, with l a fixed integer, such that*

$$R_{lm}(x) \sim w(x) \tag{3.1}$$

and

$$|R'_{lm}(x)|\, \varphi(x) \le \mathcal{C} \frac{m}{\sqrt{a_m}}\, w(x) \tag{3.2}$$

for $x \in [\varepsilon_m, a_m]$, where $\varepsilon_m = \varepsilon_m(w)$ and $a_m = a_m(w)$ are defined by (2.1) *and* (2.2)*. The constants in "$\sim$" and $\mathcal{C}$ are independent of m.*

By Lemma 3.1 and Remark 2.1, using arguments analogous to those in [8, 11, 12] we can reduce the problem of the polynomial inequalities related to the weight u on $(0, +\infty)$, to inequalities on bounded intervals with Jacobi weights.

Theorem 3.2. *Let $0 < p \le \infty$. Then, for any $P_m \in \mathbb{P}_m$, we have*

$$\|P_m' \varphi u\|_p \le \mathcal{C} \frac{m}{\sqrt{a_m}} \|P_m u\|_p \tag{3.3}$$

and

$$\|P_m' u\|_p \le \mathcal{C} \frac{m}{\sqrt{\varepsilon_m a_m}} \|P_m u\|_p , \tag{3.4}$$

where $\mathcal{C} \neq \mathcal{C}(m, P_m)$.

We want to emphasize that the presence of the algebraic factor x^γ in the definition of u allows us to iterate the Bernstein inequality (3.3) as follows

$$\left\| P_m^{(r)} \varphi^r u \right\|_p \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}} \right)^r \|P_m u\|_p ,$$

for $1 \le r \in \mathbb{Z}$.

Also, the factor

$$\frac{m}{\sqrt{\varepsilon_m a_m}} \sim \left(\frac{m}{\sqrt{a_m}} \right)^{\frac{2\alpha+2}{2\alpha+1}} = \left(\frac{m}{\sqrt{a_m}} \right)^{1+\frac{1}{2\alpha+1}} .$$

in the Markoff inequality (3.4) is smaller than that in the analogous inequality (see [10])

$$\|P_m' w_\beta\|_p \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}} \right)^2 \|P_m w_\beta\|_p$$

with the generalized Laguerre weight $w_\beta(x) = \mathrm{e}^{-x^\beta}$ on $(0, +\infty)$. Whereas, the factors of the Bernstein inequalities for the weights u and w_β are the same.

Using standard arguments, the Markoff inequality (3.4) can be deduced from the Bernstein inequality (3.3) and the Schur inequality stated in next theorem.

Theorem 3.3. *Let $0 < p \le \infty$. Then, for any $P_m \in \mathbb{P}_m$, we have*

$$\|P_m u\|_p \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}} \right)^{\frac{\delta}{\alpha+1/2}} \|P_m v_\delta u\|_p$$

with $\mathcal{C} \neq \mathcal{C}(m, P_m)$.

In analogy with the Bernstein and Markoff inequalities, we give two versions of the Nikolskii inequality.

Theorem 3.4. *Let $0 < p < q \le \infty$. Then, for any $P_m \in \mathbb{P}_m$, we get*

$$\left\| P_m \varphi^{\frac{1}{p}-\frac{1}{q}} u \right\|_q \le \mathcal{C} \left(\frac{m}{\sqrt{a_m}} \right)^{\frac{1}{p}-\frac{1}{q}} \|P_m u\|_p ,$$

and

$$\|P_m u\|_q \le \mathcal{C} \left(\frac{m}{\sqrt{\varepsilon_m a_m}} \right)^{\frac{1}{p}-\frac{1}{q}} \|P_m u\|_p \tag{3.5}$$

where $\mathcal{C} \neq \mathcal{C}(m, P_m)$.

In analogy with different weighted polynomial inequalities, the factor $\frac{m}{\sqrt{\varepsilon_m a_m}}$ in the second Nikolskii inequality is the same as that in the Markoff inequality.

4. Proofs

First of all we are going to prove Remark 2.1.

Proof of Remark 2.1. We are going to prove inequality (2.6) only for $\gamma > 0$, since the case $\gamma < 0$ is similar.

For any $P_m \in \mathbb{P}_m$, $0 < p \le \infty$, letting $n = m + \lceil\gamma\rceil$, we can write

$$\begin{aligned}\|P_m\, u\|_p &\le \|P_m\, u\|_{L^p[\varepsilon_n, a_n]} + \|P_m\, u\|_{L^p[0,\varepsilon_n]} + \|P_m\, u\|_{L^p[a_n,+\infty)}\\ &=: I_1 + I_2 + I_3\,.\end{aligned}$$

So, in order to prove (2.6), it suffices to show that I_2 and $I_3 \le \mathcal{C} I_1$.

For I_2, since $v_\gamma(x) = x^\gamma$ is monotone increasing, using (2.3), we have

$$\begin{aligned}I_2 = \|P_m\, u\|_{L^p[0,\varepsilon_n]} &= \|P_m v_\gamma w\|_{L^p[0,\varepsilon_n]}\\ &\le \varepsilon_n^\gamma \|P_m w\|_{L^p[0,\varepsilon_n]}\\ &\le \mathcal{C}\varepsilon_n^\gamma \|P_m w\|_{L^p[\varepsilon_n,a_n]}\\ &\le \mathcal{C}\, \|P_m v_\gamma w\|_{L^p[\varepsilon_n,a_n]}\,.\end{aligned}$$

Analogously, for I_3, since $x^{\gamma-\lceil\gamma\rceil}$ is monotone decreasing, using again (2.3), we get

$$\begin{aligned}I_3 = \|P_m\, u\|_{L^p[a_n,+\infty)} &= \|P_m v_\gamma w\|_{L^p[a_n,+\infty)}\\ &\le a_n^{\gamma-\lceil\gamma\rceil} \left\|P_m v_{\lceil\gamma\rceil} w\right\|_{L^p[a_n,+\infty)}\\ &\le \mathcal{C} a_n^{\gamma-\lceil\gamma\rceil} \left\|P_m v_{\lceil\gamma\rceil} w\right\|_{L^p[\varepsilon_n,a_n]}\\ &\le \mathcal{C}\, \|P_m v_\gamma w\|_{L^p[\varepsilon_n,a_n]}\,.\end{aligned}$$

We omit the proof of inequality (2.7), which is analogous to the first part of this proof. □

Now, we prove Lemma 3.1, dividing the proof into four steps. To be more precise, in the second and the third steps, we are going to show that for the two factors of the weight w, i.e. $\mathrm{e}^{-x^{-\alpha}}$ and e^{-x^β}, there exist polynomials satisfying inequalities of the form (3.1) and (3.2). To this aim, we will use the Lagrange interpolation based on Chebyshev zeros. So, in the first step, we recall the formula of the error of this interpolation process for analytic functions. Finally, in the fourth step, we will complete the proof of Lemma 3.1.

Proof of Lemma 3.1 *– first step.* Let us consider an interval of the form $[\varepsilon, a]$, where $\varepsilon > 0$ and $a > 2\varepsilon$. Let σ be a function, defined in the complex plane, which is analytic inside an ellipse Γ with foci at ε, a and semiaxes

$$\frac{a-\varepsilon}{4}\left(\rho \pm \frac{1}{\rho}\right), \qquad \rho > 1\,,$$

intersecting the real line at $\varepsilon/2$ and $a+\varepsilon/2$, whence

$$\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)=1+\frac{\varepsilon}{a-\varepsilon}\,. \tag{4.1}$$

It follows that

$$\begin{aligned}\rho &= 1+\frac{\varepsilon}{a-\varepsilon}+\sqrt{\frac{2\varepsilon}{a-\varepsilon}+\frac{\varepsilon^2}{(a-\varepsilon)^2}}\\ &> 1+\frac{\varepsilon}{a}+\sqrt{\frac{\varepsilon}{a}}\,. \end{aligned}\tag{4.2}$$

Now, let P be the Lagrange polynomial interpolating σ at the zeros of $T_{lm}\left(\psi(x)\right)$, where $T_{lm}\left(\psi(x)\right)$ is the Chebyshev polynomial in $[\varepsilon,a]$, $\psi:[\varepsilon,a]\to[-1,1]$ is a linear transformation and l is a fixed integer. For $x\in[\varepsilon,a]$ we can write (see for instance [9, p. 55] or [1, p. 124])

$$\begin{aligned}|\sigma(x)-P(x)| &= \left|\frac{1}{2\pi\mathrm{i}}\int_\Gamma \frac{\sigma(\zeta)T_{lm}\left(\psi(x)\right)}{(\zeta-x)T_{lm}\left(\psi(\zeta)\right)}\mathrm{d}\zeta\right|\\ &\le \frac{1}{2\pi}\frac{\operatorname{length}(\Gamma)\max_{\zeta\in\Gamma}|\sigma(\zeta)|}{\min_{\zeta\in\Gamma}|\zeta-x|\min_{\zeta\in\Gamma}|T_{lm}\left(\psi(\zeta)\right)|}\\ &\le \left[\frac{\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)}{\frac{1}{2}\left(\rho+\frac{1}{\rho}\right)-1}\right]\frac{\max_{\zeta\in\Gamma}|\sigma(\zeta)|}{\frac{1}{2}\left(\rho^{lm}-\frac{1}{\rho^{lm}}\right)}\,.\end{aligned}$$

Whence, using (4.1) and (4.2), we obtain

$$\begin{aligned}|\sigma(x)-P(x)| &\le \frac{4a}{\varepsilon}\max_{\zeta\in\Gamma}|\sigma(\zeta)|\ \rho^{-lm}\\ &\le \frac{4a}{\varepsilon}\max_{\zeta\in\Gamma}|\sigma(\zeta)|\ \mathrm{e}^{-lm\log\left(1+\frac{\varepsilon}{a}+\sqrt{\frac{\varepsilon}{a}}\right)}\\ &\le \frac{4a}{\varepsilon}\max_{\zeta\in\Gamma}|\sigma(\zeta)|\ \mathrm{e}^{-lm\sqrt{\frac{\varepsilon}{a}}}\,. \end{aligned}\tag{4.3}$$

□

Proof of Lemma 3.1 – *second step.* Let us consider the function $\sigma(x)=\mathrm{e}^{-x^{-\alpha}}$ with $\alpha>0$. We are going to use (4.3) in order to prove that there exists a polynomial $P_1\in\mathbb{P}_{lm}$, with $m\ge m_0$ and l fixed, such that

$$P_1(x)\sim \mathrm{e}^{-x^{-\alpha}} \tag{4.4}$$

and

$$|P_1'(x)|\sqrt{a_m x}\le \mathcal{C}m\,\mathrm{e}^{-x^{-\alpha}} \tag{4.5}$$

for $x\in[\varepsilon_m,a_m]$, where the constants in "$\sim$" and $\mathcal{C}$ are independent of m,σ,P_1.

Let us first show that (4.4) holds in $\left[\frac{1}{2}\varepsilon_m,2a_m\right]\supset[\varepsilon_m,a_m]$. To this aim it suffices to prove that

$$|\sigma(x)-P_1(x)|\le\frac{1}{2}\sigma\left(\frac{1}{2}\varepsilon_m\right),\qquad x\in\left[\frac{1}{2}\varepsilon_m,2a_m\right]. \tag{4.6}$$

In fact (4.6) implies

$$\frac{1}{2}\,\sigma(x) \le P_1(x) \le \frac{3}{2}\,\sigma(x)\,,$$

i.e. (4.4) for $x \in \left[\frac{1}{2}\,\varepsilon_m, 2a_m\right]$.

Note that the function $\sigma(z)$, with $z^{-\alpha} = \mathrm{e}^{-\alpha \mathrm{Log}(z)}$ and Log the principal value of the logarithm, is holomorphic for $\Re z > 0$. Hence $\sigma(z)$ is analytic inside the ellipse Γ defined as in the previous step, with $\varepsilon = \frac{1}{2}\,\varepsilon_m$ and $a = 2a_m$. Thus, letting P_1 be the Lagrange polynomial interpolating $\sigma(x)$ at the Chebyshev zeros in $[\frac{1}{2}\,\varepsilon_m, 2a_m]$, by (4.3), we have

$$\begin{aligned}|\sigma(x) - P_1(x)| &\le \frac{16a_m}{\varepsilon_m} \max_{\zeta\in\Gamma} |\sigma(\zeta)| \mathrm{e}^{-lm\sqrt{\frac{\varepsilon_m}{4a_m}}} \\ &\le \frac{16a_m}{\varepsilon_m} \max_{\zeta\in\Gamma} \mathrm{e}^{|\zeta|^{-\alpha}} \mathrm{e}^{-lm\sqrt{\frac{\varepsilon_m}{4a_m}}} \\ &\le \frac{16a_m}{\varepsilon_m} \mathrm{e}^{4^\alpha \varepsilon_m^{-\alpha} - lm\sqrt{\frac{\varepsilon_m}{4a_m}}} \\ &\le \frac{1}{2}\mathrm{e}^{-2^\alpha \varepsilon_m^{-\alpha}}\,,\end{aligned}$$

choosing

$$l \ge \frac{\log\left(\frac{32a_m}{\varepsilon_m}\right) + (4^\alpha + 2^\alpha)\varepsilon_m^{-\alpha}}{m\sqrt{\frac{\varepsilon_m}{4a_m}}} \sim 1 + \varepsilon_m^\alpha \log m\,,$$

by (2.1) and (2.2). Note that the quantity on the right-hand side is $\sim 1 + o(1)$ as $m \to \infty$ and then (4.4) follows choosing l sufficiently large but fixed.

Let us now prove that inequality (4.5) holds in $[\varepsilon_m, a_m]$. We can write

$$\begin{aligned}|P_1'(x)\sqrt{a_m x}| &\le |[P_1'(x) - \sigma'(x)]\sqrt{a_m x}| + |\sigma'(x)|\sqrt{a_m x} \\ &=: A_1 + A_2\,.\end{aligned} \tag{4.7}$$

In order to estimate the term A_1, we are going to use arguments similar to the first part of this step, using the formula

$$\begin{aligned}|P_1'(x) - \sigma'(x)|\sqrt{a_m x} \le{}& \left|\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\sigma(\zeta) T_{lm}\left(\psi(x)\right)}{(\zeta - x)^2 T_{lm}\left(\psi(\zeta)\right)} \mathrm{d}\zeta\right| \sqrt{a_m x} \\ &+ \left|\frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\sigma(\zeta) \frac{\mathrm{d}}{\mathrm{d}x} T_{lm}\left(\psi(x)\right)}{(\zeta - x) T_{lm}\left(\psi(\zeta)\right)} \mathrm{d}\zeta\right| \sqrt{a_m x}\,.\end{aligned}$$

Both terms on the right-hand side can be handled in analogy with the first part of this step. For the second term, with $\psi(x) = (4x - 4a_m - \varepsilon_m)/(4a_m - \varepsilon_m)$ and $x \in [\varepsilon_m, a_m]$, we observe that

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}x} T_{lm}\left(\psi(x)\right)\right| \sqrt{a_m x} &\le m \frac{\psi'(x)}{\sqrt{1 - \left(\psi(x)\right)^2}} \sqrt{a_m x} \\ &\le \mathcal{C} m \frac{\sqrt{a_m x}}{\sqrt{(2a_m - x)(x - \varepsilon_m/2)}} \le \mathcal{C} m\,,\end{aligned}$$

since

$$\sqrt{a_m x} \sim \sqrt{(2a_m - x)(x - \varepsilon_m/2)}\,, \qquad x \in [\varepsilon_m, a_m]\,. \tag{4.8}$$

In fact, the last relation follows from

$$1 \le \frac{2a_m - x}{a_m} \le 2$$

and

$$1 \le \frac{x}{x - \varepsilon_m/2} = 1 + \frac{\varepsilon_m}{2x - \varepsilon_m} \le 2\,.$$

Then, proceeding as in the first part of this step, we get

$$A_1 = |P_1'(x) - \sigma'(x)|\sqrt{a_m x} \le \mathcal{C} m \mathrm{e}^{-x^{-\alpha}}\,. \tag{4.9}$$

Now, consider the term A_2 in (4.7). Since $x^{-\alpha-1/2}$ is monotone decreasing and $\varepsilon_m^{-\alpha-1/2} \sim m/\sqrt{a_m}$, by (2.1), we have

$$\begin{aligned} A_2 = |\sigma'(x)|\sqrt{a_m x} &= \alpha x^{-\alpha-1/2}\sqrt{a_m}\sigma(x) \\ &\le \mathcal{C}\varepsilon_m^{-\alpha-1/2}\sqrt{a_m}\sigma(x) \\ &\le \mathcal{C}\frac{m}{\sqrt{a_m}}\sqrt{a_m}\sigma(x)\,. \end{aligned} \tag{4.10}$$

Combining (4.9) and (4.10) in (4.7), inequality (4.5) follows. □

Proof of Lemma 3.1 – *third step.* From [5, 6] (see also [2, 3, 4]) we deduce that there exists $P_2 \in \mathbb{P}_{lm}$, with $m \ge m_0$ and l fixed, such that

$$P_2(x) \sim \mathrm{e}^{-x^\beta}$$

and

$$|P_2'(x)|\,\sqrt{a_m x} \le \mathcal{C} m \mathrm{e}^{-x^\beta}$$

for $x \in [a_m/m^2, a_m] \supset [\varepsilon_m, a_m]$, where the constants in "$\sim$" and $\mathcal{C}$ are independent of m. □

Proof of Lemma 3.1 – *fourth step.* Combining the results in the second and the third step, the polynomial $P_1 \cdot P_2$ satisfies (3.1) and (3.2), since

$$P_1(x)P_2(x) \sim w(x)$$

and

$$\begin{aligned} |(P_1P_2)'(x)|\,\sqrt{a_m x} &\le P_1(x)\,|P_2'(x)|\,\sqrt{a_m x} + |P_1'(x)|\,P_2(x)\sqrt{a_m x} \\ &\le \mathcal{C} m w(x) \end{aligned}$$

for $x \in [\varepsilon_m, a_m]$. Hence Lemma 3.1 is completely proved. □

Proof of Theorem 3.2. For any $P_m \in \mathbb{P}_m$, by Remark 2.1 and Lemma 3.1, we get

$$\begin{aligned}\|P_m' \varphi v_\gamma w\|_p &\leq \mathcal{C} \|P_m' \varphi v_\gamma w\|_{L^p[\varepsilon_n, a_n]} \\ &\leq \mathcal{C} \|P_m' \varphi v_\gamma R_{lm}\|_{L^p[\varepsilon_n, a_n]} \\ &\leq \mathcal{C} \left\|(P_m R_{lm})' \varphi v_\gamma\right\|_{L^p[\varepsilon_n, a_n]} + \mathcal{C} \|P_m R_{lm}' \varphi v_\gamma\|_{L^p[\varepsilon_n, a_n]} \\ &\leq \mathcal{C} \left\|(P_m R_{lm})' \varphi v_\gamma\right\|_{L^p[\varepsilon_n, a_n]} + \mathcal{C} \frac{m}{\sqrt{a_m}} \|P_m v_\gamma w\|_p ,\end{aligned}$$

where $n = m - 1 + \lceil \gamma + 1/2 \rceil$.

Now, in order to prove that the first term at the right-hand side is bounded by the second term, we want to use the known Bernstein inequality related to Jacobi weights on the bounded interval $[\varepsilon_{2n}, a_{2n}]$. To this aim we observe that, for $x \in [\varepsilon_n, a_n]$, the relations

$$\sqrt{(x - \varepsilon_{2n})(a_{2n} - x)} \sim \sqrt{a_m} \varphi(x)$$

and

$$(x - \varepsilon_{2n})^\gamma \sim v_\gamma(x)$$

hold, applying the same arguments as in (4.8), by (2.2) and (2.1). Hence, we can use the Bernstein inequality related to the interval $[\varepsilon_{2n}, a_{2n}]$ with the Jacobi weight $(x - \varepsilon_{2n})^\gamma$, obtaining

$$\begin{aligned}&\left\|(P_m R_{lm})' \varphi v_\gamma\right\|_{L^p[\varepsilon_n, a_n]} \\ &\leq \frac{\mathcal{C}}{\sqrt{a_m}} \left\|(P_m R_{lm})' \sqrt{(\cdot - \varepsilon_{2n})(a_{2n} - \cdot)} (\cdot - \varepsilon_{2n})^\gamma\right\|_{L^p[\varepsilon_n, a_n]} \\ &\leq \frac{\mathcal{C}}{\sqrt{a_m}} \left\|(P_m R_{lm})' \sqrt{(\cdot - \varepsilon_{2n})(a_{2n} - \cdot)} (\cdot - \varepsilon_{2n})^\gamma\right\|_{L^p[\varepsilon_{2n}, a_{2n}]} \\ &\leq \mathcal{C} \frac{m}{\sqrt{a_m}} \|P_m R_{lm} (\cdot - \varepsilon_{2n})^\gamma\|_{L^p[\varepsilon_{2n}, a_{2n}]} \\ &\leq \mathcal{C} \frac{m}{\sqrt{a_m}} \|P_m v_\gamma w\|_p ,\end{aligned}$$

by Lemma 3.1 and since $(x - \varepsilon_{2n})^\gamma \leq x^\gamma$. □

The Markoff inequality (3.4) in Theorem 3.2 can be deduced from (3.3) and Theorem 3.3. Therefore, we are going to prove Theorem 3.3, omitting the proof of (3.4).

Proof of Theorem 3.3. For any $P_m \in \mathbb{P}_m$, letting $n = m + \lceil \gamma \rceil$, by Remark 2.1, we have

$$\begin{aligned}\|P_m u\|_p &\leq \mathcal{C} \|P_m u\|_{L^p[\varepsilon_n, a_n]} \\ &\leq \mathcal{C} \varepsilon_n^{-\delta} \|P_m v_\delta u\|_{L^p[\varepsilon_n, a_n]} ,\end{aligned}$$

whence our claim follows, taking into account (2.1). □

Proof of Theorem 3.4. Let $P_m \in \mathbb{P}_m$. By Remark 2.1 and Lemma 3.1 we have

$$\left\| P_m \, \varphi^{\frac{1}{p}-\frac{1}{q}} v_\gamma \, w \right\|_q \le \mathcal{C} \left\| P_m \, \varphi^{\frac{1}{p}-\frac{1}{q}} v_\gamma \, w \right\|_{L^q[\varepsilon_n, a_n]} \le \mathcal{C} \left\| P_m \, \varphi^{\frac{1}{p}-\frac{1}{q}} v_\gamma \, R_{lm} \right\|_{L^q[\varepsilon_n, a_n]},$$

where $n = m + \lceil \gamma + (1/p - 1/q)/2 \rceil$.

In analogy with the proof of Theorem 3.2, we are going to apply the Nikolskii inequality related to Jacobi weights on the bounded interval $[\varepsilon_{2n}, a_{2n}]$. Since, for $x \in [\varepsilon_n, a_n]$, we have

$$\sqrt{(x-\varepsilon_{2n})(a_{2n}-x)} \sim \sqrt{a_m}\varphi(x)$$

and

$$(x-\varepsilon_{2n})^\gamma \sim v_\gamma(x)\,,$$

using the Nikolskii inequality with the weight $(x-\varepsilon_{2n})^\gamma$, and by Lemma 3.1, we obtain

$$\begin{aligned}
&\left\| P_m R_{lm} \, \varphi^{\frac{1}{p}-\frac{1}{q}} v_\gamma \right\|_{L^q[\varepsilon_n, a_n]} \\
&\le \frac{\mathcal{C}}{\left(\sqrt{a_m}\right)^{\frac{1}{p}-\frac{1}{q}}} \left\| P_m R_{lm} \left(\sqrt{(\cdot - \varepsilon_{2n})(a_{2n} - \cdot)} \right)^{\frac{1}{p}-\frac{1}{q}} (\cdot - \varepsilon_{2n})^\gamma \right\|_{L^q[\varepsilon_n, a_n]} \\
&\le \frac{\mathcal{C}}{\left(\sqrt{a_m}\right)^{\frac{1}{p}-\frac{1}{q}}} \left\| P_m R_{lm} \left(\sqrt{(\cdot - \varepsilon_{2n})(a_{2n} - \cdot)} \right)^{\frac{1}{p}-\frac{1}{q}} (\cdot - \varepsilon_{2n})^\gamma \right\|_{L^q[\varepsilon_{2n}, a_{2n}]} \\
&\le \mathcal{C} \left(\frac{m}{\sqrt{a_m}} \right)^{\frac{1}{p}-\frac{1}{q}} \| P_m R_{lm} (\cdot - \varepsilon_{2n})^\gamma \|_{L^p[\varepsilon_{2n}, a_{2n}]} \\
&\le \mathcal{C} \left(\frac{m}{\sqrt{a_m}} \right)^{\frac{1}{p}-\frac{1}{q}} \| P_m \, v_\gamma w \|_p \,,
\end{aligned}$$

which completes the proof. □

We omit the proof of (3.5) which follows from Theorems 3.4 and 3.3.

5. Appendix

In this section we show that the weight w defined in (1.2) can be reduced to a weight belonging to the class $\mathcal{F}(C^2+)$ in [7].

First of all, for the reader's convenience, we recall the definition of the class $\mathcal{F}(C^2+)$, given by Levin and Lubinsky in [7, pp. 7–8] .

Let $I = (c, d)$ be an interval, with $-\infty \le c < 0 < d \le +\infty$, and $\varrho : I \in \mathbb{R}$ be a weight function, with $\varrho = \mathrm{e}^{-Q}$, $Q : I \in [0, +\infty)$, satisfying the following properties:

(i) Q' is continuous in I and $Q(0) = 0$;
(ii) Q'' exists and is positive in $I \setminus \{0\}$;
(iii) $\lim_{x \to c^+} Q(x) = \lim_{x \to d^-} Q(x) = \infty$;

(iv) the function

$$T(x) = \frac{xQ'(x)}{Q(x)}\,, \qquad x \in I \setminus \{0\}\,,$$

is quasi-decreasing in $(c, 0)$ and quasi-increasing in $(0, d)$, with

$$T(x) \geq \Lambda > 1\,, \qquad x \in I \setminus \{0\}\,;$$

(v) there exist $C_1, C_2 > 0$ and a compact subinterval $J \subseteq I$, such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}\,, \qquad \text{a.e.} \quad x \in I \setminus \{0\}\,,$$

and

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}\,, \qquad \text{a.e.} \quad x \in I \setminus J\,.$$

Then, following Levin and Lubinsky, we say $\varrho \in \mathcal{F}(C^2+)$.

The related Mhaskar–Rahmanov–Saff numbers $\bar{a}_{-t} = \bar{a}_{-t}(\varrho)$ and $\bar{a}_t = \bar{a}_t(\varrho)$, with $\bar{a}_{-t} < 0 < \bar{a}_t$ and $t > 0$, are uniquely defined by the equations (see [7, p. 13])

$$t = \frac{1}{\pi} \int_{\bar{a}_{-t}}^{\bar{a}_t} \frac{xQ'(x)}{\sqrt{(x - \bar{a}_{-t})(\bar{a}_t - x)}}\,\mathrm{d}x \tag{5.1}$$

and

$$0 = \frac{1}{\pi} \int_{\bar{a}_{-t}}^{\bar{a}_t} \frac{Q'(x)}{\sqrt{(x - \bar{a}_{-t})(\bar{a}_t - x)}}\,\mathrm{d}x\,. \tag{5.2}$$

Moreover, $\bar{a}_t$ is an increasing function of $t \in \mathbb{R}$, with

$$\lim_{t \to -\infty} \bar{a}_t = c\,, \qquad \lim_{t \to +\infty} \bar{a}_t = d\,.$$

In particular, if $\bar{a}_t \geq |\bar{a}_{-t}|$ for t sufficiently large, the relations (see [7, p. 27])

$$Q(\bar{a}_t) \sim \frac{t}{\sqrt{T(\bar{a}_t)}} \tag{5.3}$$

and

$$Q(\bar{a}_{-t}) \sim \frac{t}{\sqrt{T(\bar{a}_{-t})}} \sqrt{\frac{|\bar{a}_{-t}|}{\bar{a}_t}} \tag{5.4}$$

lead to approximations of $\bar{a}_{\pm t}$.

Now, let us consider the weight w defined by (1.2), i.e.

$$w(x) = \mathrm{e}^{-x^{-\alpha} - x^{\beta}}\,, \qquad \alpha > 0\,,\ \beta > 1\,, \qquad x \in (0, +\infty)\,.$$

By using a linear transformation this weight can be reduced to

$$\widetilde{w}(y) = \mathrm{e}^{-Q(y)}\,, \qquad y \in (-\lambda, +\infty)\,, \tag{5.5}$$

where

$$Q(y) = \frac{1}{(\lambda + y)^{\alpha}} + (\lambda + y)^{\beta} - \lambda^{-\alpha} - \lambda^{\beta}\,, \qquad \lambda := \left(\frac{\alpha}{\beta}\right)^{\frac{1}{\alpha+\beta}}\,, \tag{5.6}$$

with α, β as above.

Proposition 5.1. *The weight* $\widetilde{w}$ *in* (5.5) *belongs to the class* $\mathcal{F}(C^2+)$.

Proof. With Q given by (5.6), it is easily seen that

$$Q(0)=0\,, \qquad \lim_{y\to-\lambda^+} Q(y)=\lim_{y\to+\infty} Q(y)=\infty\,.$$

Concerning Q', we have

$$Q'(y)=\frac{\beta(\lambda+y)^{\alpha+\beta}-\alpha}{(\lambda+y)^{\alpha+1}}\,.$$

Hence Q' is continuous in $(-\lambda,+\infty)$, and $\operatorname{sgn} Q'(y)=\operatorname{sgn} y$.

Moreover, $Q'(0)=0$ and so $Q:(-\lambda,+\infty)\to[0,+\infty)$ is monotone decreasing in $(-\lambda,0]$, and monotone increasing in $[0,+\infty)$. Furthermore,

$$Q''(y)=\frac{\alpha(\alpha+1)+\beta(\beta-1)(\lambda+y)^{\alpha+\beta}}{(\lambda+y)^{\alpha+2}}>0, \qquad -\lambda<y<+\infty.$$

Then, properties (i), (ii) and (iii) are fulfilled.

Now, let us show that the function

$$T(y):=\frac{yQ'(y)}{Q(y)}\,, \qquad -\lambda<y<+\infty\,,$$

fulfills the first part of property (iv), namely that $T(y)$ is quasi-monotone in $(-\lambda,0)$ and $(0,+\infty)$. By definition we have

$$T(y)=\frac{1}{\lambda+y}\cdot\frac{y[\beta(\lambda+y)^{\alpha+\beta}-\alpha]}{1+(\lambda+y)^{\alpha+\beta}-(\lambda^{-\alpha}+\lambda^{\beta})(\lambda+y)^{\alpha}}\,.$$

Let us consider the case $-\lambda<y\le 0$. We set

$$\tilde T(y):=\frac{y[\beta(\lambda+y)^{\alpha+\beta}-\alpha]}{1+(\lambda+y)^{\alpha+\beta}-(\lambda^{-\alpha}+\lambda^{\beta})(\lambda+y)^{\alpha}}\,.$$

This is a non-negative continuous function in the interval $(-\lambda,0)$, and

$$\tilde T(-\lambda)=\alpha\lambda>0\,.$$

Moreover, by using l'Hospital rule twice, we get

$$\begin{aligned}
&\lim_{y\to0}\tilde T(y)\\
&=\lim_{y\to0}\frac{2\beta(\alpha+\beta)(\lambda+y)^{\alpha+\beta-1}+y\beta(\alpha+\beta)(\alpha+\beta-1)(\lambda+y)^{\alpha+\beta-2}}{(\alpha+\beta)(\alpha+\beta-1)(\lambda+y)^{\alpha+\beta-2}-\alpha(\alpha-1)(\lambda^{-\alpha}+\lambda^{\beta})(\lambda+y)^{\alpha-2}}\\
&=\frac{2\beta(\alpha+\beta)\lambda^{\alpha+\beta+1}}{(\alpha+\beta)(\alpha+\beta-1)\lambda^{\alpha+\beta}-\alpha(\alpha-1)(\lambda^{-\alpha}+\lambda^{\beta})\lambda^{\alpha}}\\
&=\frac{2\alpha(\alpha+\beta)\lambda}{(\alpha+\beta)(\alpha+\beta-1)\frac{\alpha}{\beta}-\alpha(\alpha-1)(1+\frac{\alpha}{\beta})}\\
&=2\lambda>0.
\end{aligned}$$

It follows that $0<c_1\le\tilde T(y)\le c_2<\infty$ in $(-\lambda,0)$, i.e.

$$\frac{c_1}{\lambda+y}\le T(y)\le\frac{c_2}{\lambda+y}\,, \qquad -\lambda<y<0\,.$$

Hence $T(y)$ is quasi-monotone decreasing in $(-\lambda,0)$.

Let us now consider the case $0 < y < +\infty$. Since T is continuous on $(0,+\infty)$ and

$$T(0) = \frac{\lim_{y\to 0} \tilde{T}(y)}{\lambda} = \frac{2\lambda}{\lambda} = 2$$

and

$$\lim_{y\to+\infty} T(y) = \beta\,,$$

we get

$$0 < c_3 \le T(y) \le c_4 < \infty, \qquad 0 \le y < +\infty\,,$$

which implies that $T(y)$ is quasi-monotone increasing (or decreasing) in $(0,+\infty)$.

To show that the function

$$T(y) := \frac{yQ'(y)}{Q(y)}\,, \qquad -\lambda < y < +\infty\,,$$

fulfills the second part of property (iv), we consider

$$f(y) := yQ'(y) - Q(y)\,.$$

Since $f(0) = 0$ and, by property (ii),

$$\operatorname{sgn} f'(y) = \operatorname{sgn}(yQ''(y)) = \operatorname{sgn} y\,,$$

f is strictly decreasing in $(-\lambda, 0)$ and strictly increasing in $(0,+\infty)$. Thus $f(y) > 0$ if $y \in (-\lambda,+\infty)\setminus\{0\}$, which implies

$$T(y) > 1\,, \qquad y \in (-\lambda,+\infty)\setminus\{0\}\,.$$

Moreover, we have already shown that $\lim_{y\to0} T(y) = 2$ whence

$$T(y) \ge \Lambda > 1\,, \qquad y \in (-\lambda,+\infty)\,.$$

Finally, we prove property (v), namely

$$0 < \frac{Q(y)Q''(y)}{Q'(y)^2} \le c_5 < \infty\,, \qquad -\lambda \le y < +\infty\,.$$

Let

$$\begin{aligned} U(y) &:= \frac{Q(y)Q''(y)}{Q'(y)^2} \\ &= \frac{\alpha(\alpha+1) + \beta(\beta-1)(\lambda+y)^{2\alpha+2\beta} + [\alpha(\alpha+1)+\beta(\beta-1)](\lambda+y)^{\alpha+\beta}}{\alpha^2 + \beta^2(\lambda+y)^{2\alpha+2\beta} - 2\alpha\beta(\lambda+y)^{\alpha+\beta}}\,. \end{aligned}$$

Hence

$$U(-\lambda) = 1 + \frac{1}{\alpha}, \quad \lim_{y\to+\infty} U(y) = 1 - \frac{1}{\beta} > 0\,,$$

and using l'Hospital rule,

$$U(0) = Q''(0)\lim_{y\to0}\frac{Q(y)}{Q'(y)^2} = Q''(0)\lim_{y\to0}\frac{Q'(y)}{2Q'(y)Q''(y)} = \frac{1}{2}\,. \qquad \square$$

Since $\widetilde{w}$ is nonsymmetric, so are also its associated Mhaskar–Rakhmanov–Saff numbers, $\widetilde{a}_{-t}$ and $\widetilde{a}_t$, $t > 0$. These numbers are uniquely defined by the equations (5.1)-(5.2) (see [7, p. 13]). Moreover, $\widetilde{a}_t$ is an increasing function of t, with

$$\lim_{t\to+\infty} \widetilde{a}_{-t} = -\lambda\,, \qquad \lim_{t\to+\infty} \widetilde{a}_t = +\infty\,.$$

To give more explicit expressions for a_{-t} and a_t, by (5.3) and (5.4), we get

$$\sqrt{Q(\widetilde{a}_{-t})Q'(\widetilde{a}_{-t})} \sim \frac{t}{\sqrt{\widetilde{a}_t}} \sim \sqrt{Q(\widetilde{a}_t)Q'(\widetilde{a}_t)}\,.$$

Since

$$Q(\widetilde{a}_{-t}) \sim \frac{1}{(\lambda+\widetilde{a}_{-t})^{\alpha}}\,, \qquad Q'(\widetilde{a}_{-t}) \sim \frac{1}{(\lambda+\widetilde{a}_{-t})^{\alpha+1}}\,,$$

and

$$Q(\widetilde{a}_t) \sim \widetilde{a}_t^{\beta}\,, \qquad Q'(\widetilde{a}_t) \sim \widetilde{a}_t^{\beta-1}\,,$$

we obtain

$$\lambda+\widetilde{a}_{-t} \sim \left(\frac{\sqrt{\widetilde{a}_t}}{t}\right)^{\frac{1}{\alpha+1/2}} \tag{5.7}$$

and

$$\widetilde{a}_t \sim t^{1/\beta}\,. \tag{5.8}$$

Coming back to the weight w in (1.2), we denote by $\varepsilon_t = \varepsilon_t(w)$ and $a_t = a_t(w)$ the Mhaskar–Rakhmanov–Saff numbers related to w, with

$$\lim_{t\to+\infty} \varepsilon_t = 0\,, \qquad \lim_{t\to+\infty} a_t = +\infty\,.$$

Then, from (5.7) and (5.8), we deduce

$$\varepsilon_t \sim \left(\frac{\sqrt{a_t}}{t}\right)^{\frac{1}{\alpha+1/2}}$$

and

$$a_t \sim t^{1/\beta}\,.$$

Hence the restricted range inequalities (2.3) and (2.4) can be deduced from Theorem 4.2 in [7, p. 96]. In particular, from the results in [7] we get

$$\|P_m\,\widetilde{w}\|_{L^p((-\lambda,+\infty)\setminus[\widetilde{a}_{-sm},\widetilde{a}_{sm}])} \le \mathcal{C}\mathrm{e}^{-AH(m)}\,\|P_m\,\widetilde{w}\|_p\,,$$

where

$$H(m) = \min\left\{\frac{m}{\sqrt{T(\widetilde{a}_{-m})}}\sqrt{\frac{2|\widetilde{a}_{-m}|}{|\widetilde{a}_{-m}|+\widetilde{a}_m}}\,, \frac{m}{\sqrt{T(\widetilde{a}_m)}}\sqrt{\frac{2\widetilde{a}_m}{|\widetilde{a}_{-m}|+\widetilde{a}_m}}\right\},$$

whence inequality (2.4) follows, taking into account that $T(\widetilde{a}_{-m}) \sim (\lambda + \widetilde{a}_{-m})^{-1}$ and $T(\widetilde{a}_m) \sim 1$.

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