On groups whose subnormal subgroups are inert *

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Abstract. A subgroup H of a group G is said to be inert if $H \cap H^g$ has finite index in both H and H^g for any $g \in G$. We study hyper-(abelian or finite) groups in which subnormal subgroups are inertial.
Keywords: *commensurable, inert, subnormal subgroup.*

1 Introduction

The class $\mathbf{T}$ of groups in which subnormal subgroups are normal and its generalizations received much attention in the literature. In particular classes $\mathbf{T}_*$ and $\mathbf{T}^*$ of groups G in which subnormal subgroups H have property $|H : H_G| < \infty$ and $|H^G : H| < \infty$ (resp.) were studied in [6] and [3] resp. To see those classes in the same framework we consider the class $\mathbf{T}*$ in which subnormal subgroups are *cn*, that is *commensurable to a normal subgroup.* Here two subgroups X and Y of a group are told *commensurable* iff the index of $X \cap Y$ in both X and Y is finite. Commensurability is an equivalence relation that we denote by $\sim$. Clarly *cn*-subgroups are *inert*, that is *commensurable to their conjugates.* Thus we consider the *class* $\tilde{\mathbf{T}}$ *of groups whose subnormal subgroups are inert.*

A group whose all subgroups are inert is said *totally inertial* (or just *inertial*) . Such groups also received attention in the context of locally finite groups (see [1] for example). In the context of generalized soluble groups, a description of a class of groups whose subgroups are strongly inertial is

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given in [5] while a characterization of inertial groups with some finiteness conditions was given by D. Robinson in [8].

Theorem A of [8] states that *if G is a hyper-(abelian or finite) inertial group, then it is abelian or dihedral, provided it has no non-trivial periodic normal subgroups.* Here we obtain a similar characterization of $\tilde{\mathbf{T}}$-groups by substituting the class of dihedral by that of *semi-dihedral* groups, see Theorem $\tilde{A}$.

In Theorem B of [8] it is shown that a finitely generated hyper-(abelian or finite) *group is inertial iff it has a finite index torsion-free normal subgroup in which elements of G induce power automorphisms.* In our Theorem $\tilde{B}$ we show that a corresponding statement holds for $\tilde{\mathbf{T}}$-groups.

For terminology, notation and basic facts we refer to [8] and [7]

2 Main results

Let us give examples of $\tilde{\mathbf{T}}$-groups.

Lemma 1 *If $G_1 \lhd G_0 \lhd G$, with G_1 and G/G_0 finite, and if any subnormal subgroup of G_0/G_1 is inert (resp. cn) in G/G_1, then G is $\tilde{\mathbf{T}}$(resp. $\mathbf{T}*$).*

An automorphism of a group is said to be *inertial automorphism* iff it maps subgroups to commensurable subgroups (setwise). Inertial automorphisms fill a subgroup $\mathcal{I}\mathrm{Aut}(G)$ of $\mathrm{Aut}(G)$. Inertial automorphisms of abelian groups A have been studied in [4]. In the case A a torsion-free abelian group they are *rational power automorphisms*, according to the following definition. Recall that if $0 \neq n, m \in \mathbb{Z}$, for each $a \in A$, there is at most one $b \in A$ such that

$$(*) \qquad b^n = a^m$$

If such a unique b exists, we write $b = a^{\frac{m}{n}}$. Thus if m, n are coprime and $A = A^m = A^n$, then $(*)$ defines an automorphism $a \mapsto b$ that we call the rational power automorphism $q = \frac{m}{n}$. It is *inertial iff either $\frac{m}{n} = \pm 1$ or A has finite rank.* Therefore, when A is torsion-free abelian, $\mathcal{I}\mathrm{Aut}(A)$ is isomorphic to a subgroup of $\mathbb{Q}^*$, the multiplicative group of the rationals.

We introduce now a class of $\mathbf{S}\tilde{\mathbf{T}}$-groups.

Definition *A group G is said to be* semidihedral *on a torsion-free abelian subgroup A if $G = A \rtimes K$ and K acts faithfully on A by means of inertial automorphims.*

Thus elements of K induce *rational power* automorphisms. If all elements of K induce power automorphism, then G is abelian or dihedral, and it is an inertial group. Otherwhise, by Theorem 2 of [4], *A has finite rank.*

Proposition *Let A be a torsion-free abelian normal subgroup of a group G.*
i) G is semidihedral on A iff $A = C_G(A)$ and G acts on A by means of inertial automorphisms. In this case $A = \mathrm{Fit}(G)$
ii) If G is semidihedral on A and G_0 is a non-abelian subgroup of finite index in G, then G_0 is semidihedral on $A_0 = A \cap G_0$.

Proof. (i) Assume G semidihedral on A. Since non-trivial inertial automorphisms of A are fixed-point-free, then $A = C_G(A) = Fit(G)$. Conversely, if $A = C_G(A)$, for any $x \in G \setminus A$, the subgroup $N := \langle x, A\rangle = A \rtimes \langle x\rangle$ has trivial centre. By 11.4.21 of [7], up to equivalence there exists a unique extension of N by $Q = G/N$ with coupling the natural omomorphism $Q \to OutN$, and so G is isomorphic to the subgroup $A \rtimes G/A$ of the holomorph of A.

(ii) Every element of $G_0 \setminus A_0$ acts fixed-point-free on A_0 and so $C_{G_0}(A_0) = A_0$. Hence G_0 is semidihedral on A_0 by (i). □

Denote by $\mathbf{S\tilde{T}}$ the class of hyper-(abelian or finite) $\mathbf{\tilde{T}}$-groups.

Theorem $\mathbf{\tilde{A}}$ *A group G without non-trivial periodic normal subgroups is a $\mathbf{S\tilde{T}}$-group iff it is semidihedral on a torsion-free abelian subgroup.*

Proof. Suppose G is a $\mathbf{S\tilde{T}}$-group. By Corollary 5.1 of [8], any torsion-free nilpotent normal subgroup N of G is abelian. Thus $A = \mathrm{Fit}(G)$ is abelian and by Theorem 2 in [4] it follows that $G/C_G(A)$ is abelian, too. Suppose, by contradiction, that $A \neq C := C_G(A)$. Since G is hyper-(abelian or finite), there exists a G-invariant subgroup U of C properly containing A and such that U/A is finite or abelian. If U/A is abelian, then U is nilpotent and so $U = A$, a contradiction. Then U/A is finite, so U is central-by-finite and U' is finite. Then $U' = 1$, a contradiction again. Hence $A = C$ and G is semidihedral on A by Proposition 2(i).

Conversely, let G be semidihedral on A, $H \not\leq A$ a subnormal subgroup of G with defect i. If $|H/(A \cap H)| = |AH : A|$ is finite, then $|H^G : H|$ is finite as G/A is abelian, so H is inert in G. Otherwise by Theorem 2 in [4], there is $g \in H \setminus A$ acting on A as a non-power rational power automorphism and A has finite rank. Say $g = q$, with $q \in \mathbb{Q}^* \setminus \{1, -1\}$. Hence $H \geq [A,_i g] = A^{(q-1)^i}$.

Therefore $|A/(A \cap H)| = |AH : H|$ is finite and again H is inert in G. Therefore G is a $\mathbf{S\tilde{T}}$-group. □

Theorem $\tilde{\mathbf{B}}$ *Let G be a finitely generated group. The following conditions are equivalent:*
i) G is a $\mathbf{S\tilde{T}}$-group;
ii) G has a semidihedral normal subgroup with finite index G_0 such that G acts by means of rational power automorphisms on $A_0 = Fit(G_0)$ and power automorphisms on G_0/A_0;
iii) G has a finite normal subgroup F such that G/F is semidihedral.

Proof. $(i) \Rightarrow (ii)$ As finitely generated semidihedral groups are finitely presented, we may assume that our claim holds for every proper quotient of G. By Theorem $\tilde{A}$, we may also assume that G has a periodic normal non-trivial subgroup N, which is either finite or an infinite elementary abelian p-group. In the latter case by [4] N has a subgroup of finite index on which elements of G act as power automorphisms and so N contains is proper normal subgroup K of G with N/K finite. So we may assume that N is finite with order say $n > 1$. Then there are normal subgroups in G

$$(*) \quad N \leq A_1 \leq G_1 \leq G$$

such that $A_1 = C_{G_1}(A_1/N)$, G_1 has finite index in G and A_1/N and G_1/A_1 are torsion-free abelian. If we intersect every subgroup of the chain $(*)$ with $C := C_G(N)$, we obtain the chain $N_2 \leq A_2 \leq G_2 \leq C$, where G_2 has finite index in G and A_2/N_2 and G_2/A_2 are torsion-free abelian. If $G_2 = A_2$, we get the claim by taking $G_0 := (A_2)^n$, which has finite index in the finitely generated group G and is torsion-free abelian since A_2 is nilpotent of class 2. Otherwise, let us check that $A_2 = C_{G_2}(A_2/N_2)$. Suppose that $x \in G_2$ and $[x, A_2] \leq N_2$. Since $[x, A_2N] \leq N$, we have that x centralizes a non-trivial subgroup of A_1/N. On the other hand every element of G_1 induces on A_1/N either a fixed-point free automorphism or the identity map. Thus $x \in A_1$ and so $x \in A_1 \cap C = A_2$ and $A_2 = C_{G_2}(A_2/N_2)$.

Let again $A_0 := (A_2)^n$, which is abelian and torsion-free. As G_2/A_0 is finitely generated and finite-by-abelian, it has an abelian normal torsion-free subgroup of finite index, say G_0/A_0. Finally, if $x \in G_0$ and $[x, A_0] = 1$, then x centralizes a non-trivial subgroup of A_2/N_2 and so $x \in A_2$. Since A_2/A_0 is periodic, then $A_2 \cap G_0 = A_0$, and we get $C_{G_0}(A_0) = A_0$ and G_0 is semidihedral and $A_0 = Fit(G_0)$.

Finally, as elements of G acts as inertial automorphisms of the abelian torsion-free group A_0, they are rational power on it, and as periodic inertial automorphisms of the abelian torsion-free group G_0/A_0, they are power on it, by Theorem 2 of [4].

$(ii) \Rightarrow (i)$ Let first $H \leq G_0$ be a subnormal subgroup of G with defect i and $H \not\leq A_0$. Suppose $|H/(A_0 \cap H)| = |A_0 H : A_0|$ is finite. Since elements of G act as power automorphisms on G_0/A_0, then $|H^G : H|$ is finite and H is inert in G. Otherwise by Theorem 2 in [4], there is $g \in H \setminus A$ acting on A_0 as a non-power rational power automorphism, say $g = q$, with $q \in \mathbb{Q}^* \setminus \{1, -1\}$ and A_0 has finite rank. Hence $H \geq [A_{0,i}\, g] = A_0^{(q-1)^i}$. Therefore $|A_0/(A_0 \cap H)| = |A_0 H : H|$ is finite and again H is inert in G. Finally if H is any subnormal subgroup of G, it is commensurable to $H \cap G_0$, which is inert by the above.

$(ii) \Rightarrow (iii)$ Let $C := C_G(A_0)$. Since $C \cap G_0 = A_0$, we have that C/A_0 is finite. It follows that C' and $F/C' := tor(C/C')$ are finite as well. Thus F is finite and G/F is semidihedral on C/F, since if $x \in C_G(C/F)$ then $[x, A_0] \leq A_0 \cap F = 1$.

$(iii) \Rightarrow (i)$ This is trivial. $\square$

3 Groups in which being commensurable to a normal subgroup is a transitive relation

If we define H *an* G (almost normal) iff $|H^G : H| < \infty$ then
$\mathbf{T}^*$-groups are groups in which:
* $\forall H \leq K \leq G$, $HanK$ and $KanG \Rightarrow HanG$; (see [3])
similarly, if we define H cf G (core finite) iff $|H : H_G| < \infty$ then
$\mathbf{T}_*$-groups are groups in which:
$_*$ $\forall H \leq K \leq G$, $HcfK$ and $KcfG \Rightarrow HcfG$; (see [6])

For $\mathbf{T}*$-groups the picture is similar as wee see in next Proposition. Recall that $H \,\mathrm{sn}\, G$ means H is subnormal in G, that is there is $i \in \mathbb{N}$ such that $[H,_i G] \leq H$.

Proposition 1 *For a group G the following conditions are equivalent:*
i) $\forall H \leq G$, $H \,\mathrm{sn}\, G \Rightarrow HcnG$;
ii) $\forall H \leq K \leq G$, $HcnK$ *and* $KcnG \Rightarrow HcnG$.

Proof. $(i) \Rightarrow (ii)$ Suppose there are subgroups H_1 and K_1 such that $H \sim H_1 \lhd K \sim K_1 \lhd G$. The subgroup $K_2 := (K \cap K_1)_{KK_1}$ has finite index in KK_1 and so $H \cap K_2$ has finite index in H. Hence $H \sim H \cap K_2 \sim H_1 \cap K_2$. On the other hand $H_1 \cap K_2 \lhd K_2 \lhd K_1 \lhd G$. Hence $HcnG$ and (ii) holds.

The converse is obvious. □

Corollary 1 *For a finitely generated hyper(finite-or-abelian) group G the following are equivalent:*
i) G is a $\mathbf{T}*$-group
ii) G acts by means of power automorphisms on a finite index abelian normal subgroup A.
iii) $\forall H \leq G$, $H \,\mathrm{sn}\, G$ *implies* H^G/H_G *is finite, that is H is both* an *and* cf.

Clearly the A of (ii) can be taken to be free abelian of finite rank and G induces on A either the identity or the inversion map.

Proof. If (i) holds, then by Theorem B* there is a normal series $A \leq G_0 \leq G$ such that G/G_0 is finite and G acts by means of inertial automorphisms on the torsion-free subgroup $A = C_{G_0}(A)$. Since $H \,\mathrm{cn}\, \mathrm{G}$ for every subgroup of A, G induces on A either the identity or the inversion map and $|G_1/C_{G_1(A)}| \leq 2$. Thus G/A is finite and (ii) holds.

If (ii) holds and H is any subgroup of G, it is plain that $H \,\mathrm{cn}\, \mathrm{G}$ as H is finite mod $(A \cap H) \lhd G$. If $[A, H] = 1$ then H has finitely many conjugates, as $A \leq C_G(H)$. Thus $H^G/(A \cap H)$ is finite by Dietzman Lemma and so H^G/H_G is finite. Otherwise, if H is subnormal with defect i, then $H \geq [A,_i H] = A^{2^i}$, whence $|G : H| \leq |G/A^{2^i}|$ is finite and (iii) holds.

It is trivial that (iii) implies (i). □

References

[1] V.V. Belayev, M. Kuzucuoğlu and E. Seckin, Totally inert groups, *Rend. Sem. Mat. Univ. Padova* **102** (1999), 151-156.

[2] C. Casolo, Groups with finite conjugacy classes of subnormal subgroups, *Rend. Sem. Mat. Univ. Padova* **81** (1989), 107-149.

[3] C. Casolo, Subgroups of Finite Index in Generalized T-groups, *Rend. Sem. Mat. Univ. Padova* **80** (1988), 265-277.

[4] U. Dardano and S. Rinauro, Inertial automorphisms of an abelian group, *Rend. Sem. Mat. Univ. Padova* **127** (2012), 213-233.

[5] M. De Falco, F. de Giovanni, C. Musella and N. Trabelsi, Strongly inertial groups, to appear.

[6] S. Franciosi, F. de Giovanni and M.L. Newell, Groups whose subnormal subgroups are normal-by-finite, *Comm. Alg.* **23(14)** (1995), 5483-5497.

[7] D.J.S. Robinson, "A Course in the Theory of Groups", Springer V., Berlin, 1982.

[8] D.J.S. Robinson, On inert subgroups of a group, *Rend. Sem. Mat. Univ. Padova* **115** (2006), 137-159.

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