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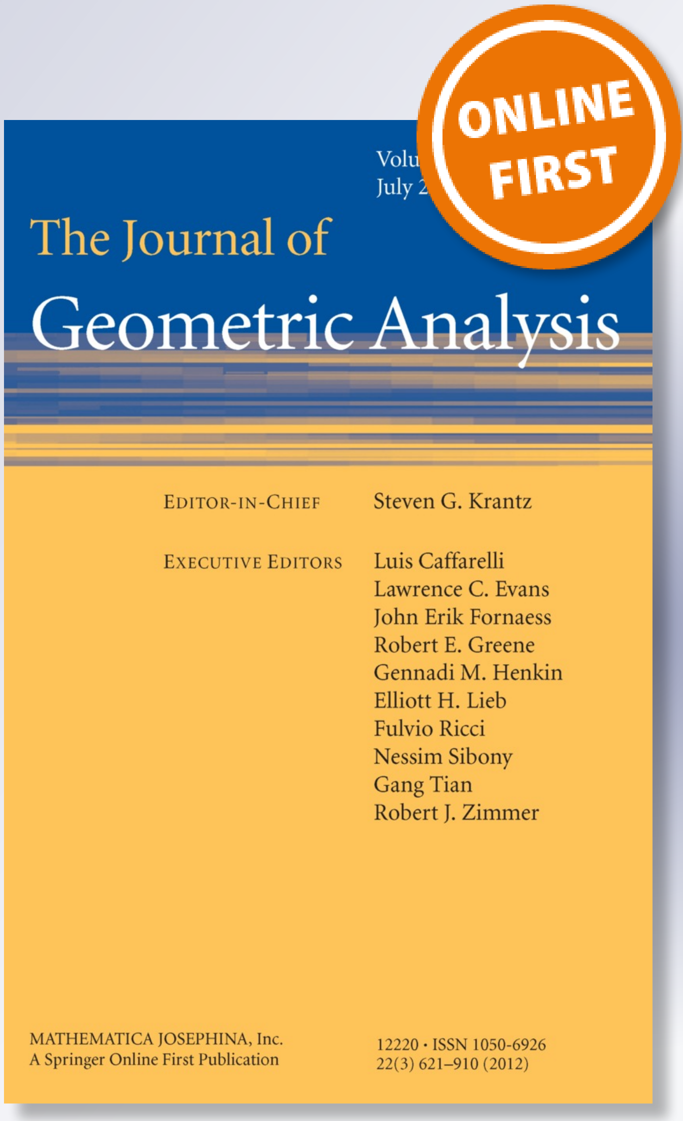

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Levi Harmonic Maps of Contact Riemannian Manifolds

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Abstract We study *Levi harmonic* maps, i.e., C^∞ solutions $f: M \to M'$ to $\tau_{\mathcal{H}}(f) \equiv \operatorname{trace}_g(\Pi_{\mathcal{H}} \beta_f) = 0$, where (M, η, g) is an (almost) contact (semi) Riemannian manifold, M' is a (semi) Riemannian manifold, β_f is the second fundamental form of f, and $\Pi_{\mathcal{H}} \beta_f$ is the restriction of β_f to the Levi distribution $\mathcal{H} = \operatorname{Ker}(\eta)$. Many examples are exhibited, e.g., the Hopf vector field on the unit sphere S^{2n+1}, immersions of Brieskorn spheres, and the geodesic flow of the tangent sphere bundle over a Riemannian manifold of constant curvature 1 are Levi harmonic maps. A CR map f of contact (semi) Riemannian manifolds (with spacelike Reeb fields) is pseudoharmonic if and only if f is Levi harmonic. We give a variational interpretation of Levi harmonicity. Any Levi harmonic morphism is shown to be a Levi harmonic map.

Keywords Contact Riemannian manifold · CR map · Levi harmonic map · Pseudoharmonic map · Parabolic geodesic · Parabolic exponential map · Brieskorn sphere

Mathematics Subject Classification (2000) 53C43 · 53D15 · 53D35 · 32V05 · 53C25 · 53C40

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1 Introduction

The present paper is devoted to the study of a class of variational principles whose corresponding Euler–Lagrange equations are degenerate elliptic and generalize ordinary harmonic map theory in the spirit of sub-Riemannian geometry (cf. [49]), i.e., given a smooth map $f : M \to M'$ of (semi) Riemannian manifolds (M, g) and (M', g') one replaces the Hilbert–Schmidt norm of df by the trace with respect to g of the restriction of $f^* g'$ to a given codimension-one distribution $\mathcal{H}$ on M (rather than applying the same construction to the full $f^* g'$). Omitting a direction (the conormal η to $\mathcal{H}$) has far-reaching consequences, e.g., the tension tensor is the trace of the restriction to $\mathcal{H}$ of the second fundamental form β_f of f (rather than the trace of the full β_f), the principal part in the Euler–Lagrange system is a second-order differential operator $\Delta_{\mathcal{H}}$ whose ellipticity degenerates precisely in the missed direction η. When M is a strictly pseudoconvex CR manifold and $\mathcal{H}$ is its Levi distribution, the operator $\Delta_{\mathcal{H}}$ is the sublaplacian (cf. [21]) and one is led to subelliptic problems. Indeed, E. Barletta et al. introduced (cf. [6]) pseudoharmonic maps $f : M \to M'$ from a nondegenerate CR manifold M endowed with a contact form θ into a Riemannian manifold M', as a global manifestation of J. Jost & C-J. Xu's subelliptic harmonic maps (from an open set in $\mathbb{R}^{2n+1}$ endowed with a Hörmander system of vector fields, cf. [33]). When M' is itself a nondegenerate CR manifold carrying the contact form θ', a result in [6] describes pseudoharmonicity of CR maps $f : M \to M'$. R. Petit, [47], considered the following (pseudohermitian analog to the) second fundamental form

$$\beta_f(X, Y) = \nabla'^f_X f_* Y - f_* \hat{\nabla}_X Y, \quad X, Y \in \mathfrak{X}(M), \tag{1.1}$$

where $\hat{\nabla}$ is the Tanaka–Webster connection of M and $\nabla'^f = f^{-1}\nabla'$ is the pullback of the Levi-Civita connection ∇' of M' (a connection in the pullback bundle $f^{-1}TM' \to M$). The approach in [6] is to replace ∇' by an arbitrary linear connection D' on M', consider the restriction $\Pi_{\mathcal{H}}\beta_f$ of (1.1) to the Levi distribution $\mathcal{H} = \mathrm{Ker}(\theta)$, and take the trace with respect to the Levi form G_θ. Then f is *pseudoharmonic* (with respect to the data (θ, D')) if $\tau(f) \equiv \mathrm{trace}_{G_\theta}(\Pi_{\mathcal{H}}\beta_f) = 0$.

Recently, R. Petit et al. [20], studied *contact harmonic* maps, i.e., C^∞ maps $f : M \to M'$ from a compact strictly pseudoconvex CR manifold M into a contact metric manifold M' which are critical points of the functional

$$E(f) = \frac{1}{2}\int_M \left\|(df)_{\mathcal{H},\mathcal{H}'}\right\|^2 \theta \wedge (d\theta)^n,$$

where θ is a contact form on M and $(df)_{\mathcal{H},\mathcal{H}'} = \mathrm{pr}_{\mathcal{H}'} \circ f_* : \mathcal{H} \to f^{-1}\mathcal{H}'$.

J. Konderak & R. Wolak, [37], introduced transversally harmonic maps as foliated maps $f : (M, \mathcal{F}, g) \to (M', \mathcal{F}', g')$ between foliated Riemannian manifolds satisfying a condition similar to the vanishing of the tension field in Riemannian geometry. The approach there is to use the canonical adapted connections (in the transverse bundles of the given Riemannian foliations $\mathcal{F}$ and $\mathcal{F}'$) to build a foliated analog β_f of the second fundamental form and then take the trace of β_f with respect to the given holonomy invariant Riemannian bundle metric g_Q in the transverse bundle

$Q \equiv \nu(\mathcal{F}) \to M$. A variational interpretation of the resulting equations

$$\mathrm{trace}_{g_Q}(\beta_f) = 0 \tag{1.2}$$

was given by A. Tommasoli et al. [22], who considered the variational principle $\delta E_T(f) = 0$ for the transverse energy functional

$$E_T(f) = \frac{1}{2} \int_M \|d_T f\|^2 dv_g \tag{1.3}$$

and showed that the corresponding Euler–Lagrange equations agree with (1.2) if and only if the source foliation $\mathcal{F}$ is harmonic. We note that, for the special case primarily considered in this paper, i.e., that of a contact metric manifold (M, η, g), the flow $\mathcal{F}_\xi$ defined by the Reeb vector field ξ is a Riemannian foliation (equivalently, g is bundle-like) if and only if M is a K-contact metric manifold, i.e., ξ is a Killing vector field (and then $\mathcal{F}_\xi$ is totally geodesic).

As a natural continuation of the ideas in [6], we introduce the notion of a *Levi harmonic* map $f : M \to M'$ from an almost contact semi-Riemannian manifold into a semi-Riemannian manifold, and study the Levi harmonicity for CR maps between two almost contact semi-Riemannian manifolds. Following the ideas of B. Fuglede (who started the study of the semi-Riemannian case within harmonic map theory, cf. [26] and [2], pp. 427–455) we allow *a priori* for the case $\mathfrak{s} = -1$ (cf. notation in Sect. 2). This is perhaps the most general geometric setting (metrics are but semi-Riemannian and in general the contact condition (2.3) is not satisfied and the underlying almost CR structures are not integrable).

The paper is organized as follows. In Sects. 2.1–2.2 we recall the notions of contact semi-Riemannian geometry and pseudohermitian CR geometry needed through the paper (basing the exposition on G. Calvaruso et al. [12], and G. Tomassini et al. [21]). Also in Sect. 2.3 we set the basis of a parabolic exponential map theory (and corresponding contact normal coordinates) on a contact Riemannian manifold. This is similar to the work by D. Jerison & J.M. Lee, [32], in CR geometry, with the additional difficulty that the almost CR structure $T_{1,0}(M)$ is not parallel with respect to the generalized Tanaka–Webster connection $\hat{\nabla}$ (so that *special coframes* in the sense of [32], p. 311, may not be produced).

In Sect. 3 for each C^∞ map $f : (M, \varphi, \xi, \eta, g) \to (M', g')$ from an almost contact semi-Riemannian manifold into a semi-Riemannian manifold, we consider the ordinary second fundamental form β_f given by (1.1) where $\hat{\nabla}$ is replaced by the Levi-Civita connection of (M, g). The following notion is central in the present paper. f is *Levi harmonic* with respect to $\mathcal{H} = \mathrm{Ker}(\eta)$ if

$$\tau_{\mathcal{H}}(f) \equiv \mathrm{trace}_g(\Pi_{\mathcal{H}} \beta_f) = 0,$$

where $\Pi_H \beta_f$ is the restriction of β_f to $\mathcal{H} \otimes \mathcal{H}$. Moreover, we compute the tension field $\tau_{\mathcal{H}}(f)$ for a CR map $f : M \to M'$ between two almost contact semi-Riemannian manifolds satisfying the so-called *φ-condition*. While the φ-condition (cf. (3.8) in Sect. 3) looks rather artificial, we emphasize successively that the class of almost contact semi-Riemannian manifolds obeying (3.8) is quite large. For instance, contact semi-Riemannian manifolds are known to satisfy the φ-condition (and if this is the case then $\tau_{\mathcal{H}}(f) = 2n\mathfrak{s}\varphi' f_* \xi$ (cf. Theorem 3.9)). As an application of parabolic exponential map theory we study the geometry of fixed point sets of isopseudohermitian transformations of a contact Riemannian manifold M (cf. Theorem 3.6 and Corollary 3.7 in Sect. 3.2).

Sections 3.3 to 3.5 are devoted to CR maps from contact semi-Riemannian manifolds with additional geometric properties. In Sect. 3.3 we consider the case where M is an orientable real hypersurface of an (indefinite) Kähler manifold $\overline{M}$, equipped with the almost contact (semi-) Riemannian structure induced by $\overline{M}$, and we compute $\tau_{\mathcal{H}}(f)$ in terms of the mean curvature of M. If M is a ruled real hypersurface of a complex space form $\overline{M}(c)$, $c \in \mathbb{R} \setminus \{0\}$, then any CR map $f : M \to M'$ is shown to be Levi harmonic. In Sect. 3.4 we show that any CR map $f : M \to M'$ between two quasi-cosymplectic manifolds is Levi harmonic. In Sect. 3.5 we study Levi harmonic maps defined by the Reeb vector field of a K-contact manifold. The Hopf vector field on the unit sphere S^{2n+1} and the geodesic flow of the unit tangent sphere bundle of a Riemannian manifold of constant curvature $+1$ are shown to be Levi harmonic maps (cf. Corollaries 3.13 and 3.14).

For each C^∞ map $f : M \to M'$ of contact semi-Riemannian manifolds, we consider the second fundamental form $\hat{\beta}_f$ given by (1.1) where ∇' is replaced by the generalized Tanaka–Webster connection of M' (cf. [51]). Let $\hat{\tau}_{\mathcal{H}}(f)$ be the trace with respect to g of the restriction $\Pi_{\mathcal{H}}\hat{\beta}_f$ of $\hat{\beta}(f)$ to $\mathcal{H} \otimes \mathcal{H}$. We call f *pseudoharmonic* if $\hat{\tau}_{\mathcal{H}}(f) = 0$. Although the notion in [6] (*pseudoharmonicity* with respect to the data (θ, D'), where D' is an arbitrary linear connection on M') is sufficiently general to include (for $D' = \hat{\nabla}'$) pseudoharmonicity as meant in this paper, the main results in [6] are confined (except for Theorem 1.1 in [6], p. 724) to the case of a Riemannian target manifold ($D' = \nabla'$). Our finding in Sect. 3.6 is that a CR map $f : M \to M'$ of contact semi-Riemannian manifolds is pseudoharmonic if and only if f is Levi harmonic.

In Sect. 4 we study Levi harmonic maps defined by isometric immersions $i : M \to \overline{M}$, where M is an invariant submanifold of an almost contact metric manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. In particular, if $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is a contact metric manifold or a quasi-cosymplectic manifold, then the inclusion $i : M \to \overline{M}$ is Levi harmonic and minimal (cf. Theorem 4.1).

Section 5 is devoted to maps from Brieskorn spheres. The inclusion of the Brieskorn sphere $\Sigma^{2n-1}(2, \ldots, 2)$ into S^{2n+1} is Levi harmonic. Moreover, as a consequence of Corollary 3.7 (on the geometry of a connected component of the fixed point set of an isopseudohermitian transformation) we show that certain immersions of Brieskorn spheres $\Sigma^{2n-3}(a') \to \Sigma^{2n-1}(2k, a')$ are Levi harmonic.

A variational interpretation of Levi harmonicity is formulated in Sect. 6 together with a discussion of the resulting Euler–Lagrange equations in the framework of degenerate elliptic equations theory.

Section 7 discusses several ramifications of the theory of harmonic morphisms, within contact Riemannian geometry, such as Levi harmonic and pseudoharmonic morphisms. Any Levi harmonic morphism $f : M \to M'$ of an almost contact Riemannian manifold M into a Riemannian manifold M' is a Levi harmonic map (cf. Theorem 7.7). A similar result for pseudoharmonic morphisms holds only for Sasakian target manifolds M' (cf. Theorem 7.8).

2 Contact Metric Geometry

In Sect. 2.1 we collect a few basic facts on contact Riemannian manifolds (cf. [10] and [12] for the Riemannian and semi-Riemannian cases, respectively). Sect. 2.2 is devoted to the needed notions of CR and pseudohermitian geometry (cf. [21] and [19]). In Sect. 2.3 we establish the existence and main properties of the parabolic exponential map and corresponding contact normal coordinates at a point on a contact Riemannian manifold. The material in Sect. 2.3 is new and adapts (to the context of contact Riemannian geometry) a technique introduced by G.B. Folland & E.M. Stein (cf. [25]) and successively refined by D. Jerison & J.M. Lee (cf. [32]) within pseudohermitian geometry (i.e., by making use of a distinguished linear connection associated with a fixed contact form on a strictly pseudoconvex CR manifold, the Tanaka–Webster connection). Our almost CR structures are in general non-integrable, and the use of the Tanaka–Webster connection is replaced by its generalization due to S. Tanno, [51].

2.1 Contact Semi-Riemannian Manifolds

Let M be a real $(2n+1)$-dimensional C^∞ manifold. An *almost contact structure* (φ, ξ, η) on M consists of a $(1,1)$-tensor field φ, a tangent vector field $\xi \in \mathfrak{X}(M)$ (the *characteristic*, or *Reeb*, *field*), and a differential 1-form $\eta \in \Omega^1(M)$ such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{2.1}$$

In particular, $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. Let $\mathfrak{s} \in \{\pm 1\}$. Given an almost contact structure (φ, ξ, η) on M, a *compatible metric* is a semi-Riemannian metric g on M such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \mathfrak{s}\eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M). \tag{2.2}$$

Then $\eta(X) = \mathfrak{s} g(\xi, X)$ and $g(\xi, \xi) = \mathfrak{s}$. Therefore, the characteristic vector field ξ is either spacelike or timelike (ξ is never lightlike). Also if $\Phi(X, Y) = g(X, \varphi Y)$ then (by (2.2)) $\Phi \in \Omega^2(M)$. The synthetic object (φ, ξ, η, g) is an *almost contact semi-Riemannian structure*. If in addition the *contact condition*

$$d\eta = \Phi \tag{2.3}$$

is satisfied then η is a contact form, i.e., $\eta \wedge (d\eta)^n$ is a volume form on M (and (φ, ξ, η, g) is referred to as *contact semi-Riemannian structure* on M). On each contact semi-Riemannian manifold the tensor field $h = (1/2)\mathcal{L}_\xi \varphi$ (where $\mathcal{L}$ is the Lie derivative) is symmetric and satisfies

$$\nabla \xi = -\mathfrak{s}\varphi - \varphi \circ h, \qquad \nabla_\xi \varphi = 0, \qquad h \circ \varphi + \varphi \circ h = 0, \qquad h(\xi) = 0. \tag{2.4}$$

Here ∇ is the Levi-Civita connection of the semi-Riemannian manifold (M, g). Moreover (by Lemma 4.3 in [12]),

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = 2g(X, Y)\xi - \eta(Y)\big\{\mathfrak{s}X + \mathfrak{s}\eta(X)\xi + h(X)\big\}. \tag{2.5}$$

A contact semi-Riemannian manifold is a *K-contact semi-Riemannian manifold* if ξ is a Killing vector field (equivalently, $h = 0$). The result in [10] that K-contact Riemannian manifolds are characterized by the Ricci curvature condition $\varrho(\xi, \xi) = 2n$ holds in the positive definite case and fails in general (for arbitrary contact semi-Riemannian manifolds, cf. [12]). A contact semi-Riemannian structure (φ, ξ, η, g) is *Sasakian* if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \mathfrak{s}\eta(Y)X.$$

2.2 Almost CR Structures

Let M be a real $(2n + 1)$-dimensional manifold. An *almost CR structure* on M is a complex subbundle $T_{1,0}(M)$, of complex rank n, of the complexified tangent bundle $T(M) \otimes \mathbb{C}$ such that $T_{1,0}(M) \cap T_{0,1}(M) = (0)$ where $T_{0,1}(M) = \overline{T_{1,0}(M)}$ (overbars denote complex conjugates). The integer n is the *CR dimension*. An almost CR structure $T_{1,0}(M)$ is *integrable*, and then $T_{1,0}(M)$ is referred to as a *CR structure*, if $Z, W \in C^\infty(U, T_{1,0}(M))$ yields $[Z, W] \in C^\infty(U, T_{1,0}(M))$ for any open set $U \subset M$. The *Levi* (or *maximally complex*) *distribution* is the real rank $2n$ distribution on M given by $\mathcal{H} \equiv H(M) = \mathrm{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$. It carries the complex structure

$$J : \mathcal{H} \to \mathcal{H}, \quad J(Z + \overline{Z}) = i(Z - \overline{Z}), \quad Z \in T_{1,0}(M) \quad (i = \sqrt{-1}).$$

Then $T_{1,0}(M) = \{X - iJX : X \in \mathcal{H}\}$, i.e., $T_{1,0}(M)$ is the eigenbundle of $J^{\mathbb{C}}$ (the $\mathbb{C}$-linear extension of J to $\mathcal{H} \otimes \mathbb{C}$) corresponding to the eigenvalue i. The pair $(\mathcal{H}, J)$ (the real manifestation of $T_{1,0}(M)$) is often referred to as an almost CR structure on M, as well. A *pseudohermitian structure* is a differential 1-form $\theta \in \Omega^1(M)$ such that $\mathrm{Ker}(\theta) = \mathcal{H}$. Given a pseudohermitian structure θ on M the *Levi form* G_θ is given by

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in \mathcal{H}.$$

An almost CR structure $(\mathcal{H}, J)$ is *nondegenerate* if the Levi form G_θ is nondegenerate for some θ. If this is the case, θ is a contact form (i.e., $\theta \wedge (d\theta)^n$ is a volume form). Let $E_x \subset T_x^*(M)$ be the subspace consisting of the values at $x \in M$ of all pseudohermitian structures on M. Then $E = \bigcup_{x \in M} E_x$ is (the total space of) a real line subbundle of the cotangent bundle $T^*(M)$ and the pseudohermitian structures are the globally defined nowhere zero C^∞ sections in E. If M is oriented (an assumption adopted throughout this paper) then E is trivial, i.e., $E \approx M \times \mathbb{R}$ (a vector bundle isomorphism). Therefore, any other pseudohermitian structure $\hat{\theta} \in C^\infty(E)$ is related to θ by $\hat{\theta} = \lambda\theta$ for some C^∞ function $\lambda : M \to \mathbb{R} \setminus \{0\}$. Then $G_{\hat{\theta}} = \lambda G_\theta$, hence nondegeneracy is a CR invariant. An almost CR structure $(\mathcal{H}, J)$ is *strictly pseudoconvex* if G_θ is positive definite for some θ. If this is the case then $G_{-\theta}$ is negative definite (hence strict pseudoconvexity is not a CR invariant). It should be observed that G_θ is Hermitian, i.e., $G_\theta(JX, JY) = G_\theta(X, Y)$ (equivalently G_θ is symmetric, i.e., $G_\theta(X, Y) = G_\theta(Y, X)$) if and only if

$$[JX, Y] + [X, JY] \in C^\infty(\mathcal{H}), \quad X, Y \in C^\infty(\mathcal{H}). \tag{2.6}$$

Integrability of $T_{1,0}(M)$ is known to be equivalent to (2.6) together with

$$J\{[JX, Y] + [X, JY]\} = [JX, JY] - [X, Y], \quad X, Y \in C^\infty(\mathcal{H}).$$

Thus for any CR structure G_θ is Hermitian. Let $(M, \mathcal{H}, J)$ be a nondegenerate almost CR manifold and θ a fixed contact form on M. Let us extend J to an endomorphism φ of the tangent bundle by requesting that $\varphi = J$ on $\mathcal{H}$ and $\varphi(T) = 0$. Here $T \in \mathfrak{X}(M)$ is the unique nowhere zero tangent vector field on M determined by $\theta(T) = 1$ and $t \rfloor d\theta = 0$. Then $\varphi^2 = -I + \theta \otimes T$. The integrability condition on $\mathcal{H}$ is relaxed throughout to the requirement (2.6). Then the *Webster metric* is the semi-Riemannian metric g_θ given by

$$g_\theta(X, Y) = (d\theta)(X, JY), \qquad g_\theta(X, T) = 0, \qquad g_\theta(T, T) = 1,$$

for any $X, Y \in \mathcal{H}$. Then $(\varphi, \xi = -T, \eta = -\theta, g = g_\theta)$ is a contact semi-Riemannian structure on M. If G_θ is positive definite, the Webster metric g_θ is a Riemannian metric (and (φ, ξ, η, g) is a contact metric structure on M). Conversely, any almost contact manifold (M, φ, ξ, η) carries the almost CR structure given by $\mathcal{H} = \mathrm{Ker}(\eta)$ and $J = \varphi|_{\mathcal{H}}$. By a result of S. Ianuş, [29], if (φ, ξ, η) is *normal* (i.e., $[\varphi, \varphi] + 2(d\eta) \otimes \xi = 0$) then $(\mathcal{H}, J)$ is integrable.

2.3 Contact Normal Coordinates

Let $(M, (\varphi, \xi, \eta, g))$ be a contact semi-Riemannian manifold. Let $\hat{\nabla}$ be the *generalized Tanaka–Webster* connection, i.e., the linear connection given by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi(Y) - \eta(Y)\nabla_X \xi + \{(\nabla_X \eta)Y\}\xi \tag{2.7}$$

for any $X, Y \in \mathfrak{X}(M)$. $\hat{\nabla}$ is due to S. Tanno, [51] (though confined to the positive definite case) and admits (when ξ is spacelike) an axiomatic description similar to that of the ordinary Tanaka–Webster connection (cf. N. Tanaka, [50]) except for the property $\hat{\nabla}\varphi = 0$ (the vanishing of $\hat{\nabla}\varphi$ is equivalent to the integrability of the underlying almost CR structure). Indeed, if (φ, ξ, η, g) is a contact semi-Riemannian structure with $\mathfrak{s} = 1$, then (2.7) may be described (cf. [51], p. 354) as the unique linear connection $\hat{\nabla}$ on M obeying the axioms

$$\hat{\nabla}\eta = 0, \qquad \hat{\nabla}\xi = 0, \qquad \hat{\nabla} g = 0, \tag{2.8}$$

$$T_{\hat{\nabla}}(\xi, \varphi X) + \varphi T_{\hat{\nabla}}(\xi, X) = 0, \quad X \in \mathfrak{X}(M), \tag{2.9}$$

$$T_{\hat{\nabla}}(X, Y) = 2(d\eta)(X, Y)\xi, \quad X, Y \in \mathcal{H} = \mathrm{Ker}(\eta), \tag{2.10}$$

$$(\hat{\nabla}_X \varphi)Y = Q(Y, X), \quad X, Y \in \mathfrak{X}(M). \tag{2.11}$$

Here $T_{\hat{\nabla}}$ is the torsion tensor field of $\hat{\nabla}$. Also, Q is the *Tanno tensor*, i.e.,

$$Q(Y, X) = (\nabla_X \varphi)Y + \{(\nabla_X \eta)\varphi Y\}\xi + \eta(Y)\varphi(\nabla_X \xi).$$

By a result in [51], $Q = 0$ if and only if $(\mathcal{H}, J)$ is integrable (and then $\hat{\nabla}$ is the ordinary Tanaka–Webster connection of (M, η)).

Definition 2.1 A class C^2 curve $\gamma : (-\epsilon, \epsilon) \to M$ is said to be a *parabolic geodesic* if there is $\lambda \in \mathbb{R}$ such that

$$(\hat{\nabla}_{\dot{\gamma}} \dot{\gamma})_{\gamma(s)} = 2\lambda \xi_{\gamma(s)}, \quad |s| < \epsilon. \tag{2.12}$$

To keep track of the parameter λ, a solution γ to (2.12) is also referred to as a *λ-parabolic geodesic*.

Let $N = 2n + 1$ and let $\chi = (x^1, \dots, x^N) : U \to \mathbb{R}^N$ be a local coordinate system on M. We set $\Omega = \chi(U) \times \mathbb{R}^N \subset \mathbb{R}^{2N}$. Let $\hat{\Gamma}^i_{jk} \in C^\infty(U)$ be the local coefficients of $\hat{\nabla}$ with respect to (U, x^i) and let us set

$$F^i(\mathbf{u}, \mathbf{v}, \lambda) = 2\lambda \xi^i\big(\chi^{-1}(\mathbf{u})\big) - \hat{\Gamma}^i_{jk}\big(\chi^{-1}(\mathbf{u})\big) v^j v^k$$

so that $F^i \in C^\infty(\Omega \times \mathbb{R})$ for any $1 \le i \le N$. By classical results in ODE theory, for each $(\mathbf{u}_1, \mathbf{v}_1) \in \Omega$ there is an open neighborhood $\omega \subset \Omega$ of $(\mathbf{u}_1, \mathbf{v}_1)$ and a number $\delta > 0$ such that for any $(\mathbf{u}_0, \mathbf{v}_0) \in \omega$ and any $\lambda \in \mathbb{R}$ the system

$$\frac{d^2 u^i}{ds^2}(s) = F^i\left(\mathbf{u}(s), \frac{d\mathbf{u}}{ds}(s), \lambda\right), \quad 1 \le i \le N, \tag{2.13}$$

has a unique solution $\mathbf{u}_\lambda = \mathbf{u}(\cdot, \lambda) : (-\delta, \delta) \to \mathbb{R}^N$ such that

$$\mathbf{u}(0, \lambda) = \mathbf{u}_0, \qquad \frac{\partial \mathbf{u}}{\partial s}(0, \lambda) = \mathbf{v}_0. \tag{2.14}$$

Also (by classical results on the dependence of the solution to the Cauchy problem (2.13)–(2.14) on initial conditions and parameters) the map $(\mathbf{u}_0, \mathbf{v}_0, s, \lambda) \mapsto \mathbf{u}(s, \lambda)$ is of class C^∞ in all its $2N + 2$ variables. For the remainder of Sect. 2.3 the discussion is confined to the case of contact metric structures (i.e., g is positive definite and $\mathfrak{s} = 1$). We establish

Theorem 2.2 *For every point $x_0 \in M$ there is an open neighborhood $U \subset M$ of x_0 and a number $\epsilon > 0$ such that for each $x \in U$ and each $X \in \mathcal{H}_x$ of length $\|X\| = g_x(X, X)^{1/2} < \epsilon$ and each $\lambda \in \mathbb{R}$ there is a unique λ-parabolic geodesic $\gamma_{X,\lambda} : (-2, 2) \to M$ satisfying the conditions $\gamma_{X,\lambda}(0) = x$ and $\dot{\gamma}_{X,\lambda}(0) = X$.*

Proof The existence and uniqueness theorem above implies that the statement is true if we replace the interval $(-2, 2)$ by an arbitrarily small interval. Precisely, there is a neighborhood U of x_0 in M and there exist numbers $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for each $x \in U$ and each $Y \in \mathcal{H}_x$ of length $\|Y\| < \epsilon_1$ and each $\mu \in \mathbb{R}$ there is a unique μ-parabolic geodesic $\gamma_{Y,\mu} : (-2\epsilon_2, 2\epsilon_2) \to M$ satisfying the required initial conditions

$$\gamma_{Y,\mu}(0) = x, \qquad \dot{\gamma}_{Y,\mu}(0) = Y.$$

Let $0 < \epsilon < \epsilon_1 \epsilon_2$ and let $X \in \mathcal{H}_x$ with $\|X\| < \epsilon$ and let $\lambda \in \mathbb{R}$. If $r = 1/\epsilon_2$ then $\|rX\| = r\|X\| < r\epsilon < \epsilon_1$ while if $|s| < 2$ then $|\epsilon_2 s| < 2\epsilon_2$. Let then $\gamma : (-2, 2) \to M$

be the curve given by

$$\gamma(s) = \gamma_{rX, r^2\lambda}(s/r), \quad |s| < 2.$$

For each fixed $|s| < 2$ let (V, x^i) be a local coordinate system on M such that $\gamma(s) \in V$. Then

$$(\hat{\nabla}_{\dot\gamma}\dot\gamma)_{\gamma(s)} = \left\{ \frac{d^2\gamma^i}{ds^2}(s) + \hat{\Gamma}^i_{jk}\big(\gamma(s)\big)\frac{d\gamma^j}{ds}(s)\frac{d\gamma^k}{ds}(s) \right\}\left(\frac{\partial}{\partial x^i}\right)_{\gamma(s)}$$
$$= \frac{1}{r^2}(\hat{\nabla}_{\dot\gamma_{rX,r^2\lambda}}\dot\gamma_{rX,r^2\lambda})_{\gamma(s)} = 2\lambda\xi_{\gamma(s)},$$

hence γ is a λ-parabolic geodesic (of initial data (x, X)). □

Let $x_0 \in M$. By Theorem 2.2 there exist an open neighborhood $U \subset M$ of x_0 and a number $\epsilon > 0$ such that for any $x \in U$, $X \in \mathcal{H}_x$ of length $\|X\| < \epsilon$, and $\lambda \in \mathbb{R}$ there is a unique λ-parabolic geodesic $\gamma_{X,\lambda} : (-2, 2) \to M$ of initial data (x, X).

Definition 2.3 The *parabolic exponential* map at x is given by

$$\mathrm{Exp}_x : N_\epsilon(0_x) \to M, \quad \mathrm{Exp}_x(X + \lambda\xi_x) = \gamma_{X,\lambda}(1),$$
$$X \in \mathcal{H}_x, \quad \lambda \in \mathbb{R}, \quad X + \lambda\xi_x \in N_\epsilon(0_x).$$

Here $N_\epsilon(0_x) = \{Y + \mu\xi_x \in T_x(M) : Y \in \mathcal{H}_x, \|Y\| < \epsilon, |\mu| < \epsilon\}$.

0-parabolic geodesics are merely geodesics of the generalized Tanaka–Webster connection $\hat{\nabla}$. The terminology λ-*parabolic* is only motivated when $\lambda \neq 0$, for given any $X \in \mathcal{H}_x$ and any $\lambda \in \mathbb{R} \setminus \{0\}$ one may consider the curve $\mathcal{P}_{X,\lambda} : \mathbb{R} \to T_x(M)$ given by $\mathcal{P}_{X,\lambda}(s) = sX + s^2\lambda\xi_x$ (a *parabola* tangent to M at x) and observe that the parabolic exponential maps $\mathcal{P}_{X,\lambda}$ onto the λ-parabolic geodesic of initial data (x, X). Indeed, as a byproduct of the proof of Theorem 2.2,

$$\gamma_{rX,r^2\lambda}(s) = \gamma_{X,\lambda}(rs) \tag{2.15}$$

whenever either side is defined. Then (by (2.15))

$$\mathrm{Exp}_x\big[\mathcal{P}_{X,\lambda}(s)\big] = \gamma_{sX,s^2\lambda}(1) = \gamma_{X,\lambda}(s).$$

Theorem 2.4 *The parabolic exponential* Exp_x *is a diffeomorphism on a sufficiently small neighborhood of the origin in* $T_x(M)$.

Proof To prove Theorem 2.4, we need to recall a few elements of the geometry of the tangent bundle over M. By eventually replacing U furnished by Theorem 2.2 with a smaller open set, we may assume that U is the domain of a local coordinate system (x^i) such that $x^i(x_0) = 0$. Let $(\pi^{-1}(U), x^i, y^i)$ be the induced local coordinates on $T(M)$ and set $\xi^i = y^i \circ \xi \in C^\infty(U)$. Here $\pi : T(M) \to M$ is the projection. Let $\pi^{-1}T(M) \to T(M)$ be the pullback of $T(M)$ by π (the *diagonal bundle*). The *natural lift* of $X \in \mathfrak{X}(M)$ is the section $X^\pi = X \circ \pi \in C^\infty(\pi^{-1}TM)$. Let $D = \pi^{-1}\hat{\nabla}$ be the pullback connection, i.e., the connection in $\pi^{-1}T(M) \to T(M)$ induced by $\hat{\nabla}$. This is most easily described in local coordinates as

$$D_{\partial_i} X_j = \big(\hat{\Gamma}^k_{ij} \circ \pi\big) X_k, \quad D_{\dot{\partial}_i} X_j = 0,$$

where we have set $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial y^i$ (for simplicity) while $X_i = (\partial/\partial x^i)^\pi \in C^\infty(\pi^{-1}(U), \pi^{-1}T(M))$. A tangent vector field $V \in \mathfrak{X}(T(M))$ is *horizontal* if $D_V \mathcal{L} = 0$, where $\mathcal{L} \in C^\infty(\pi^{-1}T(M))$ is the *Liouville vector*, i.e., $\mathcal{L}(v) = (v, v)$ for any $v \in T(M)$. Let $H \subset T(M)$ be the horizontal distribution, i.e., H_v consists of the values at $v \in T(M)$ of all horizontal vector fields on $T(M)$. Then H is a *nonlinear connection* on $T(M)$, i.e.,

$$T_v\big(T(M)\big) = H_v \oplus \mathrm{Ker}(d_v \pi), \quad v \in T(M).$$

Also if $N^i_j = (\hat{\Gamma}^i_{jk} \circ \pi) y^k \in C^\infty(\pi^{-1}(U))$ (the local coefficients of the nonlinear connection H) then

$$\delta_i = \frac{\delta}{\delta x^i} = \partial_i - N^j_i \dot{\partial}_j, \quad 1 \le i \le 2n+1,$$

is a local frame in $\pi^{-1}T(M) \to T(M)$ defined on the open set $\pi^{-1}(U)$. The *vertical lift* is the vector bundle morphism $\gamma : \pi^{-1}T(M) \to \mathrm{Ker}(d\pi)$ given by

$$\gamma_v(v, w) = \dot{a}(0) \in T_v\big(T(M)\big), \quad v \in T(M),\ w \in T_{\pi(v)}(M),$$
$$a(t) = v + tw, \quad |t| < \delta\ (\delta > 0).$$

Locally $\gamma(X_i) = \dot{\partial}_i$. The *horizontal lift* is the vector bundle morphism $\beta : \pi^{-1}T(M) \to H$ given by

$$\beta_v = \big[L_v : H_v \to \big(\pi^{-1}T(M)\big)_v\big]^{-1},$$
$$L_v : T_v\big(T(M)\big) \to \big(\pi^{-1}T(M)\big)_v, \quad v \in T(M),$$
$$L_v(A) = \big(v, (d_v\pi)A\big), \quad A \in T_v\big(T(M)\big).$$

Locally $\beta(X_i) = \delta_i$. Next we consider the (globally defined) vector field $\mathfrak{X}$ on the product manifold $T(M) \times \mathbb{R}$ given by

$$\mathfrak{X} = \beta(\mathcal{L}) + 2t\gamma\big(\xi^\pi\big), \tag{2.16}$$

where t is the Cartesian coordinate on the $\mathbb{R}$ factor and $\xi^\pi \in C^\infty(\pi^{-1}T(M))$ is the natural lift of ξ (i.e., $\xi^\pi = \xi \circ \pi$). If $\pi^{-1}(U) \times \mathbb{R}$ is endowed with the local coordinates (x^i, y^i, t) then (2.16) may be written $\mathfrak{X} = y^i \delta_i + 2t\xi^i \dot{\partial}_i$. Let $\{\Phi_s\}_{|s|<\epsilon}$ be the local 1-parameter group of local transformations gotten by integrating $\mathfrak{X}$ so that $\Phi(s, v, \lambda) = \Phi_s(v, \lambda)$ is well defined and C^∞ in a neighborhood of the origin $(-\epsilon, \epsilon) \times \mathcal{U} \times (-\epsilon, \epsilon) \subset \mathbb{R} \times \pi^{-1}(U) \times \mathbb{R}$. Let $p_1 : T(M) \to \mathbb{R} \to T(M)$ be the projection on the first factor. Given $X \in \mathcal{U} \cap \mathcal{H}_x$ (with $x = \pi(X) \in U$) and $|\lambda| < \epsilon$, let us set

$$\gamma(s) = (\pi \circ p_1)\big[\Phi_s(X, \lambda)\big] \in U. \tag{2.17}$$

We also adopt the notation

$$C_{(X,\lambda)}(s) = \Phi_s(X,\lambda), \quad |s| < \epsilon,$$

so that

$$\dot{C}_{(X,\lambda)}(s) = \mathfrak{X}_{C_{(X,\lambda)}(s)}. \tag{2.18}$$

On the other hand,

$$\dot{C}_{(X,\lambda)}(s) = \frac{d\gamma^i}{ds}(s)\partial_{i,C(s)} + \frac{dv^i}{ds}(s)\dot{\partial}_{i,C(s)} + \frac{da}{ds}\left(\frac{\partial}{\partial t}\right)_{C(s)},$$

where C is short for $C_{(X,\lambda)}$ and

$$\gamma^i = x^i \circ \gamma, \qquad v^i = y^i \circ C, \qquad a = t \circ C.$$

Then (by (2.16)–(2.18))

$$\begin{aligned} &\frac{d\gamma^i}{ds}(s) = v^i(s), \qquad \frac{da}{ds}(s) = 0, \\ &\frac{dv^i}{ds}(s) = 2\lambda\xi^i\big(\gamma(s)\big) - N^i_j\big(p_1\big(C(s)\big)\big)v^j(s), \end{aligned}$$

so that $a(s) = \lambda$ and

$$\frac{d^2\gamma^i}{ds^2}(s) = 2\lambda\xi^i\big(\gamma(s)\big) - \Gamma^i_{jk}\big(\gamma(s)\big)\frac{d\gamma^j}{ds}(s)\frac{d\gamma^k}{ds}(s),$$

that is, γ is the λ-parabolic geodesic of initial data (x, X). Thus

$$\gamma_{X,\lambda}(s) = (\pi \circ p_1)\big[\Phi_s(X,\lambda)\big], \tag{2.19}$$

where either side is defined. Let $|s_0| < \epsilon$, $X \in \mathcal{U} \cap \mathcal{H}_x$, and $|\lambda| < \epsilon$. Then (by (2.15) with $s = s_0$ and $r = r_0 = 1/s_0$)

$$\begin{aligned} \mathrm{Exp}_x(X + \lambda\xi_x) &= \gamma_{X,\lambda}(1) = \gamma_{X,\lambda}(r_0 s_0) = \gamma_{r_0 X, r_0^2\lambda}(s_0) \\ &= (\pi \circ p_1)\big[\Phi_{s_0}\big(r_0 X, r_0^2\lambda\big)\big]. \end{aligned}$$

As $\Phi_{s_0}(v,\mu)$ is defined and C^∞ for all (v,μ) in a neighborhood of the origin, it follows that Exp_x is C^∞ in a sufficiently small neighborhood of the zero vector $0_x \in T_x(M)$. Let $y^i_x = y^i|_{\pi^{-1}(x)}$ be the natural coordinates on $T_x(M)$ (induced by (U, x^i) with $x \in U$). Also, for each $v \in T_x(M)$ let $F_v : T_v(T_x(M)) \to T_x(M)$ be the natural linear isomorphism (i.e., $F_v(\partial/\partial y^i_x)_v = (\partial/\partial x^i)_x$). We ought to compute

$$(d_{0_x}\mathrm{Exp}_x)F^{-1}_{0_x}(X), \qquad (d_{0_x}\mathrm{Exp}_x)F^{-1}_{0_x}(\xi_x).$$

Let us set $\Psi^j = x^j \circ \mathrm{Exp}_x$. If $X \in \mathcal{U} \cap \mathcal{H}_x$ then we consider the curve $\gamma(s) = \mathrm{Exp}_x(sX) \in M$ so that (by (2.15)) $\gamma(s) = \gamma_{sX,0}(1) = \gamma_{X,0}(s)$. Therefore,

$$X = \dot{\gamma}_{X,0}(0) = \dot{\gamma}(0) = \frac{d\gamma^i}{ds}(0)\left(\frac{\partial}{\partial x^i}\right)_x$$
$$= \frac{\partial \Psi^i}{\partial y^j_x}(0_x) y^j(X)\left(\frac{\partial}{\partial x^i}\right)_x = (d_{0_x}\mathrm{Exp}_x) F_{0_x}^{-1}(X).$$

Similarly, let $a(s) \in M$ be an integral curve of ξ such that $a(0) = x$ and let $|\lambda| < \epsilon$. We set $b(r) = a(r^2\lambda) \in M$ and observe that

$$\dot{b}(r) = 2r\lambda \xi_{b(r)}. \tag{2.20}$$

In particular (by (2.8)),

$$(\hat{\nabla}_{\dot{b}}\dot{b})_{b(r)} = 2\lambda \xi_{b(r)} + 2r\lambda(\hat{\nabla}_{\dot{b}}\xi)_{b(r)} = 2\lambda\xi_{b(r)}$$

so that b is the λ-parabolic geodesic of initial data $(x, 0_x)$, i.e., $b(r) = \gamma_{0_x,\lambda}(r)$. Then for $r = 1$

$$a(\lambda) = \gamma_{0_x,\lambda}(1), \quad |\lambda| < \epsilon.$$

Finally, we may compute the tangent vector to $a(\lambda) = \mathrm{Exp}_x(\lambda\xi_x)$ to yield

$$(d_{0_x}\mathrm{Exp}_x) F_{0_x}^{-1}(\xi_x) = \xi_x.$$

Therefore, $(d_{0_x}\mathrm{Exp}_x) \circ F_{0_x}^{-1}$ is the identity on a neighborhood of $0_x \in T_x(M)$ (and the statement follows from the implicit function theorem). Theorem 2.4 is proved. □

Let $x_0 \in M$. Special frames as in [32], p. 311 (gotten by parallel displacement of a linear basis $\{w_\alpha : 1 \le \alpha \le n\} \subset T_{1,0}(M)_{x_0}$) may not be built because the generalized Tanaka–Webster connection does not descend to a connection in $T_{1,0}(M)$. Indeed (by (2.8)) $\hat{\nabla}$ parallelizes $\mathcal{H} = \mathrm{Ker}(\eta)$ yet (by (2.11)) $\hat{\nabla}$ does not parallelize the complex structure along $\mathcal{H}$.

Let $\{v_a : 1 \le a \le 2n\} \subset \mathcal{H}_{x_0}$ be a g_{x_0}-orthonormal basis. By Theorem 2.4 there is an open neighborhood $U \subset M$ of x_0 and a number $\epsilon > 0$ such that $\mathrm{Exp}_{x_0} : N_\epsilon(0_{x_0}) \to U$ is a diffeomorphism. Hence for any $x \in U$ there exist $X \in \mathcal{H}_{x_0}$ and $\lambda \in \mathbb{R}$ such that $X + \lambda\xi_{x_0} \in N_\epsilon(0_{x_0})$ and $\gamma_{X,\lambda}(1) = \mathrm{Exp}_{x_0}(X + \lambda\xi_{x_0}) = x$. Let $\tau_x : T_{x_0}(M) \to T_x(M)$ be the parallel displacement operator along $\gamma_{X,\lambda}$ associated with $\hat{\nabla}$. As $\hat{\nabla}\xi = 0$ and $\mathcal{H}$ is ∇-parallel, one has $\tau_x(\xi_{x_0}) = \xi_x$ and $\tau_{x_0}(\mathcal{H}_{x_0}) = \mathcal{H}_x$.

Lemma 2.5 *Let us set $\xi_a(x) = \tau_x(v_a)$. Then $\{\xi_a : 1 \le a \le 2n\}$ is a C^∞ orthonormal local frame of $\mathcal{H}$ defined on the open set U such that $\hat{\nabla}_{\dot{\gamma}_{X,\lambda}}\xi_a = 0$ for any $X \in \mathcal{H}_{x_0}$ and $\lambda \in \mathbb{R}$ with $X + \lambda\xi_{x_0} \in N_\epsilon(0_{x_0})$.*

Proof Let $u : \mathbb{R}^N \to T_{x_0}(M)$ be a linear frame tangent to M at x_0 (an $\mathbb{R}$-linear isomorphism). We set

$$\varphi = \left(x^1, \ldots, x^N\right) = u^{-1} \circ \left(\mathrm{Exp}_{x_0} : N_\epsilon(0_{x_0}) \to U\right)^{-1} : U \to \mathbb{R}^N \tag{2.21}$$

so that (U, φ) is a local coordinate system about x_0 (and $\varphi(x_0) = 0$). Let $v \in \mathcal{H}_x$ and let us set $Y_x = \tau_x(v)$. We start by showing that Y is C^∞. Indeed, let $\varphi = (x^1, \ldots, x^N)$ be a contact normal coordinate system on U centered at x_0 and let us set $Y = Y^j \partial/\partial x^j$. By definition, Y is $\hat{\nabla}$-parallel along each parabolic geodesic $\lambda_{X,\lambda}$ with $X + \lambda\xi_{x_0} \in \mathcal{H}_{x_0}$. This means that $\mathbf{u}(t, X, \lambda) = Y^j(\gamma_{X,\lambda}(t))e_j$ is C^∞ with respect to t and satisfies

$$\frac{\partial u^j}{\partial t}(t, X, \lambda) = -\Gamma^j_{ik}\big(\gamma_{X,\lambda}(t)\big)\frac{d\gamma^i_{X,\lambda}}{dt}(t)u^k(t, X, \lambda), \tag{2.22}$$

$$u^j(0, X, \lambda) = Y^j(x_0). \tag{2.23}$$

By standard results in ODE theory, the solution to the Cauchy problem (2.22)–(2.23) depends smoothly on parameters. Also, for any $\mathbf{x} \in \varphi(U)$

$$\big(Y^j \circ \varphi^{-1}\big)(\mathbf{x}) = X^j\big(\gamma_{X,\lambda}(1)\big) = u^j(1, X, \lambda),$$

where $X = \Pi_{\mathcal{H},x_0} u(\mathbf{x})$ and $\lambda = \eta_{x_0}(u(\mathbf{x}))$, hence Y^j is C^∞ in a neighborhood of $x = \varphi^{-1}(\mathbf{x})$. Then each ξ_a is C^∞ on U. Finally (by the Ricci condition $\hat{\nabla} g = 0$) each τ_x is a linear isometry of $(\mathcal{H}_{x_0}, g_{x_0})$ into $(\mathcal{H}_x, g_x)$. □

Definition 2.6 A local frame $\{\xi_a : 1 \le a \le 2n\} \subset C^\infty(U, \mathcal{H})$ is *special* if (i) $g(\xi_a, \xi_b) = \delta_{ab}$ for any $1 \le a, b \le 2n$ and (ii) each ξ_a is $\hat{\nabla}$-parallel along any parabolic geodesic $\gamma_{X,\lambda}$.

Let $\{\xi_a : 1 \le a \le 2n\}$ be a special local frame of $\mathcal{H}$, defined on an open neighborhood $U \subset M$ of a point $x_0 \in M$ such that the parabolic exponential map $\operatorname{Exp}_{x_0} : N_\epsilon(0_{x_0}) \to U$ is a diffeomorphism. Let $\{\eta^a : 1 \le a \le 2n\} \subset \Omega^1(U)$ be the local 1-forms on M defined by $\eta^a(\xi_b) = \delta^a_b$ and $\eta^a(\xi) = 0$. Let us consider the linear frame $u : \mathbb{R}^N \to T_{x_0}(M)$ given by $u^{-1}(v) = \eta^i_{x_0}(v)e_i$ for any $v \in T_{x_0}(M)$ where $\eta^0 = \eta$ and $\{e_i : 0 \le i \le 2n\} \subset \mathbb{R}^N$ is the canonical linear basis. Finally, let us define $\varphi : U \to \mathbb{R}^N$ by (2.21).

Definition 2.7 $\varphi = (x^1, \ldots, x^N) : U \to \mathbb{R}^N$ is referred to as a *contact normal* local coordinate system centered at x_0.

When $T_{1,0}(M)$ is integrable (and $\varphi_{x_0} u(e_\alpha) = u(e_{\alpha+n})$ for any $1 \le \alpha \le n$) $(x^1, \ldots, x^N)$ are the pseudohermitian normal coordinates introduced by D. Jerison & J.M. Lee, [32], p. 311.

Lemma 2.8 *Let (U, x^i) be a local system of contact normal coordinates on M, associated with the special local frame $\{\xi_a : 1 \le a \le 2n\}$ of $\mathcal{H}$, centered at the point $x \in U$. Then*

(i) *any λ-parabolic geodesic of initial data (x, X) with $X \in \mathcal{H}_x$ is locally given by*

$$\gamma^0(t) = \lambda t^2, \qquad \gamma^a(t) = \lambda^a t, \quad 1 \le a \le 2n, \tag{2.24}$$

where $X = \lambda^a \xi_{a,x}$. Conversely, any local coordinate system (U, x^i) possessing the above property is the contact normal coordinate system determined by the frame $\{\xi_a : 1 \le a \le 2n\}$.

(ii) *If $\hat{\Gamma}^i_{jk} \in C^\infty(U)$ are the local coefficients of the generalized Tanaka–Webster connection $\hat{\nabla}$ with respect to the contact normal coordinate system (U, x^i) centered at x, then*

$$\hat{\Gamma}^i_{ab}(x) + \hat{\Gamma}^i_{ba}(x) = 0. \tag{2.25}$$

Formula (2.25) should be paralleled to Proposition 8.4 in [36], Vol. I, p. 148. Further identities (satisfied by $\hat{\Gamma}^i_{jk}$ at the center x) are derived in Lemma 7.9.

Proof of Lemma 2.8 (i) Let $\gamma_{X,\lambda}(t) = \mathrm{Exp}_x(tX + t^2\lambda\xi_x)$ and $\gamma^i = x^i \circ \gamma_{X,\lambda}$ for any $0 \le i \le 2n$ so that (2.24) holds, and conversely.

(ii) For any $(\lambda^0, \dots, \lambda^{2n}) \in \mathbb{R}^{2n+1}$ the curve locally given by (2.24) with respect to a contact normal coordinate system is a λ-parabolic geodesic (with $\lambda = \lambda^0$). Then (by (2.12))

$$\frac{d^2\gamma^i}{dt^2} + \big(\hat{\Gamma}^i_{jk} \circ \gamma\big)\frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt} = 2\lambda\big(\xi^i \circ \gamma\big) \tag{2.26}$$

or (by (2.24))

$$\lambda^a\lambda^b\hat{\Gamma}^0_{ab}\big(\gamma_{X,\lambda}(t)\big) = 2\lambda\big\{\xi^0\big(\gamma_{X,\lambda}(t)\big) - 1\big\} + O(t),$$
$$\lambda^a\lambda^b\hat{\Gamma}^c_{ab}\big(\gamma_{X,\lambda}(t)\big) = 2\lambda\xi^c\big(\gamma_{X,\lambda}(t)\big) + O(t),$$

hence (for $t = 0$)

$$\lambda^a\lambda^b\hat{\Gamma}^0_{ab}(x) = 2\lambda\big\{\xi^0(x) - 1\big\}, \qquad \lambda^a\lambda^b\Gamma^c_{ab}(x) = 2\lambda\xi^c(x)$$

yielding

$$\xi^0(x) = 1, \qquad \xi^a(x) = 0, \quad 1 \le a \le 2n, \tag{2.27}$$

and then (2.25). □

As a byproduct, one obtained (2.27), i.e., $\xi_x = (\partial/\partial x^1)_x$. One may also show that $\xi_{a,x} = (\partial/\partial x^a)_x$ for any $1 \le a \le 2n$. Indeed,

$$d_{\varphi(x)}\varphi^{-1} = d_{\varphi(x)}(\mathrm{Exp}_x \circ u) = (d_{0_x}\mathrm{Exp}_x) \circ (d_{\varphi(x)}u) = F_{0_x} \circ (d_{\varphi(x)}u),$$

hence (as $u^i(X^0, \dots, X^{2n}) = X^j\xi^i_j(x)$ with $\xi_j = \xi^i_j, \partial/\partial x^i$ and $\xi_0 = \xi$)

$$\big(\partial/\partial x^a\big)_x = \big(d_{\varphi(x)}\varphi^{-1}\big)\big(\partial/\partial X^a\big)_{\varphi(x)} = \xi^i_a(x)F_{0_x}\big(\partial/\partial y^i_x\big)_{0_x} = \xi_{a,x}. \qquad \square$$

3 Pseudoharmonic and Levi Harmonic Maps

3.1 Basic Notions

Let $(M, (\varphi, \xi, \eta, g))$ be a real $(2n+1)$-dimensional almost contact semi-Riemannian manifold and (M', g') a semi-Riemannian manifold. Let $f : M \to M'$ be a C^∞ map and $f^{-1}T(M') \to M$ the pullback of $T(M')$ by f. Let $\nabla'^f = f^{-1}\nabla'$ be the pullback of the Levi-Civita connection ∇' of (M', g'), i.e., the connection in the vector bundle $f^{-1}T(M') \to M$ induced by ∇'. If (U, x^i) and (V, y^α) are local coordinate systems on M and N such that $f(U) \subset V$ then ∇'^f is locally described by

$$\nabla'^f_{\partial/\partial x^j} \big(\partial/\partial y^\beta\big)^f = \frac{\partial f^\alpha}{\partial x^j} \big(\Gamma'^\gamma_{\alpha\beta} \circ f\big)\big(\partial/\partial y^\gamma\big)^f$$

where $Y^f = Y \circ f \in C^\infty(f^{-1}(V), f^{-1}T(M'))$ denotes the natural lift of $Y \in \mathfrak{X}(V)$ and $\Gamma'^\gamma_{\alpha\beta}$ are the Christoffel symbols of (M', g'). Similarly, any linear connection D on M' admits a well-defined pullback $D^f = f^{-1}D$ to a connection in the vector bundle $f^{-1}T(M')$. Let $\mathcal{H} = \mathrm{Ker}(\eta)$ and $J = \varphi|_{\mathcal{H}}$ be the almost CR structure underlying (φ, ξ, η, g). The second fundamental form β_f of f is given by

$$\beta_f(X, Y) = \nabla'^f_X f_* Y - f_* \nabla_X Y, \quad X, Y \in \mathfrak{X}(M). \tag{3.1}$$

Here ∇ is the Levi-Civita connection of (M, g). Also $f_* X \in C^\infty(f^{-1}T(M'))$ is given by $(f_* X)(x) = (d_x f)X_x \in T_{f(x)}(M')$ for any $x \in M$ and any $X \in \mathfrak{X}(M)$. Next let $\tau_{\mathcal{H}}(f) \in C^\infty(f^{-1}T(M'))$ be given by

$$\tau_{\mathcal{H}}(f) = \mathrm{trace}_g(\Pi_{\mathcal{H}} \beta_f), \tag{3.2}$$

where $\Pi_{\mathcal{H}} \beta_f$ is the restriction of β_f to $\mathcal{H} \otimes \mathcal{H}$.

Definition 3.1 A C^∞ map $f : M \to M'$ is *Levi harmonic* with respect to $\mathcal{H} = \mathrm{Ker}(\eta)$ if $\tau_{\mathcal{H}}(f) = 0$.

This is similar to the construction in [6] (where ∇ is replaced by $\hat{\nabla}$) except that we consider the ordinary second fundamental form of f (cf., e.g., [23]). The contact analog $\tau_{\mathcal{H}}(f)$ of the tension tensor $\tau(f)$ in [23] is defined in terms of covariant derivatives along $\mathcal{H}$ (the derivatives in the "bad" real direction ξ are dropped) and traces are taken with respect to the Levi form (rather than the full metric g). As another generalization of the ideas in [6] we set

$$\hat{\beta}_f(X, Y) = \big(f^{-1}\hat{\nabla}'\big)_X f_* Y - f_* \hat{\nabla}_X Y, \quad X, Y \in \mathfrak{X}(M), \tag{3.3}$$

$$\hat{\tau}_{\mathcal{H}}(f) = \mathrm{trace}_g(\Pi_{\mathcal{H}} \hat{\beta}_f), \tag{3.4}$$

and adopt

Definition 3.2 A C^∞ map $f : M \to M'$ is *pseudoharmonic* if $\hat{\tau}_{\mathcal{H}}(f) = 0$.

3.2 Pseudohermitian Maps

By a celebrated result of A. Lichnerowicz, [38], any holomorphic map of compact Kählerian manifolds is harmonic and an absolute minimum within its homotopy class. As emphasized by H. Urakawa, [52], Lichnerowicz's theorem lacks an appropriate CR analog. Indeed, CR maps of strictly pseudoconvex CR manifolds are not harmonic (with respect to the Webster metrics) in general (cf. [52]). Another attempt to bridge Kählerian and pseudohermitian geometry is to parallel CR and subelliptic harmonic maps (cf. [6]). As it turns out, a CR map f is not subelliptic harmonic either, unless f is pseudohermitian. In this section we consider CR, pseudohermitian, and isopseudohermitian maps and prove a result imitative of [6] (cf. Theorem 3.9 below). Also, by using the parabolic exponential formalism built in Sect. 2.3, we prove a pseudohermitian analog to a result by K. Nomizu, [30], p. 113 (on fixed point sets of isometries of a Riemannian manifold). Only the ideas in [6] are extended since, as explained in Sect. 1, pseudo and Levi harmonicity are logically inequivalent notions.

Definition 3.3 A C^∞ map $f : M \to M'$ of almost CR manifolds is a *CR map* if

$$(d_x f)\mathcal{H}_x \subset H\big(M'\big)_{f(x)}, \qquad (d_x f) \circ J_x = J'_{f(x)} \circ (d_x f), \tag{3.5}$$

for any $x \in M$.

Typical examples of CR maps are gotten as traces of holomorphic maps of Kählerian manifolds on real hypersurfaces. Precisely, let $\overline{M}$ be a Kählerian manifold. Any orientable real hypersurface $M \subset \overline{M}$ admits a natural almost contact metric structure (cf., e.g., [10]). If $M' \subset \overline{M}'$ is another oriented real hypersurface in the Kählerian manifold $\overline{M}'$ and $F : \overline{M} \to \overline{M}'$ is a holomorphic map such that $F(M) \subset M'$ then $f \equiv F|_M : M \to M'$ is a CR map. It should be emphasized that, in spite of our metric approach (where the wealth of additional first-order geometric structure (φ, ξ, η, g) is meant to "compensate" for the lack of integrability of $T_{1,0}(M)$) the property (3.5) is tied to the almost CR structures alone. In particular, the statements above hold true for traces of holomorphic maps among indefinite Kählerian manifolds (cf. E. Barros & A. Romero, [8], for definitions and examples). Indeed, let $\overline{M}$ be an indefinite Kählerian manifold and $M \subset \overline{M}$ an orientable real hypersurface. The indefinite Kähler structure of $\overline{M}$ induces on M an almost contact semi-Riemannian structure (cf. A. Bejancu & K.L. Duggal, [9]).

Let θ and θ' be pseudohermitian structures on the almost CR manifolds M and M', respectively. If $f : M \to M'$ is a CR map then $f^*\theta' = \mu\theta$ for some $\mu \in C^\infty(M)$.

Definition 3.4 A CR map f is *pseudohermitian* if $\mu = c$ for some $c \in \mathbb{R}$. Also f is *isopseudohermitian* if $c = 1$.

We shall need

Lemma 3.5 (i) *Let M be a nondegenerate almost CR manifold and η a contact form on M. Let $(M', \varphi', \xi', \eta', g')$ be a contact Riemannian manifold and $f : M \to M'$ an isopseudohermitian immersion of (M, η) into (M', η') (i.e., f is an immersion, a CR map, and $f^*\eta' = \eta$) such that $f(M)$ is tangent to ξ'. Let $T_x(M)^\perp$ be the orthogonal* *complement of* $(d_x f)T_x(M)$ *in* $T_{f(x)}(M')$ *with respect to* $g'_{f(x)}$ *for all* $x \in M$. *If* $\tan_x : T_{f(x)}(M') \to T_x(M)$ *is the natural projection with respect to the direct sum decomposition* $T_{f(x)}(M') = [(d_x f)T_x(M)] \oplus T_x(M)^\perp$ *then* $\hat{\nabla}$ *defined by*

$$(\hat{\nabla}_X Y)_x = \tan_x \big(\hat{\nabla}'_X Y\big)_{f(x)}, \quad X, Y \in \mathfrak{X}(M),\ x \in M, \tag{3.6}$$

is the generalized Tanaka–Webster connection of $(M, \varphi, \xi, \eta, g)$ *where* $\varphi = \varphi'|_{\mathrm{Ker}(\eta)}$, $\xi = \xi' \circ f$ *and* $g = f^* g'$.

(ii) *Let* $(M, (\varphi, \xi, \eta, g))$ *be a contact metric manifold and* $f : M \to M$ *a pseudohermitian transformation, i.e., a CR diffeomorphism such that* $f^*\eta = c\eta$ *for some* $c \in \mathbb{R}$. *If* $\gamma : (-\delta, \delta) \to M$ *is a* λ*-parabolic geodesic then* $f \circ \gamma : (-\delta, \delta) \to M$ *is a* λ'*-parabolic geodesic with* $\lambda' = c\lambda$.

In the case where the almost CR structure $T_{1,0}(M)$ is integrable, part (ii) in Lemma 3.5 is Lemma 11.9 in E. Barletta et al. [7], p. 205. Cf. also J. Masamune et al. [18].

Proof of Lemma 3.5 (i) If $X, Y \in \mathfrak{X}(M)$ there exist C^∞ extensions $X', Y' \in \mathfrak{X}(M')$, i.e., $X' \circ f = X$ and $Y' \circ f = Y$. Then $(\hat{\nabla}'_{X'} Y') \circ f$ does not depend upon the choice of extensions, and the notation $(\hat{\nabla}'_X Y) \circ f$ is legitimate. A calculation shows that

$$\big(\hat{\nabla}'_X Y\big) \circ f = \big(f^{-1}\hat{\nabla}'\big)_X f_* Y. \tag{3.7}$$

Let $\hat{\nabla}$ be defined by (3.6). Arguments similar to the proof of Theorem 6 in [16], p. 187, show that $\hat{\nabla}$ satisfies (2.8)–(2.11), hence (by the uniqueness part in Tanno's result, cf. [51]) $\hat{\nabla}$ is precisely the generalized Tanaka–Webster connection.

(ii) The proof closely follows the calculations in [7], pp. 205–206. One defines the maps $f_\rightarrow : \mathfrak{X}(M) \to \mathfrak{X}(M)$ and $f_\rightarrow \hat{\nabla} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by setting

$$(f_\rightarrow X)_y = (d_{f^{-1}(y)} f) X_{f^{-1}(y)}, \quad y \in M,$$
$$(f_\rightarrow \hat{\nabla})_X Y = (f_\rightarrow)^{-1} \hat{\nabla}_{f_\rightarrow X} f_\rightarrow Y, \quad X, Y \in \mathfrak{X}(M),$$

so that $f_\rightarrow$ is a module isomorphism and $f_\rightarrow \hat{\nabla}$ a linear connection on M. Exploiting the identity

$$f^* g = cg + c(c-1)\eta \otimes \eta,$$

one readily checks that (2.8)–(2.11) hold with $\hat{\nabla}$ replaced by $f_\rightarrow \hat{\nabla}$, hence $f_\rightarrow \hat{\nabla} = \hat{\nabla}$. Finally, if γ is a λ-parabolic geodesic and $\gamma_f \equiv f \circ \gamma$ then

$$\hat{\nabla}_{\dot{\gamma}_f} \dot{\gamma}_f = 2c\lambda(\xi \circ \gamma_f). \qquad \square$$

Given a contact metric manifold M, let $f : M \to M$ be an isopseudohermitian map and $\mathrm{Fix}(f) = \{x \in M : f(x) = x\}$ the set of its fixed points. Let $d_T f \equiv (df)_{\mathcal{H},\mathcal{H}} = f_* : \mathcal{H} \to f^{-1}\mathcal{H}$. For each fixed point $x \in \mathrm{Fix}(f)$ we set

$$\mathrm{Fix}(d_T f)_x = \big\{X \in \mathcal{H}_x : (d_x f)X = X\big\},$$
$$\mathrm{Fix}(d_x f) = \big\{X + \lambda\xi_x \in T_x(M) : X \in \mathrm{Fix}(d_T f)_x\ \lambda \in \mathbb{R}\big\}.$$

Then $\mathrm{Fix}(d_x f)$ is a linear subspace of $T_x(M)$. Also, as $f : M \to M$ is a CR map, $\mathrm{Fix}(d_T f)_x$ is J_x-invariant where $J_x = \varphi|_{\mathcal{H}_x}$. In particular, $\dim_{\mathbb{R}} \mathrm{Fix}(d_T f)_x = 2r$ for some $r \geq 0$ (eventually depending on x). By the proof of Theorem 2.4, there is $\epsilon > 0$ and an open neighborhood $U \subset M$ of x such that $\mathrm{Exp}_x : N_\epsilon(0_x) \to U$ is a diffeomorphism, where $N_\epsilon(0_x) = \{X + \lambda\xi_x : X \in \mathcal{H}_x, \|X\| < \epsilon, |\lambda| < \epsilon\}$. Then

Theorem 3.6 *Let $f : M \to M$ be an isopseudohermitian transformation.*

(i) *If $X \in \mathcal{H}_x$ and $\lambda \in \mathbb{R}$ are such that $X + \lambda\xi_x \in \mathrm{Fix}(d_x f) \cap N_\epsilon(0_x)$ then the λ-parabolic geodesic $\gamma : (-2, 2) \to M$ of initial data (x, X) consists of fixed points of f, i.e., $\gamma(t) \in \mathrm{Fix}(f)$ for any $|t| < 2$.*
(ii) *The parabolic exponential map Exp_x is a diffeomorphism of $\mathrm{Fix}(d_x f) \cap N_\epsilon(0_x)$ onto $\mathrm{Fix}(f) \cap U$.*
(iii) *Each connected component L of the fixed point set $\mathrm{Fix}(f)$ is a $(2r + 1)$-dimensional submanifold of M, where $2r = \dim_{\mathbb{R}} \mathrm{Fix}(d_T f)_x$ for any $x \in L$. Moreover, L is tangent to the Reeb field ξ. Let us set*

$$\varphi_{L,x} = \varphi_x|_{\mathrm{Fix}(d_x f)}, \qquad \eta_L = i^*\eta, \qquad g_L = i^*g,$$

where $i : L \to M$ is the inclusion. Then either L is a maximal integral curve of ξ (when $r = 0$) or $(\varphi_L, \xi, \eta_L, g_L)$ is a contact metric structure on L (when $r \geq 1$) such that $i : L \to M$ is an isopseudohermitian immersion.
(iv) *Assume that $r \geq 1$. Each λ-parabolic geodesic of L such that $\dot\gamma(0) \in H(L)_{\gamma(0)}$ is a λ-parabolic geodesic of M.*

Proof (i) Let us set $\beta(t) = f(\gamma(t))$ so that (by Lemma 3.5) β is a λ-parabolic geodesic. Note that $\beta(0) = x$ and $\dot\beta(0) = (d_x f)\dot\gamma(0) = X$, hence $f \circ \gamma = \beta = \gamma_{X,\lambda} = \gamma$.

(ii) It suffices to check that $\mathrm{Exp}_x(\mathrm{Fix}(d_x f) \cap N_\epsilon(0_x)) = \mathrm{Fix}(f) \cap U$. To this end let $y \in \mathrm{Exp}_x(\mathrm{Fix}(d_x f) \cap N_\epsilon(0_x))$. Then there exist $X \in \mathcal{H}_x$ and $\lambda \in \mathbb{R}$ such that $X + \lambda\xi_x \in \mathrm{Fix}(d_x f) \cap N_\epsilon(0_x)$ and $y = \mathrm{Exp}_x(X + \lambda\xi_x) = \gamma_{X,\lambda}(1) \in \mathrm{Fix}(f) \cap U$ (by part (i) in Theorem 3.6). Vice versa, let $y \in \mathrm{Fix}(f) \cap U$ and $X + \lambda\xi_x \in N_\epsilon(0_x)$ such that $\mathrm{Exp}_x(X + \lambda\xi_x) = y$. Let $\gamma \equiv \gamma_{X,\lambda}$ be the λ-parabolic geodesic of initial data (x, X) and let us set

$$\beta(t) = f\big(\gamma(t)\big), \qquad \alpha(t) = \mathrm{Exp}_x\big(t(d_x f)X + t^2\lambda\xi_x\big).$$

By Lemma 3.5 and the proof of Theorem 2.2 both α and β are λ-parabolic geodesics of initial data $(x, (d_x f)X)$, hence $\alpha = \beta$. Finally, as y is a fixed point of f one has

$$\begin{aligned}\mathrm{Exp}_x(X + \lambda\xi_x) &= y = f(y) = f\big(\mathrm{Exp}_x(X + \lambda\xi_x)\big) = f\big(\gamma(1)\big)\\ &= \beta(1) = \alpha(1) = \mathrm{Exp}_x\big((d_x f)X + \lambda\xi_x\big)\end{aligned}$$

so that $X + \lambda\xi_x \in \mathrm{Fix}(d_x f)$.

(iii) Let $x \in \mathrm{Fix}(f)$ and let L be the connected component of $\mathrm{Fix}(f)$ through x. The set $N_\epsilon(0_x) \cap \mathrm{Fix}(d_x f)$ is open in $\mathrm{Fix}(d_x f)$ (as a topological subspace of $T_x(M)$). On the other hand $N_\epsilon(0_x) \cap \mathrm{Fix}(d_x f) = \{X + \lambda\xi_x \in \mathrm{Fix}(d_x f) : \|X\| < \epsilon, |\lambda| < \epsilon\}$, hence $N_\epsilon(0_x) \cap \mathrm{Fix}(d_x f)$ is a connected subset of $\mathrm{Fix}(d_x f)$. By part (ii) in Theorem 3.6 the map $\mathrm{Exp}_x : N_\epsilon(0_x) \cap \mathrm{Fix}(d_x f) \to U \cap \mathrm{Fix}(f)$ is a diffeomorphism, hence $U \cap \mathrm{Fix}(f)$ is a connected subset of $\mathrm{Fix}(f)$ passing through the point x. The maximality of L then yields $U \cap \mathrm{Fix}(f) \subset L$. In particular, $U \cap \mathrm{Fix}(f)$ is the trace of $U \cap \mathrm{Fix}(f)$ on L and $U \cap \mathrm{Fix}(f)$ is open in $\mathrm{Fix}(f)$, hence $U \cap \mathrm{Fix}(f)$ is open in L too. At this point one may build a local chart on L with domain $U \cap \mathrm{Fix}(f)$, as follows. Let $u : \mathbb{R}^{2n+1} \to T_x(M)$ be an arbitrary linear frame tangent to M at x and

$$\varphi = \left(x^1, \ldots, x^{2n+1}\right) \equiv u^{-1} \circ \left(\mathrm{Exp}_x : N_\epsilon(0_x) \to U\right)^{-1}$$

the corresponding local coordinate system on M about x. For each $y \in U$ the $\mathbb{R}$-linear isomorphism $h_y : T_y(M) \to \mathbb{R}^{2n+1}$ induces on $T_y(M)$ the same topology as the norm $\|v\| = g_y(v, v)^{1/2}$, $v \in T_y(M)$. As f is a diffeomorphism $r = \frac{1}{2}\dim_{\mathbb{R}} \mathrm{Fix}(d_T f)_x =$ const. Hence, by eventually relabeling the coordinates $h_x(\mathrm{Fix}(d_x f)) = \mathbb{R}^{2r+1} = \{\xi \in \mathbb{R}^{2n+1} : \xi^a = 0, 2r + 2 \le a \le 2n + 1\}$. Then

$$\psi \equiv h_x \circ \left(\mathrm{Exp}_x : \mathrm{Fix}(d_x f) \cap N_\epsilon(0_x) \to \mathrm{Fix}(f) \cap U\right)^{-1}$$

is a local coordinate system on L about x. For each $X \in \mathcal{H}_x$ and $\lambda \in \mathbb{R}$ such that $X + \lambda\xi_x \in \mathrm{Fix}(d_x f) \cap N_\epsilon(0_x)$ we set as above

$$\gamma_{X,\lambda}(t) = \mathrm{Exp}_x\left(tX + t^2\lambda\xi_x\right), \quad |t| < 2,$$

and consider the curve $\alpha : (-\epsilon, \epsilon) \to L$ given by

$$\alpha(\lambda) = \gamma_{0_x,\lambda}(1), \quad |\lambda| < \epsilon.$$

The fact that α lies in L is a consequence of part (i) in Theorem 3.6. If $a(\lambda) = \lambda\xi_x$ then (with the notation in the proof of Theorem 2.4) $\dot{a}(\lambda) = F^{-1}_{a(\lambda)}(\xi_x)$, hence (again by the proof of Theorem 2.4)

$$T_x(L) \ni \dot{\alpha}(0) = (d_{0_x} \mathrm{Exp}_x) \circ F^{-1}_{0_x}(\xi_x) = \xi_x$$

so that $\xi \in \mathfrak{X}(L)$. Similarly, $T_x(L) = \mathrm{Fix}(d_x f)$ and L is an almost CR manifold with the almost CR structure $H(L)_x = \mathrm{Fix}(d_T f)_x$.

(iv) Let γ be a λ-parabolic geodesic of L such that $\gamma(0) = x \in L$ and $X \equiv \dot{\gamma}(0) \in H(L)_x = \mathrm{Fix}(d_T f)_x$. Next, for sufficiently small $\delta > 0$ we consider the λ-parabolic geodesic in M given by $\overline{\gamma}(t) = \mathrm{Exp}(tX + t^2\lambda\xi_x)$, $|t| < \delta$, so that (by Theorem 3.6) $\overline{\gamma}(t) \in \mathrm{Fix}(f) \cap U$. If $\hat{\nabla}^L$ is the generalized Tanaka–Webster connection of L then (by (3.6))

$$(\hat{\nabla}^L_{\dot{\overline{\gamma}}}\dot{\overline{\gamma}})_{\overline{\gamma}(t)} = \tan_{\overline{\gamma}(t)}(\hat{\nabla}_{\dot{\overline{\gamma}}}\dot{\overline{\gamma}})_{\overline{\gamma}(t)} = 2\lambda \tan_{\overline{\gamma}(t)}(\xi_{\overline{\gamma}(t)}) = 2\lambda\xi_{\overline{\gamma}(t)}$$

so that $\overline{\gamma}$ is also a λ-parabolic geodesic of L, of the same initial data (x, X). Therefore $\gamma = \overline{\gamma}$, i.e., γ is a λ-parabolic geodesic in M. □

Corollary 3.7 *Let $f : M \to M$ be an isopseudohermitian transformation of the contact metric manifold $(M, (\varphi, \xi, \eta, g))$ and L a connected component of the fixed point set of f. If $\dim(L) \geq 3$ then the inclusion $i : L \to M$ is pseudoharmonic*, *i.e.*, $\hat{\tau}_{H(L)}(i) = 0$.

Proof Let $x \in L$ and $X \in H(L)_x = \mathrm{Fix}(d_T f)_x$ such that $X \neq 0$. There are $\epsilon > 0$ and $\delta > 0$ such that for any $v \in T_x(L)$ with $\|v\| < \epsilon$ there is a unique geodesic $(-\delta, \delta) \to L$ of $\hat{\nabla}^L$ with initial conditions (x, v). Let us choose $0 < r < \epsilon/\|X\|$ and consider the unique geodesic $\gamma : (-\delta, \delta) \to L$ of $\hat{\nabla}^L$ such that $\gamma(0) = x$ and $\dot{\gamma}(0) = rX$. By (iv) in Theorem 3.6 the curve γ is also a geodesic of $\hat{\nabla}$, hence (by (3.7))

$$r^2 \hat{\beta}_{i,x}(X, X) = \hat{\beta}_{i,\gamma(0)}\big(\dot{\gamma}(0), \dot{\gamma}(0)\big) = \big\{\big(f^{-1}\hat{\nabla}\big)_{\dot{\gamma}} f_* \dot{\gamma} - f_* \hat{\nabla}^L_{\dot{\gamma}} \dot{\gamma}\big\}_{\gamma(0)} = 0. \qquad \square$$

Next we adopt

Definition 3.8 We say $(M, (\varphi, \xi, \eta, g))$ satisfies the *φ-condition* if

$$\nabla_{\varphi X} \varphi X + \nabla_X X = \varphi[\varphi X, X] \tag{3.8}$$

for any $X \in \mathcal{H}$.

We establish the following

Theorem 3.9 *Let $(M, \varphi, \xi, \eta, g)$ and $(M', \varphi', \xi', \eta', g')$ be two almost contact semi-Riemannian manifolds with $\dim(M) = 2n + 1$, satisfying the φ-condition. For each CR map $f : M \to M'$*

$$\tau_{\mathcal{H}}(f) = -\,\mathrm{trace}(\varphi \nabla \xi) \varphi'^f f_* \xi. \tag{3.9}$$

Here $\varphi'^f : f^{-1}T(M') \to f^{-1}T(M')$ is the pullback of φ' by f. If additionally (φ, ξ, η, g) is a contact semi-Riemannian structure then

$$\tau_{\mathcal{H}}(f) = -2n\mathfrak{s}\varphi'^f f_* \xi, \tag{3.10}$$

where $\mathfrak{s} = g(\xi, \xi)$. Hence f is Levi harmonic if and only if $f_* \xi = c\xi'$ for some $c \in \mathbb{R}$. If this is the case then f is a pseudohermitian map.

Proof The tangent bundle to any $(2n + 1)$-dimensional almost contact semi-Riemannian manifold M admits a local semi-orthonormal frame (a *φ-basis*), i.e., a frame of the form $\{\xi, E_\alpha, \varphi E_\alpha : 1 \leq \alpha \leq n\}$. By (2.2) if E_α is a spacelike (respectively, timelike) then φE_α is spacelike (respectively, timelike). In particular, a semi-Riemannian metric compatible with an almost contact structure has either signature $(2p+1, 2n-2p)$ or signature $(2p, 2n-2p+1)$, according to whether ξ is spacelike or timelike.

Let $\{\xi, E_\alpha, \varphi E_\alpha : 1 \leq \alpha \leq n\}$ be a φ-basis and let us set $\mathfrak{s}_\alpha = g(E_\alpha, E_\alpha) \in \{\pm 1\}$. As f is a CR map one has

$$\tau_{\mathcal{H}}(f) = \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{\nabla'^f_{E_\alpha} f_* E_\alpha - f_* \nabla_{E_\alpha} E_\alpha + \nabla'^f_{\varphi E_\alpha} f_* \varphi E_\alpha - f_* \nabla_{\varphi E_\alpha} \varphi E_\alpha\big\}$$

$$= \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ \nabla'^f_{E_\alpha} f_* E_\alpha + \nabla'^f_{\varphi E_\alpha} \varphi'^f f_* E_\alpha - f_*(\nabla_{E_\alpha} E_\alpha + \nabla_{\varphi E_\alpha} \varphi E_\alpha) \big\}.$$

Next (as both M and M' satisfy the φ-condition)

$$\tau_{\mathcal{H}}(f) = \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big(\varphi'^f f_* - f_* \varphi \big) [\varphi E_\alpha, E_\alpha].$$

Note that $[\varphi E_\alpha, E_\alpha] = -\varphi^2 [\varphi E_\alpha, E_\alpha] + g([\varphi E_\alpha, E_\alpha], \xi)\xi$ and $\varphi'^f f_* - f_* \varphi = 0$ on $\mathcal{H}$. Also

$$\begin{aligned} \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha g\big([\varphi E_\alpha, E_\alpha], \xi\big) &= \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ g(\nabla_{\varphi E_\alpha} E_\alpha, \xi) - g(\nabla_{E_\alpha} \varphi E_\alpha, \xi) \big\} \\ &= \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ -g(\nabla_{\varphi E_\alpha} \xi, E_\alpha) + g(\nabla_{E_\alpha} \xi, \varphi E_\alpha) \big\} \\ &= -\sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ g(\varphi \nabla_{\varphi E_\alpha} \xi, \varphi E_\alpha) + g(\varphi \nabla_{E_\alpha} \xi, E_\alpha) \big\} \\ &= -\operatorname{trace}(\varphi \nabla \xi) + \mathfrak{s} g(\varphi \nabla_\xi \xi, \xi) = -\operatorname{trace}(\varphi \nabla \xi), \end{aligned}$$

yielding (3.9). Let us assume from now on that M is a contact semi-Riemannian manifold. Then (by (2.5)) for any $X, Y \in \mathcal{H}$

$$\nabla_X \varphi Y - \varphi \nabla_X Y - \nabla_{\varphi X} Y - \varphi \nabla_{\varphi X} \varphi Y = 2g(X, Y)\xi.$$

In particular, for $Y = \varphi X$ one derives (3.8). Hence any contact pseudo metric manifold satisfies the φ-condition and then (3.9) must hold. Then (by (2.4))

$$\begin{aligned} \operatorname{trace}(\varphi \nabla \xi) &= \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ g(\varphi \nabla_{\varphi E_\alpha} \xi, \varphi E_\alpha) + g(\varphi \nabla_{E_\alpha} \xi, E_\alpha) \big\} \\ &= \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ g(\mathfrak{s} E_\alpha + h E_\alpha, E_\alpha) + g(\mathfrak{s} \varphi E_\alpha + h \varphi E_\alpha, \varphi E_\alpha) \big\} \\ &= 2n\mathfrak{s} + \operatorname{trace}(h) = 2n\mathfrak{s}, \end{aligned}$$

yielding (3.10). Consequently $\tau_{\mathcal{H}}(f) = 0$ if and only if $f_* \xi = c(\xi')^f$ for some $c \in \mathbb{R}$. Necessity is immediate. Conversely $\tau_{\mathcal{H}}(f) = 0$ implies $f_* \xi = \lambda (\xi')^f$ for some $\lambda \in C^\infty(M, \mathbb{R})$. Yet f is a CR map, hence $f^* \eta' = \lambda \eta$. Then

$$\begin{aligned} \big(d f^* \eta'\big)(\xi, \cdot) &= \xi(\lambda)\eta - d\lambda, \\ \big(d f^* \eta'\big)(\xi, X)_x &= \big(f^* d\eta'\big)(\xi, X) \\ &= \big(d\eta'\big)_{f(x)} \big((d_x f)\xi_x, (d_x f) X_x\big) \\ &= \lambda(x) \big(d\eta'\big)_{f(x)} \big(\xi'_{f(x)}, (d_x f) X_x\big) = 0, \end{aligned}$$

for any $X \in \mathfrak{X}(M)$ and $x \in M$ so that

$$d\lambda = \xi(\lambda)\eta. \tag{3.11}$$

Finally (using (3.11) twice)

$$\xi(\lambda)\eta \wedge d\eta = d\lambda \wedge d\eta = -d(d\lambda \wedge \eta) = -d\big(\xi(\lambda)\eta \wedge \eta\big) = 0$$

so that for any $X \in \mathcal{H}$

$$-\xi(\lambda)g(X,X) = \xi(\lambda)(d\eta)(X,\varphi X) = \xi(\lambda)(\eta \wedge d\eta)(\xi, X, \varphi X) = 0,$$

i.e., $d\lambda = 0$. □

3.3 Levi Harmonic Maps from Real Hypersurfaces of Indefinite Kähler Manifolds

Let M be an orientable real hypersurface of an (indefinite) Kähler manifold $(\overline{M}, J, g)$ of real dimension $2n+2$ equipped with the induced almost contact (semi-) Riemannian structure (φ, ξ, η, g). We recall the Gauss and Weingarten formulas

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y), \qquad \overline{\nabla}_X N = -AX, \tag{3.12}$$

where $\overline{\nabla}$ and ∇ are respectively the Levi-Civita connection of $\overline{M}$ and M. Also B is the second fundamental form and A the shape operator. Then

$$(\nabla_X \varphi)Y = \mathfrak{s}g(AX,Y)\xi - \eta(Y)AX, \qquad \nabla_X \xi = -\varphi AX. \tag{3.13}$$

ξ is commonly referred to as the *Hopf vector* field of M. Cf. [9], p. 551 (ξ in the current paper and that in [9] or [41] differ by a sign). M is a *Hopf hypersurface* (cf. [41, 44]) if ξ is an eigenvector of the shape operator, i.e., $A\xi = \alpha\xi$. Note that $\nabla_\xi \xi = -\varphi A\xi$ yields $\nabla_\xi \xi = 0 \Longleftrightarrow A\xi = \alpha\xi$. Next (by (3.13)) for any $X \in \operatorname{Ker}(\eta)$

$$(\nabla_X \varphi)\varphi X = \mathfrak{s}g(AX, \varphi X)\xi,$$

that is,

$$\nabla_X X + \varphi \nabla_X \varphi X = -\mathfrak{s}g(AX,\varphi X)\xi. \tag{3.14}$$

Replacing X by φX gives

$$\nabla_{\varphi X}\varphi X - \varphi(\nabla_{\varphi X} X) = \mathfrak{s}g(A\varphi X, X)\xi = \mathfrak{s}g(AX,\varphi X)\xi \tag{3.15}$$

for any $X \in \operatorname{Ker}(\eta)$. As a consequence of (3.14)–(3.15) the almost contact semi-Riemannian structure of an orientable real hypersurfaces in an indefinite Kähler manifold satisfies the φ-condition. Let $\{\xi, E_\alpha, \varphi E_\alpha : 1 \le \alpha \le n\}$ be a φ-*basis*. Then (by (3.13))

$$\operatorname{trace}(\varphi\nabla\xi) = \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ g(\varphi\nabla_{\varphi E_\alpha}\xi, \varphi E_\alpha) + g(\varphi\nabla_{E_\alpha}\xi, E_\alpha)\big\}$$

$$= \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ g(\nabla_{\varphi E_\alpha} \xi, E_\alpha) - g(\nabla_{E_\alpha} \xi, \varphi E_\alpha) \big\}$$

$$= \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ -g(\varphi A \varphi E_\alpha, E_\alpha) + g(\varphi A E_\alpha, \varphi E_\alpha) \big\}$$

$$= \sum_{\alpha=1}^{n} \mathfrak{s}_\alpha \big\{ g(A \varphi E_\alpha, \varphi E_\alpha) + g(A E_\alpha, E_\alpha) \big\}.$$

Therefore

$$\operatorname{trace}(\varphi \nabla \xi) = H - \mathfrak{s} g(A\xi, \xi) \tag{3.16}$$

where $H = \operatorname{trace}(A)$ is mean curvature of M in $\overline{M}$.

Let M be an orientable real hypersurface of a complex space form $\overline{M}(c)$, $c \neq 0$, equipped with the almost contact metric structure (φ, ξ, η, g) induced by the Kählerian structure of $\overline{M}(c)$. If $\mathcal{H} = \operatorname{Ker}(\eta) = (\mathbb{R}\xi)^\perp$ is integrable and each integral manifold is totally geodesic in $\overline{M}(c)$ then M is a *ruled hypersurface* (cf. M. Kimura, [34], M. Kimura & S. Maeda, [35]). If this is the case then (cf. [35])

$$A\xi = \mu\xi + \nu E, \quad \nu \neq 0, \quad AE = \nu\xi, \quad AX = 0,$$

for any X orthogonal to ξ and E, where $E \in (\mathbb{R}\xi)^\perp$ is a unit vector field and $\mu, \nu \in C^\infty(M, \mathbb{R})$. Then

$$H = \operatorname{trace}(A) = \mu = g(A\xi, \xi). \tag{3.17}$$

Finally (by (3.16)–(3.17) and Theorem 3.9)

Theorem 3.10 *Let M be an orientable real hypersurface of an (indefinite) Kähler manifold $\overline{M}$, equipped with the induced almost contact (semi-) Riemannian structure. Let $(M', \varphi', \xi', \eta', g')$ be an almost contact semi-Riemannian manifold which satisfying the φ-condition. Then for any CR map $f : M \to M'$*

$$\tau_{\mathcal{H}}(f) = \big(H - \mathfrak{s} g(A\xi, \xi)\big) \varphi'^{f} f_* \xi. \tag{3.18}$$

If M is a ruled real hypersurface of a complex space form $\overline{M}(c)$ with $c \neq 0$ then any CR map $f : M \to M'$ is Levi harmonic.

3.4 Levi Harmonic Maps from Generalized Cosymplectic Manifolds

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold. M is *almost cosymplectic* if the differential forms η and Φ are closed (cf. G. Goldberg & K. Yano, [27]). M is *cosymplectic* if it is almost cosymplectic and the almost contact structure (φ, ξ, η) is normal (equivalently $\nabla\phi = 0$). Cf. [10], p. 77. (Almost) cosymplectic manifolds have been extensively studied (cf. [14, 24, 42, 46]). An almost contact Riemannian manifold M is *quasi-cosymplectic* (cf. [11], and [15], p. 666) if

$$(\nabla_X \varphi)Y + (\nabla_{\varphi X}\varphi)\varphi Y = \eta(Y)\nabla_{\varphi X}\xi, \quad X, Y \in \mathfrak{X}(M). \tag{3.19}$$

The class of quasi-cosymplectic manifolds contains the classes of cosymplectic and almost cosymplectic manifolds. By a result of Z. Olszak (cf. [42], Lemma 2.2, p. 240) any almost cosymplectic manifold satisfies (3.19). Examples of quasi-cosymplectic manifolds which are not almost cosymplectic do exist (cf. [11], and [15], p. 668). By a result in [11] the Calabi quasi-Kähler structure on $\mathbb{R}^4 \times S^2$ yields a quasi-cosymplectic structure on $\mathbb{R}^2 \times S^2 \times \mathbb{R}$ which is not almost cosymplectic. Moreover, a result in [15] shows that $\mathbb{R}^3 \times S^2$ with the almost contact metric structure induced by the quasi-Kähler structure on $\mathbb{R}^4 \times S^2$ via the embedding of $\mathbb{R}^3 \times S^2$ in $\mathbb{R}^4 \times S^2$ provides another examples of the sort. Also $\mathbb{R}^5 \times \mathbb{C}P^3$ admits an almost contact metric structure which is quasi-cosymplectic but not almost cosymplectic (cf. [15], p. 668).

Let $(M, \varphi, \xi, \eta, g)$ be a quasi-cosymplectic manifold. Let us set $Y = \xi$ in (3.19). Then

$$\varphi \nabla_X \xi = -\nabla_{\varphi X}\xi, \quad \nabla_\xi \xi = 0. \tag{3.20}$$

Moreover, let $X \in \mathrm{Ker}(\eta)$ and let us set $Y = \varphi X$ in (3.19). Then $(\nabla_X \varphi)\varphi X = (\nabla_{\varphi X}\varphi)X$ or $\nabla_X X + \nabla_{\varphi X}\varphi X = \varphi[\varphi X, X]$. Thus any quasi-cosymplectic manifold satisfies the φ-condition. Besides (by (3.20))

$$\begin{aligned}
\mathrm{trace}(\varphi\nabla\xi) &= \sum_{\alpha=1}^{n}\big\{g(\varphi\nabla_{E_\alpha}\xi, E_\alpha) + g(\varphi\nabla_{\varphi E_\alpha}\xi, \varphi E_\alpha)\big\} \\
&= \sum_{\alpha=1}^{n}\big\{g(-\nabla_{\varphi E_\alpha}\xi, E_\alpha) + g(\nabla_{E_\alpha}\xi, \varphi E_\alpha)\big\} \\
&= -\sum_{\alpha=1}^{n}\big\{g(\nabla_{\varphi E_\alpha}\xi, E_\alpha) + g(\varphi\nabla_{E_\alpha}\xi, E_\alpha)\big\} = 0.
\end{aligned}$$

Consequently (by Theorem 3.9)

Theorem 3.11 *Any CR map $f : M \to M'$ among two quasi-cosymplectic manifolds is Levi harmonic.*

3.5 Levi Harmonicity of the Reeb Field

We need to recall the standard contact metric structure on (the total space of) the unit tangent sphere bundle (cf., e.g., [10]). Let (M, g) be an n-dimensional Riemannian manifold. With the notation in Sect. 2.3 the *Sasaki metric* G on $T(M)$ is given by

$$G(\beta X, \beta Y) = G(\gamma X, \gamma Y) = g(X, Y) \circ \pi, \qquad G(\beta X, \gamma Y) = G(\gamma X, \beta Y) = 0,$$

for any $X, Y \in C^\infty(\pi^{-1}TM)$. The *geodesic flow* $\xi \in \mathfrak{X}(T(M))$ is defined by $\xi = \beta(\mathcal{L})$. The *tangent sphere bundle* is $U(M, g) = \{v \in T(M) : g_{\pi(v)}(v, v) = 1\}$. The

Sasaki metric on $U(M,g)$ is $\tilde{G} = \iota^* G$ where $\iota : U(M,g) \to T(M)$ is the inclusion. Let $\tilde{\xi}$ be the pointwise restriction of ξ to $U(M,g)$. Then $\tilde{\xi} \in \mathfrak{X}(U(M,g))$ (any horizontal vector field on $T(M)$ is tangent to $U(M,g)$). Also $N = \gamma(\mathcal{L})$ is a unit normal field on $U(M,g)$. The *tangential lift* of $X \in C^\infty(\pi^{-1} TM)$ is defined as $\gamma X - g^\pi(X, \mathcal{L}) N \in \mathfrak{X}(U(M,g))$. For each tangent vector field $X \in \mathfrak{X}(M)$ we adopt the notation

$$X^H = \beta(X^\pi), \qquad X^V = \gamma(X^\pi), \qquad X^T = X^V - g^\pi(X^\pi, \mathcal{L}) N.$$

Clearly

$$T\big(U(M,g)\big) = H \oplus \gamma(\mathbb{R}\mathcal{L})^\perp$$

where $(\mathbb{R}\mathcal{L})_v^\perp$ is the orthogonal complement of $\mathbb{R}\mathcal{L}(u)$ in the inner product space $((\pi^{-1}TM)_v, g_v^\pi)$ for any $v \in U(M,g)$. The Sasaki metric $\tilde{G}$ on $U(M,g)$ is explicitly described by

$$\tilde{G}\big(X^H, Y^H\big) = g(X,Y) \circ \pi, \qquad \tilde{G}\big(X^H, Y^T\big) = 0,$$
$$\tilde{G}\big(X^T, Y^T\big) = g(X,Y) \circ \pi - g^\pi\big(X^\pi, \mathcal{L}\big) g^\pi\big(Y^\pi, \mathcal{L}\big),$$

for any $X, Y \in \mathfrak{X}(M)$. The standard contact metric structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ on $U(M,g)$ is given by

$$\overline{\varphi}\big(X^H\big) = X^T, \qquad \overline{\varphi}\big(X^T\big) = -X^H + (1/2) g^\pi\big(X^\pi, \mathcal{L}\big)\beta(\mathcal{L}),$$
$$\overline{\xi} = 2\tilde{\xi} = 2\beta(\mathcal{L}),$$
$$\overline{\eta}\big(X^H\big) = (1/2) g^\pi\big(X^\pi, \mathcal{L}\big), \qquad \overline{\eta}\big(X^T\big) = 0,$$
$$\overline{g} = (1/4)\tilde{G}.$$

Let $t \in (0, +\infty)$. The synthetic object

$$\big(\overline{\varphi}_t \equiv \overline{\varphi},\ \overline{\xi}_t \equiv (1/t)\overline{\xi},\ \overline{\eta}_t \equiv t\overline{\eta},\ \overline{g}_t \equiv t\overline{g} + \big(t^2 - t\big)\overline{\eta} \otimes \overline{\eta}\big) \tag{3.21}$$

is a *D-homothetic deformation* of $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. Then (3.21) is a g-natural contact metric structure on $U(M,g)$ (in the sense of [1] and [45]) and $\overline{g}_t$ is the g-natural metric defined by the parameters

$$a = t/4, \qquad b = c = 0, \qquad d = \big(t^2 - t\big)/4 = 4a^2 - a.$$

Let $(M, \varphi, \xi, \eta, g)$ be a K-contact manifold. Then (by Theorem 6.2 in [45])

$$\xi : (M, g, \eta) \to \big(U(M,g), \overline{g}_t, \overline{\eta}_t\big)$$

is a CR map. Moreover,

$$(d_x \xi) v = \xi_v^H + (\nabla_\xi \xi)_v^V = (1/2)\overline{\xi}_v = (t/2)(\overline{\xi}_t)_v, \quad v \equiv \xi_x,$$
$$\xi^* \overline{\eta}_t = \mu \eta, \quad \mu \equiv \big(\xi^* \overline{\eta}_t\big)(\xi) = t/2.$$

Thus (by Theorem 3.9)

Theorem 3.12 *Let $(M, \varphi, \xi, \eta, g)$ be a K-contact manifold and let $(\overline{\varphi}_t, \overline{\xi}_t, \overline{\eta}_t, \overline{g}_t)$ a D-homothetic deformation of the standard contact Riemannian structure of $U(M, g)$. Then $\xi : (M, g, \eta) \to (U(M, g), \overline{g}_t, \overline{\eta}_t)$ is Levi harmonic for any $t > 0$. Moreover, ξ is a pseudohermitian map (isopseudohermitian for $t = 2$).*

Note that $(\overline{\varphi}_t, \overline{\xi}_t, \overline{\eta}_t, \overline{g}_t)$ is K-contact if and only if M has constant sectional curvature $+1$ (and if this is the case the structure is Sasakian, cf. [10], p. 144).

Let $S^{2n+1} \subset \mathbb{C}^{n+1}$ be the sphere endowed with the canonical Sasakian structure $(\varphi_0, \xi_0, \eta_0, g_0)$. ξ_0 is a Hopf vector field on S^{2n+1}, hence

Corollary 3.13 *$\xi_0 : (S^{2n+1}, g_0, \eta_0) \to (U(S^{2n+1}, g_0), \overline{g}_t, \overline{\eta}_t)$ is a Levi harmonic map for any $t > 0$.*

If (M, g) is a Riemannian manifold of constant sectional curvature $+1$ the contact metric structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ on $U(M, g)$ is K-contact. Let $(\overline{g}'_t, \overline{\eta}'_t)$ be a D-homothetic deformation the corresponding contact metric structure on $T_1 T_1 M$ obtained by using the standard contact metric structure on $U(U(M, g), \overline{g})$. Then (by Theorem 3.12)

Corollary 3.14 *Let (M, g) be a Riemannian manifold of constant curvature $+1$. The geodesic flow $\overline{\xi}$ of $U(M, g)$ is a Levi harmonic map of $(U(M, g), \overline{g}, \overline{\eta})$ into $(U(U(M, g), \overline{g}), \overline{g}'_t, \overline{\eta}'_t)$.*

3.6 On a Theorem by E. Barletta

Given a contact metric manifold M, S. Tanno defined (cf. [51]) the generalized Tanaka–Webster connection $\hat{\nabla}$ by (2.7). Also $\hat{\nabla}$ may be described by the axioms (2.8)–(2.11). In Sect. 2.3 Tanno's formalism is readily transferred to the case of contact semi-Riemannian structures (φ, ξ, η, g) with $\mathfrak{s} = 1$. If $X \in \mathrm{Ker}(\eta)$ and $Y = \varphi X$ then (by (2.11))

$$(\hat{\nabla}_X \varphi)\varphi X = (\nabla_X \varphi)\varphi X + g(\nabla_X \xi, -X)\xi,$$
$$(\hat{\nabla}_{\varphi X} \varphi)X = (\nabla_{\varphi X} \varphi)X + g(\nabla_{\varphi X} \xi, \varphi X)\xi.$$

Hence (by (2.4))

$$(\hat{\nabla}_X \varphi)\varphi X - (\hat{\nabla}_{\varphi X} \varphi)X = (\nabla_X \varphi)\varphi X - (\nabla_{\varphi X} \varphi)X$$

so that (by (2.5))

$$(\hat{\nabla}_X \varphi)\varphi X - (\hat{\nabla}_{\varphi X} \varphi)X = 0, \quad X \in \mathcal{H}, \tag{3.22}$$

or

$$\hat{\nabla}_X X + \hat{\nabla}_{\varphi X} \varphi X = \varphi(\hat{\nabla}_{\varphi X} X - \hat{\nabla}_X \varphi X) = \varphi\big(\hat{T}(\varphi X, X) + [\varphi X, X]\big).$$

Then (by (2.9))

$$\hat{\nabla}_X X + \hat{\nabla}_{\varphi X} \varphi X = \varphi[\varphi X, X], \quad X \in \mathcal{H}, \tag{3.23}$$

i.e., $\hat{\nabla}$ satisfies the φ-condition. Let $f : M \to M'$ be a C^∞ map between two contact semi-Riemannian manifolds with $\mathfrak{s} = 1$ and consider the fundamental form $\hat{\beta}_f$ and the tension field $\hat{\tau}_{\mathcal{H}}(f)$ given by (3.3)–(3.4) and leading to the notion of pseudoharmonicity. By exploiting (3.23) a *verbatim* repetition of the proof of Theorem 3.9 leads to

$$\hat{\tau}_{\mathcal{H}}(f) = \text{trace}(\varphi\nabla\xi)\varphi'^f f_*\xi = \tau_{\mathcal{H}}(f).$$

We proved (similarly to Theorem 1.1 in [6])

Theorem 3.15 *Let M and M' be two contact semi-Riemannian manifolds whose Reeb fields are spacelike. For any CR map $f : M \to M'$*

$$\hat{\tau}_{\mathcal{H}}(f) = 2n\varphi'^f f_*\xi = \tau_{\mathcal{H}}(f).$$

Thus f is pseudoharmonic if and only if it is $\mathcal{H}$-harmonic. In particular, f is pseudoharmonic if and only if $f_*\xi = c\xi'^f$ for some $c \in \mathbb{R}$. If this is the case f is pseudohermitian.

4 Invariant Submanifolds and Levi Harmonicity

Let M be a submanifold of a $(2\overline{n}+1)$-dimensional almost contact metric manifold $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. M is an *invariant submanifold* of $\overline{M}$ if $\overline{\varphi}_p T_p(M) \subset T_p(M)$ for any $p \in M$. Two extreme cases may be distinguished (cf. [53]) as I) $\overline{\xi}$ is tangent to M (and then M is odd-dimensional, i.e., $\dim(M) = 2n+1$) or II) $\overline{\xi}$ is transverse to M (and then M is even-dimensional). When $\overline{M}$ is a contact Riemannian manifold case II does not occur (cf. [10], p. 122). In this section we only consider case I. Then M carries the induced almost contact Riemannian structure (φ, ξ, η, g) given by

$$\overline{\varphi} \circ i_* = i_* \circ \varphi, \qquad \eta = i^*\overline{\eta}, \qquad g = i^*\overline{g},$$

where $i : M \to \overline{M}$ is the inclusion. In particular, i is a CR map. By a result of G.D. Ludden (cf. [39]) if $\overline{M}$ is cosymplectic then M is cosymplectic, too. A result by D. Chinea (cf., e.g., Theorem 8.1 in [10]) shows that when $\overline{M}$ is a contact Riemannian manifold $i : M \to \overline{M}$ is a minimal isometric immersion. We establish

Theorem 4.1

(a) *Let $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ be an almost contact Riemannian manifold satisfying the φ-condition. Then $(M, \varphi, \xi, \eta, g)$ is an almost contact Riemannian manifold satisfying the φ-condition. Moreover, the map $i : M \to \overline{M}$ is Levi harmonic. The mean curvature of i is given by*

$$H = \big[1/(2n+1)\big]\alpha(\xi, \xi)$$

where α is the second fundamental form of i. If additionally $\overline{\xi}$ is geodesic then i is minimal.

(b) *Let* $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *be a quasi-cosymplectic manifold. Then* $(M, \varphi, \xi, \eta, g)$ *is a quasi-cosymplectic manifold and* i *is both Levi harmonic and minimal.*
(c) *Let* $(\overline{M}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ *be a contact Riemannian manifold. Then* $(M, \varphi, \xi, \eta, g)$ *is a contact Riemannian manifold and* i *is both Levi harmonic and minimal.*

Proof (a) Assume $\overline{M}$ satisfies the φ-condition. Then for any $X \in \mathcal{H}$

$$\overline{\nabla}_{\overline{\varphi}X}\overline{\varphi}X + \overline{\nabla}_X X = \overline{\varphi}[\overline{\varphi}X, X],$$

or (by the Gauss formula)

$$\nabla_{\varphi X}\varphi X + \alpha(\varphi X, \varphi X) + \nabla_X X + \alpha(X, X) = \varphi[\varphi X, X]. \tag{4.24}$$

The tangential and normal components of (4.24) are

$$\nabla_{\varphi X}\varphi X + \nabla_X X = \varphi[\varphi X, X], \tag{4.25}$$

$$\alpha(\varphi X, \varphi X) + \alpha(X, X) = 0, \tag{4.26}$$

for any $X \in \mathcal{H}$. Thus (by (4.25)) M satisfies the φ-condition. Therefore (as i is a CR map) equation (3.9) yields $\tau_{\mathcal{H}}(i) = \mathrm{trace}(\varphi\nabla\xi)\varphi\xi = 0$, that is, i is Levi harmonic. Let $\{\xi, E_\beta, \varphi E_\beta : 1 \le \beta \le n)$ be a φ-basis. Then (by (4.26)) the mean curvature of i is

$$H = \frac{1}{2n+1}\left\{\alpha(\xi, \xi) + \sum_{\beta=1}^{n}\big(\alpha(\varphi E_\beta, \varphi E_\beta) + \alpha(E_\beta, E_\beta)\big)\right\} = \frac{1}{2n+1}\alpha(\xi, \xi)$$

where $\alpha(\xi, \xi) = \overline{\nabla}_\xi \xi - \nabla_\xi \xi$. When $\overline{\xi}$ is geodesic, $\alpha(\xi, \xi) = 0$ and i is minimal.

(b) Assume $\overline{M}$ to be quasi-cosymplectic. Then (by (3.19))

$$(\overline{\nabla}_X\overline{\varphi})Y + (\overline{\nabla}_{\overline{\varphi}X}\overline{\varphi})\overline{\varphi}Y = \overline{\eta}(Y)\overline{\nabla}_{\overline{\varphi}X}\xi \tag{4.27}$$

for any $X, Y \in \mathfrak{X}(M)$. Next (by the Gauss formula)

$$\begin{aligned}(\overline{\nabla}_X\overline{\varphi})Y &= \overline{\nabla}_X\overline{\varphi}Y - \overline{\varphi}\overline{\nabla}_X Y = \overline{\nabla}_X\varphi Y - \overline{\varphi}\overline{\nabla}_X Y \\ &= \nabla_X\varphi Y + \alpha(X, \varphi Y) - \overline{\varphi}\big(\nabla_X Y + \alpha(X, Y)\big),\end{aligned}$$

that is,

$$(\overline{\nabla}_X\overline{\varphi})Y = (\nabla_X\varphi)Y + \alpha(X, \varphi Y) - \overline{\varphi}\alpha(X, Y). \tag{4.28}$$

Hence

$$(\overline{\nabla}_{\overline{\varphi}X}\overline{\varphi})\overline{\varphi}Y = (\nabla_{\varphi X}\varphi)\varphi Y + \alpha\big(\varphi X, \varphi^2 Y\big) - \overline{\varphi}\alpha(\varphi X, \varphi Y). \tag{4.29}$$

Besides

$$\overline{\eta}(Y)\overline{\nabla}_{\overline{\varphi}X}\xi = \eta(Y)\big(\nabla_{\varphi X}\xi + \alpha(\xi, \varphi X)\big). \tag{4.30}$$

Moreover, $W \in T_p(M)^\perp \subset T_p(\overline{M})$ implies $\overline{\varphi} W \in T_p(M)^\perp$ due to

$$\overline{g}(\overline{\varphi} W, X) = -\overline{g}(W, \overline{\varphi} X) = -\overline{g}(W, \varphi X) = 0.$$

Finally (by (4.28)–(4.30)) the tangential component of (4.27) is

$$(\nabla_X \varphi) Y + (\nabla_{\varphi X} \varphi)\varphi Y = \eta(Y) \nabla_{\varphi X} \xi$$

for any $X, Y \in \mathfrak{X}(M)$. Thus M is a quasi-cosymplectic manifold. Yet quasi-cosymplectic manifolds satisfy the φ-condition and their characteristic vector fields are geodesic. Therefore (by statement (a) in Theorem 4.1) i is Levi harmonic and minimal.

(c) If $\overline{M}$ is a contact Riemannian manifold then any invariant submanifold M inherits a contact Riemannian structure by (cf., e.g., [10], p. 122). Then (c) follows from (a). □

To give an example, let $M^{2n+3}(c)$ be a complete simply connected Sasakian manifold of constant ϕ-sectional curvature c. As is well known, $M^{2n+3}(c)$ is (up to an isometry) one of the Sasakian manifolds S^{2n+3}, $\mathbb{R}^{2n+3}$, or $D^{n+1} \times \mathbb{R}$ equipped with Sasakian structures of φ-sectional curvature $c > -3$, $c = -3$, and $c < -3$ respectively, where $D^{n+1} \subset \mathbb{C}^{n+1}$ is a simply connected bounded domain. Then $M^{2n+1}(c)$ is an invariant submanifold of $M^{2n+3}(c)$ (cf. [54], p. 328), hence the inclusion $i : M^{2n+1}(c) \to M^{2n+3}(c)$ is Levi harmonic.

5 Brieskorn Spheres

Let $\mathbb{C}^{n+1}$ with the Cartesian complex coordinates $z = (z_0, \dots, z_n)$ and $a = (a_0, \dots, a_n) \in \mathbb{Z}^{n+1}$ such that $a_j \geq 2$. Let us consider the polynomial $P_a(z) = \sum_{j=0}^{n} z_j^{a_j} \in \mathbb{C}[z]$. Then $B^{2n}(a) = \{z \in \mathbb{C}^{n+1} : P_a(z) = 0\}$ is an algebraic hypersurface in $\mathbb{C}^{n+1}$ and $B^{2n}(a) \setminus \{0\}$ is an n-dimensional complex submanifold. Let us set $\Sigma^{2n-1}(a) = B^{2n}(a) \cap S^{2n+1}$ (the *Brieskorn sphere* determined by a). By a result in [54], pp. 303–305, S^{2n+1} admits a Sasakian structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ (distinct from the standard Sasakian structure) such that $\Sigma^{2n-1}(2, \dots, 2)$ is an invariant submanifold of $(S^{2n+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$. Thus (by Theorem 4.1)

Corollary 5.1 *The inclusion $\Sigma^{2n-1}(2, \dots, 2) \to S^{2n+1}$ is Levi harmonic.*

Let $Q^n = \pi_0(B^{2n+2}(2, \dots, 2))$ be the complex quadric, where $\pi_0 : \mathbb{C}^{n+2} \setminus \{0\} \to \mathbb{C}P^{n+1}$ is the projection. Let $\pi : S^{2n+3} \to \mathbb{C}P^{n+1}$ be the Hopf fibration. The saturated set $P = \pi^{-1}(Q^n)$ is the total space of the circle bundle $S^1 \to P \to Q^n$. Then P is an invariant submanifold of the sphere S^{2n+3} equipped with the standard Sasakian structure (cf. [54], p. 328), hence

Corollary 5.2 *The inclusion $P \to S^{2n+3}$ is Levi harmonic.*

Let us consider the map $F:\mathbb{C}^{n+1}\to\mathbb{C}^{n+1}$ given by $F(z)=(-z_0,z_1,\dots,z_n)$ for any $z=(z_0,\dots,z_n)\in\mathbb{C}^{n+1}$. If $k\in\mathbb{Z}$, $k\geq 1$, and $a_0=2k$ then F descends to an isometric transformation $f:M\to M$ where $M\equiv\Sigma^{2n-1}(2k,a')$ and $a'=(a_1,\dots,a_n)$. Also M is a CR manifold of hypersurface type (of CR dimension $n-1$ and CR codimension 1) embedded in $\mathbb{C}^{n+1}$ as a CR submanifold of codimension 3, hence f is a CR isomorphism (as the trace of a holomorphic map). A proof that M is a contact manifold is given in [48]. Here we give a simpler proof (avoiding the explicit calculation of the form $\eta\wedge(d\eta)^{n-1}$). Indeed, let us consider the functions $f_j\in C^\infty(U)$ given by

$$f_j(z)=\frac{a_j}{a_0}\frac{z_j^{a_j-1}}{z_0^{a_0-1}},\quad 1\leq j\leq n\ (a_0=2k)$$

where $U=\{z\in B^{2n}(a): z_0\neq 0\}$ (an open subset of $B^{2n}(a)\setminus\{0\}$). The Cauchy–Riemann equations on U are

$$\frac{\partial u}{\partial\overline{z}_j}-\overline{f}_j(z)\frac{\partial u}{\partial\overline{z}_0}=0,\quad 1\leq j\leq n. \tag{5.31}$$

As a_0-1 is odd $f_j\circ F=-f_j$, hence the equations (5.31) are invariant under the transformation F (which is the same as saying that F maps the holomorphic tangent bundle over $B^{2n}(a)\setminus\{0\}$ in itself). Next let $\Omega=\{z\in U\cap M: z_n-\overline{f}_n(z)z_0\neq 0\}$ (an open subset of M) and let us consider the functions $G_\alpha,H_\alpha\in C^\infty(\Omega)$ given by

$$G_\alpha(z)=\frac{f_\alpha(z)\overline{z}_n-\overline{z}_\alpha f_n(z)}{\overline{z}_n-f_n(z)\overline{z}_0},\qquad H_\alpha(z)=\frac{\overline{z}_\alpha-f_\alpha(z)\overline{z}_0}{\overline{z}_n-f_n(z)\overline{z}_0},\quad 1\leq\alpha\leq n-1.$$

If $T_\alpha\equiv\partial/\partial z_\alpha-G_\alpha\partial/\partial z_0-H_\alpha\partial/\partial z_n$ then the tangential Cauchy–Riemann equations on Ω (induced by (5.31)) are

$$\overline{T}_\alpha(u)=0,\quad 1\leq\alpha\leq n-1. \tag{5.32}$$

As $G_\alpha\circ f=-G_\alpha$ and $H_\alpha\circ f=H_\alpha$ the equations (5.32) are invariant under f (equivalently, f is a CR map). Moreover, the (pullback to M of the) real 1-form

$$\eta=-i\left\{\overline{z}_0dz_0+\sum_{\alpha=1}^{n-1}\overline{z}_\alpha dz_\alpha+\overline{z}_ndz_n\right\}+\text{complex conjugate}$$

($i=\sqrt{-1}$) is a pseudohermitian structure on M. Indeed, $\overline{z}_0G_\alpha+\overline{z}_nH_\alpha=\overline{z}_\alpha$ yields $\eta(T_\alpha)=0$ for any $1\leq\alpha\leq n-1$. Then $f^*\eta=\eta$, i.e., f is isopseudohermitian. Also $\mathrm{Fix}(f)=\Sigma^{2n-3}(a')$ so that

Corollary 5.3 *Let $n\geq 3$ and $a=(a_0,\dots,a_n)\in\mathbb{Z}^{n+1}$ such that $a_0=2k$, $k\geq 1$, and $a_j\geq 2$ for any $1\leq j\leq n$. The inclusion $\Sigma^{2n-3}(a')\to\Sigma^{2n-1}(2k,a')$ is Levi harmonic.*

For instance, $\Sigma^3(2,2,2)\to\Sigma^5(2,2,2,2)$ is Levi harmonic.

Proof of Corollary 5.3 As $d\eta = 2i(dz_0 \wedge d\overline{z}_0 + \sum_{\alpha=1}^{n-1} dz_\alpha \wedge d\overline{z}_\alpha + dz_n \wedge d\overline{z}_n)$ the Levi form of $M = \Sigma^{2n-1}(a)$ is given by

$$L_\theta(Z, \overline{Z}) = -i(d\eta)(Z, \overline{Z}) = \sum_{\alpha=1}^{n-1} \{ |Z^\alpha|^2 + |G_\alpha Z^\alpha|^2 + |H_\alpha Z^\alpha|^2 \},$$

hence M is strictly pseudoconvex and Corollary 3.7 applies, i.e., $\iota : \Sigma^{2n-3}(a') \to M$ is pseudoharmonic. Finally (by Theorem 3.15) as ι is a CR map, if $\mathcal{H} = H(\Sigma^{2n-3}(a'))$ then $\tau_{\mathcal{H}}(\iota) = \hat{\tau}_{\mathcal{H}}(\iota) = 0$. □

6 Variational Treatment of Levi Harmonicity

Let $(M, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact Riemannian manifold and (M', g') a Riemannian manifold. If $\Omega \subset M$ is a relatively compact domain we set

$$E_\Omega(f) = \frac{1}{2} \int_\Omega \operatorname{trace}_g \left(\Pi_{\mathcal{H}} f^* g' \right) dv_g \tag{6.33}$$

for any $f \in C^\infty(M, M')$. Let us derive the Euler–Lagrange equations for the energy functional (6.33). To this end let $\{f_t\}_{|t|<\epsilon}$ be a smooth 1-parameter variation of $f = f_0$ supported in Ω and let

$$F : M \times (-\epsilon, \epsilon) \to M', \quad F(p,t) = f_t(p), \quad p \in M, \quad |t| < \epsilon.$$

Let $V \in C^\infty(f^{-1}TM')$ be the corresponding infinitesimal variation, i.e.,

$$V_p = (d_{(p,0)}F)(\partial/\partial t)_{(p,0)}, \quad p \in M,$$

and $\operatorname{Supp}(V) \subset \Omega$. Let $\{E_a : 1 \le a \le 2n\}$ be a local orthonormal frame of $\mathcal{H}$ defined on the open set $U \subset M$. Let $\tilde{M} = M \times (-\epsilon, \epsilon)$. For any $X \in \mathfrak{X}(M)$ we define $\tilde{X} \in \mathfrak{X}(\tilde{M})$ by setting

$$\tilde{X}_{(p,t)} = (d_p \alpha_t) X_p,$$
$$\alpha_t : M \to \tilde{M}, \quad \alpha_t(p) = (p,t), \quad (p,t) \in \tilde{M}.$$

Let $u \in C^\infty(\tilde{M})$ be given by

$$u(p,t) \equiv \operatorname{trace}_g \left(\Pi_{\mathcal{H}} f_t^* g' \right)_p$$
$$= \sum_{a=1}^{2n} g'_{f_t(p)} \left((d_p f_t) E_{a,p}, (d_p f_t) E_{a,p} \right) = \sum_{a=1}^{2n} {g'}^F (F_* \tilde{E}_a, F_* \tilde{E}_a)_{(p,t)}$$

for any $p \in U$ and $|t| < \epsilon$, where ${g'}^F = F^{-1} g'$ is the pullback of g' by F (a Riemannian bundle metric in $F^{-1}TM' \to \tilde{M}$). Let also ${\nabla'}^F = F^{-1} \nabla'$ be the pullback of ∇'

by F. Next (as $\nabla'^F g'^F = 0$)

$$\frac{\partial u}{\partial t} = \sum_{a=1}^{2n} \frac{\partial}{\partial t}\{g'^F(F_*\tilde{E}_a, F_* E_a)\} = 2\sum_{a=1}^{2n} g'^F\big(\nabla'^F_{\partial/\partial t} F_*\tilde{E}_a, F_*\tilde{E}_a\big). \tag{6.34}$$

As $[\partial/\partial t, \tilde{E}_a] = 0$ and ∇' is torsion free

$$\begin{aligned} g'^F\big(\nabla'^F_{\partial/\partial t} F_*\tilde{E}_a, F_*\tilde{E}_a\big) &= g'^F\big(\nabla'^F_{\tilde{E}_a} F_*(\partial/\partial t), F_*\tilde{E}_a\big) \\ &= \tilde{E}_a\big\{g'^F\big(F_*(\partial/\partial t), F_*\tilde{E}_a\big)\big\} - g'^F\big(F_*(\partial/\partial t), \nabla'_{\tilde{E}_a} F_*\tilde{E}_a\big). \end{aligned} \tag{6.35}$$

For each $|t| < \epsilon$ let $X_t \in \mathcal{H}$ be defined by

$$g(X_t, Y) = g'^F\big(F_*(\partial/\partial t), F_*\tilde{Y}\big) \circ \alpha_t, \quad Y \in \mathcal{H}. \tag{6.36}$$

Then for any $p \in U$ and $|t| < \epsilon$

$$\begin{aligned} &\sum_{a=1}^{2n} \tilde{E}_{a,(p,t)}\big\{g'^F\big(F_*(\partial/\partial t), F_*\tilde{E}_a\big)\big\} \\ &\quad = \sum_{a=1}^{2n} E_{a,p}\big\{g(X_t, E_a)\big\} \\ &\quad = \sum_{a=1}^{2n} \big\{g(\nabla_{E_a} X_t, E_a) + g(\nabla_{E_a} E_a, X_t)\big\}(p). \end{aligned}$$

Let div be the divergence operator, i.e., $\mathcal{L}_X dv_g = \operatorname{div}(X) dv_g$ for any $X \in \mathfrak{X}(M)$. Then

$$\begin{aligned} &\sum_{a=1}^{2n} \tilde{E}_{a,(p,t)}\big\{g'^F\big(F_*(\partial/\partial t), F_*\tilde{E}_a\big)\big\} \\ &\quad = \big\{\operatorname{div}(X_t) - g(\nabla_\xi X_t, \xi)\big\}(p) + \sum_{a=1}^{2n} g(X_t, \nabla_{E_a} E_a)_p \end{aligned} \tag{6.37}$$

(by $\nabla g = 0$, $g(X_t, \xi) = 0$, $\nabla_\xi \xi \in \mathcal{H}$ and (6.36))

$$= \operatorname{div}(X_t)(p) + g'^F\big(F_*(\partial/\partial t), F_*\tilde{Y}\big)_{(p,t)}$$

where

$$Y \equiv \nabla_\xi \xi + \sum_{a=1}^{2n} \Pi_{\mathcal{H}} \nabla_{E_a} E_a. \tag{6.38}$$

On the other hand

$$\Pi_{\mathcal{H}} \nabla_{E_a} E_a = \nabla_{E_a} E_a - \eta(\nabla_{E_a} E_a)\xi = \nabla_{E_a} E_a + g(E_a, \nabla_{E_a} \xi)\xi \tag{6.39}$$

so that (6.38) becomes

$$Y = \nabla_\xi \xi + \operatorname{div}(\xi)\xi + \sum_{a=1}^{2n} \nabla_{E_a} E_a$$

and we may conclude that

$$\frac{1}{2}\frac{\partial u}{\partial t}(\cdot, 0) = \operatorname{div} X_0 - g'^f\big(V, \tau_{\mathcal{H}}(f) - f_*\{\nabla_\xi \xi + \operatorname{div}(\xi)\xi\}\big). \tag{6.40}$$

We obtain the following

Theorem 6.1 *Let $(M, \varphi, \xi, \eta, g)$ and (M', g') be respectively an almost contact Riemannian manifold and a Riemannian manifold. Let $\Omega \subset M$ be a relatively compact domain. A C^∞ map $f : M \to M'$ is a critical point of $E_\Omega : C^\infty(M, M') \to \mathbb{R}$ if and only if*

$$\tau_{\mathcal{H}}(f) = f_*\{\nabla_\xi \xi + \operatorname{div}(\xi)\xi\}. \tag{6.41}$$

Let $f : M \to M'$ be an immersion and a critical point of E_Ω. Then f is Levi harmonic if and only if the Reeb field ξ is geodesic and divergence free.

Proof By (6.40) and Green's lemma

$$\frac{d}{dt}\{E_\Omega(f_t)\}_{t=0} = -\int_\Omega g'^f\big(V, \tau_{\mathcal{H}}(f) - f_*\{\nabla_\xi \xi + \operatorname{div}(\xi)\xi\}\big) dv_g$$

for any smooth 1-parameter variation $\{f_t\}_{|t|<\epsilon}$ of f supported in Ω (the first variation formula for E_Ω), hence the Euler–Lagrange equations sought after are (6.41). As $\nabla_\xi \xi \in \mathcal{H}$ the last statement follows from (6.41). □

Corollary 6.2 *Let $(M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold such that* (i) *M satisfies the φ-condition and* (ii) *ξ is a geodesic vector field. Let $f : M \to M'$ be a C^∞ map of M into a Riemannian manifold (M', g'). Then f is Levi harmonic if and only if f is a critical point of $E_\Omega : C^\infty(M, M') \to \mathbb{R}$ for any relatively compact domain $\Omega \subset M$.*

Proof Let $\{E_a : 1 \le a \le 2n\} \equiv \{e_\alpha, \varphi e_\alpha : 1 \le \alpha \le n\}$ be a local orthonormal frame of $\mathcal{H}$. As M satisfies the φ-condition

$$\sum_{a=1}^{2n} \nabla_{E_a} E_a = \sum_{\alpha=1}^{n} (\nabla_{e_\alpha} e_\alpha + \nabla_{\varphi e_\alpha} \varphi e_\alpha) = \sum_{\alpha=1}^{n} \varphi[\varphi e_\alpha, e_\alpha] \in \mathcal{H}.$$

Hence (by (6.39)) $\operatorname{div}(\xi) = 0$ which together with the assumption $\nabla_\xi \xi = 0$ yields

$$\frac{d}{dt}\{E_\Omega(f_t)\}_{t=0} = -\int_\Omega g'^f\big(V, \tau_{\mathcal{H}}(f)\big) dv_g$$

thus leading to the variational interpretation of Levi harmonicity. □

The many ramifications of harmonicity (subelliptic harmonic, contact harmonic, Levi harmonic, and pseudoharmonic maps) seem to indicate that the theory of harmonic maps has reached a stage of mannerism. It is, however, the authors' opinion that the mentioned ramifications (to which one may add p-harmonic and exponentially harmonic maps (cf. [40] and [43]), Gromov's tangentially harmonic maps (cf. [28]), and harmonic maps from Finslerian manifolds (cf., e.g., [13] and [55, 56])) are but a measure of the enormous success enjoyed by the theory. As a more specific comment, the ramifications of harmonicity in the present paper do possess one common feature: the relevant equations (cf., e.g., (6.41)) are nonlinear *degenerate elliptic* PDE systems. Indeed, let $(M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold and let $\{E_a : 1 \le a \le 2n\}$ be a local orthonormal frame of $\mathcal{H}$ defined on $U \subset M$ and let (U', x'^α) be a local coordinate system on M' such that $f(U) \subset U'$. Let us consider the second-order differential operator locally given by

$$\Delta_{\mathcal{H}} u \equiv -\sum_{a=1}^{2n}\big\{E_a\big(E_a(u)\big) - (\nabla_{E_a} E_a)(u)\big\} = \Delta u + \xi(\xi u) - (\nabla_\xi \xi)(u)$$

for any $u \in C^2(M)$. Here Δ is the Laplace–Beltrami operator of (M, g). Then

$$\tau_{\mathcal{H}}(f) = \big\{-\Delta_{\mathcal{H}} f^\alpha + \big(\Gamma'^\alpha_{\beta\gamma} \circ f\big)\big(\nabla^{\mathcal{H}} f^\beta\big)\big(f^\gamma\big)\big\}\big(\partial/\partial x'^\alpha\big)^f$$

where $f^\alpha = x'^\alpha \circ f$ and $\nabla^{\mathcal{H}} u = \sum_{a=1}^{2n} E_a(u) E_a$ (the $\mathcal{H}$-*gradient*). On the other hand let us compute the symbol

$$\begin{aligned}\sigma_2(\Delta_{\mathcal{H}}) &\in \operatorname{Smbl}_2(E, E)\\ &\equiv \big\{\sigma \in \operatorname{Hom}\big(\pi^{-1}E, \pi^{-1}E\big) : \sigma_{t\omega} = t^2\sigma_\omega, \omega \in T'(M), t > 0\big\}\end{aligned}$$

where $E = M \times \mathbb{R}$ (the trivial line bundle). Also $T'(M) = T^*(M) \setminus (0)$ and $\pi : T'(M) \to M$ is the projection. Let $u \in C^\infty(M)$ and $\omega \in T'(M)$ with $x = \pi(\omega)$. Then

$$\sigma_2(\Delta_{\mathcal{H}})_\omega s_x = \Delta_{\mathcal{H}}\left(\frac{i^2}{2!}\big[a - a(x)\big]^2 s\right)(x)$$

where $a \in C^\infty(M)$ and $s \in C^\infty(E)$ are such that $(da)_x = \omega$ and $s(x) = (x, u(x))$. A calculation based on $\Delta_{\mathcal{H}}(uv) = u\Delta_{\mathcal{H}} v + v\Delta_{\mathcal{H}} u - g(\nabla^{\mathcal{H}} u, \nabla^{\mathcal{H}} v)$ leads to

$$\sigma_2(\Delta_{\mathcal{H}})_\omega s(x) = \|\nabla^{\mathcal{H}} a\|_x^2 s(x). \tag{6.42}$$

Therefore (by (6.42)) $\operatorname{Ker}[\sigma_2(\Delta_{\mathcal{H}})_\omega] \ne (0)$ if and only if $\omega \in \mathbb{R}\eta_x$, i.e., the ellipticity of $\Delta_{\mathcal{H}}$ degenerates precisely in the direction η. In the contact Riemannian case $\Delta_{\mathcal{H}}$

is due to S. Tanno, [51], p. 363. If this is the case then (by Theorem 3 in [5], p. 76) $\Delta_{\mathcal{H}}$ is subelliptic of order $\frac{1}{2}$ and hence hypoelliptic. Similarly, the principal part of the pseudoharmonic map system $\hat{\tau}_{\mathcal{H}}(f) = 0$, i.e., locally

$$\hat{\Delta}_{\mathcal{H}} f^{\alpha} - \sum_{a=1}^{2n} \big(\hat{\Gamma}'^{\alpha}_{\beta\gamma} \circ f\big) E_a\big(f^{\beta}\big) E_a\big(f^{\gamma}\big) = 0,$$

is the second-order differential operator $\hat{\Delta}_{\mathcal{H}}$ locally given by

$$\hat{\Delta}_{\mathcal{H}} u = -\sum_{a=1}^{2n} \big\{E_a(E_a u) - (\hat{\nabla}_{E_a} E_a)(u)\big\}$$

coinciding with the sublaplacian Δ_b (cf., e.g., [21], p. 111) in the integrable case. By (2.7)

$$\Delta_{\mathcal{H}} = \hat{\Delta}_{\mathcal{H}} + 2n H_{\mathcal{H}} \tag{6.43}$$

where $H_{\mathcal{H}} = (1/2n)\,\mathrm{trace}_g(B_{\mathcal{H}})$ is the mean curvature vector of the Levi distribution (and $B_{\mathcal{H}}(X,Y) = (\nabla_X Y)^{\perp} = \eta(\nabla_X Y)\xi$ for any $X, Y \in \mathcal{H}$). On an almost contact Riemannian manifold M the operators $\Delta_{\mathcal{H}}$ and $\hat{\Delta}_{\mathcal{H}}$ are distinct in general. However if M is contact Riemannian then M satisfies the φ-condition and consequently the Levi distribution is minimal (the contact Riemannian analog to Theorem 1.4 in [21], p. 37), hence $\Delta_{\mathcal{H}} = \hat{\Delta}_{\mathcal{H}}$ ($= \Delta_b$ when $T_{1,0}(M)$ is integrable).

We close Sect. 6 with a comparison among the functionals (1.3) and (6.33). Let M be a contact Riemannian manifold and $f : M \to M'$ a C^{∞} map into a foliated Riemannian manifold $(M', g', \mathcal{F}')$. Let $Q' = \nu(\mathcal{F}') = T(M)/T(\mathcal{F}')$ be the transverse bundle and $\pi' : T(M') \to Q'$ the natural projection. If $\{E_a : 1 \le a \le 2n\}$ is a local g-orthonormal frame in $\mathcal{H}$ defined on $U \subset M$ then $\{s_a \equiv \pi E_a : 1 \le a \le 2n\}$ is a local g_Q-orthonormal frame in $Q = \nu(\mathcal{F}_{\xi})$. Here $\mathcal{F}_{\xi}$ is the Reeb foliation (tangent to ξ), $g_Q(r,s) = g(\sigma_g(r), \sigma_g(s))$ for any $r, s \in Q$, and $\sigma_g : Q \to T(\mathcal{F}_{\xi})^{\perp} = \mathcal{H}$ is the natural bundle isomorphism. Let us assume that f is a foliated map of $(M, \mathcal{F}_{\xi})$ into $(M', \mathcal{F}')$. Then for each $x \in U$

$$\begin{aligned}\|d_T f\|(x)^2 &= \sum_{a=1}^{2n} g'^{f}_{Q'}\big((d_T f)s_a, (d_T f)s_a\big)_x \\ &= \sum_a g'_{f(x)}\big(\big((d_x f)E_{a,x}\big)^{\perp}, \big((d_x f)E_{a,x}\big)^{\perp}\big)\end{aligned}$$

where $v^{\perp}$ is the $T(\mathcal{F}')^{\perp}_{f(x)}$-component of $v \in T_{f(x)}(M')$. Assume from now on that M' is a contact Riemannian manifold as well, and $\mathcal{F}' = \mathcal{F}_{\xi'}$. Then $(d_x f)\xi_x = \lambda_f(x)\xi'_{f(x)}$ for some $\lambda_f \in C^{\infty}(M)$ and any $x \in M$. Also

$$\|d_T f\|^2 = \mathrm{trace}_g\big(\Pi_{\mathcal{H}} f^* g'\big) + \|f^*\eta'\|^2 - \lambda_f^2 \tag{6.44}$$

with $\lambda_f \equiv (f^*\eta')(\xi)$, hence Levi and transverse harmonicity (in the sense of [22]) do not coincide, in general. However if additionally f is a contact map (i.e., $f_*\mathcal{H} \subset$

$f^{-1}\mathcal{H}'$) or an isometry (i.e., $f^*g' = g$) then $\|f^*\eta'\|^2 = \lambda_f^2$ (and by (6.44) f is Levi harmonic if and only if f is transversally harmonic).

7 Levi Harmonic Morphisms

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact Riemannian manifold and $f : M \to M'$ a continuous map into a Riemannian manifold (M', g').

Definition 7.1 f is a *Levi harmonic morphism* if for any local harmonic function $v : U' \subset M' \to \mathbb{R}$ the function $u \equiv v \circ f : U \to \mathbb{R}$ satisfies $\Delta_{\mathcal{H}} u = 0$ in $U \equiv f^{-1}(U')$.

The equation $\Delta_{\mathcal{H}} u = 0$ in Definition 7.1 is meant in distribution sense. However combining existence of local harmonic coordinates about each point of M' with the hypoellipticity of $\Delta_{\mathcal{H}}$ leads to the following regularity result.

Proposition 7.2 *Every Levi harmonic morphism from a contact Riemannian manifold is C^∞.*

Replacing $\Delta_{\mathcal{H}}$ by $\hat{\Delta}_{\mathcal{H}}$ in Definition 7.1 leads to the following natural notion.

Definition 7.3 f is a *pseudoharmonic morphism* if the pullback by f of any local harmonic function $v : U' \to \mathbb{R}$ is a solution to $\hat{\Delta}_{\mathcal{H}} u = 0$.

When M is a strictly pseudoconvex CR manifold endowed with a fixed contact form η, pseudoharmonic morphisms are due to E. Barletta, [3]–[4], who proved a Fuglede–Ishihara type theorem (cf. also Theorem 6 in [17], p. 435). The nonintegrable case (where M is but (almost) contact Riemannian) has not been studied so far. The proof of Fuglede–Ishihara's theorem (cf. Theorem 4.2.2 in [2], p. 108) makes essential use of normal coordinate systems and of the existence of local harmonic functions with prescribed gradient and Hessian at a point (a result known as *Ishihara's lemma*, cf. [31]) on the target Riemannian manifold M' (and Barletta's Theorem 6 in [17], p. 435, is proved along the same lines).

The notion of morphism is of course very general and may be given for any pair of sheaf spaces $(M, \mathcal{S})$ and $(M', \mathcal{S}')$, i.e., a continuous map $f : M \to M'$ is a *morphism* if $f^*\mathcal{S}' \subset \mathcal{S}$. As a natural ramification of Definitions 7.1 and 7.3 one may consider the sheaves $\mathcal{S}$ and $\mathcal{S}'$ of local harmonics of $\hat{\Delta}_{\mathcal{H}}$ and $\hat{\Delta}_{\mathcal{H}'}$. Precisely,

Definition 7.4 Let $(M, \varphi, \xi, \eta, g)$ and $(M', \varphi', \xi', \eta')$ be two contact Riemannian manifolds. A continuous map $f : M \to M'$ is a $(\mathcal{H}, \mathcal{H}')$-*harmonic morphism* if the pullback by f of any local solution to $\hat{\Delta}_{\mathcal{H}'} v = 0$ is a (distribution) solution to $\hat{\Delta}_{\mathcal{H}} u = 0$.

A moment's thought exhibits the additional difficulties appearing in a theory of $(\mathcal{H}, \mathcal{H}')$-harmonic morphisms. For instance, it is unknown whether points on contact Riemannian manifolds M admit local coordinate neighborhoods (U, x^i) such that

$\hat{\Delta}_{\mathcal{H}} x^i = 0$ (and then no analog to the regularity result in Proposition 7.2 is *a priori* available). Also a subelliptic analog to Ishihara's lemma is missing in present time subelliptic theory. The following result is straightforward.

Lemma 7.5 *Let $f : M \to M'$ be a C^∞ map of an almost contact Riemannian manifold $(M, \varphi, \xi, \eta, g)$ into a manifold with linear connection (M', D'). Let $\{\xi_a : 1 \le a \le 2n\}$ be a local g-orthonormal frame of $\mathcal{H}$ on $U \subset M$ and (U', x'^α) a local coordinate system on M' such that $f(U) \subset U'$.*

$$L(v \circ f) = \left\{ L f^\alpha - \sum_{a=1}^{2n} \left(\Gamma'^\alpha_{\beta\gamma} \circ f\right) E_a\left(f^\beta\right) E_a\left(f^\gamma\right) \right\} (v_\alpha \circ f) - \sum_{a=1}^{2n} (v_{\alpha,\beta} \circ f) E_a\left(f^\alpha\right) E_a\left(f^\beta\right) \tag{7.45}$$

for any $L \in \{\Delta_{\mathcal{H}}, \hat{\Delta}_{\mathcal{H}}\}$ and any C^2 function $v : M' \to \mathbb{R}$, where $f^\alpha = x'^\alpha \circ f$, $v_\alpha \equiv \partial v / \partial x'^\alpha$, $v_{\alpha,\beta} = v_{\alpha\beta} - \Gamma'^\gamma_{\alpha\beta} v_\gamma$, and $v_{\alpha\beta} \equiv \partial^2 v / \partial x'^\alpha \partial x'^\beta$. Also $\Gamma'^\alpha_{\beta\gamma} \in C^\infty(U')$ are the coefficients of D' with respect to (U', x'^α).

We shall also need (cf. T. Ishihara, [31])

Lemma 7.6 *Let (M', g') be a ν-dimensional Riemannian manifold and $C_\alpha, C_{\alpha\beta} \in \mathbb{R}$, $1 \le \alpha, \beta \le \nu$, such that $C_{\alpha\beta} = C_{\beta\alpha}$ and $\sum_{\alpha=1}^{\nu} C_{\alpha\alpha} = 0$. Let $y_0 \in M'$ and (U', x'^α) a normal coordinate system on M' centered at y_0 such that $x'^\alpha(y_0) = 0$. There is a harmonic function $v : U' \to \mathbb{R}$ such that $v_\alpha(y_0) = C_\alpha$ and $v_{\alpha\beta}(y_0) = C_{\alpha\beta}$ for any $1 \le \alpha, \beta \le \nu$.*

We establish

Theorem 7.7 *Let (M, φ, ξ, g) be an almost contact Riemannian manifold and $f : M \to M'$ a C^∞ map into a Riemannian manifold (M', g'). If f is a Levi harmonic morphism then f is a Levi harmonic map.*

Proof Let $x_0 \in M$ and $y_0 = f(x_0) \in M'$. Let $\{E_a : 1 \le a \le 2n\}$ be a local g-orthonormal frame of $\mathcal{H}$ defined on the open set $U \subset M$ and (U', x'^α) a normal coordinate system on M' centered at y_0 such that $f(U) \subset U'$. Let $\nu = \dim(M')$ and $\alpha_0 \in \{1, \dots, \nu\}$ and let us set $C_\alpha = \delta_{\alpha\alpha_0}$ and $C_{\alpha\beta} = 0$. By Ishihara's lemma there is $v : U' \to \mathbb{R}$ such that $\Delta' v = 0$ in U' and $v_\alpha(y_0) = \delta_{\alpha\alpha_0}$ and $v_{\alpha\beta}(y_0) = 0$. Let us apply Lemma 7.5 with $D' = \nabla'$ (together with $\Gamma'^\alpha_{\beta\gamma}(y_0) = 0$) to derive (as f is a Levi harmonic morphism)

$$\left(\Delta_{\mathcal{H}} f^{\alpha_0} - \sum_{a=1}^{2n} \left(\Gamma'^{\alpha_0}_{\beta\gamma} \circ f\right) E_a\left(f^\beta\right) E_a\left(f^\gamma\right) \right)_{x_0} = 0,$$

i.e., f is a Levi harmonic map. □

Theorem 7.8 *Let $f : M \to M'$ be a C^∞ map of an almost contact Riemannian manifold $(M, \varphi, \xi, \eta, g)$ into a Sasakian manifold $(M', \varphi', \xi', \eta', g')$. If f is a pseudoharmonic morphism then f is a pseudoharmonic map.*

A treatment similar to the proof of Theorem 7.7 requires the use of special local coordinates such that (7.59) holds. The natural choice appears to be that of contact normal coordinates at $y_0 = f(x_0)$, which make sense on any contact Riemannian manifold M'. The local coefficients of the generalized Tanaka–Webster connection $\hat{\nabla}'$ evaluated at the center y_0 are indeed simple (cf. Lemma 7.9 below) yet (7.59) does not hold in general (as $\hat{\nabla}'$ has torsion). Consequently the result in Theorem 7.8 is confined to the class of pseudoharmonic morphism with values in a Riemannian manifold M' whose underlying metric is Sasakian.

Lemma 7.9 *Let $(M, \varphi, \xi, \eta, g)$ be a contact Riemannian manifold. Let $\hat{\Gamma}^i_{jk}$ be the coefficients of the generalized Tanaka–Webster connection with respect to a contact normal coordinate system (U, x^i) centered at $x \in U$. Then*

$$\hat{\Gamma}^0_{a0}(x) = \Gamma^0_{0a}(x) = 0, \quad 1 \le a \le 2n, \tag{7.46}$$

$$\hat{\Gamma}^a_{b0}(x) = -\frac{1}{3}\tau^a_b(x), \qquad \hat{\Gamma}^a_{0b}(x) = \frac{2}{3}\tau^a_b(x), \quad 1 \le a, b \le 2n, \tag{7.47}$$

$$\hat{\Gamma}^i_{00}(x) = 0, \quad 0 \le i \le 2n, \tag{7.48}$$

where $\tau^i_j \in C^\infty(U)$ are given by $\tau(\partial/\partial x^j) = \tau^i_j \partial/\partial x^i$ and τ is the pseudohermitian torsion, i.e., $\tau(X) = T_{\hat{\nabla}}(\xi, X)$ for any $X \in \mathfrak{X}(M)$.

This is an addition to (2.25) in Lemma 2.8 and the proof is similar. Indeed, (2.26) may be written

$$\lambda^a\lambda^b\hat{\Gamma}^0_{ab}\big(\gamma(t)\big) + 2\lambda\lambda_a t\big\{\hat{\Gamma}^0_{a0}\big(\gamma(t)\big) + \hat{\Gamma}^0_{0a}\big(\gamma(t)\big)\big\} + 4\lambda^2t^2\hat{\Gamma}^0_{00}\big(\gamma(t)\big)$$
$$= 2\lambda\big\{\xi^0\big(\gamma(t)\big) - 1\big\}, \tag{7.49}$$

$$\lambda^a\lambda^b\hat{\Gamma}^c_{ab}\big(\gamma(t)\big) + 2\lambda\lambda^a t\big\{\hat{\Gamma}^c_{a0}\big(\gamma(t)\big) + \hat{\Gamma}^c_{0a}\big(\gamma(t)\big)\big\} + 4\lambda^2t^2\hat{\Gamma}^c_{00}\big(\gamma(t)\big)$$
$$= 2\lambda\xi^c\big(\gamma(t)\big). \tag{7.50}$$

Let us differentiate with respect to t in (7.49). We obtain

$$2\lambda\lambda^a\big\{\hat{\Gamma}^0_{a0}\big(\gamma(t)\big) + \hat{\Gamma}^0_{0a}\big(\gamma(t)\big)\big\} + \lambda^a\lambda^b\lambda^c\frac{\partial\hat{\Gamma}^0_{ab}}{\partial x^c}\big(\gamma(t)\big) = 2\lambda\lambda^a\frac{\partial\xi^0}{\partial x^a}\big(\gamma(t)\big) + O(t),$$

hence (for $t = 0$)

$$2\lambda\lambda^a\big\{\hat{\Gamma}^0_{a0}(x) + \hat{\Gamma}^0_{0a}(x)\big\} + \lambda^a\lambda^b\lambda^c\frac{\partial\hat{\Gamma}^0_{ab}}{\partial x^c}(x) = 2\lambda\lambda^a\frac{\partial\xi^0}{\partial x^a}(x) \tag{7.51}$$

for any $\lambda \in \mathbb{R}$. For $\lambda = 0$ one has $\lambda^a \lambda^b \lambda^c (\partial \hat{\Gamma}^0_{ab}/\partial x^c)(x) = 0$ and (7.51) yields $\lambda^a \{\hat{\Gamma}^0_{a0}(x) + \hat{\Gamma}^0_{0a}(x) - (\partial \xi^0/\partial x^a)(x)\} = 0$ for any $(\lambda^1, \dots, \lambda^{2n}) \in \mathbb{R}^{2n}$ or

$$\hat{\Gamma}^0_{a0}(x) + \hat{\Gamma}^0_{0a}(x) = \frac{\partial \xi^0}{\partial x^a}(x). \tag{7.52}$$

On the other hand (by (2.8)) ξ is parallel with respect to $\hat{\nabla}$, i.e., $\partial \xi^i/\partial x^j + \hat{\Gamma}^i_{jk} \xi^k = 0$ which evaluates at the center x as

$$\hat{\Gamma}^i_{j0}(x) = -\frac{\partial \xi^i}{\partial x^j}(x). \tag{7.53}$$

Also $\hat{\nabla}_{\partial/\partial x^0} \partial/\partial x^j = \hat{\nabla}_{\partial/\partial x^j} \partial/\partial x^0 + T_{\hat{\nabla}}(\partial/\partial x^0, \partial/\partial x^j)$. Let us evaluate at x and use $(\partial/\partial x^0)_x = \xi_x$ to derive

$$\hat{\Gamma}^i_{0j}(x) = \hat{\Gamma}^i_{j0}(x) + \tau^i_j(x). \tag{7.54}$$

Formulae (7.52)–(7.53) lead to

$$\hat{\Gamma}^0_{a0}(x) = -\frac{\partial \xi^0}{\partial x^a}(x), \qquad \hat{\Gamma}^0_{0a}(x) = 2\frac{\partial \xi^0}{\partial x^a}(x).$$

As τ_x is $\mathcal{H}_x$-valued and $(\partial/\partial x^0)_x = \xi_x$ and $(\partial/\partial x^a)_x = \xi_{a,x}$ it must be that $\tau^0_a(x) = 0$. Finally (by (7.54))

$$\frac{\partial \xi^0}{\partial x^a}(x) = 0, \qquad \hat{\Gamma}^0_{a0}(x) = \hat{\Gamma}^0_{0a}(x) = 0,$$

and (7.46) is proved. To prove (7.47) we differentiate twice with respect to t in (7.49) and obtain

$$\begin{aligned} &8\lambda^2 \hat{\Gamma}^0_{00}\big(\gamma(t)\big) + 4\lambda\lambda^a\lambda^b \left\{ \frac{\partial \hat{\Gamma}^0_{a0}}{\partial x^b}\big(\gamma(t)\big) + \frac{\partial \hat{\Gamma}^0_{0a}}{\partial x^b}\big(\gamma(t)\big) \right\} \\ &\quad + \lambda^a\lambda^b\lambda^c\lambda^d \frac{\partial^2 \hat{\Gamma}^0_{ab}}{\partial x^c \partial x^d}\big(\gamma(t)\big) + 2\lambda\lambda^a\lambda^b \frac{\partial \hat{\Gamma}^0_{ab}}{\partial x^0}\big(\gamma(t)\big) \\ &= 2\lambda\lambda^a\lambda^b \frac{\partial^2 \xi^0}{\partial x^a \partial x^b}\big(\gamma(t)\big) + 4\lambda^2 \frac{\partial \xi^0}{\partial x^0}\big(\gamma(t)\big) + O(t). \end{aligned} \tag{7.55}$$

Thus, for $t = 0$ and $\lambda = 0$ one has $\lambda^a\lambda^b\lambda^c\lambda^d (\partial^2 \hat{\Gamma}^0_{ab}/\partial x^c \partial x^d)(x) = 0$ and the identity (7.55) evaluated at $t = 0$ becomes

$$\begin{aligned} &4\lambda \hat{\Gamma}^0_{00}(x) + 2\lambda^a\lambda^b \left\{ \frac{\partial \hat{\Gamma}^0_{a0}}{\partial x^b}(x) + \frac{\partial \hat{\Gamma}^0_{0a}}{\partial x^b}(x) \right\} + \lambda\lambda^a\lambda^b \frac{\partial \hat{\Gamma}^0_{ab}}{\partial x^0}(x) \\ &= \lambda^a\lambda^b \frac{\partial^2 \xi^0}{\partial x^a \partial x^b}(x) + 2\lambda \frac{\partial \xi^0}{\partial x^0}(x). \end{aligned}$$

In particular, for $\lambda^a = 0$

$$2\hat{\Gamma}^0_{00}(x) = \frac{\partial \xi^0}{\partial x^0}(x),$$

hence (by (7.53))

$$\frac{\partial \xi^0}{\partial x^0}(x) = 0, \qquad \hat{\Gamma}^0_{00}(x) = 0, \tag{7.56}$$

thus proving (7.48) for $i = 0$. Next we differentiate in (7.50) twice with respect to t to get

$$\begin{aligned} &8\lambda^2 t \hat{\Gamma}^c_{00}\big(\gamma(t)\big) + 2\lambda\lambda^a\big\{\hat{\Gamma}^c_{a0}\big(\gamma(t)\big) + \hat{\Gamma}^c_{0a}\big(\gamma(t)\big)\big\} \\ &\quad + 2\lambda\lambda^a\lambda^b t\left\{\frac{\partial \hat{\Gamma}^c_{a0}}{\partial x^b}\big(\gamma(t)\big) + \frac{\partial \hat{\Gamma}^c_{0a}}{\partial x^b}\big(\gamma(t)\big)\right\} + \lambda^a\lambda^b\lambda^d \frac{\partial \hat{\Gamma}^c_{ab}}{\partial x^d}\big(\gamma(t)\big) \\ &\quad + 2\lambda\lambda^a\lambda^b t \frac{\partial \hat{\Gamma}^c_{ab}}{\partial x^0}\big(\gamma(t)\big) \\ &= 2\lambda\lambda^a \frac{\partial \xi^c}{\partial x^a}\big(\gamma(t)\big) + 4\lambda^2 t \frac{\partial \xi^c}{\partial x^0}\big(\gamma(t)\big) + O\big(t^2\big), \end{aligned} \tag{7.57}$$

$$\begin{aligned} &8\lambda^2 \hat{\Gamma}^c_{00}\big(\gamma(t)\big) + 4\lambda\lambda^a\lambda^b\left\{\frac{\partial \hat{\Gamma}^c_{a0}}{\partial x^b}\big(\gamma(t)\big) + \frac{\partial \hat{\Gamma}^c_{0a}}{\partial x^b}\big(\gamma(t)\big)\right\} \\ &\quad + \lambda^a\lambda^b\lambda^d\lambda^e \frac{\partial^2 \hat{\Gamma}^c_{ab}}{\partial x^d \partial x^e}\big(\gamma(t)\big) + 2\lambda\lambda^a\lambda^b \frac{\partial \hat{\Gamma}^c_{ab}}{\partial x^0}\big(\gamma(t)\big) \\ &= 2\lambda\lambda^a\lambda^b \frac{\partial^2 \xi^c}{\partial x^a \partial x^b}\big(\gamma(t)\big) + 4\lambda^2 \frac{\partial \xi^c}{\partial x^0}\big(\gamma(t)\big) + O(t). \end{aligned} \tag{7.58}$$

Let us evaluate (7.57) at $t = 0$ and $\lambda = 0$ to derive $\lambda^a\lambda^b\lambda^d(\partial\hat{\Gamma}^c_{ab}/\partial x^d)(x) = 0$. Then (7.57) at $t = 0$ yields

$$\hat{\Gamma}^c_{a0}(x) + \hat{\Gamma}^c_{0a}(x) = \frac{\partial \xi^c}{\partial x^a}(x),$$

hence (by (7.53))

$$\hat{\Gamma}^c_{a0}(x) = -\frac{\partial \xi^c}{\partial x^a}(x), \qquad \hat{\Gamma}^c_{0a}(x) = 2\frac{\partial \xi^c}{\partial x^a}(x),$$

or (by (7.54))

$$\hat{\Gamma}^c_{a0}(x) = -\frac{1}{3}\tau^c_a(x), \qquad \hat{\Gamma}^c_{0a}(x) = \frac{2}{3}\tau^c_a(x),$$

proving (7.47). Finally (by (7.58) with $t = 0$ and $\lambda^a = 0$)

$$2\hat{\Gamma}^c_{00}(x) = \frac{\partial \xi^c}{\partial x^0}(x)$$

or (by (7.53))

$$\frac{\partial \xi^c}{\partial x^0}(x) = 0, \qquad \hat{\Gamma}^c_{00}(x) = 0,$$

completing the proof of (7.48). □

Proof of Theorem 7.8 As M' is assumed to be Sasakian its pseudohermitian torsion vanishes ($\tau'^{\alpha}_{\beta} = 0$). We proceed as in the proof of Theorem 7.7 except that we choose a *contact* normal coordinate system (U', x'^{α}) centered at y_0 and apply Lemma 7.5 for $L = \hat{\Delta}_{\mathcal{H}}$ and $D' = \hat{\nabla}'$. By (2.25) in Lemma 2.8 and (7.46)–(7.48) in Lemma 7.9 (applied to $\hat{\nabla}'$) it follows that

$$\hat{\Gamma}'^{\alpha}_{\beta\gamma}(y_0) v^{\beta} v^{\gamma} = 0, \quad \left(v^1, \ldots, v^{\nu}\right) \in \mathbb{R}^{\nu}. \tag{7.59}$$

Thus $(\hat{\Delta}_{\mathcal{H}} f^{\alpha_0} - \sum_{a=1}^{2n} (\hat{\Gamma}'^{\alpha_0}_{\beta\gamma} \circ f) E_a(f^{\beta}) E_a(f^{\gamma}))_{x_0} = 0$, i.e., f is a pseudoharmonic map. □

The study of $(\mathcal{H}, \mathcal{H}')$-harmonic morphisms is left as an open problem. An approach similar to the proofs of Theorems 7.7 and 7.8 requires a subelliptic analog to Lemma 7.6.

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