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A global method for solving second-kind Volterra–Fredholm integral equations

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Abstract

The paper presents a Nyström-type method to approximate the solution of second-kind Volterra–Fredholm integral equations. Two forms are considered, that is the disjoint form, in which the Volterra and Fredholm operators are additive integrals; and the mixed one, in which the two integrals appear in a single term through composition. In both situations, the right-hand side and the kernel functions may have algebraic singularities at ± 1 and hence equations are treated in suitable weighted spaces equipped with the uniform norm. The proposed methods, based on product and Gauss rules, are stable and convergent. The error is of the order of the best polynomial approximation of the given functions. Numerical examples are presented to illustrate the accuracy of the method.

Keywords Volterra–Fredholm integral equations · Nyström method · Gauss quadrature formulae · Product quadrature rules

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1 Introduction

This paper deals with the numerical treatment of Volterra–Fredholm integral equations (VFIEs) of the second-kind [27] arising in several applicative contexts: parabolic boundary integral equations as well as mathematical models related to epidemic evolution, physical and biological problems; see, for instance, [7, 16–18, 20, 23]. In the literature, these equations appear in two forms, namely

$$(I + \mu_1 V + \mu_2 K) f = g, \tag{1}$$

and

$$(I + \mu V K) f = g. \tag{2}$$

where $\mu, \mu_1, \mu_2 \in \mathbb{R} \setminus \{0\}$, I is the identity operator, V and K are defined as

$$(Vf)(y) = \int_{-1}^{y} h(x, y) f(x)(y - x)^{\rho}(1 + x)^{\sigma} dx, \qquad \rho, \sigma > -1, \tag{3}$$

$$(Kf)(y) = \int_{-1}^{1} k(x, y) f(x)(1 - x)^{\alpha}(1 + x)^{\beta} dx, \qquad \alpha, \beta > -1, \tag{4}$$

being h, k known kernels, g a given function, and f the unknown solution we are looking for. From now on, we will refer to equation (1) as a *Volterra–Fredholm Integral equation* (VFIE) and to equation (2) as a *mixed* VFIE.

Let us note that if the kernel k vanishes in the whole domain $[-1, 1] \times [-1, 1]$, equation (1) reduces to the Volterra integral equation

$$(I + \mu_1 V) f = g, \tag{5}$$

whereas if the function h is identically zero, then (1) leads to the Fredholm integral equation

$$(I + \mu_2 K) f = g. \tag{6}$$

Nowadays there is a wide literature on the numerical solution of equations (5) and (6); see, for instance, [4, 5, 12, 14, 15] for the first one, and [3, 11, 22] for the latter one. Equation (1) has been studied both in terms of the existence and uniqueness of solution [1] and numerical solution. For instance, in [6] the authors compare a collocation method with the well-known fixed point theorem, in [8] a Taylor expansion method is developed, and in [13] a Nyström method, involving quadrature rules based on Generalized Bernstein polynomials, is proposed, in the case where the equation contains a fast oscillating kernel. However, we know that only the case $\alpha = \beta = \rho = \sigma = 0$ has been treated. Concerning equation (2), several methods have been developed for the case $\alpha = \beta = \rho = \sigma = 0$, also in the presence of a nonlinear term. Collocation methods are proposed in [17, 18], Nyström methods have been explored in [7, 16], and iterative methods are recently investigated in [23].

In this paper, we assume that the right-hand side and the kernels k and h may have algebraic singularities in the set $\{\pm 1\}$ with respect to the variable y. According to our knowledge, this case has never been treated in the literature. In this situation, the solution inherits the same properties of the known functions. Therefore, f is a function with a low smoothness at the endpoints, which is the reason why Eqs. (1) and (2) are considered in suitable weighted spaces.

For both equations, we present global approximation methods of Nyström type. In the disjoint form, the Fredholm operator K is approximated by a suitable Gauss quadrature formula and the Volterra integral V is discretized by a product integration rule. Such schemes are suitably combined to approximate the mixed integral operator VK of Eq. (2). The methods are proved to be stable and convergent in suitable spaces of weighted continuous function, both with the order of the best polynomial approximation error of the involved known functions.

The paper is structured as follows. In Sect. 2, we describe the spaces in which we consider Eqs. (1) and (2) and we recall the well-known Gaussian quadrature rule. In Sect. 3, we introduce novel product integration schemes we need to approximate the integral operators of our equations. Sections 4 and 5 are devoted to describing and testing the numerical methods for the two equations. Section 6 contains the proofs of our theoretical results and Sect. 7 some conclusions and research perspectives are outlined.

2 Notations and preliminary results

Throughout the whole paper, we will denote by $\mathcal{C}$ any positive constant having different meanings at various occurrences, and the notation $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ will be used to underline that $\mathcal{C}$ does not depend on $a, b, \dots$. Moreover, if $A, B > 0$ are quantities depending on some parameters, the writing $A \sim B$, has to be understood as there exists a constant $\mathcal{C} \neq \mathcal{C}(A, B)$ such that $\mathcal{C}^{-1}B \leq A \leq \mathcal{C}B$.

$\mathbb{P}_m$ denotes the space of the algebraic polynomials of degree less than or equal to m, whereas $\mathbb{P}_{m,m}$ denotes the space of the bivariate algebraic polynomials of degree at most m in each variable. Moreover, for any bivariate function $q(x, z)$, we will use the notation $q_z(x)$ and $q_x(z)$ for referring to $q(x, z)$ as a function of the variable x and z, respectively.

Finally, we use the notation

$$v^{\eta,\theta}(x) := (1-x)^\eta(1+x)^\theta, \quad x \in (-1, 1), \quad \eta, \theta \in \mathbb{R}.$$

2.1 Function spaces

Let $u := v^{\gamma,\delta}$ with $\gamma, \delta \geq 0$ and let C_u be the space of the locally continuous functions f on $(-1, 1)$ satisfying the limit conditions

$$\lim_{x\to 1^-} f(x)u(x) = 0, \quad \text{if } \gamma > 0, \qquad \text{and} \qquad \lim_{x\to -1^+} f(x)u(x) = 0, \quad \text{if } \delta > 0. \tag{7}$$

In the case $\gamma = \delta = 0$, C_u coincides with the space of continuous functions in $[-1, 1]$, i.e. $C^0 := C^0([-1, 1])$.

C_u equipped with the norm

$$\|f\|_{C_u} := \|fu\|_\infty = \sup_{x\in[-1,1]} |(fu)(x)|$$

is a Banach space.

We point out that the limit conditions (7) are necessary to assure that C_u is a Banach space.

For smoother functions, we recall the Sobolev-type subspaces of order $r \in \mathbb{N}$ defined as

$$W_r(u) = \left\{ f \in C_u \ : \ f^{(r-1)} \in \mathcal{AC}((-1, 1)), \ \|f^{(r)}\varphi^r u\|_\infty < \infty \right\}, \quad r \in \mathbb{N},$$

where $\mathcal{AC}$ denotes the space of all absolutely continuous functions in $(-1, 1)$, and $\varphi(x) := \sqrt{1 - x^2}$. The space $W_r(u)$ is equipped with the norm

$$\|f\|_{W_r(u)} := \|fu\|_\infty + \|f^{(r)}\varphi^r u\|_\infty.$$

In particular, if $u \equiv 1$, we set $W_r := W_r(u)$.

For functions in C_u, the error of the best polynomial approximation is defined as

$$E_m(f)_u := \inf_{P\in\mathbb{P}_m} \|f - P\|_{C_u},$$

where in case $u \equiv 1$, we will set $E_m(f) := E_m(f)_u$.

To estimate $E_m(f)_u$ for functions in $W_r(u)$, we recall the following Favard estimate [21, p. 172]

$$E_m(f)_u \le \frac{\mathcal{C}}{m^r} \|f\|_{W_r(u)}, \quad f \in W_r(u), \quad \mathcal{C} \neq \mathcal{C}(m, f). \tag{8}$$

2.2 Lagrange interpolating polynomials and Gauss–Jacobi rule

For a given Jacobi weight $w(x) = v^{\alpha,\beta}(x)$ with $\alpha, \beta > -1$, let $\{p_m(w)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. Moreover, we will denote by $\{x_k\}_{k=1}^m$ the zeros of $p_m(w)$.

For any function $f \in C_u$, let $L_m(w, f) \in \mathbb{P}_{m-1}$ be the Lagrange polynomial interpolating f at the zeros of $p_m(w)$,

$$L_m(w, f, x) = \sum_{k=1}^m \ell_{m,k}(x) f(x_k), \tag{9}$$

with

$$\ell_{m,k}(x) = \prod_{\substack{j=1 \\ j\neq k}}^{m} \frac{x - x_j}{x_k - x_j} = \lambda_{m,k} \sum_{j=0}^{m-1} p_j(w, x) p_j(w, x_k), \tag{10}$$

where $\{\lambda_{m,k}\}_{k=1}^{m}$ are the Christoffel numbers w.r.t. the weight w.

Based on the univariate Lagrange polynomial, we recall the Gauss–Jacobi rule

$$I_1(f) := \int_{-1}^{1} f(x)w(x)dx = \sum_{k=1}^{m} \lambda_{m,k} f(x_k) + R_m^G(f), \tag{11}$$

where the remainder term $R_m^G(f)$ satisfies $R_m^G(f) = 0$, for any $f \in \mathbb{P}_{2m-1}$. Moreover, for any $f \in C_u$, with $u = v^{\gamma,\delta}$, under the assumptions

$$0 \leq \gamma < \alpha + 1, \qquad 0 \leq \delta < \beta + 1,$$

the quadrature error behaves like the best polynomial approximation error [21, §5.1.5.2]

$$|R_m^G(f)| \leq \mathcal{C} E_{2m-1}(f)_u, \qquad \mathcal{C} \neq \mathcal{C}(m, f). \tag{12}$$

Given a bounded function $h(x, y)$ in $(-1, 1) \times (-1, 1)$, we recall the bivariate Lagrange polynomial interpolating h at the grid $\{(x_i, x_j)\}_{\substack{i=1,\dots,m, \\ j=1,\dots,m}}$, where $\{x_i\}_{i=1}^{m}$ are the zeros of $p_m(w)$, [25]

$$\mathcal{L}_m(w, h; z, x) = \sum_{i=1}^{m} \sum_{j=1}^{m} \ell_{m,i}(z)\ell_{m,j}(x) f(x_i, x_j), \tag{13}$$

with $\{\ell_{m,k}(z)\}_{k=1}^{m}$ defined in (10). The polynomial $\mathcal{L}_m(f) \in \mathbb{P}_{m-1,m-1}$, and shares the following invariance property $\mathcal{L}_m(w, P) \equiv P, \;\forall\, P \in \mathbb{P}_{m-1,m-1}$.

3 Product integration formulae

In this section, we consider and study two different kinds of product integration rules by polynomials, to approximate integrals of the types

$$I_2(d, y) = \int_{-1}^{y} d(x)\psi(x, y)dx, \quad y \in (-1, 1],$$
$$I_3(q, y) = \int_{-1}^{y} q(z) \left[\int_{-1}^{1} s(x, z)w(x)dx\right] \psi(z, y)dz, \quad y \in (-1, 1].$$

Here, q and d are defined in $(-1,1)$, s and ψ are given on $(-1,1)\times(-1,1)$. Moreover, the kernel ψ can have possible algebraic singularities at $x=-1$, and/or along $x=y$. For instance, $\psi(x,y)=\frac{\log(y-x)}{\sqrt{1+x}}$. Also, the kernel $s(x,z)$ may be singular at $x=\pm1$.

3.1 A product rule for $I_2(d)$

Consider $I_2(d)$. By approximating d by the Lagrange polynomial $L_m(w,d)$ interpolating d at the zeros of $p_m(w)$, we have

$$I_2(d,y)=\mathcal{P}_m(d,y)+R_m^{\mathcal{P}}(d,y),$$

where

$$\mathcal{P}_m(d,y)=\sum_{k=1}^{m}c_k(y)d(x_k),\quad c_k(y)=\int_{-1}^{y}\ell_{m,k}(x)\psi(x,y)dx. \tag{14}$$

The rule (14) generalizes a formula studied in [15], where the specific case $\psi(x,y)=(y-x)^\alpha(1+x)^\beta$ was treated. Of course, $R_m^{\mathcal{P}}(d,y)=0,\ \forall d\in\mathbb{P}_{m-1}$. Let us now assess the stability of the rule (14) in C_u, and the convergence. Indeed, setting

$$\|\mathcal{P}_m(y)\|_{C_u}:=\sup_{\|d\|_{C_u}=1}|\mathcal{P}_m(d,y)|=\sup_{\|d\|_{C_u}=1}\sum_{k=1}^{m}\frac{|c_k(y)|}{u(x_k)},\quad \forall y\in(-1,1), \tag{15}$$

next theorem states sufficient conditions assuring $\|\mathcal{P}_m(y)\|_{C_u}$ is uniformly bounded w.r.t. m. Moreover, the quadrature error is estimated in terms of the best approximation error in C_u.

Theorem 1 *For any $d\in C_u$ with $u=v^{\gamma,\delta}$ such that*

$$0\le\gamma<\frac{\alpha}{2}+\frac{5}{4},\qquad 0\le\delta<\frac{\beta}{2}+\frac{5}{4}, \tag{16}$$

under the assumptions

$$\sup_{y\in(-1,1]}\int_{-1}^{y}\frac{|\psi(x,y)|}{u(x)}\log\left(2+\frac{|\psi(x,y)|}{u(x)}\right)dx<\infty,$$
$$\sup_{y\in(-1,1]}\int_{-1}^{y}\frac{|\psi(x,y)|}{\sqrt{w(x)\varphi(x)}}dx<\infty, \tag{17}$$

one has

$$\sup_{y\in(-1,1]}\sup_{m}\|\mathcal{P}_m(y)\|_{C_u}\le\mathcal{C},\quad \mathcal{C}\neq\mathcal{C}(m), \tag{18}$$

and

$$\sup_{y\in(-1,1]} |R_m^{\mathcal{P}}(d,y)| \le \mathcal{C} E_{m-1}(d)_u, \quad \mathcal{C} \ne \mathcal{C}(m,d). \tag{19}$$

Remark 1 For $d \in W_r(u)$, in view of (8), under the assumption of Theorem 1

$$\sup_{y\in(-1,1]} |R_m^{\mathcal{P}}(d,y)| \le \mathcal{C} \frac{\|d\|_{W_r(u)}}{m^r}, \quad \mathcal{C} \ne \mathcal{C}(m,d).$$

3.2 A product rule for $I_3(q)$

Consider now the integral

$$I_3(q,y) = \int_{-1}^{y} q(z) \left[\int_{-1}^{1} s(x,z) w(x) dx \right] \psi(z,y) dz, \quad y \in (-1,1].$$

Approximating $s(x,z)q(z)$ by the bivariate Lagrange polynomial $\mathcal{L}_m(w, sq; x, z)$ in (13), we obtain the following formula

$$I_3(q,y) = \Sigma_m(q,y) + R_m^{G\mathcal{P}}(q,y), \tag{20}$$

being

$$\Sigma_m(q,y) := \sum_{k=1}^{m} c_k(y) q(x_k) \sum_{j=1}^{m} \lambda_{m,j} s(x_j, x_k), \quad c_k(y) = \int_{-1}^{y} \ell_{m,k}(z) \psi(z,y) dz, \tag{21}$$

and $R_m^{G\mathcal{P}}(q,y)$ the remainder term.

Note that formula (21) can be equivalently obtained by approximating the internal integral by the m-th Gauss–Jacobi rule (11), i.e.

$$I(s,z) := \int_{-1}^{1} s(x,z) w(x) dx \sim \sum_{j=1}^{m} \lambda_{m,j} s(x_j, z) =: G_m(s,z), \qquad \forall z \in [-1,1], \tag{22}$$

and the integrals

$$\sum_{j=1}^{m} \lambda_{m,j} \int_{-1}^{y} s(x_j, z) q(z) \psi(z,y) dz,$$

by the m-th product rule (14) with $d(z) = s(x_j, z) q(z)$.

For this reason, from now on we refer to the rule (20) as *mixed Gauss-product formula*. By virtue of the exactness degrees of the two native quadrature formulae, under the assumption $s_z(x) \in \mathbb{P}_{2m-1}$ for any $z \in [-1, 1]$, we have

$$I_3(q, y) = \Sigma_m(q, y), \text{ if } \quad q(z)s(x, z) \in \mathbb{P}_{m-1}, \forall x \in [-1, 1].$$

Hence, we have in conclusion, that the mixed Gauss-product formula is exact if $s(x, z)q(z) \in \mathbb{P}_{2m-1,m-1}$. About the stability of the rule and the error estimate, we are able to prove the following

Theorem 2 *Let be $w = v^{\alpha,\beta}$. Under the assumptions $\sup_z s_z \in C_u$ with $u = v^{\gamma,\delta}$ and*

$$0 \le \gamma < \alpha + 1, \qquad 0 \le \delta < \beta + 1, \tag{23}$$

$$\sup_{y\in(-1,1]} \int_{-1}^{y} |\psi(z, y)| \log(2 + |\psi(z, y)|)dz < \infty, \tag{24}$$

$$\sup_{y\in(-1,1]} \int_{-1}^{y} \frac{|\psi(z, y)|}{\sqrt{w(z)\varphi(z)}}dz < \infty, \tag{25}$$

formula (21) is stable in C^0, i.e.

$$\sup_{y\in(-1,1]} \sup_m \|\Sigma_m(y)\|_{C^0} < \infty.$$

Moreover, for any $q \in C^0$ the following error estimate holds

$$\sup_{y\in(-1,1]} |R_m^{G\mathcal{P}}(q, y)| \le \mathcal{C} \sup_{z\in[-1,1]} \left(\|q\|_\infty E_{\lfloor \frac{m-1}{2} \rfloor}(s_z)_u + \|s_z u\|_\infty E_{\lfloor \frac{m-1}{2} \rfloor}(q) \right), \tag{26}$$

where $\mathcal{C} \ne \mathcal{C}(m)$.

4 The numerical method for the VFIE

In this section, we introduce a Nyström-type method to approximate the solution of Eq. (1) in the weighted space C_u. We emphasize that the choice of this setting allows us to treat right-hand sides g or kernels h and k presenting singularities at the endpoints w.r.t. the external variable y.

Let us first approximate the operator V in (3) by means of the product rule (14) with $d(x) = f(x)h_y(x)$ and $\psi(x, y) = (y - x)^\rho(1 + x)^\sigma$, i.e.

$$(V_m f)(y) = \sum_{k=1}^{m} c_k(y)h(x_k, y)f(x_k), \quad c_k(y) = \int_{-1}^{y} \ell_{m,k}(x)(y - x)^\rho(1 + x)^\sigma dx.$$

$$(27)$$

Note that by using the linear transformation $\phi_y : [-1, y] \to [-1, 1]$ defined as

$$\phi_y(x) := 2\,\frac{1+x}{1+y} - 1,$$

and applying (10), the coefficients $\{c_k(y)\}_{k=1}^m$ can be rewritten as follows

$$\begin{aligned} c_k(y) &= \lambda_{m,k} \sum_{j=0}^{m-1} p_j(w, x_k) \int_{-1}^{y} p_j(w, x)(y-x)^{\rho}(1+x)^{\sigma}\, dx \\ &= \left(\frac{1+y}{2}\right)^{\rho+\sigma+1} \lambda_{m,k} \sum_{j=0}^{m-1} p_j(w, x_k) \left[\int_{-1}^{1} p_j\left(w, \phi_y^{-1}(z)\right) v^{\rho,\sigma}(z) dz\right], \end{aligned}$$

where $\phi_y^{-1}(z) = z\left(\frac{1+y}{2}\right) + \frac{y-1}{2}$. Consequently, setting $n := \left\lfloor \frac{m}{2} \right\rfloor$, by approximating the integral by the exact n-point Gauss rule (11) with respect to the Jacobi weight $v^{\rho,\sigma}$, we get

$$c_k(y) = \left(\frac{1+y}{2}\right)^{\rho+\sigma+1} \lambda_{m,k} \sum_{j=0}^{m-1} p_j(w, x_k) \left[\sum_{i=1}^{n} \widetilde{\lambda}_{n,i}\, p_j\left(w, \phi_y^{-1}(z_i)\right)\right],$$

with z_i the i-th zero of $p_n(v^{\rho,\sigma})$ and $\widetilde{\lambda}_{n,i}$ the related Christoffel number.

The following propositions provide conditions assuring the compactness of the integral operator V and the collectively compactness of the approximating sequence of discrete operators $\{V_m\}_m$.

Proposition 1 *Let V be the linear operator defined in (3) and assume that*

$$\sup_{x\in[-1,1]} \|h_x\|_{W_1(u)} < \infty, \qquad \sup_{y\in(-1,1]} \int_{-1}^{y} (y-x)^{\rho-1} \frac{(1+x)^{\sigma}}{u(x)}\, dx < \infty. \tag{28}$$

Then, $V : C_u \to C_u$ is a compact operator.

Proposition 2 *Let V_m be the operator defined in (27). Assume that the kernel h is such that*

$$\sup_{y\in(-1,1]} u(y)\|h_y\|_{W_1} < \infty,$$

with the weight u satisfying

$$0 \le \gamma < \frac{\alpha}{2} + \frac{5}{4}, \qquad 0 \le \delta < \frac{\beta}{2} + \frac{5}{4}, \tag{29}$$

the parameters ρ *and* σ *fulfilling*

$$\rho > \frac{\alpha}{2} + \frac{1}{4}, \qquad \sigma > \max\left\{-1, \frac{\beta}{2} - \frac{3}{4}\right\}, \tag{30}$$

and

$$\int_{-1}^{y} \frac{(y-x)^{\rho-1}(1+x)^{\sigma}}{u(x)} \log\left(2 + \frac{(y-x)^{\rho-1}(1+x)^{\sigma}}{u(x)}\right) dx < \infty. \tag{31}$$

Then, the sequence $\{V_m\}_m$*, with* $V_m : C_u \to C_u$*, is uniformly bounded, collectively compact, and pointwise convergent to* V.

Let us now consider the Fredholm operator K defined in (4). By using the Gauss–Jacobi rule (11), we introduce the discrete operator

$$(K_m f)(y) = K(L_m(k_x f))(y) = \sum_{k=1}^{m} \lambda_{m,k} k(x_k, y) f(x_k), \tag{32}$$

whose main properties are designated in the next proposition.

Proposition 3 *Let* K *and* K_m *be the linear operators defined in* (4) *and* (32)*, respectively. Assume*

$$0 \le \gamma < \alpha + 1, \qquad 0 \le \delta < \beta + 1, \tag{33}$$

and for some $r \ge 1$

$$\mathcal{M}_r^k := \sup_{x \in [-1,1]} \|k_x\|_{W_r(u)} < \infty, \qquad \mathcal{N}_r^k := \sup_{y \in (-1,1]} u(y) \|k_y\|_{W_r} < \infty. \tag{34}$$

Then, $K : C_u \to C_u$ *is a compact operator. Moreover,* $K_m : C_u \to C_u$ *is bounded, and* $\forall f \in C_u$ *it is* $K_m f \in W_r(u)$*. Finally, the sequence* $\{K_m\}_m$*,* $K_m : C_u \to C_u$ *is collectively compact, and pointwise convergent to* K.

Let us now consider the following finite-dimensional equation

$$(I + \mu_1 V_m + \mu_2 K_m) f_m = g \tag{35}$$

in the unknown f_m. Next theorem states the unisolvence of Eq. (35) and the stability of the Nyström method, thanks to the properties of the operators V_m and K_m.

Theorem 3 *Under the assumptions of Propositions 1, 2, and 3, for* m *sufficiently large, say* $m \ge m_0$*, the inverse operators* $(I + \mu_1 V_m + \mu_2 K_m)^{-1}$ *exist and are uniformly bounded with respect to* m.

In order to solve Eq. (35) in the space C_u, let us multiply both the members by the weight u and by collocating the equation at the points $\{x_i\}_{i=1}^m$, we get the following linear system of order m

$$\sum_{j=1}^{m}\left[\delta_{i,j}+\mu_1 c_j(x_i)\frac{u(x_i)}{u(x_j)}h(x_j,x_i)+\mu_2\lambda_{m,j}\frac{u(x_i)}{u(x_j)}k(x_j,x_i)\right]a_j=b_i,\quad i=1,\ldots,m, \tag{36}$$

where $\delta_{i,j}$ is the Kronecker delta, $a_j := (f_m u)(x_j)$, $j = 1, 2, \ldots, m$ are the unknowns, and $b_i := (gu)(x_i)$, $i = 1, 2, \ldots, m$. In a more compact matrix form, the system (36) can be rewritten as

$$\left(\mathcal{I}_m + \mathcal{U}_m \mathcal{A}_m \mathcal{U}_m^{-1}\right) \boldsymbol{a}_m = \boldsymbol{b}_m, \tag{37}$$

where $\boldsymbol{a}_m = [a_1, \ldots, a_m]^T$, $\boldsymbol{b}_m = [b_1, \ldots, b_m]^T$, I_m is the identity matrix of order m, and

$$\mathcal{U}_m = \text{diag}(u(x_1), \ldots, u(x_m)), \qquad \mathcal{A}_m = \mu_1 \mathcal{H}_m + \mu_2 \mathcal{K}_m \Lambda_m,$$

with

$$(\mathcal{H}_m)_{i,j} = c_j(x_i)h(x_j, x_i), \quad (\mathcal{K}_m)_{i,j} = k(x_j, x_i), \quad \Lambda_m = \text{diag}(\lambda_{m,1}, \ldots, \lambda_{m,m}).$$

System (36) admits a unique solution $\boldsymbol{a}_m^* = [a_1^*, \ldots, a_m^*]^T$ that allows to construct the so-called Nyström interpolant

$$f_m(y)u(y) = g(y)u(y) - u(y)\sum_{j=1}^{m}\left[\mu_1\frac{h(x_j,y)}{u(x_j)}c_j(y) + \mu_2\lambda_j\frac{k(x_j,y)}{u(x_j)}\right]a_j^*. \tag{38}$$

Theorem 4 *Assume* $\ker\{I + \mu_1 V + \mu_2 K\} = \{0\}$ *in* C_u*, with* u *such that*

$$0 \le \gamma < \min\left\{\frac{\alpha}{2} + \frac{5}{4}, \alpha + 1\right\}, \qquad 0 \le \delta < \min\left\{\frac{\beta}{2} + \frac{5}{4}, \beta + 1\right\},$$

and denote by f^* *and* f_m^* *the unique solutions of Eqs.* (1) *and* (35)*, respectively. Assume that the assumptions in* (30) *and* (31) *are fulfilled, the kernel* k *satisfies* (34)*, the kernel* h *is such that*

$$\sup_{x\in[-1,1]} \|h_x\|_{W_r(u)} < \infty, \qquad \sup_{y\in(-1,1]} u(y)\|h_y\|_{W_r} < \infty,$$

and the right-hand side $g \in W_r(u)$*. Then,*

$$\|(f^* - f_m^*)\|_{C_u} \le \frac{\mathcal{C}}{m^r}\|f^*\|_{W_r(u)},$$

where $\mathcal{C} \neq \mathcal{C}(m, f^*)$.

Remark 2 The proposed numerical method is easy to implement, since the construction of the linear systems (37) only requires the computation of the known functions h, k, g at zeros of $p_m(w)$.

4.1 Numerical tests

In this section, we show the performance of the numerical method described in the previous section by some numerical tests.

In each example, we first fix the weighted space C_u in which we consider the equation, according to Theorem 4. Then, we solve the linear system (36) and compute the weighted Nyström interpolant (38). To test the accuracy, if the exact solution f^* is known, we compute the relative errors

$$\epsilon_m = \frac{\|(f^* - f_m)u\|}{\|f^* u\|}, \tag{39}$$

where $\|\cdot\|$ denotes the discrete infinity norm taken on a grid of 10^3 equispaced points in $[-1, 1]$. If f^* is unknown, we consider as exact the approximated solution with $m = 1024$ and compute the relative errors

$$\varepsilon_m = \frac{\|(f_{1024} - f_m)u\|}{\|f_{1024} u\|}. \tag{40}$$

We point out that this choice does not affect the results since by virtue of Theorem 3 and Theorem 4 our method is stable and convergent.

Moreover, when the exact solution is unknown we evaluate the Estimated Order of Convergence

$$EOC_m = \frac{\log(\varepsilon_m / \varepsilon_{2m})}{\log 2}, \tag{41}$$

and in each test we show the condition number related to linear system (36), i.e.

$$\kappa(M_m) := \|M_m\| \, \|M_m^{-1}\|, \qquad M_m = \mathcal{I}_m + \mathcal{U}_m \mathcal{A}_m \mathcal{U}_m^{-1}, \tag{42}$$

where, here, $\|\cdot\|$ is the matrix infinity norm.

All the computations are performed on an Intel Xeon E-2244 G system with 16Gb RAM, running Matlab R2024a. The software developed VFIE_toolbox is distributed as an archive file and available for download at the web page https://bugs.unica.it/cana/software.

Example 1 First, to test the algorithm, let us consider the following equation

$$f(y) - \frac{1}{2\pi} \int_{-1}^{y} y e^{-x} f(x)(y-x)(1+x)\, dx + \frac{1}{\pi} \int_{-1}^{1} (x+y^2) f(x) \sqrt{1-x^2}\, dx = g(y),$$

Table 1 Numerical results for Example 1

m	ϵ_m	$\kappa(M_m)$
4	1.30e−04	1.72
8	1.32e−08	2.02
16	2.68e−16	2.14

where g is computed so that the exact solution is $f^*(y) = y \sin y$. In particular,

$$h(x, y) = ye^{-x}, \quad k(x, y) = x + y^2, \quad \rho = \sigma = 1, \quad \alpha = \beta = \frac{1}{2},$$

and

$$g(y) = y \sin y + y^2 J_2(1) - \frac{ye^{-y}}{4\pi} \Big[(y^2 + 3y + 1) \cos y \\ + e^{y+1} (\cos 1 + (y - 1) \sin 1) - (2y + 4) \sin y \Big],$$

with J_2 the Bessel function of the first kind. Table 1 reports the relative errors for the weighted solution in C_u with $u \equiv 1$, and the condition number of system (36) for different values of m. As we can note, the convergence is very fast because the kernels and right-hand side are analytic functions in $[-1, 1]$ and the system is well conditioned.

Example 2 Let us consider the following equation

$$f(y) + \frac{1}{2} \int_{-1}^{y} x \cos y f(x) \sqrt{(y - x)(1 + x)}\, dx \\ + \frac{1}{3} \int_{-1}^{1} \frac{\log(x + y + 4)}{\sqrt[4]{1 + x}} f(x)\, dx = |y - 1|^{\frac{3}{2}},$$

in the weighted space C_u with $u(y) = \sqrt[4]{1 - y^2}$, according to the conditions given in Theorem 4 and being $\rho = \sigma = \frac{1}{2}$, $\alpha = 0$, $\beta = -\frac{1}{4}$. In this space, the kernels $h(x, y) = x \cos y$ and $k(x, y) = \log(x + y + 4)$ are smooth functions whereas the right-hand side $g(y) = |y - 1|^{\frac{3}{2}}$ belongs to $W_3(u)$. Hence, according to Theorem 4, we expect an error of the order $\mathcal{O}(m^{-3})$. Figure 1 displays the solution f_{1024} assumed as exact to compute the relative errors shown in Table 2 to the left. The numerical errors are better than the theoretical estimates, as also confirmed by the EOC_m given in the last column. Moreover, by inspecting the third column, we can see that the condition number of system (36) does not increase with m.

Example 3 Let us now focus on an example in which the kernel of the Volterra operator has a low smoothness

$$f(y) + \frac{1}{10} \int_{-1}^{y} (xy + 3) \sin(\sqrt{1 + x}) f(x)(y - x)^{\frac{3}{5}}\, dx$$

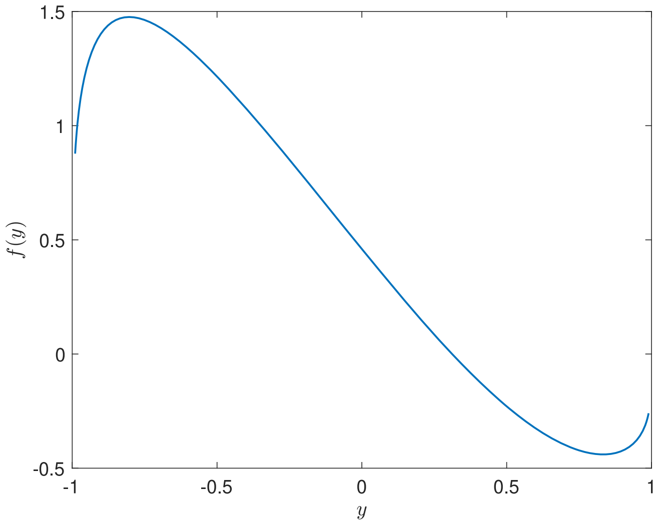

Fig. 1 Graph of the solution f_{1024} of Example 2

Table 2 Numerical results for Example 2 (to the left) and Example 3 (to the right)

m	ε_m	$\kappa(M_m)$	EOC_m	m	ε_m	$\kappa(M_m)$	EOC_m
4	3.13e−04	2.70	6.29	4	7.05e−05	4.06	2.72
8	4.01e−06	3.02	5.65	8	1.07e−05	4.73	2.82
16	7.97e−08	3.25	5.13	16	1.51e−06	5.12	2.91
32	2.28e−09	3.37	5.01	32	2.01e−07	5.37	2.95
64	7.06e−11	3.43	4.99	64	2.60e−08	5.56	2.98
128	2.22e−12	3.46	4.98	128	3.29e−09	5.70	3.01
256	7.04e−14	3.48	4.01	256	4.09e−10	5.80	3.16
512	4.36e−15	3.48		512	4.56e−11	5.88	

$$+\frac{1}{12}\int_{-1}^{1}\sin(x+y+2)f(x)\sqrt[3]{1-x}\,dx=e^y.$$

In fact, the function $h(x,y)=(xy+3)\sin(\sqrt{1+x})\in W_2(u)$, while the kernel $k(x,y)=\sin(x+y+2)\in W_r(u)\ \forall r\in\mathbb{N}$ with $u(y)=(1+y)^{\frac{4}{5}}$. Table 2 reports the results we obtain whereas in Fig. 2, we compare the numerical errors with the theoretical one which is of the order m^{-2}, given that $\rho=\frac{3}{5},\ \sigma=0,\ \alpha=\frac{1}{3},\ \beta=0$. Also in this case, the results show a faster convergence with respect to the theoretical estimate.

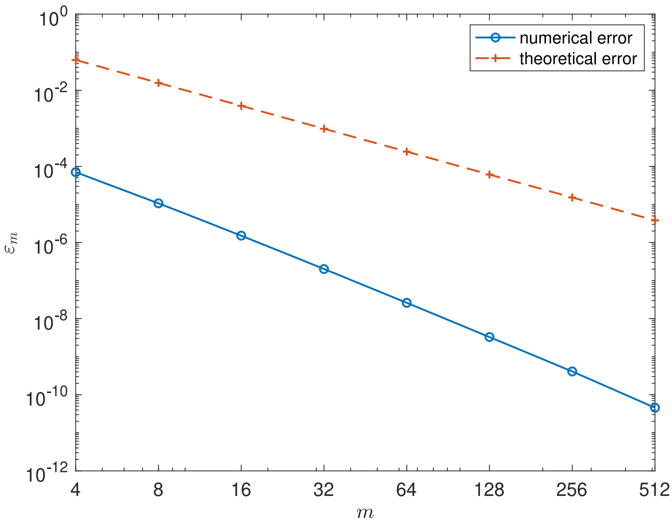

Fig. 2 Benchmark analysis of theoretical ($\mathcal{O}(m^{-2})$) and numerical errors in Example 3

5 The numerical method for the mixed VFIE

In this section, we introduce a Nyström-type method to numerically treat the mixed Eq. (2) which explicitly takes the following expression

$$f(y) + \mu \int_{-1}^{y} \left[\int_{-1}^{1} k(x, z) f(x)(1-x)^{\alpha}(1+x)^{\beta} dx \right] h(z, y)(y-z)^{\rho}(1+z)^{\sigma} dz = g(y),$$

and we look for its solution in the weighted space C_u. Before going on, let us mention a feature of the operator VK.

Theorem 5 *Let us assume that conditions of Proposition 1 and 3 holds true. Then, the operator $VK : C_u \to C_u$ is compact.*

The method is based on the approximation of the mixed Volterra–Fredholm operator VK by the mixed Gauss-product rule (21) for $q(z) = h_y(z)$, $s(x, z) = k(x, z) f(x)$, $\psi(z, y) = (y - z)^{\rho}(1 + z)^{\sigma}$. Hence, we consider the following finite-dimensional equation

$$(I + \mu\, \Omega_m) f_m = g, \tag{43}$$

where f_m is the unknown solution, and Ω_m is the following discrete operator obtained by applying (21)

$$(\Omega_m f)(y) = \sum_{k=1}^{m} c_k(y) h(x_k, y) \sum_{j=1}^{m} \lambda_{m,j} k(x_j, x_k) f(x_j).$$

Remark 3 Note that $\Omega_m f = V_m(K_m f)$ where V_m is the discrete operator (27), and K_m is given in (32).

The following theorem establishes sufficient conditions assuring the existence of the operators $(I + \mu\Omega_m)^{-1}$, for m sufficiently large, as well as the stability of the proposed Nyström method.

Theorem 6 *Under the assumptions of Proposition 1, 2, and 3, for m sufficiently large, say $m \geq m_0$, the operators $(I + \mu\Omega_m)^{-1}$ exist and are uniformly bounded with respect to m.*

Multiplying the discrete equation (43) by the weight u and collocating it at the quadrature points $\{x_i\}_{i=1}^m$, we obtain the following linear system of order m

$$\sum_{j=1}^{m} \left[\delta_{i,j} + \mu\lambda_{m,j} \frac{u(x_i)}{u(x_j)} \sum_{k=1}^{m} c_k(x_i) k(x_j, x_k) h(x_k, x_i) \right] a_j = b_i, \quad i = 1, \dots, m, \tag{44}$$

with $a_j := f_m(x_j)u(x_j)$ the unknowns and $b_i := g(x_i)u(x_i)$ the right-hand side terms. The unique solution of (44) leads to the weighted Nyström interpolant

$$f_m(y)u(y) = g(y)u(y) - u(y) \sum_{j=1}^{m} \left[\mu \frac{\lambda_{m,j}}{u(x_j)} \sum_{k=1}^{m} c_k(y) k(x_j, x_k) h(x_k, y) \right] a_j^*. \tag{45}$$

Note that system (43) can be written also in the matrix form

$$\left(\mathcal{I}_m + \mathcal{U}_m \mathcal{B}_m \mathcal{U}_m^{-1} \right) \boldsymbol{a}_m = \boldsymbol{b}_m,$$

where $\boldsymbol{a}_m = [a_1, \dots, a_m]^T$, $\boldsymbol{b}_m = [b_1, \dots, b_m]^T$, $\mathcal{I}_m$ is the identity matrix of order m, and

$$\mathcal{U}_m = \mathrm{diag}(u(x_1), \dots, u(x_m)), \quad \mathcal{B}_m = \mu \mathcal{H}_m \mathcal{K}_m \Lambda_m,$$

with

$$(\mathcal{H}_m)_{i,j} = c_j(x_i) h(x_j, x_i), \quad (\mathcal{K}_m)_{i,j} = k(x_j, x_i), \quad \Lambda_m = \mathrm{diag}(\lambda_{m,1}, \dots, \lambda_{m,m}).$$

Next theorem states sufficient conditions assuring that the solution of Eq. (43) uniformly converges in C_u to the solution of (2), and the error behaves as the best polynomial approximation of the known functions.

Theorem 7 *Assume* $\ker\{I + \mu V K\} = \{0\}$ *in C_u, with u such that*

$$0 \leq \gamma < \min\left\{ \frac{\alpha}{2} + \frac{5}{4}, \alpha + 1 \right\}, \qquad 0 \leq \delta < \min\left\{ \frac{\beta}{2} + \frac{5}{4}, \beta + 1 \right\}.$$

Table 3 Numerical results for Example 4

m	ϵ_m	$\kappa(M_m)$
4	1.24e−04	1.54
8	1.97e−09	1.83
16	3.64e−16	1.96

Assume that condition (30) *and* (31) *are fulfilled, the kernel k satisfies*

$$\sup_{x\in[-1,1]} \|k_x\|_{W_r(u)} < \infty, \qquad \sup_{z\in(-1,1]} \|k_z\|_{W_r} < \infty, \tag{46}$$

the kernel h is such that

$$\sup_{z\in[-1,1]} \|h_z\|_{W_r(u)} < \infty, \qquad \sup_{y\in(-1,1]} u(y)\|h_y\|_{W_r} < \infty, \tag{47}$$

and the right-hand side $g \in W_r(u)$. *Then,*

$$\|(f^* - f_m^*)u\|_\infty \le \frac{\mathcal{C}}{m^r}\|f^*\|_{W_r(u)},$$

where $\mathcal{C} \neq \mathcal{C}(m, f^*)$.

5.1 Numerical tests

Similarly to what done in Sect. 4.1, in this paragraph, we show the performance of our method by three examples, displaying the errors (39) or (40) with f_m as in (45), the estimated order of convergence (41), and the condition number (42) where here the matrix $M_m = \mathcal{I}_m + \mathcal{U}_m\mathcal{B}_m\mathcal{U}_m^{-1}$. The software used in the computation is contained in the package mentioned in Sect. 4.1.

Example 4 First, to test the algorithm, let us consider the following equation in C_u with $u \equiv 1$

$$f(y) + \frac{1}{10}\int_{-1}^{y}\left[\int_{-1}^{1}(x+z)f(x)dx\right]e^{z+y}(y-z)dz = g(y),$$

where

$$g(y) = e^{y+1} + \frac{e^{y-1}}{10}\left[-9 - e^{y+1}(y-4) + e^{3+y}(y-2) - 4y + e^2(5+2y)\right],$$

$$k(x,z) = x+z, \quad h(z,y) = e^{z+y}, \quad \alpha = \beta = 0, \quad \rho = 1, \ \sigma = 0,$$

so that the exact solution is $f(y) = e^{y+1}$. By Table 3, we can appreciate a fast convergence of the approximated solution to the exact one. This is certainly due to the regularity of the involved functions.

Table 4 Numerical results for Example 5 (to the left) and Example 6 (to the right)

m	ε_m	$\kappa(M_m)$	EOC_m	m	ε_m	$\kappa(M_m)$	EOC_m
4	1.46e−05	1.00	7.25	4	1.35e−02	11.5	1.88
8	9.57e−08	1.00	5.05	8	3.67e−03	14.9	2.16
16	2.89e−09	1.00	4.45	16	8.21e−04	16.5	1.97
32	1.32e−10	1.00	4.47	32	2.10e−04	17.3	2.06
64	5.94e−12	1.00	4.49	64	5.04e−05	17.7	2.00
128	2.65e−13	1.00	4.50	128	1.26e−05	18.0	2.06
256	1.17e−14	1.00	4.14	256	3.02e−06	18.2	1.81
512	6.65e−16	1.00		512	8.59e−07	18.3	

Example 5 Let us apply our method to the mixed equation

$$f(y) - \frac{1}{17}\int_{-1}^{y}\left[\int_{-1}^{1}\frac{|x|^{\frac{7}{2}}\sin(x+z)}{(x+z+5)}\frac{f(x)}{\sqrt[5]{1-x}}dx\right] \times |z|^{\frac{7}{2}}e^{-\cos^2(z+y)}\sqrt[3]{(y-z)^2(1+z)}\,dz = \cos y,$$

to approximate its solution in C_u with $u(y) = (1-y)^{\frac{1}{2}}(1+y)^{\frac{3}{5}}$. By Theorem 7, we expect a theoretical error of the order m^{-3} since $\alpha = -\frac{1}{5},\ \beta = 0,\ \rho = \frac{2}{3},\ \sigma = \frac{1}{3}$ and both kernels $k(x,z) = \frac{|x|^{\frac{7}{2}}\sin(x+z)}{(x+z+5)}$ and $h(z,y) = |z|^{\frac{7}{2}}e^{-\cos^2(z+y)}$ are functions of the subspace $W_3(u)$. In Table 4 to the left, we report the numerical errors which show a better performance of our method. We also remark that in this case system (44) has an optimal conditioning. In Fig. 3, we show the approximated solution f_{1024}.

Example 6 Let us now consider equation

$$f(y) - \frac{1}{20}\int_{-1}^{y}\left[\int_{-1}^{1}(3+2^{x+z})\frac{f(x)}{\sqrt{1+x}}dx\right]e^{|z+y|}\sqrt[3]{(y-z)(1+z)}\,dz = (1+y)^{\frac{5}{6}}.$$

Here $k(x,z) = 3 + 2^{x+z},\ h(z,y) = e^{|z+y|},\ \alpha = 0,\ \beta = -\frac{1}{2},\ \rho = \sigma = \frac{1}{3}$.

We look for the solution in the weighted space C_u with $u(y) = \sqrt[5]{1-y^2}$ in which the right-hand side $(1+y)^{\frac{5}{6}} \in W_2(u)$. The numerical results reported in Table 4 to the right, confirm the theoretical expectation $\mathcal{O}(m^{-2})$, as also displayed in Fig. 4.

6 Proofs

In this section, we collect all the proofs of the above announced theoretical results. Before going on, we recall a lemma needed in the successive proofs.

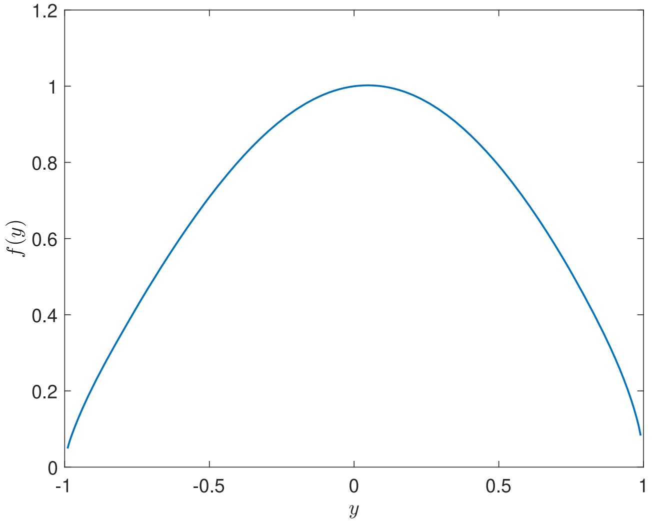

Fig. 3 Plot of the solution f_{1024} of Example 5

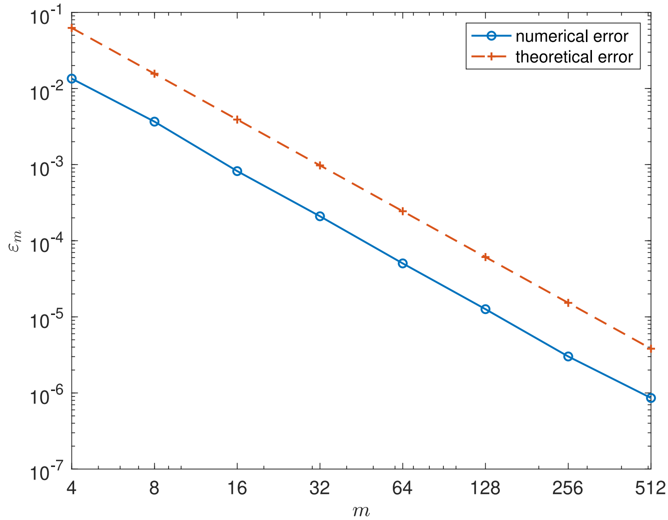

Fig. 4 Benchmark analysis of the theoretical ($\mathcal{O}(m^{-2})$) and numerical error of Example 6

Lemma 1 *For any* $f \in C_u$ *and* $g \in C^0([-1, 1])$, *it follows that*

$$E_{2m}(fg)_u \leq \|fu\|_\infty E_m(g) + 2\|g\|_\infty E_m(f)_u. \tag{48}$$

Proof Let P_m and Q_m be the best polynomials approximating f and g, respectively. Then,

$$E_{2m}(fg) \leq \|(fg - P_m Q_m)u\|_\infty \leq \|g - Q_m\|_\infty \|fu\|_\infty + \|(f - P_m)u\|_\infty \|Q_m\|_\infty.$$

The assertion follows noting that

$$\|Q_m\|_\infty \le \|Q_m - g\|_\infty + \|g\|_\infty \le 2\|g\|_\infty.$$

□

Proof of Theorem 1. By (14), for any $d \in C_u$ we have

$$|\mathcal{P}_m(d, y)| \le \int_{-1}^{y} |L_m(w, d, x)|\,|\psi(x, y)|dx = \left\| L_m(w, d)\widetilde{\psi}_y \right\|_1,$$

where $\|\cdot\|_1$ is the usual L_1-norm, and

$$\widetilde{\psi}(x, y) = \begin{cases} \psi(x, y), & y \ge x, \\ 0, & y < x. \end{cases} \tag{49}$$

Therefore, by Nevai's theorem [24] (see e.g. [21, Theorem 5.1.11]), holding under the assumptions (17) and (16), we deduce

$$|\mathcal{P}_m(d, y)| \le \left\| L_m(w, d)\widetilde{\psi}_y \right\|_1 \le \mathcal{C}\|d\|_{C_u}, \qquad \mathcal{C} \ne \mathcal{C}(m, d),$$

and hence (18), i.e.

$$\sup_{y\in(-1,1]} \sup_m \left(\sup_{\|du\|_\infty=1} |\mathcal{P}_m(d, y)| \right) \le \mathcal{C}.$$

Let us now prove the convergence estimate. Let $P_{m-1} \in \mathbb{P}_{m-1}$, by the exactness of the product rule, we have

$$\begin{aligned} |R_m^{\mathcal{P}}(d, y)| &= \left| \int_{-1}^{y} \psi(x, y)(d(x) - P_{m-1}(x))dx - \sum_{k=1}^{m} c_k(y)(d(x_k) - P_{m-1}(x_k)) \right| \\ &\le \|(d - P_{m-1})u\|_\infty \left(\int_{-1}^{1} \frac{|\widetilde{\psi}(x, y)|}{u(x)}dx + \sum_{k=1}^{m} \frac{|c_k(y)|}{u(x_k)} \right) \end{aligned}$$

being $\widetilde{\psi}$ defined in (49). Hence, by the stability of the formula and the assumptions (17) we have

$$\sup_{y\in(-1,1)} |R_m^{\mathcal{P}}(d, y)| \le \mathcal{C}E_{m-1}(d)_u.$$

□

Proof of Theorem 2. First we prove the stability of the quadrature rule $\Sigma_m(q, y)$. For $q \in C^0$,

$$|\Sigma_m(q, y)| \le \|q\|_\infty \sum_{k=1}^{m} |c_k(y)| \left| \sum_{j=1}^{m} \lambda_{m,j} s(x_j, x_k) \right|$$

$$\leq \|q\|_\infty \sup_{z\in[-1,1]} \|s_z u\|_\infty \sum_{k=1}^m |c_k(y)| \sum_{j=1}^m \frac{\lambda_{m,j}}{u(x_j)}$$
$$< \mathcal{C}\|q\|_\infty \sum_{k=1}^m |c_k(y)|$$

under the assumptions $w/u \in L_1$ and $\sup_z s_z \in C_u$. Now, we observe that the hypotheses (23–25) assure the stability of the formula in C^0 and hence we can conclude

$$|\Sigma_m(q, y)| \leq \mathcal{C}\|q\|_\infty, \quad \mathcal{C} \neq \mathcal{C}(q, m),$$

i.e. the stability of the rule follows.

Let us now prove the convergence. With $G_m(s,z)$, $I(s,z)$ defined in (22) we can write

$$R_m^{G\mathcal{P}}(q, y) = \int_{-1}^y q(z)[I(s,z) - G_m(s,z)]\psi(z,y)dz + \int_{-1}^y q(z)G_m(s,z)\psi(z,y)dz$$
$$- \sum_{k=1}^m c_k(y)q(x_k)G_m(s,x_k),$$

that is, recalling that R_m^G is the error of the Gauss rule and $R_m^{\mathcal{P}}$ the remainder term of the product rule (14), it is

$$R_m^{G\mathcal{P}}(q, y) = \int_{-1}^y q(z)R_m^G(s,z)\psi(z,y)dz + R_m^{\mathcal{P}}(qG_m(s), y) =: D_1(y) + D_2(y). \tag{50}$$

By

$$|D_1(y)| \leq \sup_{z\in[-1,1]} |R_m^G(s,z)| \int_{-1}^y |q(z)\psi(z,y)|dz$$

and using

$$|R_m^G(h,z)| \leq \mathcal{C}E_{2m-1}(h)_u,$$

holding for any function $h \in C_u$ under the assumption $w/u \in L_1$, it follows

$$|D_1(y)| \leq \mathcal{C}\|q\| \int_{-1}^y |\psi(z,y)|dz \times \sup_{z\in[-1,1]} E_{2m-1}(s_z)_u$$

and under the assumptions on ψ, we get

$$|D_1(y)| \leq \mathcal{C}\|q\|_\infty \sup_{z\in[-1,1]} E_{2m-1}(s_z)_u. \tag{51}$$

To estimate D_2, since the assumptions (23–25) assure those of Theorem 1 are fulfilled in C^0, we have

$$|D_2(y)| \leq \mathcal{C} \sup_{z\in[-1,1]} E_{m-1}(s_z q)_u$$

and by Lemma 1

$$|D_2(y)| \leq \mathcal{C} \sup_{z\in[-1,1]} \left(\|q\|_\infty E_{\left\lfloor \frac{m-1}{2} \right\rfloor}(s_z)_u + \|s_z u\|_\infty E_{\left\lfloor \frac{m-1}{2} \right\rfloor}(q) \right).$$

Estimate (26) follows combining the last inequality with (51) and (50). □

Proof of Proposition 1. Under the assumption (28), the map $V : C_u \to C_u$ is bounded, i.e.

$$|(Vf)(y)u(y)| \leq \|fu\|_\infty \sup_{x\in[-1,1]} \|h_x u\|_\infty \int_{-1}^{y} (y-x)^\rho \frac{(1+x)^\sigma}{u(x)} dx \leq \mathcal{C}\,\|fu\|_\infty.$$

Let us now prove that, for each $f \in C_u$, $Vf \in W_1(u)$.

$$\begin{aligned} |(V'f)(y)|u(y)\varphi(y) &\leq \|fu\|_\infty u(y)\varphi(y) \int_{-1}^{y} \left| \frac{\partial}{\partial y} h(x,y) \right| (y-x)^\rho \frac{(1+x)^\sigma}{u(x)} dx \\ &\quad + \|fu\|_\infty u(y)\varphi(y)\rho \int_{-1}^{y} |h(x,y)|(y-x)^{\rho-1} \frac{(1+x)^\sigma}{u(x)} dx \\ &=: A_1(y) + A_2(y). \end{aligned}$$

Setting $h'_x(y) := \frac{\partial}{\partial y} h(x,y)$, under the assumption $\sup_{|x|\leq 1} \|h'_x \varphi u\|_\infty < \infty$, one has

$$A_1(y) \leq \|fu\|_\infty \sup_{x\in[-1,1]} \|h'_x \varphi u\|_\infty \int_{-1}^{y} (y-x)^\rho \frac{(1+x)^\sigma}{u(x)} dx \leq \mathcal{C}\|fu\|_\infty,$$

and similarly,

$$A_2(y) \leq \mathcal{C}\|fu\|_\infty \sup_{x\in[-1,1]} \|h_x u\|_\infty \int_{-1}^{y} (y-x)^{\rho-1} \frac{(1+x)^\sigma}{u(x)} dx \leq \mathcal{C}\|fu\|_\infty,$$

from which we can see that $Vf \in W_1(u)$. Consequently,

$$E_m(Vf)_u \leq \frac{\mathcal{C}}{m} \|Vf\|_{W_1(u)},$$

and in conclusion [26, p. 44, §2.5.1]

$$\lim_m \left(\sup_{\|fu\|_\infty = 1} E_m(Vf)_u \right) = 0, \qquad \forall f \in C_u,$$

i.e. the compactness of the operator V. □

Proof of Proposition 2. Under the assumptions (29)–(31), the operator V_m, obtained by the product rule (14) with $q = hf$ and $\psi(x, y) = (y-x)^\rho(1+x)^\sigma$ is bounded, according to (18). Now we prove $V_m f \in W_1(u)$. We have

$$\begin{aligned} |(V_m' f)(y)|u(y)\varphi(y) &\le u(y)\varphi(y)\left(\int_{-1}^{y}\left|L_m\left(w, \frac{\partial}{\partial y}h(\cdot, y)f, x\right)\right|(y-x)^\rho(1+x)^\sigma dx\right.\\ &\quad\left. + \rho\int_{-1}^{y}|L_m(w, h_y f, x)|(y-x)^{\rho-1}(1+x)^\sigma dx\right)\\ &=: B_1(y) + B_2(y). \end{aligned} \tag{52}$$

The hypotheses of Theorem 1 with $\psi(x, y) = (y-x)^\rho(1+x)^\sigma$ are satisfied and

$$\begin{aligned} B_1(y) \le \mathcal{C}u(y)\varphi(y)\left\|f\frac{\partial}{\partial y}h(\cdot, y)u\right\|_\infty &\le \mathcal{C}\|fu\|_\infty \sup_{y\in(-1,1]}\left(u(y)\varphi(y)\left\|\frac{\partial}{\partial y}h(\cdot, y)\right\|_\infty\right)\\ &\le \mathcal{C}\|fu\|_\infty. \end{aligned}$$

Similarly, by Theorem 1 with $\psi(x, y) = (y-x)^{\rho-1}(1+x)^\sigma$, we have

$$\begin{aligned} B_2(y) &= \rho\, u(y)\varphi(y)\int_{-1}^{y}|L_m(w, h_y f, x)|(y-x)^{\rho-1}(1+x)^\sigma dx\\ &\le \mathcal{C}u(y)\varphi(y)\|h_y f u\|_\infty\\ &\le \mathcal{C}\|fu\|_\infty \sup_{y\in(-1,1]}\left(u(y)\|h_y\|_\infty\right) \le \mathcal{C}\|fu\|_\infty, \end{aligned}$$

where $\mathcal{C} \ne \mathcal{C}(m, h_y, f)$. Thus, by replacing the estimates of the terms $B_1(y)$ and $B_2(y)$ in (52), we deduce that $V_m f \in W_1(u)$ and consequently

$$E_n(V_m f)_u \le \frac{\mathcal{C}}{n}\|V_m f\|_{W_1(u)}.$$

Therefore [26, p. 44, §2.5.1],

$$\lim_n\left(\sup_m \sup_{\|fu\|_\infty=1} E_n(V_m f)_u\right) = 0,$$

that is the sequence $\{V_m\}_m$ is collectively compact. Finally, by (19)

$$|(Vf - V_m f)(y)u(y)| \le \mathcal{C}u(y)E_{m-1}(h_y f)_u,$$

and by Lemma 1,

$$
\begin{aligned}
&|(Vf - V_m f)(y)u(y)| \\
&\leq \mathcal{C}u(y)\left[\sup_{x\in[-1,1]} |h(x,y)| E_{\left\lfloor \frac{m-1}{2}\right\rfloor}(f)_u + |f(y)u(y)| E_{\left\lfloor \frac{m-1}{2}\right\rfloor}(h_y)\right], \qquad (53)
\end{aligned}
$$

where $\lfloor a \rfloor$ denotes the integer part of a. Hence, (53) tends to zero for $m \to \infty$, for all $f \in C_u$, under the assumptions on the kernel h. □

Proof of Proposition 3. Boundedness and compactness of the operator $K : C_u \to C_u$ follows by [9, Proposition 3.1], under the first assumption in (34) and both conditions in (33). Consider now the operators K_m defined in (32). Under the second assumption in (34) and both conditions in (33), using [10, Corollary 5.1], it follows the boundedness of $K_m : C_u \to C_u$, i.e.

$$\|(K_m f)u\|_\infty \leq \mathcal{C}\|fu\|_\infty,$$

Let us now prove that the sequence $K_m : C_u \to C_u$ is collectively compact. Recalling that [21, p. 375]

$$\sum_{k=1}^{m} \frac{\lambda_{m,k}}{u(x_k)} \leq \mathcal{C}\int_{-1}^{1}(1-x)^{\alpha-\gamma}(1+x)^{\beta-\delta}dx < \infty, \quad \mathcal{C} \neq \mathcal{C}(m),$$

by virtue of (33), we can write

$$\left|(K_m f)^{(r)}(y)\varphi^r(y)u(y)\right| \leq \|fu\|_\infty \sup_{x\in[-1,1]} \left\|k_x^{(r)}\varphi^r u\right\|_\infty \sum_{k=1}^{m}\frac{\lambda_{m,k}}{u(x_k)} \leq \mathcal{C}\|fu\|_\infty.$$

By (8), it follows

$$E_n(K_m f)_u \leq \frac{\mathcal{C}}{n^r}\|K_m f\|_{W_r(u)}, \qquad (54)$$

and hence

$$\lim_n \left(\sup_m \sup_{\|fu\|_\infty=1} E_n(K_m f)_u\right) = 0, \quad \forall f \in C_u,$$

by which, according to [26, p. 44, §2.5.1], the sequence $K_m : C_u \to C_u$ is collectively compact. Finally, by (12)

$$|(Kf - K_m f)(y)u(y)| \leq \mathcal{C}E_{2m-1}(k_y f)_u \qquad (55)$$

and by [10, Corollary 5.1], under the second assumption in (34), it follows

$$\lim_m \|(K - K_m)f\|_{C_u} = 0, \quad \forall f \in C_u.$$

□

Proof of Theorem 3. Let us introduce the operator $A_m = V_m + K_m$. By Propositions 1, 2, and 3, it follows that

(i) $A_m f$ converges to Af for any $f \in C_u$, where $A = V + K$. In fact A_m is the sum of two discrete operators V_m and K_m which converges to V and K, respectively. Consequently,

$$\sup_m \|A_m\|_{C_u \to C_u} < \infty, \tag{56}$$

(ii) the sequences $\{A_m\}$ are collectively compact being sums of sequences of collectively compact operators. Hence,

$$\lim_{m\to\infty} \|(A - A_m)A_m\|_{C_u \to C_u} = 0. \tag{57}$$

Therefore, the assertion follows by using [3, Theorem 4.1.2]. □

Proof of Theorem 4. First note that all the assumptions assure the validity of Propositions 1, 2, and 3. Let us now consider equation $(I + \mu_1 V + \mu_2 K)f = g$. The operator $I + \mu_1 V$ is a linear bounded operator having inverse bounded and K is a compact operator. Therefore, by [19, Corollary 3.6], under the assumption $\ker\{I + \mu_1 V + \mu_2 K\} = \{0\}$, equation $(I + \mu_1 V + \mu_2 K)f = g$ has a unique solution f^*. Furthermore, by the hypothesis on h, k and g, we can deduce that $f^* \in W_r(u)$.

By applying [3, Theorem 4.1.2] and using (55) and (53), one has

$$\begin{aligned}\|(f^* - f_m^*)u\|_\infty &\le \mathcal{C}(\|(V - V_m)f^*u\|_\infty + \|(K - K_m)f^*u\|_\infty)\\ &\le \mathcal{C}\left(E_{m-1}(h_y f^*)_u + E_{2m-1}(k_y f^*)_u\right),\end{aligned}$$

where $\mathcal{C}$ is a positive constant independent of m and f^*. Hence, the assertions follows by noting that by (48) we have

$$\begin{aligned}E_{m-1}(h_y f^*)_u &\le \mathcal{C}\left[\|f^*u\|_\infty E_{\left\lfloor\frac{m-1}{2}\right\rfloor}(h_y) + 2\|h_y\|_\infty E_{\left\lfloor\frac{m-1}{2}\right\rfloor}(f^*)_u\right]\\ &\le \frac{\mathcal{C}}{m^r}\|f^*u\|_{W_r(u)},\end{aligned} \tag{58}$$

and similarly

$$E_{2m-1}(k_y f^*)_u \le \mathcal{C}\left[\|f^*u\|_\infty E_{\left\lfloor\frac{2m-1}{2}\right\rfloor}(k_y) + 2\|k_y\|_\infty E_{\left\lfloor\frac{2m-1}{2}\right\rfloor}(f^*)_u\right]$$

$$\leq \frac{\mathcal{C}}{m^r} \|f^* u\|_{W_r(u)}. \tag{59}$$

□

Proof of Theorem 5. The operator VK is the composition of two bounded and compact operators $V : C_u \to C_u$ and $K : C_u \to C_u$. Therefore, it is bounded and compact too; see, for instance, [3, Lemma 1.2.2]. □

Proof of Theorem 6. The proof follows the same lines of the proof of Theorem 3, taking into account that by Remark 3 it is $\Omega_m = V_m K_m$. In fact, the operator Ω_m converges to $A = VK$, by virtue of Theorem 2, ensuring (56) with Ω_m in place of A_m. Moreover, the sequence $\{\Omega_m\}$ is collectively compact being the composition of two sequences of collectively compact operators; see, for instance, [2]. This assures (57) with Ω_m instead of A_m. □

Proof of Theorem 7. By [3, Theorem 4.1.2] and taking into account Remark 3, one has

$$\begin{aligned} \|(f^* - f_m^*)u\|_\infty &\leq \mathcal{C}\|(VK - \Omega_m) f^* u\|_\infty \\ &= \sup_{y\in(-1,1]} u(y)|R_m^{G\mathcal{P}}(h_y, y)|. \end{aligned}$$

Then, by applying Theorem 2, we have

$$\begin{aligned} \|(f^* - f_m^*)u\|_\infty \leq &\mathcal{C} \sup_{y\in(-1,1]} u(y)\|h_y\|_\infty \sup_{z\in[-1,1]} E_{\lfloor \frac{m-1}{2} \rfloor}(f^* k_z)_u \\ &+ \|f^* u\|_\infty \sup_{z\in[-1,1]} \|k_z\|_\infty \sup_{y\in(-1,1]} u(y) E_{\lfloor \frac{m-1}{2} \rfloor}(h_y), \end{aligned}$$

where $\mathcal{C} \neq \mathcal{C}(m, f^*)$. Therefore, considering that by (48) we have

$$\begin{aligned} \sup_{z\in[-1,1]} E_{\lfloor \frac{m-1}{2} \rfloor}(f^* k_z)_u \leq &\|f^* u\|_\infty \sup_{z\in[-1,1]} E_{\lfloor \frac{m-1}{4} \rfloor}(k_z) \\ &+ 2 \sup_{z\in[-1,1]} \|k_z\|_\infty E_{\lfloor \frac{m-1}{4} \rfloor}(f^*)_u, \end{aligned}$$

by using the assumptions (46) and (47), we deduce the assertion. □

7 Conclusions

In this paper, we have developed Nyström type methods to approximate the solution of Volterra–Fredholm equations whose given functions may have algebraic singularities at the endpoints ± 1. We have considered the two forms in which these appear in literature: the case where the two operators arise in an additive term and the one in which they are composed. The presence of possible singularities in the given functions implies a possible singular solution and leads to consider equations in weighted spaces equipped with uniform norm. This is, in our view, the first paper in which the possibility that the solution may have singularities is contemplated.

Volterra–Fredholm integral equations arise in several models. Parabolic boundary integral equations are usually reformulated in terms of such equations. Also, some epidemic, evolution and biological problems are modelled through equations of the types (1) and (2). In these applicative contexts, the equations are bivariate and nonlinear terms appear. Here, we have considered the linear univariate case as a preliminary step to approach more sophisticated problems. The good performance achieved now encourages us to extend the procedure to the bivariate and nonlinear case in future works to apply the method in an experimental setting.

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