Mediterr. J. Math. (2025) 22:53
https://doi.org/10.1007/s00009-025-02822-5
1660-5446/25/020001-19

Mediterranean Journal of Mathematics

Uniform and L^p Convergence of the Hermite Interpolation at Pollaczek–Laguerre Zeros

Maria Carmela De Bonis, Giuseppe Mastroianni and
Incoronata Notarangelo

Dedicated to Professor Francesco Altomare on the occasion of his retirement.

Abstract. The paper deals with the weighted polynomial approximation of functions defined on $(0, +\infty)$, which can grow exponentially both at $+\infty$ and at 0. To this aim, we introduce interpolating operators of Hermite and Hermite–Fejér-type, based at the zeros of Pollaczek–Laguerre type orthogonal polynomials. We prove that these processes converge in weighted uniform and L^p−norms and provide sharp error estimates showing that the order of convergence is the same as the best polynomial approximation, under suitable assumptions.

Mathematics Subject Classification. Primary 41A10, Secondary 41A05.

Keywords. Hermite interpolation, Hermite–Fejér interpolation, weighted polynomial approximation, orthogonal polynomials, Pollaczek–Laguerre zeros, real semiaxis.

1. Introduction

This paper concerns the weighted polynomial approximation of functions defined on the real semiaxis $\mathbb{R}^+ = (0, +\infty)$, which can grow exponentially both at 0 and at $+\infty$. To this aim the methods based on classical weight functions on $\mathbb{R}^+$, such as Laguerre weights, are inadequate and so, as far as we know, this topic has received attention in the literature only recently,

---

The first and the third author are members of "Gruppo Nazionale Calcolo Scientifico – Istituto Nazionale di Alta Matematica" (GNCS–INdAM), "Research ITalian network on Approximation" (RITA) and the UMI Group "Approximation Theory and Applications" (TAA). The third author was partially supported by INdAM–GNCS and the the University of Torino (local funds), project "Teoria dell'approssimazione, metodi analitici e numerici per equazioni funzionali e applicazioni".

when a new class of nonclassical exponential weights on the real semiaxis have been introduced (see [5,10,13,14,18]).

To be more precise, normed spaces of functions, defined in $(0,+\infty)$ and unbounded at the boundary of the interval with exponential monotonicity, have been introduced and extensively studied in [14] (see also [5,10,13,18]), where Jackson's theorem and Stechkin's inequalities have been proved in such spaces of functions using the one-sided approximation. Then concrete polynomial approximation processes have been defined and the results have been applied to numerical integration and numerical methods for integral equations (see [4,9,15]). In particular, in [16] a Lagrange-type interpolating polynomial, performing in L^p the tasks of the best polynomial approximation, has been introduced.

In this paper, generalizing the previous results, we will introduce Hermite and Hermite–Fejér type interpolating polynomials, proving their convergence and giving the corresponding error estimates. The proofs are novel because, without estimating the kernel of the operator, we will relate its behavior to that of the Lagrange polynomial based on the same interpolation knots. The presented estimates cannot be improved because they show that these processes converge with the order of the best polynomial approximation, under suitable assumptions (cf. [1,2,17,19–21] where Hermite interpolation processes have been considered, but for functions with different behavior on unbounded intervals).

The paper is organized as follows. In Sect. 2 we first introduce our Hermite type interpolation processes and then we recall the weighted function spaces where we are going to study them. In Sect. 3, we state the results dealing with the boundedness and convergence of the interpolation process, providing error estimates in weighted L^p norm with $1 < p \leq +\infty$. The proofs of the main results are given in Sect. 4.

In the sequel $c, \mathcal{C}$ will stand for positive constants which can assume different values in each formula and we shall write $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$, when $\mathcal{C}$ is independent of $a, b, \ldots$. Furthermore, $A \sim B$ means that if A and B are positive quantities depending on some parameters, then there exists a positive constant $\mathcal{C}$ independent of these parameters such that $(A/B)^{\pm 1} \leq \mathcal{C}$. Finally, we will denote by $\mathbb{P}_m$ the set of all algebraic polynomials of degree at most m. As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, will stand for the sets of all natural, integer, real numbers, while $\mathbb{Z}^+$ and $\mathbb{R}^+$ denote the sets of positive integer and positive real numbers, respectively.

2. Preliminaries

2.1. Interpolation Processes

Let

$$w(x) = x^{\gamma} e^{-(x^{-\alpha}+x^{\beta})} =: x^{\gamma}\sigma(x), \tag{1}$$

with $x \in \mathbb{R}^+ = (0,+\infty)$, $\alpha > 0$, $\beta > 1$, $\gamma \geq 0$, and let $\{p_m(w)\}_m$ be the corresponding sequence of orthonormal polynomials having positive leading coefficients. The properties of such orthonormal system have been studied in [6,14,18]. The zeros of $p_m(w)$ lie in the Mhaskar–Rahmanov–Saff interval $A_m = [\varepsilon_m, a_m]$, i.e.

$$\varepsilon_m =: x_0 < x_1 < \cdots < x_m < x_{m+1} := a_m,$$

where $\varepsilon_m = \varepsilon_m(\sqrt{w}) \sim \left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{\alpha+\frac{1}{2}}}$ and $a_m = a_m(\sqrt{w}) \sim m^{\frac{1}{\beta}}$ are the so called Mhaskar-Rahmanov-Saff (M-R-S) numbers related to the weight $\sqrt{w}$.

Letting $v(x) = (a_m - x)(x - \varepsilon_m)$ and $q_{m+2}(x) = v(x)p_m(w, x)$, for any function f which is continuously differentiable on $(0, +\infty)$, say $f' \in C^0(\mathbb{R}^+)$, we denote by

$$L_m(w, f, x) = \sum_{k=0}^{m+1} l_k(x)f(x_k), \quad l_k(x) = \frac{q_{m+2}(x)}{q'_{m+2}(x_k)(x - x_k)}$$

and

$$H_m(w, f, x) = \sum_{k=0}^{m+1} l_k^2(x)[(1 - 2l'_k(x_k)(x - x_k))f(x_k) + (x - x_k)f'(x_k)]$$

the Lagrange and Hermite interpolating polynomials, respectively, based on the zeros of q_{m+2}. The Hermite polynomial can be also written as

$$\begin{aligned} H_m(w, f, x) &= \sum_{k=0}^{m+1} l_k^2(x)(1 - 2l'_k(x_k)(x - x_k))f(x_k) \\ &\quad + \sum_{k=0}^{m+1} l_k^2(x)(x - x_k)f'(x_k) \\ &=: F_m(w, f, x) + G_m(w, f, x) \end{aligned}$$

where $F_m(w, f)$ is the m-th Hermite–Fejér polynomial.

Now, with $\theta \in (0, 1)$ and χ_θ the characteristic function of $A_{\theta m} = [\varepsilon_{\theta m}, a_{\theta m}]$, we introduce the new operators defined as follows

$$L^*_m(w, f, x) := L_m(w, \chi_\theta f, x) = \sum_{x_k \in A_{\theta m}} l_k(x)f(x_k) \tag{2}$$

and

$$\begin{aligned} H^*_m(w, f, x) &:= H_m(w, \chi_\theta f, x) \\ &= \sum_{x_k \in A_{\theta m}} l_k^2(x)(1 - 2l'_k(x_k)(x - x_k))f(x_k) \\ &\quad + \sum_{x_k \in A_{\theta m}} l_k^2(x)(x - x_k)f'(x_k) \\ &=: F^*_m(w, f, x) + G^*_m(w, f, x), \end{aligned} \tag{3}$$

where F^*_m is an Hermite–Fejér-type interpolating polynomial.

The idea of truncation have been used for the first time in [11,12] for Gauss–Laguerre quadrature rules and then by several authors to obtain convergent procedures for polynomial interpolation and numerical integration with different kind of exponential weights on bounded or unbounded intervals (see [5,8] and the references therein).

Clearly, the Hermite type operator $H_m^*(w)$ maps a differentiable function to a polynomial of degree at most $2m+3$, but in general, it does not map a polynomial to itself. Namely $H_m^*(w,f) \in \mathbb{P}_{2\,m+3}$ but, for instance, $H_m^*(w,f,x) \neq 1$ if $f \equiv 1$. Moreover, the Hermite polynomial $H_m^*(w,f)$ is such that

$$H_m^*(w,f,x_i) = \begin{cases} f(x_i), & \text{if} \quad x_i \in A_{\theta m}, \\ 0, & \qquad \text{otherwise}, \end{cases}$$

and

$$\frac{d}{dx}[H_m^*(w,f)](x_i) = \begin{cases} f'(x_i), & \text{if} \quad x_i \in A_{\theta m}, \\ 0, & \qquad \text{otherwise}, \end{cases}$$

We emphasize that $H_m^*(w,f)$ and $F_m^*(w,f)$ are not truncated polynomials but are polynomials on the real semiaxis, even if they interpolate only a finite section of the function.

2.2. Function Spaces

As already announced in the Introduction, we are going to consider the operator $H_m^*(w)$ in a space of continuous functions in $\mathbb{R}^+$, having an exponential behavior at 0 and/or $+\infty$ (for example $f(x) = e^{\frac{1}{x^2}+x^3}, x > 0$). Here we limit ourselves to recall the notions needed to state the main results of the paper. Other properties will be shown in the section dedicated to the proofs (see [10,13,14]).

Naturally, we consider the weighted approximation of the above functions. Then, with

$$u(x) = x^\delta \sigma(x), \qquad \sigma(x) = e^{-(x^{-\alpha}+x^\beta)}, \tag{4}$$

$x \in \mathbb{R}^+ = (0,+\infty)$, $\alpha > 0$, $\beta > 1$, $\delta \geq 0$, where σ is the same weight as in (1), for every $1 \leq p < +\infty$ we will write $f \in L_u^p$ if $fu \in L^p$. The space L_u^p will be endowed with the norm

$$\|f\|_{L_u^p} = \|fu\|_p = \left(\int_0^{+\infty} |fu|^p\right)^{\frac{1}{p}} < +\infty, \quad 1 \leq p < +\infty.$$

If $p = +\infty$ we consider the function space

$$L_u^\infty := C_u = \left\{ f \in C^0(\mathbb{R}^+) \quad \lim_{\substack{x\to+\infty \\ x\to 0}} (fu)(x) = 0 \right\},$$

with the norm

$$\|f\|_{L_u^\infty} = \|fu\|_\infty = \sup_{x>0} |(fu)(x)|.$$

For smoother functions we introduce the Sobolev-type spaces

$$W_s^p(u) = W_s^p = \left\{ f \in L_u^p \quad f^{(s-1)} \in AC(\mathbb{R}^+) \text{ and } \|f^{(s)}\varphi^s u\|_p < +\infty \right\},$$

with integer $s \geq 1$ and the norm

$$\|f\|_{W_s^p} = \|fu\|_p + \|f^{(s)}\varphi^s u\|_p,$$

where $\varphi(x) = \sqrt{x}$, $1 \leq p \leq \infty$ and $AC(\mathbb{R}^+)$ is the set of all absolutely continuous functions in every closed interval of $\mathbb{R}^+$.

To introduce a modulus of continuity, we consider the interval

$$I_h(c) = \left[h^{\frac{1}{\alpha+\frac{1}{2}}}, \frac{c}{h^{\frac{1}{\beta}-\frac{1}{2}}} \right]$$

with $h > 0$ sufficiently small and $c > 1$ fixed. Now, for every $f \in L^p_u$, $1 \leq p \leq \infty$, $s \geq 1$ and $t > 0$ sufficiently small, we set

$$\Omega^s_\varphi(f,t)_{u,p} = \sup_{0<h<t} \|(\Delta^s_{h\varphi} f)u\|_{L^p(I_h(c))},$$

where

$$\Delta^s_{h\varphi} f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \frac{h\varphi(x)}{2}(r-2k)\right),$$

and define the $\varphi-$modulus of smoothness ω^s_φ as follows

$$\omega^s_\varphi(f,t)_{u,p} = \Omega^s_\varphi(f,t)_{u,p} + \inf_{q\in\mathbb{P}_{s-1}} \|(f-q)u\|_{L^p\left(\left(0, t^{\frac{1}{\alpha+\frac{1}{2}}}\right)\right)}$$
$$+ \inf_{q\in\mathbb{P}_{s-1}} \|(f-q)u\|_{L^p\left(\left(t^{\frac{1}{\beta}-\frac{1}{2}}, +\infty\right)\right)}.$$

For $s = 1$ we will write ω_φ and Ω_φ instead of ω^1_φ and Ω^1_φ. We recall that the behavior of $\omega^s_\varphi(f,t)_{u,p}$ is independent of the constant c of the interval $I_h(c)$, as shown in [14].

With these moduli of smoothness we can estimate the error of best polynomial approximation in L^p_u. Letting

$$E_m(f)_{u,p} = \inf_{P_m\in\mathbb{P}_m} \|(f-P_m)u\|_p, \quad 1 \leq p \leq \infty,$$

the following Jackson

$$E_m(f)_{u,p} \leq \mathcal{C}\omega^s_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p}, \tag{5}$$

weak Jackson

$$E_m(f)_{u,p} \leq \mathcal{C}\int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^s_\varphi(f,t)_{u,p}}{t} dt, \quad \text{if } \Omega^s_\varphi(f,t)_{u,p} t^{-1} \in L^1 \tag{6}$$

and Stechkin type inequalities

$$\omega^s_\varphi\left(f, \frac{\sqrt{a_m}}{m}\right)_{u,p} \leq \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s \sum_{i=0}^{m} \left(\frac{i}{\sqrt{a_i}}\right)^s \frac{E_i(f)_{u,p}}{i}. \tag{7}$$

hold with $\mathcal{C}$ independent of f and m, as proved in [14].

From (5), (6) and (7) we deduce that the error of best approximation characterizes suitable function spaces. For instance, with $1 \leq p \leq \infty$, we have

$$f \in L^p_u \quad \Leftrightarrow \quad \lim_{t\to 0} \omega_\varphi(f,t)_{u,p} = 0 \quad \Leftrightarrow \quad \lim_{m\to\infty} E_m(f)_{u,p} = 0.$$

Moreover, for smoother functions, we have more concrete information about the order of convergence. Namely, if $f \in W_s^p(u)$, $s \geq 1$ and $1 \leq p \leq \infty$, we get

$$E_m(f)_{u,p} \leq \mathcal{C} \left(\frac{\sqrt{a_m}}{m} \right)^s \|f\|_{W_s^p(u)}$$

with $\mathcal{C} \neq \mathcal{C}(m, f)$.

Finally, the following condition

$$\int_0^1 \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+\frac{1}{p}}} dt < +\infty$$

implies the continuity of the function f in $(0, +\infty)$ (see [13]).

3. Main Results

To study the convergence of our Hermite and Hermite–Fejér type processes in weighted L^p-metric, we will use the following lemma.

Lemma 3.1. *With the notation of Theorem 3.1 we have*

$$\left\| L_m^*(w, F) \frac{uv}{\sqrt{w\sqrt{v}}} \right\|_p \leq \mathcal{C} \left(\sum_{j=\bar{k}}^{k^*} \left(|fu|(x_j) + \frac{\sqrt{a_m}}{m} |f'\varphi u|(x_j) \right)^p \Delta x_j \right)^{\frac{1}{p}}, \tag{8}$$

where $\mathcal{C}$ depends only on θ, if and only if the parameters γ and δ of the weights w and u, respectively, satisfy (11), namely

$$\frac{1}{2} + \frac{1}{p} < \gamma - \delta < \frac{3}{2} + \frac{1}{p}.$$

We are now able to state our main results.

Theorem 3.1. *For any $1 < p < \infty$, let w and u be the weights (1) and (4), namely $w(x) = x^\gamma \sigma(x)$ and $u(x) = x^\delta \sigma(x)$, where $\sigma(x) = e^{-(x^{-\alpha}+x^\beta)}$, $x > 0$, $\alpha > 0$, $\beta > 1$, and $\gamma, \delta \geq 0$. Then, for every function f such that $f' \in L_u^p$ and $\Omega_\varphi(f', t)_{u,p} t^{-1-\frac{1}{p}} \in L^1$, we have*

$$\|H_m^*(w, f)u\|_p \sim \left(\sum_{x_k \in A_{\theta m}} \Delta x_k \left| (fu)(x_k) + \frac{\sqrt{a_m}}{m} (f'\varphi u)(x_k) \right|^p \right)^{\frac{1}{p}} \tag{9}$$

and, consequently, for $f \in L_u^p$ with $\Omega_\varphi^s(f, t)_{u,p} t^{-1-\frac{1}{p}} \in L^1$, we get

$$\|F_m^*(w, f)u\|_p \sim \left(\sum_{x_k \in A_{\theta m}} \Delta x_k \, |(fu)(x_k)|^p \right)^{\frac{1}{p}} \tag{10}$$

if and only if the parameters γ and δ of the weights w and u satisfy

$$\frac{1}{2} + \frac{1}{p} < \gamma - \delta < \frac{3}{2} + \frac{1}{p}. \tag{11}$$

Here the constants in the equivalences (9) and (10) depend on θ and do not depend on m and f.

We point out that the assumption $\Omega_\varphi(f',t)_{u,p}t^{-1-\frac{1}{p}} \in L^1$ implies the continuity of f' in $(0,+\infty)$ (see [13]).

Theorem 3.2. *With the same notations and under the same assumptions of Theorem 3.1, the following error estimates hold*

$$\|[f - H_m^*(w,f)]u\|_p \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{1+\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi^s(f',t)_{u\varphi,p}}{t^{1+\frac{1}{p}}}dt + \mathcal{C}e^{-cm^\nu}\|fu\|_p \tag{12}$$

and

$$\|[f-F_m^*(w,f)]u\|_p \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+\frac{1}{p}}}dt + \mathcal{C}e^{-cm^\nu}\|fu\|_p, \tag{13}$$

where $\nu = \left(1-\frac{1}{2\beta}\right)\frac{2\alpha}{2\alpha+1}$ *and the constants* $\mathcal{C}, c$ *depend on* θ *but are independent of* m *and* f*.*

As an example, if $f \in W_s^p(u)$, $s \ge 1$, then

$$\|[f - H_m^*(w,f)]u\|_p \le \mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^s \|f\|_{W_s^p(u)}$$

and if $f \in W_1^p(u)$ then

$$\|[f - H_m^*(w,f)]u\|_p \le \mathcal{C}\frac{\sqrt{a_m}}{m}\|f\|_{W_1^p(u)}$$

and

$$\|[f - F_m^*(w,f)]u\|_p \le \mathcal{C}\frac{\sqrt{a_m}}{m}\|f\|_{W_1^p(u)}.$$

Now we study the convergence of our Hermite and Hermite–Fejér type processes in weighted uniform metric.

Theorem 3.3. *Let* w *and* u *be the weights (1) and (4) and* $f \in W_1^\infty(u)$*. Then*

$$\|H_m^*(w,f)u\|_\infty \le \mathcal{C}\left(\|fu\|_\infty + \frac{\sqrt{a_m}}{m}\|f'\varphi u\|_\infty\right)\log m \tag{14}$$

and

$$\|(f - H_m^*(w,f))u\|_\infty \le \mathcal{C}\frac{\sqrt{a_m}}{m}E_{M-1}(f')_{u\varphi,\infty}\log m + \mathcal{C}e^{-cm}\|fu\|_\infty, \tag{15}$$

with c *and* $\mathcal{C}$ *independent of* m *and* f*, but depending on* θ*, and* $M = \left\lfloor \frac{\theta}{\theta+1}m \right\rfloor \sim m$*, if and only if*

$$\frac{1}{2} < \gamma - \delta < \frac{3}{2}. \tag{16}$$

Theorem 3.4. *Let* $\bar{u}(x) = u(x)\log(1+x+x^{-1})$*,* $x > 0$*. Then, under the assumption (16), for every* $f \in C_{\bar{u}}$*, we have*

$$\|F_m^*(w,f)u\|_\infty \le \mathcal{C}\|f\bar{u}\|_\infty \tag{17}$$

and

$$\|(f - F_m^*(w,f))u\|_\infty \le \mathcal{C}\omega_\varphi\left(f, \frac{\sqrt{a_m}}{m}\log m\right)_{\bar{u},\infty} + \mathcal{C}e^{-cm^\nu}\|f\bar{u}\|_\infty, \tag{18}$$

with $\mathcal{C}, c$ *independent of* m *and* f*, but depending on* θ*, and* $\nu = \left(1 - \frac{1}{2\beta}\right)\frac{2\alpha}{2\alpha+1}$.

4. Proofs

First of all we recall some polynomial inequalities (see [6,16,18]). Letting $v(x) = (a_m - x)(x - \varepsilon_m)$, it is easy to deduce that

$$v(x) < a_m x, \;\; x \in A_m, \qquad \text{and} \qquad v(x_k) \sim a_m x_k, \;\; x_k \in A_{\theta m}. \tag{19}$$

Moreover, we have

$$\left| p_m(w,x)\sqrt{w(x)\sqrt{v(x)}} \right| \le \mathcal{C}, \quad x > 0, \tag{20}$$

$$\frac{1}{|p_m'(w,x_k)|} \sim \Delta x_k \sqrt{w(x_k)\sqrt{v(x_k)}}, \tag{21}$$

where, for $x_k \in A_{\theta m}$,

$$\Delta x_k = x_{k+1} - x_k \sim \frac{\sqrt{a_m}}{m}\sqrt{x_k}, \tag{22}$$

$$\left| p_m(w,x)\sqrt{w(x)\sqrt{v(x)}} \right| \sim \frac{|x - x_d|}{\Delta x_{d+1}}, \quad x \in (x_1, x_m). \tag{23}$$

For every polynomial, $P_m \in \mathbb{P}_m$, for any $s > 1$ and $1 \le p \le +\infty$ the following restricted range inequalities [6,18]

$$\|P_m u\|_p \le \mathcal{C}\|P_m u\|_{L^p(A_m)} \tag{24}$$

and

$$\|P_m u\|_{L^p(\mathbb{R}^+ \setminus A_{sm})} \le \mathcal{C}e^{-cm^\nu}\|P_m u\|_p, \tag{25}$$

hold with $\mathcal{C}, c$ independent of m, P_m, where $\nu = \left(1 - \frac{1}{2\beta}\right)\frac{2\alpha}{2\alpha+1}$. Moreover, the Bernstein type inequality takes the following form

$$\|P_m'\varphi u\|_p \le \mathcal{C}\frac{m}{\sqrt{a_m}}\|P_m u\|_p, \quad 1 \le p \le \infty, \tag{26}$$

with $\varphi(x) = \sqrt{x}$ and $\mathcal{C}$ independent of m, P_m.

Now, we set

$$B_k(f,x) = u(x)l_k^2(x)\left[(1 - 2l_k'(x_k)(x - x_k))f(x_k) + (x - x_k)f'(x_k)\right]$$

where l_k are the fundamental Lagrange polynomials.

Obviously, $B_k(f,x_k) = (uf)(x_k)$. If $x \in \left[x_k - \frac{\Delta x_k}{8}, x_k + \frac{\Delta x_k}{8}\right] =: I_k$ and $x_k \in A_{\theta m}$, then

$$|B_k(f,x)| \le \mathcal{C}_\theta\left(|fu|(x_k) + \frac{\sqrt{a_m}}{m}|f'\varphi u|(x_k)\right). \tag{27}$$

In fact, since $x \sim x_k$ and $|x - x_k| \le \frac{\Delta x_k}{8}$, using (19)–(22), we deduce $|l_k(x)| \sim 1$. Then, it remains to prove that

$$|1 - 2l_k'(x_k)(x - x_k)| \le \mathcal{C}_\theta.$$

Now, using (21), we get

$$\begin{aligned}|l'_k(x_k)||x-x_k| &\le \frac{v'(x_k)}{v(x_k)}\Delta x_k + \left|\frac{p''_m(w,x_k)}{p'_m(w,x_k)}\right|\Delta x_k\\ &\le \frac{\Delta x_k}{a_m-x_k}+\frac{\Delta x_k}{x_k-\varepsilon_m}+(\Delta x_k)^2\left|p''_m(w,x_k)\sqrt{w(x_k)\sqrt{v(x_k)}}\right|.\end{aligned}$$

Taking into account that the first two addenda are bounded and applying two times the Bernstein inequality (26) and, then, (20) to the third addendum, (27) follows.

Let us consider, now, the case $x_k \notin I_k$, i.e. $|x-x_k| > \frac{\Delta x_k}{8}$. We have

$$\begin{aligned}B_k(f,x) &= u(x)l_k^2(x)\frac{(x-x_k)}{\Delta x_k}\\ &\quad\times\left[\left(\frac{\Delta x_k}{(x-x_k)}-2\Delta x_k l'_k(x_k)\right)f(x_k)+\Delta x_k f'(x_k)\right],\end{aligned}$$

from which, letting $C_k = \frac{\Delta x_k}{(x-x_k)} - 2\Delta x_k l'_k(x_k)$ and making explicit $l_k(x)$, we deduce

$$\begin{aligned}B_k(f,x) &= u(x)v(x)p_m(w,x)l_k(x)\left[\frac{C_k f(x_k)+\Delta x_k f'(x_k)}{v(x_k)p'_m(w,x_k)\Delta x_k}\right]\\ &=: u(x)v(x)p_m(w,x)l_k(x)F(x_k), \qquad (28)\end{aligned}$$

where $F(x_k)$ is the expression in square brackets.

We are now able to prove Lemma 3.1. The proof is quite similar to the one in [16, Lemma 3.2]. For the reader's convenience we will report its main steps. In particular, to the sake of brevity we will only prove that conditions (11) on γ and δ are sufficient for the validity of (8). Concerning the necessity of such conditions it is sufficient to repeat step by step the proof of Lemma 3.2 in [16].

Proof of Lemma 3.1, (11)⇔(8). Taking into account that

$$\begin{aligned}\left\|L^*_{m+1}(w,F)\frac{uv}{\sqrt{w\sqrt{v}}}\right\|_p &\le \mathcal{C}\sup_{\|g\|_q=1}\int_{A_m}L^*_{m+1}(w,F,x)\frac{(uv)(x)}{\sqrt{w(x)\sqrt{v(x)}}}g(x)dx\\ &=: \mathcal{C}\sup_{\|g\|_q=1}B(g), \qquad (29)\end{aligned}$$

it results

$$B(g)=\sum_{j=\bar{k}}^{k^*}\frac{F(x_j)u(x_j)}{v(x_j)p'_m(w,x_j)u(x_j)}\int_{A_m}\frac{p_m(w,x)}{x-x_j}\frac{(uv^2)(x)}{\sqrt{w(x)\sqrt{v(x)}}}g(x)dx.$$

Since

$$\left|\frac{F(x_j)u(x_j)}{v(x_j)p'_m(w,x_j)u(x_j)}\right| \le \mathcal{C}_j\frac{\left|(fu)(x_j)+\frac{\sqrt{a_m}}{m}(f'\varphi u)(x_j)\right|}{(uv^2)(x_j)p'_m(w,x_j)^2\Delta x_j},$$

using (21), $v(x_j)\sim a_m x_j$ and $\mathcal{C}_j < \mathcal{C}_\theta$, we get

$$\left|\frac{F(x_j)u(x_j)}{v(x_j)p'_m(w,x_j)u(x_j)}\right| \le \mathcal{C}\Delta x_j\left|(fu)(x_j)+\frac{\sqrt{a_m}}{m}(f'\varphi u)(x_j)\right|\frac{x_j^{\gamma-\delta-\frac{3}{2}}}{a_m^{\frac{3}{2}}}.$$

Therefore

$$\begin{aligned}
B(g) \leq &\ \frac{\mathcal{C}}{a_m^{\frac{3}{2}}} \sum_{j=\bar{k}}^{k^*} \Delta x_j \left| (fu)(x_j) + \frac{\sqrt{a_m}}{m}(f'\varphi u)(x_j) \right| x^{\gamma-\delta-\frac{3}{2}} \\
&\times \left| \int_{A_m} \frac{p_m(w,x)}{x-x_j} \frac{(uv^2)(x)}{\sqrt{w(x)\sqrt{v(x)}}} g(x)dx \right| \\
=: &\ \frac{\mathcal{C}}{a_m^{\frac{3}{2}}} \sum_{j=\bar{k}}^{k^*} \Delta x_j \left| (fu)(x_j) + \frac{\sqrt{a_m}}{m}(f'\varphi u)(x_j) \right| x^{\gamma-\delta-\frac{3}{2}} |\Pi(x_j)|,
\end{aligned}$$

where

$$\begin{aligned}
\Pi(t) &= \int_{A_m} \frac{p_m(w,x)Q(x) - p_m(w,t)Q(t)}{x-t} \frac{(uv^2)(x)}{Q(x)\sqrt{w(x)\sqrt{v(x)}}} g(x)dx \\
&= H\left(p_m(w)\frac{uv^2 g}{\sqrt{w\sqrt{v}}}, t\right) - p_m(w,t)Q(t)H\left(\frac{uv^2}{Q\sqrt{w\sqrt{v}}}g, t\right),
\end{aligned}$$

H is the Hilbert transform on A_m and Q is a positive polynomial that will be specified later. Now, since $\gamma - \delta - \frac{3}{2} > -\frac{1}{q}$, using Hölder's inequality and Marcinkewicz's inequality, we have

$$B(g) \leq \frac{\mathcal{C}}{a_m^{\frac{3}{2}}} \left(\sum_{j=\bar{k}}^{k^*} \left| (fu)(x_j) + \frac{\sqrt{a_m}}{m}(f'\varphi u)(x_j) \right|^p \Delta x_j \right)^{\frac{1}{p}} \|x^{\gamma-\delta-\frac{3}{2}}\Pi\|_{L^q(A_m)}.$$

Estimating the $q-$norm, we deduce

$$\begin{aligned}
\|x^{\gamma-\delta-\frac{3}{2}}\Pi\|_{L^q(A_m)} \leq &\left\| x^{\gamma-\delta-\frac{3}{2}} H\left(p_m(w)\frac{uv^2 g}{\sqrt{w\sqrt{v}}}\right) \right\|_{L^q(A_m)} \\
&+ \left\| x^{\gamma-\delta-\frac{3}{2}} p_m(w) Q H\left(\frac{uv^2}{Q\sqrt{w\sqrt{v}}}g\right) \right\|_{L^q(A_m)}.
\end{aligned}$$

Taking into account that $\delta + \frac{3}{2} - \gamma > -\frac{1}{p}$, the Hilbert transform is bounded and the first norm is bounded by

$$\left\| x^{\gamma-\delta-\frac{3}{2}} \frac{uv^2 g}{\sqrt{w\sqrt{v}}} \right\|_{L^q(A_m)} \leq \mathcal{C}\|g\|_q a_m^{\frac{3}{2}},$$

using the estimate of $|p_m(w)|$ and $v(x) \leq \mathcal{C}a_m x$. Finally, choosing $Q \sim \frac{uv^2\, g}{\sqrt{w\sqrt{v}}}$, we get in a similar way

$$\begin{aligned}
\left\| x^{\gamma-\delta-\frac{3}{2}} p_m(w) Q H\left(\frac{uv^2}{Q\sqrt{w\sqrt{v}}}g\right) \right\|_{L^q(A_m)} &\leq \mathcal{C}\|H(g)\|_{L^q(A_m)} a_m^{\frac{3}{2}} \\
&\leq \mathcal{C}\|g\|_q a_m^{\frac{3}{2}}.
\end{aligned}$$

Summing up, we deduce

$$B(g) \leq \mathcal{C} \left(\sum_{j=\bar{k}}^{k^*} \left| (fu)(x_j) + \frac{\sqrt{a_m}}{m} (f'\varphi u)(x_j) \right|^p \Delta x_j \right)^{\frac{1}{p}}$$

and replacing the above estimate into (29), we deduce (8). □

Proof of Theorem 3.1. Using the restricted range inequality (24), with $1 < p < +\infty$, we have

$$\begin{aligned} \|H_m^*(w,f)u\|_p &\leq \mathcal{C}\|H_m^*(w,f)u\|_{L^p(A_m)} \\ &\leq \sup_{\|g\|_q=1} \int_{A_m} H_m(w,f,x)u(x)g(x)dx \\ &=: \sup_{\|g\|_q=1} A(g). \end{aligned}$$

Letting

$$\begin{aligned} x_{\bar{k}} &= \min\{x_k \geq \varepsilon_{\theta m}\}, \quad x_{k^*} = \max\{x_k \leq a_{\theta m}\}, \\ I_k &= \left[x_k - \frac{\Delta x_k}{8}, x_k + \frac{\Delta x_k}{8} \right], \quad I = \bigcup_{k=\bar{k}}^{k^*} I_k, \end{aligned}$$

we get

$$\begin{aligned} A(g) &= \left\{ \int_I + \int_{A_m \setminus I} \right\} \sum_{k=\bar{k}}^{k^*} B_k(f,x)g(x)dx \\ &= \sum_{j=\bar{k}}^{k^*} \int_{I_j} \sum_{k=\bar{k}}^{k^*} B_k(f,x)g(x)dx + \int_{A_m \setminus I} \sum_{k=\bar{k}}^{k^*} B_k(f,x)g(x)dx \\ &= \sum_{j=\bar{k}}^{k^*} \int_{I_j} B_j(f,x)g(x)dx + \sum_{j=\bar{k}}^{k^*} \int_{I_j} \sum_{\substack{k=\bar{k} \\ k \neq j}}^{k^*} B_k(f,x)g(x)dx \\ &\quad + \int_{A_m \setminus I} \sum_{k=\bar{k}}^{k^*} B_k(f,x)g(x)dx \\ &=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \end{aligned}$$

Concerning $\mathcal{I}_1$, we use (27) and the Hölder inequality to get

$$\begin{aligned} \mathcal{I}_1 &\leq \mathcal{C} \sum_{j=\bar{k}}^{k^*} \left(|fu|(x_j) + \frac{\sqrt{a_m}}{m} |f'\varphi u|(x_j) \right) \int_{I_j} g(x)dx \\ &\leq \mathcal{C} \sum_{j=\bar{k}}^{k^*} \left(|fu|(x_j) + \frac{\sqrt{a_m}}{m} |f'\varphi u|(x_j) \right) (\Delta x_j)^{\frac{1}{p}} \left(\int_{I_j} |g(x)|^q dx \right)^{\frac{1}{q}} \\ &\leq \mathcal{C} \left(\sum_{j=\bar{k}}^{k^*} \left(|fu|(x_j) + \frac{\sqrt{a_m}}{m} |f'\varphi u|(x_j) \right)^p \Delta x_j \right)^{\frac{1}{p}} \|g\|_q. \end{aligned}$$

For $\mathcal{I}_2+\mathcal{I}_3$, since $|x-x_i|>\frac{\Delta x_i}{8}$, we use (28) and we obtain

$$\mathcal{I}_2+\mathcal{I}_3 \le \int_{A_m} \left|u(x)v(x)p_m(w,x)\sum_{k=\bar{k}}^{k^*} l_k(x)F(x_k)\right| g(x)dx$$

from which, using (20) and the Hölder inequality, we have

$$\mathcal{I}_2+\mathcal{I}_3 \le \left\|L^*_{m+1}(w,F)\frac{uv}{\sqrt{w\sqrt{v}}}\right\|_p \|g\|_q.$$

Then, using the Lemma 3.1 together with the estimate of $\mathcal{I}_1$, and taking into account that the other inequality is trivial, (9) follows. □

To estimate the error of the previous interpolation formulas, since $H^*_m(w,P)\neq P$, we need some additional properties of the operator $H^*_m(w)$.

Let $M=\left\lfloor\frac{\theta m}{1+\theta}\right\rfloor\sim m$ and $A'_{\theta m}=A_m\setminus A_{\theta m}$, then $\forall P\in\mathbb{P}_M$ we have

$$P=H^*_m(w,P)+\Gamma(P)=F^*_m(w,P)+G^*_m(w,P)+\Gamma(P),$$

where $\Gamma(P)=H_m(w,(1-\chi_\theta)P)$ with

$$\|\Gamma(P)u\|_p\le \mathcal{C}e^{-cm^\nu}\|Pu\|_p,\quad 1\le p\le\infty,\quad \nu=\left(1-\frac{1}{2\beta}\right)\left(\frac{2\alpha}{2\alpha+1}\right), \tag{30}$$

and

$$\|G^*_m(w,P)u\|_p\le\mathcal{C}\frac{\sqrt{a_m}}{m}\|P'\varphi u\|_p,\quad 1\le p<\infty. \tag{31}$$

Moreover, we will also use

$$\sum_{x_k\in A_{\theta m}}\Delta x_k|fu|^p(x_k)\le\|fu\|_p+\left(\frac{\sqrt{a_m}}{m}\right)^{\frac{1}{p}}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi(f,t)_{u,p}}{t^{1+1/p}}dt \tag{32}$$

and, if P is a quasi-best approximation of $f\in L^p_u$,

$$\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi(f-P,t)_{u,p}}{t^{1+1/p}}dt\le\mathcal{C}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega^r_\varphi(f,t)_{u,p}}{t^{1+1/p}}dt \tag{33}$$

and

$$\left\{\|(f-P)u\|_p+\frac{\sqrt{a_m}}{m}\|P^{(r)}\varphi^r u\|_p\right\}\sim\omega^r_\varphi\left(f,\frac{\sqrt{a_m}}{m}\right)_{u,p}. \tag{34}$$

Proof of Theorem 3.2. We first prove (13). With $P\in\mathbb{P}_M$ best approximation of $f\in L^\infty_u$, using (10), (31) and (30), we have

$$\begin{aligned}\|[f-F^*_m(w,f)]u\|_p &= \|(f-P)+F^*_m(w,P-f)+G^*_m(w,P)+\Gamma(P)\|_p\\ &\le\|(f-P)u\|_p+\mathcal{C}\left(\sum_{x_k\in A_{\theta m}}\Delta x_k\,|[(f-P)u](x_k)|^p\right)^{\frac{1}{p}}\\ &\quad+\mathcal{C}\frac{\sqrt{a_m}}{m}\|P'\varphi u\|_p+\mathcal{C}e^{-cm^\nu}\|Pu\|_p\\ &\le\mathcal{C}\|(f-P)u\|_p+\mathcal{C}\frac{\sqrt{a_m}}{m}\|P'\varphi u\|_p\end{aligned}$$

$$+\mathcal{C}\left(\sum_{x_k\in A_{\theta m}}\Delta x_k\,|[(f-P)u](x_k)|^p\right)^{\frac{1}{p}}+\mathcal{C}e^{-cm^\nu}\|Pu\|_p.$$

Moreover, recalling (32)–(34), (13) follows.

Now we prove (12). Since

$$f-H_m^*(w,f)=f-F_m^*(w,f)-G_m^*(w,f)$$

and estimating $\|[f-F_m^*(w,f)]u\|_p$ we get

$$\|[f-H_m^*(w,f)]u\|_p\le\mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi u\|_p+\|G_m^*(w,f)u\|_p+\|\Gamma(P)\|_p.$$

Using (9) (with $f\equiv 0$)

$$\|G_m^*(w,f)u\|_p\le\mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi u\|_p+\mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{1+\frac{1}{p}}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi(f',t)_{u\varphi,p}}{t^{1+\frac{1}{p}}}dt$$

and

$$\|[f-H_m^*(w,f)]u\|_p\le\mathcal{C}\frac{\sqrt{a_m}}{m}\|f'\varphi u\|_p$$
$$+\mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{1+\frac{1}{p}}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi(f',t)_{u\varphi,p}}{t^{1+\frac{1}{p}}}dt+\|\Gamma(P)\|_p. \tag{35}$$

Now, let $q\in\mathbb{P}_{M-1}$ be the best approximation of $f'\in L^p_{u\varphi}$ and let Q be one of its primitives. We have

$$\|[f-H_m^*(w,f)]u\|_p\le\mathcal{C}\|[(f-Q)-H_m^*(w,f-Q)]u\|_p+\|\Gamma(Q)\|_p.$$

Using (35), we get

$$\|[f-H_m^*(w,f)]u\|_p\le\mathcal{C}\frac{\sqrt{a_m}}{m}E_M(f')_{u\varphi,p}$$
$$+\mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{1+\frac{1}{p}}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi(f'-q,t)_{u\varphi,p}}{t^{1+\frac{1}{p}}}dt$$
$$+\|\Gamma(Q)\|_p+\|\Gamma(P)\|_p.$$

By (33) the first and second addenda are dominated by

$$\mathcal{C}\left(\frac{\sqrt{a_m}}{m}\right)^{1+\frac{1}{p}}\int_0^{\frac{\sqrt{a_m}}{m}}\frac{\Omega_\varphi^r(f',t)_{u\varphi,p}}{t^{1+\frac{1}{p}}}dt$$

and

$$|\Gamma(Q)\|_p+|\Gamma(P)\|_p\le\mathcal{C}e^{-cm^\nu}\|fu\|_p.$$

Then, (12) follows. □

Proof of Theorem 3.3. Let $x\in A_{\theta m}$ and $x_d<x$ the closest zero to x. Recalling (28) and using (19), we have

$$\|H_m^*(w,f)u\|_\infty \le [\|fu\|_\infty + \|f'\varphi u\|_\infty] \times \left(1 + \sum_{\substack{x_k \le A_{\theta m}\\ k\neq d\pm 1}} \left(\frac{x}{x_k}\right)^{\delta+\frac{3}{2}-\gamma} \frac{\Delta x_k}{|x-x_k|}\right).$$

The sum can be written as

$$\sum_{\substack{x_k \le A_{\theta m}\\ k\neq d\pm 1}} = \sum_{x_k \le x_{d-1}} + \sum_{x_{d+1}<x_k} =: S_1 + S_2.$$

Then

$$\begin{aligned} S_1 &\le \mathcal{C}\int_0^{x_{d-1}} \left(\frac{x}{t}\right)^{\delta+\frac{3}{2}-\gamma} \frac{dt}{x-t}\\ &\le \mathcal{C}\int_0^{1-\frac{\Delta x_d}{x_d}} \frac{dy}{y^{\delta+\frac{3}{2}-\gamma}(1-y)}\\ &\le 2\mathcal{C}\left\{\int_0^{\frac{1}{2}} \frac{dy}{y^{\delta+\frac{3}{2}-\gamma}} + \int_{\frac{1}{2}}^{1-\frac{\Delta x_d}{x_d}} \frac{dy}{(1-y)}\right\} \sim \log m, \end{aligned}$$

being $0 < \delta + \frac{3}{2} - \gamma < 1$. Concerning S_2, since $\frac{x}{x_k} < 1$ we get

$$S_2 \le \int_{1+\frac{\Delta x_d}{x_d}}^{a_m} \frac{dy}{y-1} \sim \log m$$

and (14).

We omit the proof of (15) because it is quite similar to the one of (14). □

Proof of Theorem 3.4. We write $F_m^*(w)$ in the following form

$$u(x)F_m^*(w,f,x) = \sum_{x_k\in A_{\theta m}} u(x)l_k^2(x)f(x_k) + \sum_{x_k\in A_{\theta m}} \frac{l_k^2(x)}{u(x_k)}\frac{x-x_k}{\Delta x_k}\left[\frac{-2l_k'(x_k)\Delta x_k}{\log(1+x^{-\alpha}+x^\beta)}\right](\bar{u}f)(x_k). \tag{36}$$

We have

$$\begin{aligned} \sum_{x_k\in A_{\theta m}} u(x)l_k^2(x)f(x_k) &\le \mathcal{C}\|fu\|_\infty \left(1 + \sum_{x_k\in A_{\theta m}} \left(\frac{x}{x_k}\right)^{\delta+\frac{3}{2}-\gamma}\left(\frac{\Delta x_k}{x-x_k}\right)^2\right)\\ &\le \mathcal{C}\|fu\|_\infty \left(1 + \Delta x_d \int_{\varepsilon\theta m}^{x_{d-1}} \left(\frac{x}{t}\right)^{\delta+\frac{3}{2}-\gamma} \frac{dt}{(x-t)^2}\right.\\ &\quad \left. + \frac{\sqrt{a_m}}{m}\int_{x_{d+1}}^{a_m} \sqrt{t}\left(\frac{x}{t}\right)^{\delta+\frac{3}{2}-\gamma} \frac{dt}{(t-x)^2}\right). \end{aligned}$$

Letting $t = xy$ we have

$$\int_{\varepsilon\theta m}^{x_{d-1}} \left(\frac{x}{t}\right)^{\delta+\frac{3}{2}-\gamma} \frac{dt}{(x-t)^2} \le \mathcal{C}\frac{\Delta x}{x}\int_0^{1-\frac{\Delta x}{x}} \frac{dy}{y^{\delta+\frac{3}{2}-\gamma}(1-y)^2} \le \mathcal{C},$$

being $0 < \delta + \frac{3}{2} - \gamma < 1$. Moreover, since in the second integral $x < t$ and recalling that $\delta + \frac{3}{2} - \gamma > 0$, letting $t = xy$ we get

$$\begin{aligned}
&\int_{x_{d+1}}^{a_m} \sqrt{t} \left(\frac{x}{t}\right)^{\delta+\frac{3}{2}-\gamma} \frac{dt}{(t-x)^2} \\
&\quad \le \mathcal{C} \frac{\Delta x}{x} \int_{1+\frac{\Delta x}{x}}^{\frac{a_m}{x}} \sqrt{y} \frac{dy}{(y-1)^2} \\
&\quad \le \mathcal{C} \frac{\Delta x}{x} \left\{ \int_{1+\frac{\Delta x}{x}}^{2+\frac{\Delta x}{x}} + \int_{2+\frac{\Delta x}{x}}^{\frac{a_m}{x}} \right\} \sqrt{y} \frac{dy}{(y-1)^2} \\
&\quad \le \mathcal{C} \frac{\Delta x}{x} \left\{ \left(2 + \frac{\Delta x}{x}\right)^{\frac{1}{2}} \int_{1+\frac{\Delta x}{x}}^{2+\frac{\Delta x}{x}} \frac{dy}{(y-1)^2} + \int_2^{\infty} \frac{dy}{y^{\frac{3}{2}}} \right\} \le \mathcal{C}.
\end{aligned}$$

Summing up we have

$$\sum_{x_k \in A_{\theta m}} u(x) \frac{l_k^2(x)}{u(x_k)} (fu)(x_k) \le \mathcal{C} \|fu\|_\infty.$$

To estimate the second sum in (36), we prove that

$$\left| \frac{2 l_k'(x_k) \Delta x_k}{\log(1 + x^{-\alpha} + x^\beta)} \right| \sim \mathcal{O}\left(\frac{1}{\log m} \right). \tag{37}$$

We have

$$|2 l_k'(x_k) \Delta x_k| = \left| 2 \frac{v'(x_k)}{v(x_k)} \Delta x_k + \frac{p_m''(w, x_k)}{p_m'(w, x_k)} \Delta x_k \right|$$

and

$$\begin{aligned}
\frac{v'(x_k)}{v(x_k)} \Delta x_k &\le \frac{\sqrt{a_m}}{m} \sqrt{x_k} \frac{4\|v\|_\infty}{a_m x_k} \le \mathcal{C} \frac{\sqrt{a_m}}{m} \frac{\sqrt{x_k}}{a_m x_k} \frac{a_m + \varepsilon_m}{2} \\
&\le \mathcal{C} \frac{\sqrt{a_m}}{m} \frac{a_m}{a_m \varepsilon_m^{\frac{1}{2}}} = \left(\frac{\sqrt{a_m}}{m} \right)^{1 - \frac{1}{2\alpha+1}}.
\end{aligned}$$

Moreover, letting $\bar{Q}(x) = x^{-\alpha} + x^\beta$ (i.e. $w = e^{-\bar{Q}}$), using arguments similar to [7, Lemma 5.1] (see also [3]), we have

$$\left| \frac{p_m''(w, x_k)}{p_m'(w, x_k)} \right| \le \mathcal{C}(1 - Q'(x_k)) \le \mathcal{C} \frac{\alpha x_k^{-\alpha} + \beta x_k^\beta}{x}.$$

Then,

$$\frac{p_m''(w, x_k)}{p_m'(w, x_k)} \frac{\Delta x_k}{\log(1 + x^{-\alpha} + x^\beta)} \le \mathcal{C} \frac{\sqrt{a_m}}{m} \frac{\alpha x^{-\alpha} + \beta x^\beta}{\sqrt{x_k} \log(1 + x^{-\alpha} + x^\beta)}.$$

For x_k "close" ε_m (for example, $x < 1$)

$$\begin{aligned}
\frac{p_m''(w, x_k)}{p_m'(w, x_k)} \frac{\Delta x_k}{\log(1 + x^{-\alpha} + x^\beta)} &\le \mathcal{C} \frac{\sqrt{a_m}}{m} \frac{x^{-\alpha-\frac{1}{2}}}{\log(1 + x_k^{-1})} \\
&\le \mathcal{C} \frac{\sqrt{a_m}}{m} \frac{\varepsilon_m^{-\alpha-\frac{1}{2}}}{\log(1 + \varepsilon_m^{-1})} \sim \frac{1}{\log m}
\end{aligned}$$

and for x_k large (for example, $x \geq 1$),

$$\frac{p''_m(w,x_k)}{p'_m(w,x_k)} \frac{\Delta x_k}{\log(1+x^{-\alpha}+x^{\beta})} \leq \mathcal{C} \frac{\sqrt{a_m}}{m} \frac{x^{\beta-\frac{1}{2}}}{\log(1+x_k^{\beta})}$$
$$\leq \mathcal{C} \frac{\sqrt{a_m}}{m} \frac{a_m^{\beta-\frac{1}{2}}}{\log(1+a_m)} \sim \frac{1}{\log m}.$$

Therefore, (37) follows.

Coming back to the estimate of the second sum in (36), using (37), we get

$$\sum_{x_k \in A_{\theta m}} \frac{l_k^2(x)}{u(x_k)} \frac{x-x_k}{\Delta x_k} \left[\frac{-2l'_k(x_k)\Delta x_k}{\log(1+x^{-\alpha}+x^{\beta})} \right] (\bar{u}f)(x_k)$$
$$\leq \frac{\mathcal{C}}{\log m} \sum_{x_k \in A_{\theta m}} \left(\frac{x}{t}\right)^{\delta+\frac{3}{2}-\gamma} \Delta x_k |\bar{u}f|(x_k) \leq \mathcal{C}\|f\bar{u}\|_\infty,$$

being $0 < \delta + \frac{3}{2} - \gamma < 1$, then (17) follows.

Finally, we prove (18). Since

$$f - F_m^*(w,f) = (f-P) + F_m^*(w, P-f) + G_m^*(w,P) + \Gamma(P),$$

with $P \in \mathbb{P}_M$ best approximation of $f \in C_{\bar{u}}$, we have

$$\|[f - F_m^*(w,f)]u\|_\infty$$
$$\leq \|(f-P)u\|_\infty + \mathcal{C}\|(f-P)\bar{u}\|_\infty + \mathcal{C}\frac{\log m}{m}\|P'\varphi u\|_\infty + \|\Gamma(P)\|_\infty$$
$$\leq \left\{ \|(f-P)\bar{u}\|_\infty + \frac{\log m}{m}\|P'\varphi\bar{u}\|_\infty \right\} + \|\Gamma(P)\|_\infty$$
$$\leq \mathcal{C}\omega_\varphi\left(f, \frac{\log m}{m}\right)_{\bar{u},\infty} + \mathcal{C}e^{-Am^\nu}\|fu\|_\infty$$

and (18) follows. □

Author contributions All authors contributed equally to this work.

Funding Open access funding provided by Università degli Studi di Torino within the CRUI-CARE Agreement.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interests The authors declare no competing interests.

References

[1] De Bonis, M.C., Kubayi, D.: H ermite-Fejér and Grünwald interpolation at generalized Laguerre zeros. Filomat (2019). https://doi.org/10.2298/FIL1915855D

[2] De Bonis, M.C., Mastroianni, G.: On the Hermite-Fejér interpolation based at the zeros of generalized Freud polynomials. Mediterr. J. Math. **15**(1), 26 (2018). https://doi.org/10.1007/s00009-018-1073-4

[3] Jung, H.S., Sakai, R.: The Markov-Bernstein inequality and Hermite-Fejér interpolation for exponential-type weights. J. Approx. Theory **162**(7), 1381–1397 (2010). https://doi.org/10.1016/j.jat.2010.02.006

[4] Junghanns, P., Mastroianni, G., Notarangelo, I.: On Nyström and product integration methods for Fredholm integral equations, in: Contemporary computational mathematics – a celebration of the 80th birthday of Ian Sloan (J. Dick, F.Y. Kuo and H. Wo žniakowski, Eds), Springer Cham, 645–673, (2018). ISBN: 978-331972456-0, 978-331972455-3. https://doi.org/10.1007/978-3-319-72456-0_29

[5] Junghanns, P., Mastroianni, G., Notarangelo, I.: Weighted Polynomial Approximation and Numerical Methods for Integral Equations, Pathways in Mathematics, Birkhäuser, Cham (2021). ISBN: 978-3-030-77496-7, 978-3-030-77499-8, 978-3-030-77497-4. https://doi.org/10.1007/978-3-030-77497-4

[6] Levin, A.L., Lubinsky, D.S.: Orthogonal Polynominals for Exponential Weights, CSM Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 4. Springer-Verlag, New York (2001)

[7] Lubinsky, D.S., Rabinowitz, P.: Hermite and Hermite-Fejér interpolation and associated product integration rules on the real line: the L^1 theory. Canad. J. Math. **44**(3), 561–590 (1992). https://doi.org/10.4153/CJM-1992-036-2

[8] Mastroianni, G., Milovanović, G.V.: Interpolation Processes Basic Theory and Applications. Springer Monographs in Mathematics, Berlin (2008). (**ISBN: 978-3-540-68346-9**)

[9] Mastroianni, G., Milovanović, G.V., Notarangelo, I.: A Nyström method for a class of Fredholm integral equations on the real semiaxis. Calcolo **54**, 567–585 (2017). https://doi.org/10.1007/s10092-016-0199-7

[10] Mastroianni, G., Milovanović, G.V., Notarangelo, I.: Polynomial approximation with Pollaczeck-Laguerre weights on the real semiaxis. Survey, Electron. Trans. Numer. Anal. **50**, 36–51 (2018). https://doi.org/10.1553/etna_vol50s36

[11] Mastroianni, G., Monegato, G.: Truncated quadrature rules over $]0,+\infty[$ and Nyström type methods. SIAM J. Numer. Anal. **41**, 1870–1892 (2003). https://doi.org/10.1137/S0036142901391475

[12] Mastroianni, G., Monegato, G.: Truncated Gauss–Laguerre quadrature rules, In: Recent Trends in Numerical Analysis (D. Trigiante, Ed.), Nova Science Publishers, pp. 213–221 (2000). https://doi.org/10.1007/BF02874673

[13] Mastroianni, G., Notarangelo, I.: Embedding theorems with an exponential weight on the real semiaxis. Electron. Notes Discrete Math. **43**, 155–160 (2013). https://doi.org/10.1016/j.endm.2013.07.026

[14] Mastroianni, G., Notarangelo, I.: Polynomial approximation with an exponential weight on the real semiaxis. Acta Math. Hungarica **142**(1), 167–198 (2014). https://doi.org/10.1007/s10474-013-0348-2

[15] Mastroianni, G., Notarangelo, I., Milovanović, G.V.: Gaussian quadrature rules with an exponential weight on the real semiaxis. IMA J. Numer. Anal. **34**(4), 1654–1685 (2014). https://doi.org/10.1093/imanum/drt034

[16] Mastroianni, G., Notarangelo, I.: Lagrange interpolation at Pollaczek-Laguerre zeros on the real semiaxis. J. Approx. Theory **245**, 83–100 (2019). https://doi.org/10.1016/j.jat.2019.04.004

[17] Mastroianni, G., Notarangelo, I.: Hermite and Hermite-Fejér interpolation at Pollaczek zeros. Calcolo **62**, 6 (2025). https://doi.org/10.1007/s10092-024-00630-z

[18] Mastroianni, G., Notarangelo, I., Szabados, J.: Polynomial inequalities with an exponential weight on $(0,+\infty)$. Mediterr. J. Math. **10**(2), 807–821 (2013). https://doi.org/10.1007/s00009-012-0231-3

[19] Mastroianni, G., Notarangelo, I., Szili, L., Vértesi, P.: A note on Hermite-Fejér interpolation at Laguerre zeros. Calcolo **55**(3), 39, 12 (2018). https://doi.org/10.1007/s10092-018-0281-4

[20] Mastroianni, G., Notarangelo, I., Szili, L., Vértesi, P.: Some new results on orthogonal polynomials for Laguerre type exponential weights. Acta Math. Hungar. **155**(2), 466–478 (2018). https://doi.org/10.1007/s10474-018-0841-8

[21] Szabados, J., Vértesi, P.: Interpolation of functions. World Scientific Publishing Co. Inc, Teaneck, NJ (1990). (**ISBN: 9971-50-915-6**)

Maria Carmela De Bonis and Giuseppe Mastroianni
Department of Basic and Applied Sciences
University of Basilicata
viale dell'Ateneo Lucano, 10
85100 Potenza
Italy
e-mail: `mariacarmela.debonis@unibas.it`

Giuseppe Mastroianni
e-mail: `giuseppe.mastroianni@unibas.it`

Incoronata Notarangelo
Department of Mathematics “Giuseppe Peano”
University of Torino
via Carlo Alberto, 10
10124 Torino
Italy
e-mail: `incoronata.notarangelo@unito.it`

Received: December 16, 2024.
Revised: February 13, 2025.
Accepted: February 17, 2025.