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Research Article

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Cauchy formula for vector-valued holomorphic functions and the Cauchy-Kovalevskaja theorem

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Abstract: We establish a version of the Cauchy integral formula for holomorphic functions on a polidisc, with values in a complex Fréchet space $\mathfrak{X}$. We prove a vector-valued analog to Weierstrass's theorem on sequences of holomorphic functions $f_\nu \in O(\Omega, \mathfrak{X})$ converging uniformly on compact subsets of $\Omega \subset \mathbb{C}^n$. We obtain a Cauchy-Kovalevskaja-type theorem, i.e., prove existence of $\mathfrak{X}$-valued C^ω solutions to the Cauchy problem $P(x, D)u = f$ on $U \subset \Omega$ and $(\partial^j u/\partial x_n^j)|_{x_n=0} = \varphi_j$ on $U_0 = U \cap \{x_n = 0\}$, $j \in \{0, 1, \ldots, m-1\}$, where $P(x, D) \equiv \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha$ and $a_\alpha \in C^\omega(\Omega)$, $f \in C^\omega(\Omega, \mathfrak{X})$ and $\varphi_j \in C^\omega(\Omega_0, \mathfrak{X})$, with $0 \in \Omega \subset \mathbb{R}^n$ open and $\Omega_0 = \Omega \cap \{x_n = 0\}$. The existence of an open set $0 \in U \subset \Omega$ and of a solution $u \in C^\omega(U, \mathfrak{X})$ to the Cauchy problem requires the structural condition $a_{(0,\ldots,0,m)}(0) \ne 0$ and relies, as well as in the classical case of scalar valued solutions, on the Cauchy integral formula and on Weierstrass' theorem for $\mathfrak{X}$-valued holomorphic functions.

Keywords: vector-valued holomorphic function; Bochner integral; Cauchy formula; Cauchy-Kovalevskaja theorem

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Dedicated to Ermanno Lanconelli.

1 Introduction

The present article studies holomorphic functions $f : \Omega \to \mathfrak{X}$ and their application to linear partial differential equations with real analytic coefficients, where $\Omega \subset \mathbb{C}^n$ ($n \ge 1$) is an open set and $\mathfrak{X}$ is a complex topological vector space. The need for vector-valued holomorphic functions arises mainly in the theory of 1-parameter semigroups (cf. Arendt et al. [1]) and in analytic functional calculus (cf. Vasilescu [23]). For instance, families of holomorphic functions depending continuously on a parameter $t \in [0, 1]$ may be recast as holomorphic functions with values in the Banach space $\mathfrak{X} = C([0, 1], \mathbb{C})$ of all $\mathbb{C}$ valued continuous functions on the unit interval $[0, 1]$ Cf. also Barletta and Dragomir [2,3], Barletta et al. [4], and Dragomir and Nishikawa [14].

Holomorphic functions of one complex variable, with values in a complex locally convex space $\mathfrak{X}$, were first studied by Grothendieck [17], who characterized the topological dual $O(\Omega, \mathfrak{X})^*$ with $\Omega \subset \mathbb{C}$ open.

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Properties of $\mathfrak{X}$-valued holomorphic functions on analytic spaces were studied by Bungart [8]. Bungart's purpose in [8] is to extend Theorems A and B of Cartan (cf. [12]) to this more general class of holomorphic functions. Cartan's Theorem B for vector-valued holomorphic functions was also proved by Bishop (cf. [6]) by a different methodological approach. To compensate for the lack of generality of the considerations in the present article, let $(V, \mathcal{O})$ be a Stein space, $\mathcal{S}$ be a coherent analytic sheaf over $(V, \mathcal{O})$, and $H(V, \mathcal{S})$ be the space of cross-sections in $\mathcal{S}$. Let $(W, \mathcal{O}^W)$ be a closed subvariety of $(V, \mathcal{O})$, and let $B(W, \mathcal{O}^W)$ be the Banach space of bounded functions in $H(W, \mathcal{O}^W)$. The restriction map $H(V, \mathcal{O}) \to H(W, \mathcal{O}^W)$ admits a continuous linear right inverse $B(W, \mathcal{O}^W) \to H(V, \mathcal{O})$ and the methods developed in [8] allow for a choice of this map $f \longmapsto F$ that can be represented by an integral formula:

$$F(z) = \int_W f \, \mathrm{d}\eta_z \tag{1.1}$$

where η is a holomorphic mapping of V into the space of bounded measures on W. In particular, if $U \subset V$ is a bounded domain, then there is a holomorphic mapping η from U into the space of measures on the distinguished boundary $\partial_0 U$ of U such that

$$f(z) = \int_{\partial_0 U} f \, \mathrm{d}\eta_z, \quad z \in U, \tag{1.2}$$

for any function f holomorphic on the closure $\overline{U}$ of the domain. When $U \subset \mathbb{C}^n$ is a bounded domain with C^∞ boundary, a choice of η with values in $\mathfrak{X} = C^\infty(\Lambda^{2n-1}T^*(\partial U))$ (organized as a Fréchet space) is available. Precisely (by Theorem 20.2 in [8], p. 343) there is a holomorphic map $\alpha : U \to \mathfrak{X}$ such that

$$f(z) = \int_{\partial U} f \alpha_z, \quad z \in U, \tag{1.3}$$

for any $f \in O(\overline{U})$. A question raised in [8], i.e., whether α can be chosen to be a kernel, has been explored in [9], where the problem of building appropriate kernels that are holomorphic (anti-holomorphic) in the parameters involved (and not only real analytic, as is the case with Bochner-Martinelli formula, cf. [7] and [18]) is taken up within the line of thought started in [24] with the Cauchy-Weil formula. Cf. also [10] and [11]. It should be reminded that a related formula was discussed, together with its relationship to boundary kernels, in [5]. Both Cauchy-Weil and Bergman formulas hold solely for domains with particular boundary properties, while for existence theorems leading to representation formulas such as (1.3), holding for arbitrary smoothly bounded domains, one should rely on the findings in [8,15,16].

As a drawback of the approach in [8], the abstract existence theorems (Theorem 19.1 in [8], p. 340) underlying the representations (1.1)–(1.2) do not provide explicit representation formulas, as required by applications of sorts. In the present article, we aim to applications to the Cauchy problem:

$$P(x, D)u = f \quad \text{in } \Omega, \tag{1.4}$$

$$\left.\frac{\partial^j u}{\partial x_n^j}\right|_{x_n=0} = \varphi_j \quad \text{on } \Omega_0 = \Omega \cap \{x_n = 0\}, \quad 0 \le j \le m-1, \tag{1.5}$$

$$P(x, D) \equiv \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \quad 0 \in \Omega \subset \mathbb{R}^n,$$
$$a_\alpha \in C^\omega(\Omega), \quad f \in C^\omega(\Omega, \mathfrak{X}), \varphi_j \in C^\omega(\Omega_0, \mathfrak{X}).$$

As well as in the case of scalar valued dependent variables, the Cauchy problems (1.4) and (1.5) may be transplanted to the holomorphic category and then restated as the integro-differential equation:

$$v - Tv = \tilde{f}, \tag{1.6}$$

where $T : O(\mathscr{U}, \mathfrak{X}) \to O(\mathscr{U}, \mathfrak{X})$ is the linear operator

$$Tv \equiv -\sum_{j=0}^{m-1} \sum_{\substack{\beta \in \mathbb{Z}_+^{n-1} \\ |\beta| \le m-1}} \tilde{a}_{\beta j}(z) \frac{z_n^{m-j}}{(m-j-1)!} \int_0^1 (1-t)^{m-j-1} (D_{z'}^{\beta} v)(z', tz_n) \mathrm{d}t.$$

The main ingredient needed in the treatment of (1.6) is a Cauchy integral formula involving a Bochner integral on the distinguished boundary of a polydisc. The less general (as compared to the representation formulas (1.1) and (1.2)) Cauchy integral formula is recovered to the case of vector-valued holomorphic functions by taking a different approach (with respect to that in [8]), i.e., by extending the arguments in [21] from functions of one complex variable ($n = 1$) to the case of functions of several complex variables ($n \ge 2$). A key step is to show that weakly holomorphic functions with values in a complex Fréchet space are strongly holomorphic, and within to establish the strong continuity of weakly holomorphic functions (cf. Theorem 2.1 below). The proof makes use of a trick discovered in early works by Nicolescu (cf. [20]) and Ciorănescu (cf. [13]) itself relying on the fact that weak and strong boundedness in a locally convex space are equivalent. We would deceive the reader if we were not to mention that the solution to the Cauchy problems (1.4) and (1.5) does not rely solely on the Cauchy integral formula (2.5) but also on its more refined consequences such as Weierstrass' theorem (that we recover to the case of uniformly convergent [on compact sets] sequences of vector-valued holomorphic functions in Section 3 [cf. Theorem 3.1 there]) and on the truncated Taylor formula with integral reminder (a vector-valued version of which is recovered, with a Bochner integral reminder, in Appendix B to the present article). To solve (1.6), we use an algorithm[1] $\mathscr{A}(T, \mathscr{F})$ learning from $v_0 \in O(\mathscr{U}, \mathfrak{X})$ and producing a sequence $\{v_\nu\}_{\nu \ge 1}$ that converges to the solution. A hard analysis ingredient in the proof of the (vector-valued analog to the) Cauchy-Kovalevskaja theorem, that is worth to mention, is the estimate (cf. [5.6] in Lemma 5.3) of the error committed by $\mathscr{A}(T, \mathscr{F})$ at step ν, i.e.,

$$\mathbf{p}[w_\nu(z)] \le \frac{M(\mathbf{p})[ae^m C(m)]^\nu |z_n|^\nu}{(1 - \max_{1 \le k \le n-1} |z_k|)^{m\nu}}, \qquad \mathbf{p} \in \mathscr{P}, \nu \in \mathbb{Z}_+, z \in \mathbb{B}^n. \tag{1.7}$$

Here,

$$w_\nu \coloneqq v_{\nu+1} - v_\nu,\ v_{\nu+1} = Tv_\nu + \mathscr{F},\ \nu \ge 0,$$
$$\mathscr{F}, v_0 \in O(\mathscr{U}, \mathfrak{X}),\ \overline{\mathbb{B}^n} \subset \mathscr{U} \subset \mathbb{C}^n.$$

The frankly didactical proof of (1.7) provides the explicit calculation of the constants in (1.7) (as paralleled to the classical proof, where only existence of $M(\mathbf{p}) > 0$ and $C(m) > 0$ is inferred):

$$M(\mathbf{p}) = \sup_{z \in \overline{\mathbb{B}^n}} \mathbf{p}[w_0(z)],\ C(m) = b_m \sum_{j=1}^{m} m^j,$$
$$a = \max_{0 \le j \le m-1} \max_{\beta \in B_j} \sup_{z \in \overline{\mathbb{B}^n}} |\tilde{a}_{\beta,j}(z)|,$$
$$b_m = |B_0|,\ B_0 \coloneqq \{\beta \in \mathbb{Z}_+^{n-1} : |\beta| \le m\}.$$

2 Vector-valued holomorphic functions of several complex variables

Let $\Omega \subset \mathbb{C}^n$ be an open set ($n \ge 1$), and let $\mathfrak{X}$ be a complex topological vector space. A function $f : \Omega \to \mathfrak{X}$ is *weakly holomorphic* in Ω if $\Lambda \circ f \in O(\Omega)$, i.e., $\Lambda \circ f : \Omega \to \mathbb{C}$ is a holomorphic function, for every $\Lambda \in \mathfrak{X}^*$. Also $f : \Omega \to \mathfrak{X}$ is *strongly holomorphic* if for any $a \in \Omega$ there is a neighborhood $a \in U \subset \Omega$ and a series

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1 The same as in the proof of Banach's fixed point principle.

$\sum_{|\alpha|\geq 0}(z-a)^\alpha x_\alpha$ with $x_\alpha \in \mathfrak{X}$ such that $\sum_{|\alpha|=0}^\infty (z-a)^\alpha x_\alpha = f(z)$ for any $z \in U$. Strongly holomorphic functions are weakly holomorphic, as well (because every $\Lambda \in \mathfrak{X}^*$ is linear and continuous). As to the converse, the following facts are classically known, though only for $n = 1$ (cf. Theorem 3.31 in [21], p. 82). Let $\Omega \subset \mathbb{C}$ be an open set, $\mathfrak{X}$ a locally convex space, and $f : \Omega \to \mathfrak{X}$ a weakly holomorphic function. Then (i) f is strongly continuous. Let $\Gamma(\Omega)$ denote the set of all closed rectifiable curves $\Gamma = \{\gamma(t) : a \leq t \leq b\}$ in $\Omega \subset \mathbb{C}$, and let us set

$$\operatorname{Ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{d\zeta}{\zeta - z},\ z \in \mathbb{C}.$$

Let us assume that $\overline{\mathrm{co}}[f(\Gamma)]$ is a compact subset[2] of $\mathfrak{X}$, for any $\Gamma \in \Gamma(\Omega)$. Then (ii)

$$\int_\Gamma f(\zeta)d\zeta = 0,\ \Gamma \in \Gamma(\Omega),\ \operatorname{Ind}_\Gamma(z) = 0,\ z \in \mathbb{C}\backslash\Omega, \tag{2.1}$$

$$f(z) = \frac{1}{2\pi i}\int_\Gamma (\zeta - z)^{-1} f(\zeta)d\zeta,\ z \in \Omega,\ \operatorname{Ind}_\Gamma(z) = 1, \tag{2.2}$$

$$\int_{\Gamma_1} f(\zeta)d\zeta = \int_{\Gamma_2} f(\zeta)d\zeta, \tag{2.3}$$

$$\Gamma_1, \Gamma_2 \in \Gamma(\Omega),\ \operatorname{Ind}_{\Gamma_1}(z) = \operatorname{Ind}_{\Gamma_2}(z),\ z \in \mathbb{C}\backslash\Omega.$$

Let $\mathfrak{X}$ be a complex Fréchet space. Then (iii) f is $\mathbb{C}$-differentiable at each $z_0 \in \Omega$, i.e., the limit $\lim_{z\to z_0}(z - z_0)^{-1}\{f(z) - f(z_0)\}$ exists in the topology of $\mathfrak{X}$, for any $z_0 \in \Omega$. The formulas (2.1) and (2.2) are, respectively, Cauchy's theorem and the Cauchy integral formula for $\mathfrak{X}$-valued holomorphic functions (cf. Theorem 1.5 and formula (1.59) in [22], pp. 42–49, for the scalar valued counterpart of (2.1) and (2.3)). Strong continuity of weakly holomorphic functions follows mainly from the Cauchy formula for scalar valued holomorphic functions (cf. [21], pp. 83–84).

We recall that a subset $E \subset \mathfrak{X}$ is *bounded* if for any neighborhood V of $0 \in \mathfrak{X}$ there is a number $s > 0$ such that $E \subset tV$ for any $t \geq s$. Also E is *weakly bounded* if $\Lambda(E) \subset \mathbb{C}$ is bounded for any $\Lambda \in \mathfrak{X}^*$. It should be observed that while basic notions make sense for an arbitrary topological vector space $\mathfrak{X}$, most basic results require more specialized values. The assumption that $\mathfrak{X}$ is a locally convex space (in statement (i)) is exploited in two ways. First, any weakly bounded subset of a locally convex space is (strongly) bounded (cf. Theorem 3.18 in [21], p. 70). Second, for any locally convex space $\mathfrak{X}$ the topological dual $\mathfrak{X}^*$ separates points on $\mathfrak{X}$ (cf. Corollary to Theorem 3.4 in [21], pp. 59–60).

Let $\Omega \subset \mathbb{C}^n$ be an open set ($n \geq 2$) and let $a \in \Omega$. Let $\rho = (\rho_1, \ldots, \rho_n)$ be a polyradius ($\rho_j > 0$) such that the polydisc $\overline{P}(a, 2\rho) = \{z \in \mathbb{C}^n : |z_j - a_j| \leq 2\rho_j, 1 \leq j \leq n\}$ is contained in Ω. Let $\mathfrak{X}$ be a locally convex space. Let $f : \Omega \to \mathfrak{X}$ be a weakly holomorphic function and let $\Lambda \in \mathfrak{X}^*$. Let us set $a^{(j)} = (a_1, \ldots, a_j) \in \mathbb{C}^j$ for any $1 \leq j \leq n$. As $\Lambda \circ f : \Omega \to \mathbb{C}$ is holomorphic, for every $z \in P(a, 2\rho)$

$$\Lambda[f(z)] - \Lambda[f(a)] = \sum_{j=1}^n \{\Lambda[f(a^{(j-1)}, z_j, \ldots, z_n)] - \Lambda[f(a^{(j)}, z_{j+1}, \ldots, z_n)]\} =$$

(by the Cauchy formula)

$$= \frac{1}{2\pi i}\sum_{j=1}^n \left[\int_{|\zeta_j - z_j| = 2\rho_j} \frac{\Lambda[f(a^{(j-1)}, \zeta_j, z_{j+1}, \ldots, z_n)]}{\zeta_j - z_j} d\zeta_j - \int_{|\zeta_j - z_j| = 2\rho_j} \frac{\Lambda[f(a^{(j-1)}, \zeta_j, z_{j+1}, \ldots, z_n)]}{\zeta_j - a_j} d\zeta_j\right].$$

Let us set

$$M(\Lambda) = \sup\{|\Lambda[f(\zeta)]| : \zeta \in \overline{P}(a, 2\rho)\}.$$

2 If $A \subset \mathfrak{X}$ then $\overline{\mathrm{co}}(A)$ is the closed convex hull of A.

If $z \in \overline{P}(a, \rho)\setminus\{a\}$, then

$$\Lambda[f(z)] - \Lambda[f(a)] \leq \frac{M(\Lambda)}{4\pi} \sum_{j=1}^{n} \frac{|z_j - a_j|}{\rho_j} \int\limits_{|\zeta_j - z_j| = 2\rho_j} \frac{d\zeta_j}{|\zeta_j - a_j} \leq M(\Lambda)|z - a| \sum_{j=1}^{n} \frac{1}{\rho_j}$$

(by $|\zeta_j - z_j| \geq |\zeta_j - a_j| - |z_j - a_j| = 2\rho_j - |z_j - a_j| \geq \rho_j$). Consequently, the set

$$\{|z - a|^{-1}[f(z) - f(a)] : z \in \overline{P}(a, \rho)\setminus\{a\}\} \tag{2.4}$$

is weakly bounded in $\mathfrak{X}$ and hence strongly bounded, too. Let $V \subset \mathfrak{X}$ be an open neighborhood of the origin $0 \in \mathfrak{X}$. As the set (2.4) is bounded, there is $s > 0$ such that

$$f(z) - f(a) \in t|z - a|V, z \in \overline{P}(a, \rho)\setminus\{a\}, t \geq s.$$

We recall that a subset $B \subset \mathfrak{X}$ of a topological vector space is *balanced* if for any $\alpha \in \mathbb{C}$ with $|\alpha| < 1$ one has $\alpha B \subset B$. As every topological vector space has a balanced local base of neighborhoods of the origin (cf. [21], p. 13), it follows that f is (strongly) continuous at the point a. We established the following:

Theorem 2.1. *Let $\mathfrak{X}$ be a locally convex space and let $\Omega \subset \mathbb{C}^n$ be an open set* ($n \geq 2$). *Any weakly holomorphic function $f : \Omega \to \mathfrak{X}$ is strongly continuous.*

Let us get back to the one complex variable case ($n = 1$) and the statements (i)–(iii) above. Once statement (i) was proved, one may conclude (by Theorem 3.27 in [21], p. 78) that the integrals $\int_\Gamma f(\zeta)d\zeta$ and $\int_\Gamma(\zeta - z)f(\zeta)d\zeta$ are well-defined elements of $\overline{\text{co}}[f(\Gamma)]$ (as Bochner integrals, cf. Definition 3.26 in [21], p. 77, and Appendix A with this article). Then formulas (2.1) and (2.3) follow from the ordinary Cauchy formula and Cauchy's theorem applied to the holomorphic functions $\Lambda \circ f$ for every $\Lambda \in \mathfrak{X}^*$. Similarly, when $n \geq 2$, we may state

Theorem 2.2. *Let $\mathfrak{X}$ be a locally convex space and let $\Omega \subset \mathbb{C}^n$ be an open set* ($n \geq 2$). *Let $a \in \Omega$ and let $\rho = (\rho_1, \dots, \rho_n)$ be a polyradius such that $\overline{P}(a, \rho) \subset \Omega$. Let $f : \Omega \to \mathfrak{X}$ be a weakly holomorphic function such that the closed convex hull of $f[\partial_0 P(a, \rho)]$ is a compact subset of $\mathfrak{X}$. Then for every $z \in P(a, \rho)$,*

$$f(z) = \frac{1}{(2\pi i)^n} \int\limits_{\partial_0 P(a,\rho)} \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} f(\zeta_1, \dots, \zeta_n) d\zeta_1 \dots d\zeta_n. \tag{2.5}$$

Here, $\partial_0 P(a, \rho) = \prod_{j=1}^{n} S^1(a_j, \rho_j)$ is the essential boundary of $P(a, \rho)$ and $S^1(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$ with $z_0 \in \mathbb{C}$ and $r > 0$.

Proof. By Theorem 2.1, the function $f : \Omega \to \mathfrak{X}$ is continuous. Then (by Theorem 3.27 in [21]) the integral

$$\int\limits_{\partial_0 P(a,\rho)} f(\zeta) \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} d\zeta \in \overline{\text{co}}[f(\partial_0 P(a, \rho))]$$

is well defined, and

$$\Lambda\left(\int\limits_{\partial_0 P(a,\rho)} f(\zeta) \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} d\zeta \right) = \int\limits_{\partial_0 P(a,\rho)} \frac{\Lambda[f(\zeta)]}{\prod_{j=1}^{n} (\zeta_j - z_j)} d\zeta$$

for any $\Lambda \in \mathfrak{X}^*$. Then (2.5) follows from the Cauchy formula for $\Lambda \circ f$, and the fact that $\mathfrak{X}^*$ separates points.

Let us go back once again to case $n = 1$. Statement (iii) above required the further specialization of the values $\mathfrak{X}$, i.e., that $\mathfrak{X}$ is a complex Fréchet space. This guarantees that $\overline{\text{co}}[F(\Gamma)]$ is a compact set, where $F(z) = z^{-2} f(z)$, $\Gamma = \{\zeta \in \mathbb{C} : |\zeta| = 2r\}$. Consequently, both Cauchy's theorem (2.1) and the Cauchy integral formula (2.2) hold for the holomorphic function F (and the proof in [21], p. 84, does apply). □

Let $\mathfrak{X}$ be a complex Fréchet space and $f : \Omega \to \mathfrak{X}$ a weakly holomorphic function. For each $a \in \mathbb{C}^n$, we set

$$\Omega_{j,a} = \{z \in \mathbb{C} : (a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_n) \in \Omega\},$$
$$f_{j,a}(z) = (a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_n), z \in \Omega_{j,a}.$$

Each $f_{j,a}$ is weakly holomorphic in $\Omega_{j,a}$ hence (by statement (iii) above) $f_{j,a}$ is strongly holomorphic in $\Omega_{j,a}$. It is a natural question whether, under these conditions, the function $f : \Omega \to \mathfrak{X}$ is holomorphic. Equivalently, the question is whether Hartogs' theorem[3] holds for $\mathfrak{X}$-valued functions f with $f_{j,a}$ holomorphic. We shall establish the following:

Theorem 2.3. *Let $\mathfrak{X}$ be a complex Fréchet space, $\Omega \subset \mathbb{C}^n$ an open set ($n \geq 2$), and $f : \Omega \to \mathfrak{X}$ a weakly holomorphic function. Then f is strongly holomorphic.*

To establish Theorem 2.3, we follow the arguments in the proof of the classical Hartogs' theorem (cf. [19], pp. 43–44). The proof is however easier because weakly holomorphic functions are already continuous, while Hartogs' theorem assumes only separate analyticity. To prove Theorem 2.3, it suffices to show

Theorem 2.4. *Let $\mathfrak{X}$ be a complex Fréchet space, and let us consider the polydisc $\Omega = \{z \in \mathbb{C}^n : |z_j| < R, 1 \leq j \leq n\}$, $R > 0$. Let $f : \Omega \to \mathfrak{X}$ be a weakly holomorphic function. Then there is $0 < r < R$, and there is a power series $\sum_{|\alpha| \geq 0} z^\alpha x_\alpha$ with $x_\alpha \in \mathfrak{X}$, converging uniformly on $P(0, \mathbf{r})$ and such that $f(z) = \sum_{|\alpha|=0}^{\infty} z^\alpha x_\alpha$ for any $z \in P(0, \mathbf{r})$. Here, $\mathbf{r} = (r, \dots, r)$.*

Proof. Let $D_r(0) = \{x \in \mathbb{C} : |z| < r\}$. As $f : \Omega \to \mathfrak{X}$ is continuous (cf. Theorem 2.1), it is Bochner integrable on the product of circles

$$T_\rho = \prod_{j=1}^{n} \{\zeta_j \in \mathbb{C} : |\zeta_j| = \rho\}, \; 0 < \rho < R,$$

i.e., $f \in L^1(T_\rho, \mathfrak{X}, d\zeta)$. Let $z' = (z_1, \dots, z_{n-1})$ such that $|z_j| < R$ for any $1 \leq j \leq n-1$. Note that $D_\rho(0) \subset \Omega_{n,z'}$. Also, as argued earlier, $f_{n,z'}$ is holomorphic in $\Omega_{n,z'}$, and in particular in $D_\rho(0)$. Hence, (by statement (ii))

$$f(z', z_n) = \frac{1}{2\pi i} \int_{|\zeta_n| = \rho} (\zeta_n - z_n)^{-1} f(\zeta', z_n) d\zeta_n, \; |z_n| < \rho.$$

For fixed $z_1, \dots, z_{n-2}$ with $|z_j| < R$, $1 \leq j \leq n-2$, and fixed $\zeta_n \in D_\rho(0)$, the function $f(z_1, \dots, z_{n-1}, \zeta_n)$ is holomorphic in $|z_{n-1}| < \rho$, and hence, we may repeat the aforementioned procedure. In the end, for any $|z_j| < \rho$, $1 \leq j \leq n$, one has

$$(2\pi i)^n f(z_1, \dots, z_n) = \int_{|\zeta_n| = \rho} d\zeta_n \int_{|\zeta_{n-1}| = \rho} d\zeta_{n-1} \dots \int_{|\zeta_1| = \rho} \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} f(\zeta_1, \dots, \zeta_n) d\zeta_1. \tag{2.6}$$

Let $\overline{P}(0, \mathbf{r}) = \{z \in \mathbb{C}^n : |z_j| \leq r, 1 \leq j \leq n\}$, where $\mathbf{r} = (r, \dots, r)$ and $0 < r < \rho$. Let $z \in \overline{P}(0, \mathbf{r})$ and $\zeta \in T_\rho$. Then

$$\prod_{j=1}^{n} (\zeta_j - z_j)^{-1} = \sum_{|\alpha|=0}^{\infty} \frac{z^\alpha}{\zeta^{\alpha + \mathbf{1}}}, \tag{2.7}$$

where $\alpha + \mathbf{1} = (\alpha_1 + 1, \dots, \alpha_n + 1)$ and the series in the right-hand of (2.7) converges uniformly for $\zeta \in T_\rho$ and $z \in \overline{P}(0, \mathbf{r})$. Let $f_\alpha(z, \zeta) = (z^\alpha / \zeta^{\alpha + \mathbf{1}}) f(\zeta)$. □

Lemma 2.5. *For any $z \in \overline{P}(0, \mathbf{r})$, the series $\sum_{|\alpha| \geq 0} f_\alpha(z, \zeta)$ is convergent in the topology of $\mathfrak{X}$, uniformly with respect to $\zeta \in T_\rho$.*

Proof. Let $\mathscr{B}$ be a balanced local base (of neighborhoods of the origin in $\mathfrak{X}$). As every $\Lambda \circ f$ ($\Lambda \in \mathfrak{X}^*$) is continuous, the set $f(T_\rho)$ is weakly bounded and then bounded in $\mathfrak{X}$. Hence, for any $V \in \mathscr{B}$, there is $s > 0$ such that $f(T_\rho) \subset tV$ for any $t \geq s$. Let us set

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3 Cf. Narasimhan, [19], p. 43, for $\mathfrak{X} = \mathbb{C}$.

$$s_\nu(z,\zeta)=\sum_{|\alpha|=0}^{\nu}\frac{z^\alpha}{\zeta^{\alpha+\mathbf{1}}},\quad S_\nu(z,\zeta)=\sum_{|\alpha|=0}^{\nu}f_\alpha(z,\zeta).$$

As $\sum_{|\alpha|\geq 0}z^\alpha/\zeta^{\alpha+\mathbf{1}}$ is convergent (uniformly for $\zeta\in T_\rho$ and $z\in\overline{P}(0,\mathbf{r})$), there is $N\geq 1$ such that

$$|R_{\nu\mu}(z,\zeta)|<\frac{1}{s},\ \nu\geq\mu>N,\ \zeta\in T_\rho,\ |z_j|\leq r,$$

where $R_{\nu\mu}=s_\nu-s_\mu$. Next, as V is balanced

$$R_{\nu\mu}(z,\zeta)f(\zeta)\in R_{\nu\mu}(z,\zeta)f(T_\rho)\subset R_{\nu\mu}(z,\zeta)sV\subset V,$$
$$\nu\geq\mu>N,\ \zeta\in T_\rho,\ z\in\overline{P}(0,\mathbf{r}).$$

Hence, for each $z\in\overline{P}(0,\mathbf{r})$, the sequence $\{S_\nu(z,\zeta)\}_{\nu\geq 0}$ is Cauchy in $\mathfrak{X}$ uniformly for $\zeta\in T_\rho$. □

By Lemma 5, we may integrate $\sum_{|\alpha|\geq 0}f_\alpha(z,\zeta)$ term by term (integration with respect to $\zeta\in T_\rho$) and obtain (by (2.6)):

$$f(z)=\sum_{|\alpha|=0}^{\infty}z^\alpha x_\alpha,\ z\in\overline{P}(0,\mathbf{r}),\ 0<r\leq\rho,$$
$$x_\alpha=\frac{1}{(2\pi i)^\alpha}\int_{|\zeta_n|=\rho}\mathrm{d}\zeta_n\int_{|\zeta_{n-1}=\rho}\mathrm{d}\zeta_{n-1}\dots\int_{\zeta_1|=\rho}(1/\zeta^{\alpha+\mathbf{1}}f(\zeta)\mathrm{d}\zeta_1\in\mathfrak{X}.$$

To end the proof of Theorem 2.4, we need to establish the following:

Lemma 2.6. *The series $\sum_{|\alpha|\geq 0}z^\alpha x_\alpha$ converges in $\mathfrak{X}$ uniformly for $z\in\overline{P}(0,\mathbf{r})$.*

Proof. Let $\mathscr{P}$ be a separating family of seminorms, determining the topology of $\mathfrak{X}$ as a locally convex space. Let us set

$$V(\mathbf{p},k)=\left\{x\in\mathfrak{X}:\mathbf{p}(x)<\frac{1}{k}\right\},\ \mathbf{p}\in\mathscr{P},\ k\in\mathbb{Z},\ k\geq 1.$$

The collection $\mathscr{B}$ of all finite intersections of the sets $V(\mathbf{p},k)$ is a convex balanced local base of $\mathfrak{X}$. Let $\mathbf{p}\in\mathscr{P}$ and $k\in\mathbb{Z}$, $k\geq 1$. As $f(T_\rho)$ is bounded in $\mathfrak{X}$, there is $s>0$ such that $f(T_\rho)\subset sV(\mathbf{p},k)$, i.e., $\mathbf{p}[f(\zeta)]<s/k$ for any $\zeta\in T_\rho$. Then (by Theorem A1 in Appendix A to the present article),

$$\mathbf{p}(x_\alpha)\leq\frac{1}{(2\pi)^n}\int_{|\zeta_n|=\rho}\mathrm{d}\zeta_n\int_{|\zeta_{n-1}|=\rho}\mathrm{d}\zeta_{n-1}\dots\int_{|\zeta_1|=\rho}\frac{\mathbf{p}[f(\zeta)]}{|\zeta^{\alpha+\mathbf{1}}|}\mathrm{d}\zeta_1\leq\frac{s}{k(2\pi)^n}\int_{T_\rho}\frac{\mathrm{d}\zeta}{\zeta^{\alpha+\mathbf{1}}}=\frac{s}{k}\rho^{-|\alpha|},$$

and hence,

$$\mathbf{p}\left(\sum_{|\alpha|=\mu+1}^{\nu}z^\alpha x_\alpha\right)\leq\sum_{|\alpha|=\mu+1}^{\nu}|z^\alpha|\mathbf{p}(x_\alpha)\leq\frac{s}{k}\sum_{|\alpha|=\mu+1}^{\nu}\prod_{j=1}^{n}|z_j|^{\alpha_j}\rho^{-|\alpha|}\leq\frac{s}{k}\sum_{|\alpha|=\mu+1}^{\nu}\left(\frac{r}{\rho}\right)^{|\alpha|}<\frac{1}{k}$$

for some $N\geq 1$ and any $\nu>\mu\geq N$. □

3 Weierstrass theorem

Let $\mathfrak{X}$ be a complex Fréchet space, $\Omega\subset\mathbb{C}^n$ an open set, and $\{f_\nu\}_{\nu\geq 0}\subset O(\Omega,\mathfrak{X})$ a sequence of holomorphic functions. We establish the following vector-valued analog to Weierstrass' theorem.

Theorem 3.1. *If $\{f_\nu\}_{\nu\geq 1}$ converges uniformly on any compact subset of Ω, then the pointwise limit $f=\lim_{\nu\to\infty}f_\nu$ is holomorphic in Ω, i.e., $f\in O(\Omega,\mathfrak{X})$. Moreover, for every $\alpha\in\mathbb{Z}_+^n$, the sequence $\{D^\alpha f_\nu\}_{\nu\geq 0}$ converges to $D^\alpha f$ uniformly on compact subsets of Ω.*

Let $a \in \Omega$ and let $\mathbf{r} = (r_1, \ldots, r_n)$ be a polyradius such that $\overline{P(a, \mathbf{r})} \subset \Omega$. We need

Lemma 3.2. $f \in C(\overline{P(a, \mathbf{r})}, \mathfrak{X})$.

Proof. Any strongly holomorphic function is weakly holomorphic, hence (by Theorem 2.1) each f_ν is strongly continuous, and in particular, $f_\nu : \overline{P(a, \mathbf{r})} \to \mathfrak{X}$ is strongly continuous. We claim that $f_\nu : \overline{P(a, \mathbf{r})} \to \mathfrak{X}$ is also strongly bounded. Indeed, as any $\Lambda \in \mathfrak{X}^*$ is continuous, the (scalar valued) function $\Lambda \circ f_\nu : \overline{P(a, \mathbf{r})} \to \mathbb{C}$ is continuous on the compact set $\overline{P(a, \mathbf{r})}$, and hence, the set $\{\Lambda[f_\nu(z)] : z \in \overline{P(a, \mathbf{r})}\}$ is bounded, which is to say that the set $\{f_\nu(z) : z \in \overline{P(a, \mathbf{r})}\}$ is weakly bounded. Any weakly bounded subset of a locally convex space is strongly bounded, so the claim is proved. As $\mathfrak{X}$ is a Fréchet space, its topology is compatible with some invariant metric. So $f_\nu : \overline{P(a, \mathbf{r})} \to \mathfrak{X}$, $\nu \geq 0$, is a sequence of continuous and bounded mappings of metric spaces, converging uniformly to $f|_{\overline{P(a,\mathbf{r})}}$. Consequently, $f|_{\overline{P(a,\mathbf{r})}}$ is continuous. □

By the Cauchy integral formula,

$$2\pi i f_\nu(z) = \int_{|\zeta_n|=r_n} d\zeta_n \int_{|\zeta_{n-1}|=r_{n-1}} d\zeta_{n-1} \ldots \int_{|\zeta_1|=r_1} \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} f_\nu(\zeta_1, \ldots \zeta_n) d\zeta_1 \tag{3.1}$$

for any $z \in P(a, \mathbf{r})$.

Lemma 3.3. *For any* $z \in P(a, \mathbf{r})$,

$$\lim_{\nu\to\infty} \int_{\partial_0 P(a,\mathbf{r})} F_\nu(\zeta) d\zeta = \int_{\partial_0 P(a,\mathbf{r})} F(\zeta) d\zeta,$$

where

$$F_\nu(\zeta) = \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} f_\nu(\zeta), \; F(\zeta) = \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} f(\zeta).$$

Proof. If X and Y are metric spaces, let $C(X, Y)$ denote the metric space of all continuous and bounded functions $F : X \to Y$ with the distance function $d_\infty(F, G) = \sup_{x\in X} d_Y(F(x), G(x))$. The uniform convergence $f_\nu \to f$ on compact subsets of Ω yields convergence of F_ν to F in $C(\partial_0 P(a, \mathbf{r}), \mathfrak{X})$, i.e., $d_\infty(F_\nu, F) \to 0$ as $\nu \to \infty$. Let $\mathscr{P} = \{\mathbf{p}_m : m \geq 1\}$ be a countable separating family of seminorms on $\mathfrak{X}$ determining the topology of $\mathfrak{X}$, and let us set

$$d_{\mathfrak{X}}(x, y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\mathbf{p}_m(x - y)}{1 + \mathbf{p}_m(x - y)}, \; x, y \in \mathfrak{X},$$

so that $d_{\mathfrak{X}}$ is an invariant metric on $\mathfrak{X}$ compatible to the topology of $\mathfrak{X}$. For any $\varepsilon > 0$, there is $\nu_\varepsilon \in \mathbb{Z}_+$ such that for any $\nu \geq \nu_\varepsilon$

$$\varepsilon > d_\infty(F_\nu, F) = \sup_{z\in\partial_0 P(a,\mathbf{r})} d_{\mathfrak{X}}(F_\nu(z), F(z)),$$

i.e., for every $z \in \partial_0 P(a, \mathbf{r})$,

$$\varepsilon > d_{\mathfrak{X}}(F_\nu(z), F(z)) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\mathbf{p}_m(F_\nu(z), F(z))}{1 + \mathbf{p}_m(F_\nu(z), F(z))} = \sup\{S_m(\nu, z) : m \geq 1\},$$

where $S_m(\nu, z) := \sum_{k=1}^{m} 2^{-k} \mathbf{p}_k(F_\nu(z), F(z))[1 + \mathbf{p}_k(F_\nu(z), F(z))]^{-1}$, that is, for any $\nu \geq \nu_\varepsilon$, any $z \in \partial_0 P(a, \mathbf{r})$, and any $m \in \mathbb{N}$

$$\mathbf{p}_m(F_\nu(z) - F(z)) < \frac{2^m \varepsilon}{1 - 2^m \varepsilon}$$

yielding

$$\mathbf{p}_m\left(\int\limits_{\partial_0 P(a,\mathbf{r})} F_\nu(\zeta)\mathrm{d}\zeta - \int\limits_{\partial_0 P(a,\mathbf{r})} F(\zeta)\mathrm{d}\zeta\right) \le \int\limits_{\partial_0 P(a,\mathbf{r})} \mathbf{p}_m[F_\nu(\zeta) - F(\zeta)]\mathrm{d}\zeta \le \frac{2^m\varepsilon}{1-2^m\varepsilon}|\partial_0 P(a,\mathbf{r})|.$$

Here, $|A|$ is the Lebesgue measure of $A \subset \mathbb{R}^{2n}$. Let $N \in \mathbb{N}$ and let

$$\varepsilon(m,N) := \frac{2^{-(m+1)}}{1+N|\partial_0 P(a,\mathbf{r})|},\ \nu(m,N) := \nu_{\varepsilon(m,N)},$$

so that

$$\int\limits_{\partial_0 P(a,\mathbf{r})} F_\nu(\zeta)\mathrm{d}\zeta - \int\limits_{\partial_0 P(a,\mathbf{r})} F(\zeta)\mathrm{d}\zeta \in V(\mathbf{p}_m, N)$$

for any $m, N \in \mathbb{N}$ and any $\nu \ge \nu(m,N)$. However, as well known, $\{V(\mathbf{p},N) : \mathbf{p} \in \mathscr{P}, N \in \mathbb{N}\}$ is not a local base for the topology of $\mathfrak{X}$. Instead, the family $\mathscr{B}$ of all finite intersections $B = V(p_{m_1}, N_1) \cap \ldots \cap V(\mathbf{p}_{m_k}, N_k)$ is a local base. Let $\nu(B) := \max\{\nu(m_i, N_i) : 1 \le i \le k\}$. Then

$$\int\limits_{\partial_0 P(a,\mathbf{r})} F_\nu(\zeta)\mathrm{d}\zeta - \int\limits_{\partial_0 P(a,\mathbf{r})} F(\zeta)\mathrm{d}\zeta \in B$$

for any $\nu \ge \nu(B)$. □

Let us pass to the limit with $\nu \to \infty$ in (3.1). Then (by Lemma 3.3)

$$2\pi i f(z) = \int\limits_{|\zeta_n|=r_n} \mathrm{d}\zeta_n \int\limits_{|\zeta_{n-1}|=r_{n-1}} \mathrm{d}\zeta_{n-1} \ldots \int\limits_{|\zeta_1|=r_1} \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} f(\zeta_1, \ldots \zeta_n)\mathrm{d}\zeta_1, \tag{3.2}$$

and hence, $f \in \mathcal{O}(P(a,\mathbf{r}), \mathfrak{X})$. By (3.2), together with Lemma 8.6 [23], p. 29, f is holomorphic in each variable separately, and then (by Hartogs' theorem together with our Theorem 2.3) $f \in \mathcal{O}(P(a,\mathbf{r}), \mathfrak{X})$ for arbitrary $a \in \Omega$, and hence, $f \in \mathcal{O}(\Omega, \mathfrak{X})$, thus proving the first statement in Theorem 3.1.

Let now $K \subset\subset \Omega$ be a compact subset. There is a compact subset $A \subset \Omega$ such that $\mathring{A} \supset K$. As $f_\nu \to f$ for $\nu \to \infty$ uniformly on A, for any $\mathbf{p} \in \mathscr{P}$, and any $N \in \mathbb{N}$, there is $\nu(\mathbf{p}, N) \ge 0$ such that for any $\nu \ge \nu(\mathbf{p}, N)$, and any $z \in A$:

$$f_\nu(z) - f(z) \in V(\mathbf{p}, N)$$

or $\mathbf{p}(f_\nu(z) - f(z)) < 1/N$, which yields

$$\|f_\nu - f\|_{\mathbf{p},A} < \frac{1}{N},\ \nu \ge \nu(\mathbf{p}, N).$$

Here, for every $F \in C(\Omega, \mathfrak{X})$, we adopted the notation:

$$\|F\|_{\mathbf{p},K} = \sup_{z\in K} \mathbf{p}(F(z)).$$

We also write briefly $\|\cdot\|_{m,K} = \|\cdot\|_{\mathbf{p}_m,K}$ [so that $\{\|\cdot\|_{n,K} : m \in \mathbb{N}, K \subset\subset \Omega\}$ is a family of seminorms on $C(\Omega, \mathfrak{X})$ organizing it as a complex Fréchet space]. Let $a \in K$ and let $\mathbf{r} = (r_1, \ldots, r_n)$ be a polyradius such that $\overline{P(a,\mathbf{r})} \subset \mathring{A}$. Then

$$f_\nu, f \in \mathcal{O}(P(a,\mathbf{r}), \mathfrak{X}) \cap C(\overline{P(a,\mathbf{r})}, \mathfrak{X}).$$

Lemma 3.4. *For every $F \in \mathcal{O}(P(a,\mathbf{r}), \mathfrak{X}) \cap C(\overline{P(a,\mathbf{r})}, \mathfrak{X})$,*

$$F(z) = \sum_{\alpha\in\mathbb{Z}_+} \frac{(z-a)^\alpha}{\alpha!}(D^\alpha F)(a),\ z \in P(a,\mathbf{r}). \tag{3.3}$$

Moreover, for every $\alpha \in \mathbb{Z}_+^n$, *one has* $D^\alpha F \in O(P(a, \mathbf{r}), \mathfrak{X})$ *and*

$$\mathbf{p}_m[D^\alpha F)(a)] \le \frac{\alpha!}{\mathbf{r}^\alpha} \, \|F\|_{m,K}, \tag{3.4}$$

where $K = \overline{P(a, \mathbf{r})}$.

Statement (3.3) for $n = 1$ is Lemma 8.6 in [23], p. 29 (an immediate consequence of the Cauchy formula for $\mathfrak{X}$ valued holomorphic functions of one complex variable), while statement (3.4) for $n = 1$ is formula (8.3) in [23], p. 29. Similarly (3.3) and (3.4) follow from the Cauchy integral formula (2.5) in Theorem 2.2. The proof is left as an exercise to the reader.

By Lemma 3.4, one has $D^\alpha f_\nu, D^\alpha f \in O(P(a, \mathbf{r}), \mathfrak{X})$ and

$$\mathbf{p}_m[(D^\alpha f_\nu)(a) - (D^\alpha f)(a)] \le \frac{\alpha!}{\mathbf{r}^\alpha} \, \|f_\nu - f\|_{m,\overline{P(a,\mathbf{r})}} \le \frac{\alpha!}{\mathbf{r}^\alpha} \, \|f_\nu - f\|_{m,A} < \frac{1}{N}$$

for any $\nu \ge \nu(\mathbf{p}_m, N)$.

4 Cauchy problems *versus* integro-differential equations

Let $\mathfrak{X}$ be a complex topological vector space. Let $\Omega \subset \mathbb{R}^n$ be an open set with $0 \in \Omega$, and let $P(x, D)u \equiv \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u$ be a PDO of order m, with coefficients $a_\alpha : \Omega \to \mathbb{C}$. Let $f : \Omega \to \mathfrak{X}$ and $\varphi_j : \Omega_0 \to \mathfrak{X}$, $0 \le j \le m - 1$, where $\Omega_0 = \Omega \cap \{x_n = 0\}$. The *Cauchy problem* is to find a neighborhood of the origin $U \subset \Omega$ and a solution $u : U \to \mathfrak{X}$ to $P(x, D)u = f$ in U such that

$$\left. \frac{\partial^j u}{\partial x_n^j} \right|_{x_n = 0} = \varphi_j \quad \text{on } U_0 = U \cap \{x_n = 0\}, \, 0 \le j \le m - 1. \tag{4.1}$$

We seek to establish a vector-valued analog to the classical Cauchy-Kovalevskaja theorem, i.e.,

Theorem 4.1. *Let* $\mathfrak{X}$ *be a complex Fréchet space,* $\Omega \subset \mathbb{R}^n$ *an open neighborhood of the origin, and let* $f \in C^\omega(\Omega, \mathfrak{X})$ and $\varphi_j \in C^\omega(\Omega_0, \mathfrak{X})$, $0 \le j \le m - 1$, *be given functions. If* $a_\alpha \in C^\omega(\Omega)$ *and*

$$a_{(0,\dots,0,m)}(0) \ne 0, \tag{4.2}$$

then there is a neighborhood of the origin $0 \in U \subset \Omega$ *and a unique solution* $u \in C^\omega(\Omega, \mathfrak{X})$ *to the Cauchy problem* (4.1) *for* $P(x, D)u = f$.

As well as in the classical scalar valued case, it suffices to prove Theorem 4.1 for $\varphi_j \equiv 0$, $0 \le j \le m - 1$, and $a_{(0,\dots,0,m)} \equiv 1$. More important, the Cauchy problem (4.1) may be replaced with a formally similar problem in the holomorphic category. Indeed let $u \in C^\omega(U, \mathfrak{X})$ be a solution to

$$P(x, D)u = f \quad \text{in } U, \, \frac{\partial^j u}{\partial x_n^j} = 0 \quad \text{on } U_0, \quad 0 \le j \le m - 1. \tag{4.3}$$

There is an open set $\tilde{\Omega} \subset \mathbb{C}^n$ such that $\tilde{\Omega}_{\mathbb{R}} = \Omega$, and there are holomorphic extensions to $\tilde{\Omega}$ of the real analytic functions a_α and f, i.e., functions $\tilde{a}_\alpha \in O(\tilde{\Omega})$ and $\tilde{f} \in O(\tilde{\Omega}, \mathfrak{X})$ such that $\tilde{a}_\alpha|_\Omega = a_\alpha$ and $\tilde{f}|_\Omega = f$. Similarly, there is a connected open neighborhood of the origin $\tilde{U} \subset \tilde{\Omega}$ and a holomorphic function $\tilde{u} \in O(\tilde{U}, \mathfrak{X})$ such that $\tilde{U}_{\mathbb{R}} = U$ and $\tilde{u}|_U = u$. Throughout, if $A \subset \mathbb{C}^n$ is a set, then $A_{\mathbb{R}} = \{(z_1, \dots, z_n) \in A : z_j \in \mathbb{R}, 1 \le j \le n\}$. Then

$$P(z, D_z)\tilde{u} = \tilde{f} \quad \text{in } \tilde{U}, \, \frac{\partial^j \tilde{u}}{\partial z_n^j} = 0 \quad \text{on } \tilde{U}_0, \quad 0 \le j \le m - 1, \tag{4.4}$$

where $\tilde{U}_0 = \tilde{U} \cap \{z_n = 0\}$ and

$$P(z, D_z) \equiv \sum_{|\alpha| \le m} \tilde{a}_\alpha(z) D_z^\alpha, \, D_z^\alpha \equiv \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}.$$

Viceversa, if $\tilde{u} \in \mathcal{O}(\tilde{U}, \mathfrak{X})$ is a solution to (4.4) for some open set $0 \in \tilde{U} \subset \Omega$, then $u = \tilde{u}\,|_U$ with $U := \tilde{U}_{\mathbb{R}}$ is a solution to (4.3), where $a_\alpha = \tilde{a}_\alpha\,|_U$ and $f = \tilde{f}\,|_U$.

Let $\tilde{u} \in \mathcal{O}(\tilde{U}, \mathfrak{X})$ be a solution to (4.4) with $\tilde{a}_{(0,\ldots,m)} \equiv 1$. We may assume w.l.o.g. that the set $\tilde{U}$ is balanced in the z_n variable, i.e., $(z', tz_n) \in \tilde{U}$ for any $z \in \tilde{U}$ and any $|t| \leq 1$. Here, $z' = (z_1, \ldots, z_{n-1})$ so that $z = (z', z_n)$. Let us consider the C^∞ function $\psi : [-1, 1] \to \mathfrak{X}$ given by

$$\psi(t) = \tilde{u}(z', tz_n),\ |t| \leq 1.$$

As $(\partial^j \tilde{u}/\partial z_n^j)(z', 0) = 0$ for any $0 \leq j \leq m-1$, the truncated Taylor development of ψ (with the rest in Bochner integral form, cf. Appendix B to this article) is expressed as follows:

$$\psi(t) = \sum_{j=0}^{m-1} \frac{t^j}{j!}\frac{\mathrm{d}^j\psi}{\mathrm{d}t^j}(0) + R_{m-1}(t;\,0) = R_{m-1}(t;\,0),$$

$$R_{m-1}(t;\,0) = \frac{1}{(m-1)!}\int_0^t (t-\tau)^{m-1}\frac{\mathrm{d}^m\psi}{\mathrm{d}t^m}(\tau)\mathrm{d}\tau,\ |t| \leq 1.$$

Moreover, $\psi(1) = \tilde{u}(z)$ and $\psi(1) = R_{m-1}(1;\,0)$ imply

$$\tilde{u}(z) = \frac{z_n^m}{(m-1)!}\int_0^1 (1-t)^{m-1} v(z', tz_n)\mathrm{d}t, \tag{4.5}$$

where we have set for simplicity

$$v \in \mathcal{O}(\tilde{U}, \mathfrak{X}),\ v(z) = \frac{\partial^m \tilde{u}}{\partial z_n^m}(z),\ z \in \tilde{U}.$$

It will be useful to observe that the functions $J_s : \tilde{U} \to \mathfrak{X}$, $s \geq 0$,

$$J_s(z) = \int_0^1 (1-t)^{m-1} v(z', tz_n),\ z \in \tilde{U},$$

satisfy the following recurrence identities

Lemma 4.2. *For any $s \geq 1$,*

$$z_n \frac{\partial J_s}{\partial z_n}(z) = sJ_{s-1}(z) - (1+s)J_s(z). \tag{4.6}$$

Proof. For any fixed $z \in \tilde{U}$, let us consider the function $\varphi : [0, 1] \to \mathfrak{X}$ given by $\varphi(t) = v(z', tz_n)$. Then

$$J_s(z) = \int_0^1 (1-t)^s \varphi(t)\mathrm{d}t,\ \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) = z_n \frac{\partial v}{\partial z_n}(z', tz_n),$$

so that

$$\frac{\partial J_s}{\partial z_n}(z) = \frac{1}{z_n}\int_0^1 t(1-t)^s \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t)\mathrm{d}t.$$

Finally, integration by parts [together with the elementary identity $\frac{\mathrm{d}}{\mathrm{d}t}\{t(1-t^s)\} = (1-t)^{s-1} - (1+s)t(1-t)^{s-1}$] yields (4.6). □

Formula (4.5) reads

$$\tilde{u}(z) = \frac{z_n^m}{(m-1)!} J_{m-1}(z),\ z \in \tilde{U},$$

hence (by Lemma 4.2)

$$\frac{\partial \tilde{u}}{\partial z_n}(z) = \frac{z_n^{m-1}}{(m-2)!} J_{m-2}(z).$$

The iterative calculation of partials leads to

$$\frac{\partial^j \tilde{u}}{\partial z_n^j}(z) = \frac{z_n^{m-j}}{(m-j-1)!} J_{m-j-1}(z),\ 0 \le j \le m-1,$$

or

$$\frac{\partial^j \tilde{u}}{\partial z_n^j}(z) = \frac{z_n^{m-j}}{(m-j-1)!} \int_0^1 (1-t)^{m-j-1} v(z', tz_n)\,\mathrm{d}t \tag{4.7}$$

for any $0 \le j \le m-1$. Next, let us consider the linear operator

$$T : O(\tilde{U}, \mathfrak{X}) \to O(\tilde{U}, \mathfrak{X}),$$
$$Tg \equiv -\sum_{j=0}^{m-1} \sum_{|\beta| \le m-j} \tilde{a}_{\beta j}(z) \frac{z_n^{m-j}}{(m-j-1)!} \int_0^1 (1-t)^{m-j-1} (D_{z'}^{\beta} g)(z', tz_n)\,\mathrm{d}t$$

for any $g \in O(\tilde{U}, \mathfrak{X})$, where

$$D_{z'}^{\beta} \equiv \frac{\partial^{|\beta|}}{\partial z_1^{\beta_1} \dots \partial z_{n-1}^{\beta_{n-1}}},\ \beta \in \mathbb{Z}_+^{n-1}.$$

Then

$$\tilde{f} = P(z, D_z)\tilde{u} = \sum_{|\alpha| \le m} \tilde{a}_{\alpha}(z) \frac{\partial^{|\alpha|} \tilde{u}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} =$$

(by $\tilde{a}_{(0,\dots,0,m)} \equiv 1$)

$$= \frac{\partial^m \tilde{u}}{\partial z_n^m} + \sum_{j=0}^{m-1} \sum_{|\beta| \le m-j} \tilde{a}_{\beta j}(z) \frac{\partial^{|\beta|+j} \tilde{u}}{\partial z_1^{\beta_1} \dots \partial z_{n-1}^{\beta_{n-1}} \partial z_n^j} = v + \sum_{j=0}^{m-1} \sum_{|\beta| \le m-j} \tilde{a}_{\beta j}(z) \frac{\partial^{|\beta|}}{\partial z_1^{\beta_1} \dots \partial z_{n-1}^{\beta_{n-1}}} \left(\frac{\partial^j \tilde{u}}{\partial z_n^j} \right) =$$

(by (4.7))

$$= v + \sum_{j=0}^{m-1} \sum_{|\beta| \le m-j} \tilde{a}_{\beta j}(z) \frac{z_n^{m-j}}{(m-j-1)!} \int_0^1 (1-t)^{m-j-1} (D_{z'}^{\beta} v)(z', tz_n)\,\mathrm{d}t = v - Tv.$$

Therefore, for every solution $v \in O(\tilde{U}, \mathfrak{X})$ to the equation

$$v - Tv = \tilde{f}, \tag{4.8}$$

the function $\tilde{u}$ defined by (4.5) is a solution to the Cauchy problem (4.4) with $\tilde{a}_{(0,\dots,0,m)} \equiv 1$.

5 Cauchy-Kovalevskaja theorem

We need the following:

Lemma 5.1. *Let $m \in \mathbb{N}$ be a positive integer, and let $C > 0$ and $\gamma \in (0,1)$ be constants. There is $\varepsilon \in (0,1)$ such that*

$$C|z_n| \left(1 - \max_{1 \le j \le n} |z_j| \right)^{-m} \le \gamma \tag{5.1}$$

for any $z \in \overline{B_{\varepsilon}(0)}$.

Proof. Note that

$$z \in \overline{B_\varepsilon(z)} \Rightarrow \max_{1\le j\le n}|z_j| < \varepsilon \Rightarrow 1 - \max_{1\le j\le n}|z_j| \ge 1-\varepsilon \Rightarrow C|z_n|\left(1 - \max_{1\le j\le n}|z_j|\right)^{-m} \le C\varepsilon(1-\varepsilon)^{-m}.$$

Let us consider the monotonously increasing function $\phi : (0,1) \to (0,+\infty)$ given by $\phi(\varepsilon) = C\varepsilon(1-\varepsilon)^{-m}$. In particular ϕ is bijective, hence for any $\gamma \in (0,1)$, there is a unique $\varepsilon_\gamma \in (0,1)$ such that $\phi(\varepsilon_\gamma) = \gamma$. Therefore, for every $z \in \overline{B_\varepsilon(0)}$,

$$C|z_n|\left(1 - \max_{1\le j\le n}|z_j|\right)^{-m} \le C\varepsilon_\gamma(1-\varepsilon_\gamma)^{-m} = \phi(\varepsilon_\gamma) = \gamma.$$

□

Let $\mathscr{P}$ be a separating family of seminorms on $\mathfrak{X}$ determining its topology as a locally convex space. By Theorem 1.37 in [21], pp. 27–28

(a) Every $\mathbf{p} \in \mathscr{P}$ is continuous,

(b) A set $E \subset \mathfrak{X}$ is bounded $\Leftrightarrow$ every $\mathbf{p} \in \mathscr{P}$ is bounded on E.

Lemma 5.2. *Let* $\mathbf{p} \in \mathscr{P}$ *and let* $f \in O(\mathbb{B}^n, \mathfrak{X})$ *such that*

$$\mathbf{p}(f(z)) \le \frac{B}{(1 - \max_{1\le j\le n}|z_j|)^b} \tag{5.2}$$

for some constants $b > 0$ *and* $B > 0$, *and every* $z \in \mathbb{B}^n$. *Then for every multi-index* $\alpha \in \mathbb{Z}_+^n$ *and every point* $z \in \mathbb{B}^n$,

$$\mathbf{p}[(D_z^\alpha f)(z)] \le \frac{Be^{|\alpha|}(b+1)\ldots(b+|\alpha|)}{(1 - \max_{1\le j\le n}|z_j|)^{b+|\alpha|}}. \tag{5.3}$$

Proof. It suffices to prove (5.3) for $|\alpha| = 1$ and apply the resulting inequality to the derivatives of f. Also, we may assume w.l.o.g. that $\alpha = (0,\ldots,0,1) \in \mathbb{Z}_+^n$. For every $z \in \mathbb{B}^n$, there is a polyradius $\boldsymbol{\rho} = (\rho_1,\ldots,\rho_n)$ such that

$$\sum_{j=1}^n \rho_j^2 < 1,\ z \in P(0,\boldsymbol{\rho}),\ \overline{P(0,\boldsymbol{\rho})} \subset \mathbb{B}^n.$$

Let us set $\varepsilon_j \coloneqq \rho_j - |z_j|$ so that $\boldsymbol{\varepsilon} \coloneqq (\varepsilon_1,\ldots,\varepsilon_n)$ lies on the cube I^n, where $I = (0,1)$. Therefore, $P(z,\boldsymbol{\varepsilon}) \subset P(0,\boldsymbol{\rho})$. Next, let us set

$$m_z \coloneqq \max_{1\le j\le n}|z_j|,\ \varepsilon \coloneqq \varepsilon_n = \frac{1-m_z}{1+b},\ \rho = \rho_n = |z_n| + \frac{1-m_z}{1+b},$$

so that

$$\sum_{j=1}^{n-1}\rho_j^2 = \sum_{j=1}^{n-1}(|z_j| + \varepsilon_j)^2 < 1 - \left(|z_n| + \frac{1-m_z}{1+b}\right)^2,$$

and moreover, $z \in D_\rho(0)$ and $\overline{D_\varepsilon(z_n)} \subset \overline{D_\rho(0)} \subset D_1(0)$. Here, $D_r(\zeta_0) = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < r\}$. Let us consider the planar domain

$$D_{\rho,\varepsilon} = \{\zeta \in \mathbb{C} : |\zeta| < \rho,\ |\zeta - z_n| > \varepsilon\}.$$

Then $\overline{D_{\rho,\varepsilon}} \subset D_1(0)$ and

$$\partial D_{\rho,\varepsilon} = \{\zeta \in \mathbb{C} : |\zeta| = \rho\} \cup \{\zeta \in \mathbb{C} : |\zeta - z_n| = \varepsilon\} = \partial D_\rho(0) \cup \partial D_\varepsilon(z_n) \subset \partial D_1(0).$$

Let us consider the function $f_{z'} : \overline{D_\rho(0)} \to \mathfrak{X}$ given by $f(\zeta) = f(z',\zeta)$. Then $f_{z'} \in C(\overline{D_\rho(0)}, \mathfrak{X}) \cap O(D_\rho(0), \mathfrak{X})$ hence (by the Cauchy formula)

$$f(z',\zeta) = \frac{1}{2\pi i}\int_{\partial D_\rho(0)} \frac{1}{\zeta - z_n} f(z',\zeta)\,d\zeta,$$

and then

$$\frac{\partial f}{\partial z_n}(z) = \frac{1}{2\pi i} \int\limits_{\partial D_\rho(0)} \frac{1}{(\zeta - z_n)^2} f(z', \zeta) \mathrm{d}\zeta. \tag{5.4}$$

Let us set

$$\omega = \frac{1}{(\zeta - z_n)^2} f(z', \zeta) \mathrm{d}\zeta.$$

Then $\omega \in C^\infty(T^*(M) \otimes \mathfrak{X})$, i.e., ω is a $\mathfrak{X}$ valued differential 1-form on $M = D_1(0) \backslash \{z_n\}$. As $\zeta \mapsto \frac{1}{(\zeta - z_n)^2} f(z', \zeta)$ is holomorphic, the differential 1-form ω is closed, i.e., $\mathrm{d}\omega = 0$. Consequently (by the Stokes theorem),

$$0 = \int\limits_{D_{\rho,\varepsilon}} \mathrm{d}\omega = \int\limits_{\partial D_{\rho,\varepsilon}} \omega \Rightarrow \int\limits_{D_1(\rho)} \omega = \int\limits_{D_\varepsilon(z_n)} \omega$$

and (5.4) becomes

$$\frac{\partial f}{\partial z_n}(z) = \frac{1}{2\pi i} \int\limits_{D_\varepsilon(z_n)} \frac{1}{(\zeta - z_n)^2} f(\zeta', z_n) \mathrm{d}\zeta$$

or (with $\zeta = z_n + \varepsilon e^{i\theta}$)

$$\frac{\partial f}{\partial z_n}(z) = \frac{1}{2\pi\varepsilon} \int\limits_0^{2\pi} e^{-i\theta} f(z', z_n + \varepsilon e^{i\theta}) \mathrm{d}\theta.$$

Let us set $z_\varepsilon(\theta) = (z', z_n + \varepsilon e^{i\theta})$ for the sake of brevity. Then

$$m_{z_\varepsilon(\theta)} \leq m_z + \varepsilon \tag{5.5}$$

and (by Theorem A1 in Appendix A, the assumption (5.2) in the current lemma, and (5.5))

$$\mathbf{p}\left[\frac{\partial f}{\partial z_n}(z)\right] \leq \frac{1}{2\pi\varepsilon} \int\limits_0^{2\pi} \mathbf{p}[f(z_\varepsilon(\theta))] \mathrm{d}\theta \leq \frac{1}{2\pi\varepsilon} \int\limits_0^{2\pi} \frac{B \mathrm{d}\theta}{[1 - m_{z_\varepsilon(\theta)}]^b} \leq \frac{B}{\varepsilon(1 - m_z - \varepsilon)^b}.$$

On the other hand,

$$(1 - m_z - \varepsilon)^b = \left(1 - m_z - \frac{1 - m_z}{1 + b}\right)^b = (1 - m_z)^b \left(1 - \frac{1}{1 + b}\right)^b \geq (1 - m_z)^b \left(1 - \frac{1}{b}\right)^b \geq \frac{1}{e}(1 - m_z)^b,$$

hence,

$$\mathbf{p}\left[\frac{\partial f}{\partial z_n}(z)\right] \leq \frac{Be(1 + b)}{(1 - m_z)^{b+1}}. \qquad \square$$

Let $\mathscr{U} \subset \mathbb{C}^n$ be an open set such that $\overline{\mathbb{B}^n} \subset \mathscr{U}$, and let $\mathscr{F}, v_0 \in O(\mathscr{U}, \mathfrak{X})$. Next, let $v_\nu, w_\nu \in O(\mathscr{U}, \mathfrak{X})$, $\nu \geq 1$, be defined recurrently by

$$v_{\nu+1} = Tv_\nu + \mathscr{F}, w_\nu = v_{\nu+1} - v_\nu, \nu \geq 0.$$

Lemma 5.3. *For any seminorm $\mathbf{p} \in \mathscr{P}$, nonnegative integer $\nu \in \mathbb{Z}_+$, and point $z \in \mathbb{B}^n$*

$$\mathbf{p}[w_\nu(z)] \leq M(\mathbf{p})[ae^m C(m)]^\nu |z_n|^\nu \left(1 - \max_{1 \leq k \leq n-1} |z_k|\right)^{-m\nu}, \tag{5.6}$$

$$M(\mathbf{p}) = \sup_{z \in \overline{\mathbb{B}^n}} \mathbf{p}[w_0(z)], \; C(m) = b_m \sum_{j=1}^m m^j,$$

$$a = \max_{0 \leq j \leq m-1} \max_{\beta \in B_j} \sup_{z \in \overline{\mathbb{B}^n}} |\tilde{a}_{\beta,j}(z)|,$$

$$b_m = \max_{0 \leq j \leq m-1} |B_j| = |B_0|, \; B_j \coloneqq \{\beta \in \mathbb{Z}_+^{n-1} : |\beta| \leq m - j\}.$$

Proof. The proof of Lemma 5.3 is by induction over $\nu \in \mathbb{Z}_+$. Let $P(\mathbf{p}, \nu, z)$ be the predicate in (5.6). Let $\mathbf{p} \in \mathscr{P}$. Then $\mathbf{p}[w_0(z)] \leq M(\mathbf{p})$ is equivalent to $P(\mathbf{p}, 0, z)$, so that $P(\mathbf{p}, 0, z)$ is true for any $z \in \mathbb{B}^n$. Let $\nu \in \mathbb{Z}_+$, $\nu \geq 1$, such that $P(\mathbf{p}, \nu, z)$ is true for any $\mathbf{p} \in \mathscr{P}$ and any $z \in \mathbb{B}^n$. Note that, independently from the induction hypothesis

$$w_{\nu+1} = v_{\nu+2} - v_{\nu+1} = Tv_{\nu+1} + \mathscr{F} - (Tv_\nu + \mathscr{F}) = T(v_{\nu+1} - v_\nu) = Tw_\nu,$$

hence for every $z \in \mathbb{B}^n$ (by the very definition of T)

$$w_{\nu+1}(z) = -\sum_{j=0}^{m-1} \sum_{\substack{\beta \in \mathbb{Z}_+^{n-1} \\ |\beta| \leq m-j}} \tilde{a}_{\beta,j}(z) \frac{z_n^{m-j}}{(m-j-1)!} \int_0^1 (1-t)^{m-j-1} (D_{z'}^\beta w_\nu)(z', tz_n) \mathrm{d}t.$$

Consequently,

$$\mathbf{p}[w_{\nu+1}(z)] \leq \sum_{j=0}^{m-1} \sum_{\substack{\beta \in \mathbb{Z}_+^{n-1} \\ |\beta| \leq m-j}} |\tilde{a}_{\beta,j}(z)| \frac{|z_n|^{m-j}}{(m-j-1)!} \int_0^1 (1-t)^{m-j-1} \mathbf{p}[(D_{z'}^\beta w_\nu)(z', tz_n)] \mathrm{d}t. \tag{5.7}$$

For any $z \in \mathbb{B}^n$ and any $0 < t < 1$,

$$\|(z', tz_n)\|^2 = \|z'\|^2 + t^2 |z_n|^2 < \|z'\|^2 + |z_n|^2 = \|z\|^2 < 1,$$

hence $\varphi(z') \coloneqq w_\nu(z', tz_n)$ is well defined and (by the induction hypothesis) $P(\mathbf{p}, \nu, (z', tz_n))$ is true, i.e.,

$$\mathbf{p}[\varphi(z')] \leq \frac{M(\mathbf{p})[ae^m C(m)]^\nu t^\nu |z_n|^\nu}{(1 - \max_{1 \leq k \leq n-1} |z_k|)^{m\nu}} \tag{5.8}$$

is true. The function $D_{z'}^\beta w_\nu$ is holomorphic in a neighborhood $\mathscr{U}$ of $\overline{\mathbb{B}^n}$, so in particular $D_{z'}^\beta w_\nu \in \mathcal{O}(\mathbb{B}^n, \mathfrak{X})$ for every $\beta \in \mathbb{Z}_+^{n-1}$. At this point, it should be observed that (5.8) is equivalent to the hypothesis (5.2) in Lemma 5.2, provided the data in Lemma 5.2 is replaced as follows:

$$\begin{pmatrix} \mathbb{B}^n \\ f \\ B \\ b \end{pmatrix} \longmapsto \begin{pmatrix} \mathbb{B}^{n-1} \\ \varphi \\ M(\mathbf{p})[ae^m C(m)]^\nu t^\nu |z_n|^\nu \\ m\nu \end{pmatrix}.$$

Therefore, we may apply Lemma 5.2, so that to conclude that (5.3) holds with the new data, i.e.,

$$\mathbf{p}[(D_{z'}^\beta w_\nu)(z', tz_n)] = \mathbf{p}[(D^\beta \varphi)(z')] \leq \frac{M(\mathbf{p})[ae^m C(m)]^\nu t^\nu |z_n|^\nu e^{|\beta|} (m\nu + 1) \ldots (m\nu + |\beta|)}{(1 - \max_{1 \leq k \leq n-1} |z_k|)^{m\nu + |\beta|}}. \tag{5.9}$$

Thus, (by (5.7) and (5.9))

$$\begin{aligned} \mathbf{p}[w_{\nu+1}(z)] &\leq M(\mathbf{p})[ae^m C(m)]^\nu \sum_{j=0}^{m-1} \sum_{\substack{\beta \in \mathbb{Z}_+^{n-1} \\ |\beta| \leq m-j}} |\tilde{a}_{\beta,j}(z)| \\ &\quad \times \frac{|z_n|^{m-j+\nu} e^{|\beta|}}{(1 - \max_{1 \leq k \leq n-1} |z_k|)^{m\nu + |\beta|}} \frac{(m\nu + 1) \ldots (m\nu + |\beta|)}{(m-j-1)!} \int_0^1 t^\nu (1-t)^{m-j-1} \mathrm{d}t \\ &= M(\mathbf{p})[ae^m C(m)]^\nu \sum_{j=0}^{m-1} \sum_{\substack{\beta \in \mathbb{Z}_+^{n-1} \\ |\beta| \leq m-j}} |\tilde{a}_{\beta,j}(z)| \\ &\quad \times \frac{|z_n|^{m-j+\nu} e^{|\beta|}}{(1 - \max_{1 \leq k \leq n-1} |z_k|)^{m\nu + |\beta|}} \frac{(m\nu + 1) \ldots (m\nu + |\beta|)}{(m-j-1)!} B(\nu + 1, m - j), \end{aligned}$$

where $B(r, s) = \int_0^1 t^{r-1}(1-t)^{s-1} \mathrm{d}t$ is the Euler function of the first kind, hence

$$B(\nu+1, m-j) = \frac{\Gamma(\nu+1)\Gamma(m-j)}{\Gamma(\nu+m-j+1)} = \frac{\nu!(m-j-1)!}{(\nu+m-j)!}.$$

Also

$$|z_n|^{\nu+m-j} \le |z_n|^{\nu+1},\ e^{|\beta|} \le e^m,$$
$$\left(1 - \max_{1\le k\le n-1}|z_k|\right)^{m\nu+|\beta|} \ge \left(1 - \max_{1\le k\le n-1}|z_k|\right)^{m\nu+m},$$

so that

$$\mathbf{p}[w_{\nu+1}(z)] \le \frac{aM(\mathbf{p})[ae^m C(m)]^\nu |z_n|^{\nu+1} e^m \xi_\nu}{(1-\max_{1\le k\le m-1}|z_k|)^{m(\nu+1)}}, \tag{5.10}$$

$$\xi_\nu := \sum_{j=0}^{m-1}\sum_{\beta\in B_j} \frac{\nu!(m\nu+1)\dots(m\nu+|\beta|)}{(\nu+m-j)!},\ \nu\ge 1.$$

We claim that the sequence $\{\xi_\nu\}_{\nu\ge1} \subset (0,+\infty)$ is bounded by the constant $C(m)$. Indeed

$$\begin{aligned} 0 < \xi_\nu &\le \sum_{j=0}^{m-1} |B_j| \frac{\nu!(m\nu+1)\dots(m\nu+m-j)}{(\nu+m-j)!} \\ &\le b_m \nu! \sum_{j=0}^{m-1} \frac{(m\nu+1)\dots(m\nu+m-j)}{(\nu+m-j)!} \\ &= b_m\left\{\frac{m\nu+1}{\nu+1} + \frac{m\nu+1}{\nu+1}\frac{m\nu+2}{\nu+2} + \dots + \frac{m\nu+1}{\nu+1}\frac{m\nu+2}{\nu+2}\dots\frac{m\nu+m}{\nu+m}\right\}. \end{aligned}$$

Let us consider the monotonously increasing functions $f_j : [1,+\infty) \to \mathbb{R}$ given by $f_j(x) = (mx+j)/(x+j)$ for $x \ge 1$ and $1 \le j \le m$, and let us note that $\lim_{x\to+\infty} f_j(x) = m$. Finally,

$$0 < \xi_\nu \le b_m \sum_{j=1}^m m^j = C(m). \qquad \square$$

Then (5.10) implies $P(\mathbf{p}, \nu+1, z)$, and we are done.

Let us recall that Lemma 5.1 builds a number $0 < \varepsilon < 1$ starting with the data (m, C, γ), where $m \in \mathbb{N}$, $C > 0$, and $0 < \gamma < 1$. An inspection of the proof of Lemma 5.1 shows that its output ε depends on the constant C. Let us apply Lemma 5.1 with the input data $(m, ae^m C(m), \gamma)$, where m is the order of the given PDO and $0 < \gamma < 1$ is arbitrary. By Lemma 5.1, there is $\varepsilon = \varepsilon_\gamma(m) \in (0,1)$ such that[4]

$$\frac{ae^m C(m)|z_n|}{(1-\max_{1\le k\le n}|z_k|)^m} \le \gamma \tag{5.11}$$

for any $z \in \overline{B_\varepsilon(0)}$. By Lemma 5.3, for any nonnegative integer $\nu \in \mathbb{Z}_+$ and any $z \in \mathbb{B}^n$

$$\mathbf{p}[w_\nu(z)] \le M(\mathbf{p})[ae^m C(m)]^\nu |z_n|^\nu \left(1 - \max_{1\le k\le n-1}|z_k|\right)^{-m\nu},$$

hence (by $(1-\max_{1\le k\le n-1}|z_k|)^{-1} \le (1-\max_{1\le k\le n}|z_k|)^{-1}$ together with (5.11))

$$\mathbf{p}[w_\nu(z)] \le M(\mathbf{p})\gamma^\nu$$

for any $z \in \overline{B_\varepsilon(0)}$. As $\sum_{\nu=0}^\infty \gamma^\nu < \infty$ the series $\sum_{\nu\ge0} w_\nu$ converges uniformly on $\overline{B_\varepsilon(0)}$ (cf. also Appendix C to this article). Let us set

$$w := \sum_{\nu=0}^\infty w_\nu,\ s_\nu := \sum_{\kappa=0}^\nu w_\kappa,$$

4 Unlike the notation suggests, the constant ε depends on the full differential operator $P(z, D_z)$ (rather than on its order alone).

so that $s_\nu = \sum_{\kappa=0}^{\nu}(v_{\kappa+1} - v_\kappa) = v_{\nu+1} - v_0$. Thus, for any $\mathbf{p} \in \mathscr{P}$ and any $N \geq 1$, there is $\nu(\mathbf{p}, N) \in \mathbb{Z}_+$ such that for any $\nu \geq \nu(\mathbf{p}, N)$ and any $z \in \overline{B_\varepsilon(0)}$,

$$v_{\nu+1}(z) - (v_0 + w)(z) = s_\nu(z) - w(z) \in V(\mathbf{p}, N),$$

i.e., the sequence $\{v_\nu\}_{\nu\geq 0}$ converges to $v \coloneqq v_0 + w$ uniformly on $\overline{B_\varepsilon(0)}$. In particular, $v \in C(\overline{B_\varepsilon(0)}, \mathfrak{X})$. Let $\boldsymbol{\varepsilon} \coloneqq (\varepsilon, \dots, \varepsilon) \in \mathbb{Z}_+^n$. Then $\overline{P(0, \boldsymbol{\varepsilon})} \subset \mathscr{U}$. Therefore,

$$v_\nu \in O(P(0, \boldsymbol{\varepsilon}), \mathfrak{X}) \cap C(\overline{P(0, \boldsymbol{\varepsilon})}, \mathfrak{X}), \nu \in \mathbb{Z}_+,$$

hence (by the Cauchy integral formula),

$$v_\nu(z) = \frac{1}{(2\pi i)^n} \int\limits_{\partial_0 P(0,\boldsymbol{\varepsilon})} \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} v_\nu(\zeta_1, \dots, \zeta_n) \mathrm{d}\zeta_1 \dots \mathrm{d}\zeta_n$$

for any $z \in P(0, \boldsymbol{\varepsilon})$. Passing to the limit with $\nu \to \infty$ yields

$$v(z) = \frac{1}{(2\pi i)^n} \int\limits_{\partial_0 P(0,\boldsymbol{\varepsilon})} \prod_{j=1}^{n} (\zeta_j - z_j)^{-1} v(\zeta_1, \dots, \zeta_n) \mathrm{d}\zeta_1 \dots \mathrm{d}\zeta_n \tag{5.12}$$

for any $z \in \overline{B_\varepsilon(0)}$. On the other hand, the right-hand side of (5.12) is a holomorphic function in $B_\varepsilon(0) \subset P(0, \boldsymbol{\varepsilon})$ so that $v \in O(B_\varepsilon(0))$. By (the vector-valued analog to) Weierstrass' theorem (i.e., Theorem 3.1) for every $\alpha \in \mathbb{Z}_+^n$, the sequence $\{D^\alpha v_\nu\}_{\nu\geq 0}$ converges to $D^\alpha v \in O(B_\varepsilon(0), \mathfrak{X})$, uniformly on any compact subset of $B_\varepsilon(0)$. Hence, (by the very definition of T),

$$\lim_{\nu\to\infty} T v_\nu = T v$$

on any compact subset of of $B_\varepsilon(0)$. Let $\delta > 0$ such that $\overline{B_\delta(0)} \subset B_\varepsilon(0)$. Then

$$v = \lim_{\nu\to\infty} v_{\nu+1} = \lim_{\nu\to\infty}(T v_\nu + \mathscr{F}) = T v + \mathscr{F}$$

on $B_\delta(0)$, that is $v \in O(B_\delta(0), \mathfrak{X})$ is the unique solution to the integro-differential equation (4.8) on $B_\delta(0)$.

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Appendix

A Vector-valued integration

Let $\mathfrak{X}$ be a topological vector space such that $\mathfrak{X}^*$ separates points on $\mathfrak{X}$. Let Q be a compact Hausdorff space and let μ be a Borel probability measure on Q. If $f : Q \to \mathfrak{X}$ is a continuous function and $\overline{\text{co}}[f(Q)]$ is compact in $\mathfrak{X}$ then there is a unique $y \in \overline{\text{co}}[f(Q)]$ such that

$$\Lambda y = \int_Q (\Lambda \circ f) \mathrm{d}\mu, \Lambda \in \mathfrak{X}^*.$$

$\int_Q f \mathrm{d}\mu \coloneqq y$ is the *Bochner integral* of f.

Theorem A1. *Let $\mathfrak{X}$ be a Fréchet space, and let $\mathscr{P}$ be a separating family of semi-norms on $\mathfrak{X}$ determining the topology of $\mathfrak{X}$ as a locally convex space. Let μ be a positive Borel measure on the compact Hausdorff space Q. Then for any continuous function $f : Q \to \mathfrak{X}$,*

$$\mathbf{p}\left(\int_Q f \mathrm{d}\mu\right) \leq \int_Q (\mathbf{p} \circ f) \mathrm{d}\mu, \mathbf{p} \in \mathscr{P}.$$

Theorem A1 is proved in [20] only when $\mathfrak{X}$ is a Banach space (cf. Theorem 3.29 in [20], p. 81). We provide a proof holding for every Fréchet space $\mathfrak{X}$. We make use of the following corollary to Hahn-Banach theorem (cf. Theorem 3.3 in [20], pp. 58–59).

Corollary A2. *Let $\mathfrak{X}$ be a locally convex space and let $x_0 \in \mathfrak{X}$. Let $\mathscr{P}$ be a separating family of seminorms determining the topology of $\mathfrak{X}$. For every $\mathbf{p} \in \mathscr{P}$ there is $\Lambda \in \mathfrak{X}^*$ such that $\Lambda x_0 = \mathbf{p}(x_0)$ and $|\Lambda x| \leq \mathbf{p}(x)$ for any $x \in \mathfrak{X}$.*

Once again, the proof in [20], p. 59, is confined to the case where $\mathfrak{X}$ is a normed vector space.

Proof of Corollary A2. A function $\Lambda : \mathfrak{X} \to \mathbb{C}$ as in thesis of Corollary A2 may be built as follows. If $x_0 = 0$, we set $\Lambda = 0$. If $x_0 \neq 0$ let $\mathscr{Y}$ be the 1-dimensional space spanned by x_0, i.e., $\mathfrak{Y} = \mathbb{C}x_0$. For a fixed seminorm $\mathbf{p} \in \mathscr{P}$, let us consider

$$f : \mathscr{Y} \to \mathbb{C}, f(\lambda x_0) = \lambda \mathbf{p}(x_0), \lambda \in \mathbb{C}.$$

One checks easily that $f : \mathscr{Y} \to \mathbb{C}$ is linear. Moreover, for every $x = \lambda x_0 \in \mathscr{Y}$,

$$|f(x)| = |\lambda \mathbf{p}(x_0)| = |\lambda| \mathbf{p}(x_0) = \mathbf{p}(\lambda x_0) = \mathbf{p}(x).$$

At this point, one may apply the Hahn-Banach theorem to the data $\mathfrak{Y} = \mathbb{C}x_0$ and $f : \mathfrak{Y} \to \mathbb{C}$ and $\mathbf{p}$ to produce a linear functional $\Lambda : \mathfrak{X} \to \mathbb{C}$ extending f such that $|\Lambda x| \leq \mathbf{p}(x)$ for any $x \in \mathfrak{X}$. As $\mathbf{p} : \mathfrak{X} \to \mathbb{R}$ is continuous, it follows that Λ is continuous at the origin, and hence on $\mathfrak{X}$. Finally, $x_0 \in \mathscr{Y}$ and hence $\Lambda(x_0) = f(x_0) = \mathbf{p}(x_0)$.

Let us apply Corollary A2 for $x_0 = y = \int_Q f \mathrm{d}\mu$ and $\mathbf{p} \in \mathscr{P}$. We may then consider a linear and continuous functional $\Lambda \in \mathfrak{X}^*$ such that $\Lambda(y) = \mathbf{p}(y)$ and $|\Lambda x| \leq \mathbf{p}(x)$ for any $x \in \mathfrak{X}$. In particular, for $x \in f(Q)$,

$$|\Lambda f(s)| \leq \mathbf{p}(f(s)), s \in Q.$$

Finally,

$$\mathbf{p}\left(\int_Q f \mathrm{d}\mu\right) = \mathbf{p}(y) = \Lambda(y) = \int_Q (\Lambda \circ f) \mathrm{d}\mu = \left|\int_Q (\Lambda \circ f) \mathrm{d}\mu\right| \leq \int_Q |\Lambda \circ f| \mathrm{d}\mu \leq \int_Q (\mathbf{p} \circ f) \mathrm{d}\mu. \qquad \square$$

B Taylor's formula with Bochner integral reminder

Let $\mathfrak{X}$ be a locally convex space over $\mathbb{R}$, on which $\mathfrak{X}^*$ separates points, and let $f \in C^{n+2}(U, \mathfrak{X})$ where $U \subset \mathbb{R}$ is an open set. Let $t_0 \in U$ and let us set

$$P_n(t; f, t_0) = \sum_{k=0}^{n} \frac{1}{k!}(t - t_0)^k f^{(k)}(t_0), \tag{A1}$$

$$R_n(t; f, t_0) = f(t) - P_n(t; f, t_0). \tag{A2}$$

Lemma A3.
(*i*) *The reminder is of order* $o(|t - t_0|^n)$, *i.e.,*

$$\lim_{t \to t_0} \frac{1}{(t - t_0)^n} R_n(t; f, t_0) = 0.$$

(*ii*) *The reminder admits the integral representation formula*

$$R_n(t; f, t_0) = \frac{1}{n!} \int_{t_0}^{t} (t - s)^n f^{(n+1)}(s) \mathrm{d}s.$$

Proof. (i) Linear and continuous functionals $\Lambda \in \mathfrak{X}^*$ commute with derivatives of any order. Hence,

$$\Lambda[P_n(t; f, t_0)] = P_n(t; \Lambda(f), t_0), \tag{A3}$$

where $\Lambda(f) \coloneqq \Lambda \circ f$. As $\Lambda(f)$ is a scalar valued function of class C^{n+1}, for any $t \in U$, there is $\alpha = \alpha(t, t_0, \Lambda(f)) \in [0, 1]$ such that

$$R_n(t; \Lambda(f), t_0) = \frac{(t - t_0)^{n+1}}{(n+1)!} \frac{\mathrm{d}^{n+1}\Lambda(f)}{\mathrm{d}t^{n+1}}(\xi), \tag{A4}$$

where $\xi = (1 - \alpha)t_0 + \alpha t$. Consequently, as $\Lambda(f)$ is of class C^{n+2} the representation formula (A4) implies

$$\lim_{t \to t_0} \frac{R_n(t; \Lambda(f), t_0)}{(t - t_0)^{n+1}} = \frac{1}{(n+1)!} \frac{\mathrm{d}^{n+1}\Lambda(f)}{\mathrm{d}t^{n+1}}(t_0). \tag{A5}$$

On the other hand (by (A3)),

$$\frac{R_n(t; \Lambda(f), t_0)}{(t - t_0)^{n+1}} = \frac{1}{(t - t_0)^{n+1}} [\Lambda(f(t)) - P_n(t; \Lambda(f), t_0)] = \Lambda\left[\frac{1}{(t - t_0)^{n+1}} R_n(t; f, t_0)\right],$$

hence (by (A5)) for any sequence $\{t_\nu\}_{\nu \geq 1} \subset U$ with $\lim_{\nu \to \infty} t_\nu = t_0$ the limit

$$\lim_{\nu \to \infty} \Lambda\left[\frac{1}{(t_\nu - t_0)^{n+1}} R_n(t_\nu; f, t_0)\right]$$

exists and is finite. Consequently,

$$\left\{\Lambda\left[\frac{1}{(t_\nu - t_0)^{n+1}} R_n(t_\nu; f, t_0)\right] : \nu \geq 1\right\} \subset \mathbb{R}$$

is a bounded set for every $\Lambda \in \mathfrak{X}^*$, i.e., the set

$$E = \left\{\frac{1}{(t_\nu - t_0)^{n+1}} R_n(t_\nu; f, t_0) : \nu \geq 1\right\} \subset \mathfrak{X}$$

is weakly bounded. From now on, we assume that $\mathfrak{X}$ is a locally convex space. By Theorem 3.18 in [20], p. 70, every weakly bounded set in $\mathfrak{X}$ is also strongly bounded. Hence, for any neighborhood V of $0 \in \mathfrak{X}$, there is $s_V > 0$ such that $E \subset sV$ for any $s > s_V$, i.e.,

$$\frac{1}{(t_\nu - t_0)^n} R_n(t_\nu; f, t_0) \in (t_\nu - t_0)sV$$

for any $\nu \geq 1$. Consequently, the sequence

$$\left\{\frac{1}{(t_\nu - t_0)^n} R_n(t_\nu; f, t_0)\right\}_{\nu \geq 1}$$

is strongly convergent to 0 as $\nu \to \infty$.

(ii) Let $\mathfrak{X}$ be a topological vector space on which $\mathfrak{X}^*$ separates points. By a classical representation formula

$$R_n(t; \Lambda(f), t_0) = \frac{1}{n!}\int_{t_0}^{t} (t-s)^n \frac{\mathrm{d}^{n+1}(\Lambda \circ f)}{\mathrm{d}t^{n+1}}(s)\mathrm{d}s$$

for any $t \in U$. Then

$$\Lambda\left[R_n(t; f, t_0) - \frac{1}{n!}\int_{t_0}^{t}(t-s)^n f^{(n+1)}(s)\mathrm{d}s\right] = 0$$

for any $\Lambda \in \mathfrak{X}^*$, where $\int_{t_0}^{t}(t-s)^n f^{(n+1)}(s)\mathrm{d}s$ is a Bochner integral. As $\mathfrak{X}^*$ separates points, we may conclude that

$$R_n(t; f, t_0) = \frac{1}{n!}\int_{t_0}^{t}(t-s)^n f^{(n+1)}(s)\mathrm{d}s \tag{A6}$$

for any $t \in U$. □

Lemma A4. *Let $\mathfrak{X}$ be a locally convex space on which $\mathfrak{X}^*$ separates points. Let $U \subset \mathbb{R}$ be an open neighborhood of $t_0 \in \mathbb{R}$ and let $F \in C^{n+1}(U, \mathfrak{X})$ such that $F(t_0) = F'(t_0) = \ldots = F^{(n)}(t_0) = F^{(n+1)}(t_0) = 0$. Then*

$$\lim_{t \to t_0}\left[\frac{1}{(t-t_0)^n}F(t)\right] = 0.$$

Proof. $\Lambda(F) \in C^{n+1}(U, \mathbb{R})$ for every $\Lambda \in \mathfrak{X}'$ and (by applying repeatedly the classical l'Hôspital theorem),

$$\lim_{t\to t_0}\frac{\Lambda(F(t))}{(t-t_0)^{n+1}} = \lim_{t\to t_0}\frac{\frac{\mathrm{d}}{\mathrm{d}t}[\Lambda(F(t))]}{(n+1)(t-t_0)^n} = \ldots = \lim_{t\to t_0}\frac{\frac{\mathrm{d}^n}{\mathrm{d}t^n}[\Lambda(F(t))]}{(n+1)!(t-t_0)} = \lim_{t\to t_0}\frac{\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}[\Lambda(F(t))]}{(n+1)!} = \frac{1}{(n+1)!}\Lambda[F^{(n+1)}(t_0)] = 0,$$

hence

$$\lim_{\nu\to\infty}\frac{\Lambda(F(t_\nu))}{(t_\nu - t_0)^{n+1}} = 0$$

for any sequence $\{t_\nu\}_{\nu\geq 1} \subset U$ such that $\lim_{\nu\to\infty} t_\nu = t_0$. Consequently, the set

$$\left\{\frac{1}{(t_\nu - t_0)^{n+1}}F(t_\nu) : \nu \geq 1\right\} \subset \mathfrak{X}$$

is weakly bounded, and then strongly bounded, in $\mathfrak{X}$. Then for any neighborhood $V \subset \mathfrak{X}$ of 0, there is $s > 0$ such that

$$\frac{1}{(t_\nu - t_0)^n}F(t_\nu) \in (t - t_0)sV, \nu \geq 1,$$

yielding $\lim_{\nu\to\infty}\left[\frac{1}{(t_\nu - t_0)^n}F(t_\nu)\right] = 0$ strongly in $\mathfrak{X}$. □

Let $\mathfrak{X}$ be a Fréchet space. Let $A \subset \mathbb{R}^n$ be an open set and let $f \in C^k(A, \mathfrak{X})$ and $x_0 \in A$. There is $R > 0$ such that $\overline{B}_R(x_0) \subset A$. Let $w \in \mathbb{R}^n$ such that $\|w\| = 1$, and let us consider the function

$$F : (-R, R) \to \mathfrak{X}, F(t) = f(x_0 + tw), |t| < R.$$

Then $F \in C^k((-R, R), \mathfrak{X})$ and then by Taylor's formula with a reminder for $\mathfrak{X}$ valued functions of one real variable, cf. (A1) and (A2)]

$$F(t) = \sum_{j=0}^{k} \frac{t^j}{j!} F^{(j)}(0) + R_k(t; F, 0). \tag{A7}$$

On the other hand,

$$F^{(h)}(t) = h! \sum_{|\alpha|=h} \frac{w^\alpha}{\alpha!}(D^\alpha f)(x_0 + tw), \tag{A8}$$

and by choosing

$$w = \frac{1}{\|x - x_0\|}(x - x_0), t = \|x - x_0\|, x \in B_R(x_0),$$

the formula (A7) becomes

$$f(x) = F(\|x - x_0\|) = \sum_{j=0}^{k} \frac{\|x - x_0\|^j}{j!} F^{(j)}(0) + R_k(\|x - x_0\|; F, 0) =$$

[for (A8) with $t = 0$]

$$= \sum_{j=0}^{k} \|x - x_0\|^j \sum_{|\alpha|=j} \frac{1}{\alpha!} \frac{(x - x_0)^\alpha}{\|x - x_0\|^{|\alpha|}}(D^\alpha f)(x_0) + r_k(x; f, x_0),$$

where we have set

$$r_k(x; f, x_0) = R_k(\|x - x_0\|; F, 0),$$
$$F(t) = f\left(x_0 + \frac{t}{\|x - x_0\|}(x - x_0)\right), |t| < R.$$

The Taylor formula we seek for is

$$f(x) = \sum_{|\alpha|\le k} \frac{(x - x_0)^\alpha}{\alpha!}(D^\alpha f)(x_0) + r_k(x; f, x_0). \tag{A9}$$

Next let us assume that $f \in C^{k+2}(A, \mathfrak{X})$ so that $F \in C^{k+2}((-R, R), \mathfrak{X})$, and hence,

$$R_k(t; F, 0) = \frac{1}{k!}\int_0^t (t - s)^k F^{(k+1)}(s)\mathrm{d}s,$$

where from (by (A8) for $h = k + 1$)

$$r_k(x; f, x_0) = \frac{1}{k!}\int_0^{\|x-x_0\|} (\|x - x_0\| - s)^k (k + 1)! \sum_{|\alpha|=k+1} \frac{(x - x_0)^\alpha}{\alpha! \|x - x_0\|^{|\alpha|}}(D^\alpha f)\left(x_0 + \frac{s}{\|x - x_0\|}(x - x_0)\right)\mathrm{d}s$$

or (by a change of variable $r = \|x - x_0\| - s$)

$$r_k(x; f, x_0) = \frac{k + 1}{\|x - x_0\|^{k+1}} \sum_{|\alpha|=k+1} (x - x_0)^\alpha \int_0^{\|x-x_0\|} r^k (D^\alpha f)\left(x - \frac{r}{\|x - x_0\|}(x - x_0)\right)\mathrm{d}r. \tag{A10}$$

Let $\mathfrak{X}$ be a Fréchet space. Let $\mathscr{P}$ be a separating family of seminorms defining the topology of $\mathfrak{X}$ as a local convex space.

Lemma A5. *Let $A \subset \mathbb{R}^n$ be an open set and let $f \in C^{k+2}(A, \mathfrak{X})$. Let $x_0 \in A$ and $R > 0$ such that $B_R(x_0) \subset A$. Then*

$$\mathbf{p}[r_k(x; f, x_0)] \leq M_{k+1} \, \|x - x_0\|^{k+1} \times \max_{|\alpha|=k+1} \sup_{0\leq\tau\leq1} \mathbf{p}[(D^\alpha f)((1-\tau)x + \tau x_0)] \tag{A11}$$

for any $\mathbf{p} \in \mathscr{P}$ and any $x \in B_R(x_0)$.

Here, M_ℓ is the cardinality of the set $\{\alpha \in \mathbb{Z}_+^n : |\alpha| = \ell\}$.

Proof of Lemma A5.

$$\mathbf{p}[r_k(x; f, x_0)] \leq \frac{k+1}{\|x-x_0\|^{k+1}} \sum_{|\alpha|=k+1} \prod_{j=1}^{n} |x_j - x_j^0|^{\alpha_j} \int_0^{\|x-x_0\|} r^k \mathbf{p}\left[(D^\alpha f)\left(x - \frac{r}{\|x-x_0\|}(x-x_0)\right)\right] \mathrm{d}r \leq$$

(as $\prod_{j=1}^n |x_j - x_j^0|^{\alpha_j} \leq \|x - x_0\|^{|\alpha|}$)

$$\leq \left[(k+1)\int_0^{\|x-x_0\|} r^k \mathrm{d}r\right] \times \sum_{|\alpha|=k+1} \sup_{0\leq r\leq\|x-x_0\|} \mathbf{p}\left[(D^\alpha f)\left(x - \frac{r}{\|x-x_0\|}(x-x_0)\right)\right]$$

$$\leq M_{k+1} \, \|x-x_0\|^{k+1} \max_{|\alpha|=k+1} \sup_{0\leq r\leq\|x-x_0\|} \mathbf{p}\left[(D^\alpha f)\left(x - \frac{r}{\|x-x_0\|}(x-x_0)\right)\right].$$

Setting $\tau = r/\|x - x_0\| \in [0, 1]$, one may conclude that

$$\mathbf{p}[r_k(x; f, x_0)] \leq M_{k+1} \, \|x-x_0\|^{k+1} \max_{|\alpha|=k+1} \sup_{0\leq\tau\leq1} \mathbf{p}[(D^\alpha f)((1-\tau)x + \tau x_0)].$$ □

C Series in Fréchet spaces

Let $\mathfrak{X}$ be a topological vector space and let $\{x_\nu\}_{\nu\geq0} \subset \mathfrak{X}$.

Lemma A6. *If $\sum_{\nu\geq0} x_\nu$ is convergent, then $x_\nu \to 0$ in $\mathfrak{X}$ as $\nu \to \infty$.*

Proof. Let W be a neighborhood of the origin in $\mathfrak{X}$. As the map $(x, y) \mapsto x - y$ is continuous, there is a neighborhood of the origin $V \subset \mathfrak{X}$ such that $V - V \subset W$. Let $S_\mu = \sum_{\nu=0}^{\mu} x_\nu$ and $S = \lim_{\mu\to\infty} S_\mu$. There is $N = N(V) \geq 1$ such that $S_\nu - S \in V$ for any $\nu \geq N$. Hence, $x_\nu = (S_\nu - S) - (S_{\nu-1} - S) \in W$ for any $\nu \geq N$. □

Let $\mathfrak{X}$ be a Fréchet space and $\{x_\nu\}_{\nu\geq0} \subset \mathfrak{X}$. Let $\mathscr{P}$ be a family of seminorms defining the topology of $\mathfrak{X}$ as a locally convex space.

Lemma A7. *If $\sum_{\nu=0}^{\infty} \mathbf{p}(x_\nu) < \infty$ for every $\mathbf{p} \in \mathscr{P}$, then the series $\sum_{\nu\geq0} x_\nu$ is convergent in $\mathfrak{X}$.*

Proof. Let $\sigma_\mu(\mathbf{p}) = \sum_{\nu=0}^{\mu} \mathbf{p}(x_\nu)$ and $S_\mu = \sum_{\nu=0}^{\mu} x_\nu$. Let $\mathbf{p} \in \mathscr{P}$ and $k \in \mathbb{N}$. As $\{\sigma_\nu(\mathbf{p})\}_{\nu\geq0} \subset \mathbb{R}$ is a Cauchy sequence, there is $N = N(\mathbf{p}, k) \geq 1$ such that $|\sigma_\mu(\mathbf{p}) - \sigma_\nu(\mathbf{p})| < 1/k$ for any $\mu > \nu \geq N$. Then

$$\mathbf{p}(S_\mu - S_\nu) \leq |\sigma_\mu(\mathbf{p}) - \sigma_\nu(\mathbf{p})| < \frac{1}{k}, \mu > \nu \geq N,$$

that is $\{S_\nu\}_{\nu\geq 0} \subset \mathfrak{X}$ is a Cauchy sequence in $\mathfrak{X}$. Yet the topology of $\mathfrak{X}$ (as a Fréchet space) is compatible to a complete invariant metric, and hence, $\{S_\nu\}_{\nu\geq 0}$ is convergent in $\mathfrak{X}$. □

Let $\mathfrak{X}$ be a complex Fréchet space and $\{x_\nu\}_{\nu\geq 0} \subset \mathfrak{X}$.

Lemma A8. *If there is $z_0 \in \mathbb{C}\backslash\{0\}$ such that $\sum_{\nu\geq 0} z_0^\nu x_\nu$ is convergent, then $\sum_{\nu\geq 0} z^\nu x_\nu$ is convergent, for any $z \in D_{|z_0|}(0)$. Also $\sum_{\nu\geq 0} z^\nu x_\nu$ is uniformly convergent for $z \in D_r(0)$ for any $0 < r < |z_0|$.*

Proof. As $z_0^\nu x_\nu \to 0$ in $\mathfrak{X}$ as $\nu \to \infty$, for any $\mathbf{p} \in \mathscr{P}$ and any $k \in \mathbb{N}$, there is $N = N(\mathbf{p}, k) \geq 1$ such that $|z_0|^\nu \mathbf{p}(x_\nu) < 1/k$ for any $\nu \geq N$. If $z \in D_{|z_0|}(0)$ then $\mathbf{p}(z^\nu x_\nu) < q^\nu/k$, where $q = |z/z_0|$ so that $0 \leq q < 1$. Therefore, $\sum_{\nu=0}^\infty \mathbf{p}(z^\nu x_\nu) < \infty$ so that $\sum_{\nu\geq 0} z^n x_\nu$ is convergent in $\mathfrak{X}$. Finally, for each $0 < r < |z_0|$, one has

$$\sup_{|z|<r} \mathbf{p}(z^\nu x_\nu) \leq \frac{1}{k}\left(\frac{r}{|z_0|}\right)^\nu .$$ □

The *convergence radius* of $\sum_{\nu\geq 0} z^\nu x_\nu$ is

$$R = \sup\left\{|z_0| : \sum_{\nu\geq 0} z_0^\nu x_\nu \text{ is convergent in } \mathfrak{X}\right\}.$$

For each $\mathbf{p} \in \mathscr{P}$, we set $\ell(\mathbf{p}) = \limsup_{\nu\to\infty} \mathbf{p}(x_\nu)^{1/\nu}$. If $0 < \ell(\mathbf{p}) < \infty$ let $z \in D_{1/\ell(\mathbf{p})}(0)$ so that $|z|\ell(\mathbf{p}) < 1$. Then $|z|[\ell(\mathbf{p}) + \varepsilon] < 1$ for some $\varepsilon > 0$. As $\ell(\mathbf{p}) + \varepsilon > \ell(\mathbf{p})$, there is $N \geq 1$ such that $\ell(\mathbf{p}) + \varepsilon > \mathbf{p}(x_\nu)^{1/\nu}$ for any $\nu \geq N$. Thus,

$$\mathbf{p}(z^\nu x_\nu) = |z|^\nu \mathbf{p}(x_\nu) < (|z|[\ell(\mathbf{p}) + \varepsilon])^\nu$$

so that $\sum_{\nu=0}^\infty \mathbf{p}(z^\nu x_\nu) < \infty$.

Proposition A9.

(1) *If $0 < \ell(\mathbf{p}) < a$ for some $a > 0$ and any $\mathbf{p} \in \mathscr{P}$, and we set*

$$R = \inf\left\{\frac{1}{\ell(\mathbf{p})} : \mathbf{p} \in \mathscr{P}\right\}$$

then $R > 0$ and the series $\sum_{\nu\geq 0} z^\nu x_\nu$ is convergent (respectively divergent) for any $z \in D_R(0)$ (respectively for any $z \in \mathbb{C}\backslash\overline{D_R(0)}$).

(2) *If $0 < \ell(\mathbf{p}) < \infty$ for any $\mathbf{p} \in \mathscr{P}$ yet, there is a sequence $\{\mathbf{p}_j\}_{j\geq 1} \subset \mathscr{P}$ such that $\lim_{j\to\infty} \ell(\mathbf{p}_j) = \infty$ or $\ell(\mathbf{p}) = \infty$ for some $\mathbf{p} \in \mathscr{P}$, then $\sum_{\nu\geq 0} z^\nu x_\nu$ is divergent for any $z \in \mathbb{C}\backslash\{0\}$.*

(3) *If $\ell(\mathbf{p}) = 0$ for some $\mathbf{p} \in \mathscr{P}$, then let us set*

$$\mathscr{P}_0 = \{\mathbf{p} \in \mathscr{P} : \ell(\mathbf{p}) = 0\}, R = \inf\left\{\frac{1}{\ell(\mathbf{p})} : \mathbf{p} \in \mathscr{P}\backslash\mathscr{P}_0\right\}.$$

If $\sup\{\ell(\mathbf{p}) : \mathbf{p} \in \mathscr{P}\backslash\mathscr{P}_0\} < \infty$, then $R > 0$ and the series $\sum_{\nu\geq 0} z^\nu x_\nu$ is convergent for any $z \in D_R(0)$, while if $\sup\{\ell(\mathbf{p}) : \mathbf{p} \in \mathscr{P}\backslash\mathscr{P}_0\} = \infty$, then $R = 0$ and $\sum_{\nu\geq 0} z^\nu x_\nu$ is divergent for any $z \in \mathbb{C}\backslash\{0\}$.

Proof. (1) If $R = 0$, then for any $\varepsilon > 0$ there is $\mathbf{p}_\varepsilon \in \mathscr{P}$ such that $\ell(\mathbf{p}_\varepsilon) > 1/\varepsilon$. Hence, $\ell(\mathbf{p}_\varepsilon) \to \infty$ as $\varepsilon \to 0^+$, a contradiction. Hence, $R > 0$. Let $z \in D_R(0)$. Then $\sum_{\nu=0}^\infty \mathbf{p}(z^\nu x_\nu) < \infty$ for any $\mathbf{p} \in \mathscr{P}$ hence, $\sum_{\nu\geq 0} z^\nu x_\nu$ is convergent. If in turn $|z| > R$, then $|z|\ell(\mathbf{p}_0) > 1$ for some $\mathbf{p}_0 \in \mathscr{P}$. Hence, there is $\delta > 0$ such that $|z|[\ell(\mathbf{p}_0) - \delta] > 1$. On the other hand, $\ell(\mathbf{p}_0) - \delta < \ell(\mathbf{p}_0)$ hence, for any $n \geq 1$, there is $\nu(n) \geq n$ such that

$$\ell(\mathbf{p}_0) - \delta < \mathbf{p}_0(x_{\nu(n)})^{1/\nu(n)}.$$

Consequently, there is a sequence $\{\nu_j\}_{j\geq 1} \subset \mathbb{N}$ such that

$$1 \le \nu_1 < \nu_2 < \ldots \uparrow \infty,\ \ell(\mathbf{p}_0) - \delta < \mathbf{p}_0(x_{\nu_j})^{1/\nu_j}, \quad j \ge 1.$$

Finally,

$$\mathbf{p}_0(z^{\nu_j} x_{\nu_j}) = |z|^{\nu_j} \mathbf{p}_0(x_{\nu_j}) > |z|^{\nu_j}[\ell(\mathbf{p}_0) - \delta]^{\nu_j} > 1, j \ge 1.$$

Yet $\mathbf{p}_0$ is continuous, so $\sum_{\nu\ge0} z^\nu x_\nu$ is divergent.

(2) Let $\ell(\mathbf{p}_j) \to \infty$ as $j \to \infty$. Then, for any $A > 0$, there is $j(A) \ge 1$ such that for any $j \ge j(A)$ and any $n \ge 1$ there is $\nu \ge n$ satisfying $p_j(x_\nu)^{1/\nu} > A$. Given $z \in \mathbb{C}\setminus\{0\}$, one may pick $A \coloneqq 1/|z|$ and choose $j_0 \ge j(A)$. Consequently, there is a sequence $1 \le \nu_1 < \nu_2 < \ldots \uparrow \infty$ such that

$$\mathbf{p}_{j_0}(z^{\nu_k} x_{\nu_k}) > 1, k \ge 1,$$

so that (by the continuity of the seminorm $\mathbf{p}_{j_0}$) the series $\sum_{\nu\ge0} z^\nu x_\nu$ is divergent. A similar argument may be provided when $\ell(\mathbf{p}) = \infty$ for some $\mathbf{p} \in \mathscr{P}$.

(3) Let $\mathbf{p} \in \mathscr{P}_0$. Then for any $\varepsilon > 0$, there is $n_\varepsilon \ge 1$ such that

$$\mathbf{p}(x_\nu)^{1/\nu} < \varepsilon, \nu \ge \nu_\varepsilon.$$

Let $z \in \mathbb{C}\setminus\{0\}$ and let us choose $0 < \varepsilon < 1/|z|$. Then

$$\mathbf{p}(z^\nu x_\nu) < (\varepsilon|z|)^\nu, \nu \ge n_\varepsilon,$$

so that $\sum_{\nu=0}^{\infty} \mathbf{p}(z^\nu x_\nu) < \infty$. If $\mathscr{P}_0 = \mathscr{P}$ then $\sum_{\nu\ge0} z^\nu x_\nu$ is convergent for any $z \in \mathbb{C}$. If $\mathscr{P}\setminus\mathscr{P}_0 \ne \emptyset$, then let

$$R = \inf\left\{\frac{1}{\ell(\mathbf{p})} : \mathbf{p} \in \mathscr{P}\setminus\mathscr{P}_0\right\}.$$

The remainder of the proof is similar that of part (1) (when $\sup_{\mathbf{p}\in\mathscr{P}\setminus\mathscr{P}_0} \ell(\mathbf{p}) < \infty$) and part (2) (when $\sup_{\mathbf{p}\in\mathscr{P}\setminus\mathscr{P}_0} \ell(\mathbf{p}) = \infty$). □