# THE SCHRÖDINGER-BORN-INFELD SYSTEM: ATTRACTIVE CASE

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ABSTRACT. In this paper we consider a system describing a quantum particle self interacting with the Born Infeld electromagnetic field. The existence of a radial ground state solution is proved in the attractive case.

## 1. INTRODUCTION

In the recent years, several models have been proposed to provide a mathematical description of the interaction between a charged particle and the electromagnetic field generated by itself. A pioneering paper in this direction was [2], where, by studying in an open bounded set $\Omega \subset \mathbb{R}^3$ the Euler-Lagrange equations derived from the coupling of the Maxwell lagrangian and the Schrödinger one, Benci and Fortunato arrived to prove the existence of solutions $(u, \phi, \omega) \in H_0^1(\Omega) \times H^1(\Omega) \times \mathbb{R}$ to the problem

$$(\mathcal{SM}) \qquad -\Delta u + \omega u + \phi u = 0, \qquad -\Delta \phi = 4\pi u^2$$

with conditions $u = 0$ and $\phi = g$ on $\partial\Omega$ and normalizing condition $\int_\Omega u^2\, dx = 1$. ($\mathcal{SM}$) represents one of the two forms (precisely the repulsive case) in which the following Schrödinger - Newton (also known as Schrödinger - Poisson) system appears

$$(\mathcal{S}N) \qquad -\Delta u + \omega u + \theta \phi u = 0, \qquad -\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3,$$

according to the sign of the nunnull constant $\theta \in \mathbb{R}$.

Choosing $\theta$ positive or negative means to assign a sign to the self interaction potential, distinguishing the attractive case ($\theta < 0$) and the repulsive case ($\theta > 0$).

Attractive case was treated for example in [8,10,11] and, more recently, in [6] where an ODE reduction approach allowed to solve to ($\mathcal{S}N$) in $\mathbb{R}^N$, for all $N \geqslant 1$.

The theory developed by Born and Infeld (see [4] and [5]) inspired Yu who, in [13], proposed a relativistically consistent model where the self interaction arises from the coupling of the Klein - Gordon lagrangian and the Born - Infeld one, this latter being a nonlinear variant of Maxwell lagrangian. The idea of Yu was recovered in [1], where the dynamics of a quantum particle inside the electromagnetic field generated by itself was described by coupling nonlinear Schrödinger with Born - Infeld equations. The repulsive case was considered and the problem was solved by means of variational techniques. This result was later improved in [9].

In this paper we are interested in studying the Schrödinger - Born - Infeld system in the attractive case, looking for solutions of the following problem

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$$(\mathcal{SBI}) \qquad \begin{cases} -\Delta u + u - \phi u = 0 & \text{in } \mathbb{R}^3, \\ -\operatorname{div}\left(\dfrac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = u^2 & \text{in } \mathbb{R}^3, \\ u(x) \to 0,\ \phi(x) \to 0, & \text{as } x \to \infty. \end{cases}$$

At least formally, the system ($\mathcal{SBI}$) comes variationally from the action functional $J$ defined by

$$J(u,\phi) = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right) dx - \int_{\mathbb{R}^3} \phi u^2 \, dx + \int_{\mathbb{R}^3} \left(1 - \sqrt{1-|\nabla\phi|^2}\right) dx.$$

Dealing with this functional presents evident difficulties for several reasons, starting with the definition of the functional setting. Indeed we observe that, being on the one hand natural to consider $u \in H^1(\mathbb{R}^3)$, on the other the presence of the term $\int_{\mathbb{R}^3}\left(1 - \sqrt{1-|\nabla\phi|^2}\right)$ forces us to restrict the setting of admissible functions $\phi$.
We define

$$\mathcal{X} := \mathcal{D}^{1,2}(\mathbb{R}^3) \cap \{\phi \in C^{0,1}(\mathbb{R}^3) : \|\nabla\phi\|_\infty \le 1\}$$

where $\mathcal{D}^{1,2}(\mathbb{R}^3)$ is the completion of $C_c^\infty(\mathbb{R}^3)$ with respect to the norm $L^2$ of the gradient. We will denote by $H_r^1(\mathbb{R}^3)$ and $\mathcal{X}_r$ the sets of radial functions in, respectively $H^1(\mathbb{R}^3)$ and $\mathcal{X}$.

Our main result is

**Theorem 1.1.** *The problem* ($\mathcal{SBI}$) *possesses a radial ground state solution, namely a solution* $(u,\phi) \in H_r^1(\mathbb{R}^3) \times \mathcal{X}_r$ *minimizing the functional* $J$ *among all the nontrivial radial solutions. Moreover both* $u$ *and* $\phi$ *are of class* $C^2(\mathbb{R}^3)$.

The paper is organized as follows: in Section 2 we introduce some known results on the functional settings $\mathcal{X}$ and $\mathcal{X}_r$ which permit us to approach the problem variationally while in Section 3 we prove Theorem 1.1.

We finish this section with some notations. In the following we denote by $\|\cdot\|$ the norm in $H^1(\mathbb{R}^3)$ and by $\|\cdot\|_q$ the norm in $L^q(\mathbb{R}^3)$, for $q \in [1,+\infty]$. Moreover by $c, c_i, C, C_i$ we denote fixed positive constants which can vary from line to line.

## 2. PRELIMINARY RESULTS

A classical initial approach consists in applying the reduction method in order to deal with a one-variable functional. Consider the functional $E : H^1(\mathbb{R}^3) \times \mathcal{X} \to \mathbb{R}$ defined as

$$E(u,\phi) = \int_{\mathbb{R}^3} \left(1 - \sqrt{1-|\nabla\phi|^2}\right) dx - \int_{\mathbb{R}^3} \phi u^2 \, dx.$$

In [3] it has been studied the relation between solutions to the second equation in ($\mathcal{SBI}$) and minimizers of $E(u,\cdot)$, assumed $u \in H^1(\mathbb{R}^3)$ preliminary fixed.

The following lemma justifies why we will choose the radial setting.

**Lemma 2.1.** *For any* $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, *there exists a unique* $\phi_u \in \mathcal{X} \setminus \{0\}$ *minimizing the functional* $E(u,\cdot) : \mathcal{X} \to \mathbb{R}$.

*Moreover, if* $u \in H_r^1(\mathbb{R}^3)$, *then* $\phi_u \in \mathcal{X}_r$ *and it is the unique solution of the second equation of system* ($\mathcal{SBI}$), *in the following weak sense*

$$\int_{\mathbb{R}^3} \frac{\nabla\phi_u \cdot \nabla\psi}{\sqrt{1-|\nabla\phi_u|^2}} \, dx = \int_{\mathbb{R}^3} u^2 \psi \, dx, \text{ for all } \psi \in \mathcal{X}_r.$$

*As a consequence for any* $u \in H^1_r(\mathbb{R}^3)$ *we have* $\int_{\mathbb{R}^3} \frac{|\nabla \phi_u|^2}{\sqrt{1-|\nabla\phi_u|^2}}\,dx = \int_{\mathbb{R}^3} \phi_u u^2\,dx$ *and*

$$E(u,\phi_u) \leqslant -\frac{1}{2}\int_{\mathbb{R}^3} \phi_u u^2\,dx. \tag{1}$$

*Proof.* By Proposition 2.3 and Theorem 1.4 of [3], we only have to prove inequality (1). Indeed

$$E(u,\phi_u) = \int_{\mathbb{R}^3}\left(1-\sqrt{1-|\nabla\phi_u|^2}-\frac{|\nabla\phi_u|^2}{\sqrt{1-|\nabla\phi_u|^2}}\right)dx = \int_{\mathbb{R}^3}\left(\frac{\sqrt{1-|\nabla\phi_u|^2}-1}{\sqrt{1-|\nabla\phi_u|^2}}\right)dx$$

and our conclusion follows from inequality $\frac{1}{2}|\nabla\phi_u|^2 \leqslant 1-\sqrt{1-|\nabla\phi_u|^2}$. □

Now we introduce the one-variable functional defined on $H^1(\mathbb{R}^3)$ as

$$I(u) = J(u,\phi_u) = \|u\|^2 + E(u,\phi_u).$$

Next results, which can be proved exactly as the analogous in [1], lay the foundation of our variational approach even if up to constraining the functional setting to the subset of radial functions.

**Proposition 2.2.** *The functional* $I$ *is of class* $C^1$. *Moreover*

- *if* $(u,\phi) \in H^1(\mathbb{R}^3)\times\mathcal{X}$ *is a nontrivial solution of* ($\mathcal{SBI}$) *in a weak sense, then* $\phi = \phi_u$ *and* $u$ *is a critical point of* $I$;
- *for any* $u \in H^1_r(\mathbb{R}^3)$ : $\frac{I'(u)}{2} = -\Delta u + u - \phi_u u$ *in* $(H^1(\mathbb{R}^3))'$.

## 3. PROOF OF THE MAIN RESULT

By Lemma 2.1, Proposition 2.2 and Palais Principle of Symmetrical Criticality, we are allowed to find solutions of ($\mathcal{SBI}$) in a weak sense looking for critical points of $I|_{H^1_r(\mathbb{R}^3)}$. In the sequel we will write $I$ meaning $I|_{H^1_r(\mathbb{R}^3)}$.

First we are going to prove a new estimate to show that $I$ is unbounded below.

**Lemma 3.1.** *There exist* $c_1 > 0$ *and* $c_2 > 0$ *such that for any* $u \in C(\mathbb{R}^3)\cap H^1_r(\mathbb{R}^3)$ *and* $x \neq 0$

$$\frac{1}{\sqrt{1-|\nabla\phi_u(x)|^2}} \leqslant c_1 + \frac{c_2}{|x|^2}\|u\|^2_{L^2(B_{|x|})}. \tag{2}$$

*As a consequence* $\frac{1}{\sqrt{1-|\nabla\phi_u(x)|^2}} \in L^\infty(\mathbb{R}^3)$ *and there exist* $c_1 > 0$ *and* $c_2 > 0$ *such that for any* $u \in C(\mathbb{R}^3)\cap H^1_r(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3}\phi_u u^2\,dx \geqslant c_1\int_{\mathbb{R}^3}|u|^3\,dx - c_2\int_{\mathbb{R}^3}|\nabla u|^2\left(1+\frac{1}{|x|^2}\|u\|^2_{L^2(B_{|x|})}\right)dx. \tag{3}$$

*Proof.* Take any $u$ as in the assumption. By Lemma 2.1, we have that $\phi_u$ satisfies

$$-\operatorname{div}\left(\frac{\nabla\phi}{\sqrt{1-|\nabla\phi|^2}}\right) = u^2 \tag{4}$$

in a weak sense. Arguing as in the proof of [1, Proposition 2.8], we conclude that $\phi_u \in C^2(\mathbb{R}^3)$ and then it satisfies the previous equation pointwise in the classical sense. In particular, introducing $v : [0,+\infty) \to \mathbb{R}$ and $\psi : [0,+\infty) \to \mathbb{R}$ such that $v(0) = u(0)$, $\psi(0) = \phi_u(0)$, and $v(r) = u(x)$, $\psi(r) = \phi_u(x)$ for all $x \in \partial B_r$, we have

$$\left(r^2\frac{\psi'}{\sqrt{1-(\psi')^2}}\right)'(r) = -r^2v^2(r) \text{ in } [0,+\infty).$$

Integrating in $[0, r]$, we obtain

$$\frac{|\psi'(r)|}{\sqrt{1-(\psi'(r))^2}} = -\frac{\psi'(r)}{\sqrt{1-(\psi'(r))^2}} = \frac{1}{r^2}\int_0^r s^2 v^2(s)\, ds \text{ in } (0, +\infty). \tag{5}$$

For all $r > 0$ for which $|\psi'(r)| \leqslant \frac{1}{2}$, of course we have

$$\frac{1}{\sqrt{1-(\psi'(r))^2}} \leqslant \frac{2\sqrt{3}}{3}, \tag{6}$$

while for all $r > 0$ for which $|\psi'(r)| > \frac{1}{2}$, from (5) we deduce that

$$\frac{1}{\sqrt{1-(\psi'(r))^2}} \leqslant \frac{2}{r^2}\int_0^r s^2 v^2(s)\, ds. \tag{7}$$

Coming back to $u$ and $\phi_u$, from (6) and (7) we obtain (2). Now, to prove (3), we multiply by $|u|$ in (4) and integrate in $\mathbb{R}^3$.
By (2),

$$\begin{aligned}\int_{\mathbb{R}^3} |u|^3\, dx = \int_{\mathbb{R}^3} \frac{\nabla\phi_u \cdot \nabla|u|}{\sqrt{1-|\nabla\phi_u|^2}}\, dx &\leqslant \frac{1}{2}\int_{\mathbb{R}^3} \frac{|\nabla\phi_u|^2}{\sqrt{1-|\nabla\phi_u|^2}}\, dx + \frac{1}{2}\int_{\mathbb{R}^3} \frac{|\nabla u|^2}{\sqrt{1-|\nabla\phi_u|^2}}\, dx \\ &\leqslant \frac{1}{2}\int_{\mathbb{R}^3} \phi_u u^2\, dx + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \left(c_1 + \frac{c_2}{|x|^2}\|u\|^2_{L^2(B_{|x|})}\right) dx\end{aligned}$$

and then the conclusion. ☐

We introduce the following notation: for any $u \in H^1(\mathbb{R}^3)$, $t > 0$ and $\alpha \in \mathbb{R}$ we denote by $u_{\alpha,t}$ the function $t^\alpha u(\frac{\cdot}{t})$.

**Proposition 3.2.** *The functional $I$ is unbounded below. More precisely, for any $\alpha \in (0, 1)$ and $u \in C(\mathbb{R}^3) \cap H^1_r(\mathbb{R}^3)$ we have $\lim_{t\to+\infty} I(u_{\alpha,t}) = -\infty$.*

*Proof.* Take $\alpha$ and $u$ as in the assumptions. By (1) and (3),

$$\begin{aligned}I(u_{\alpha,t}) &\leqslant c_1 \int_{\mathbb{R}^3} (|\nabla u_{\alpha,t}|^2 + u_{\alpha,t}^2)\, dx + c_2 \int_{\mathbb{R}^3} \frac{|\nabla u_{\alpha,t}|^2}{|x|^2}\|u_{\alpha,t}\|^2_{L^2(B_{|x|})}\, dx - c_3 \int_{\mathbb{R}^3} |u|^3\, dx \\ &= c_1 t^{2\alpha+1}\|\nabla u\|_2^2 + c_1 t^{2\alpha+3}\|u\|_2^2 + c_2 t^{4\alpha-1}\int_{\mathbb{R}^3} \frac{|\nabla u(\frac{x}{t})|^2}{|\frac{x}{t}|^2}\|u\|^2_{L^2(B_{|\frac{x}{t}|})}\, dx - c_3 t^{3\alpha+3}\|u\|_3^3 \\ &= c_1 t^{2\alpha+1}\|\nabla u\|_2^2 + c_1 t^{2\alpha+3}\|u\|_2^2 + c_2 t^{4\alpha+2}\int_{\mathbb{R}^3} \frac{|\nabla u(x)|^2}{|x|^2}\|u\|^2_{L^2(B_{|x|})}\, dx - c_3 t^{3\alpha+3}\|u\|_3^3.\end{aligned}$$

We conclude passing to the limit as $t$ goes to $+\infty$. ☐

Since $I$ does not possess a minimizer, we are led to check the presence of some critical minimax level and, in the case, compactness property of the functional at that level. However, simple computations show immediately how hard is to verify boundedness of Palais - Smale sequences at any level. In view of this, we are going to exploit a result based on the monotonicity trick (see [7,12]).

Let us introduce the following family of auxiliary functionals $I_\lambda$, for $\lambda \in [1/2, 1]$:

$$I_\lambda(u) = \|u\|^2 - \lambda F(u), \ \forall u \in H^1_r(\mathbb{R}^3),$$

where $F(u) = -E(u, \phi_u)$ is a nonnegative operator.
Even if the following result is contained in [1], we prefer to state and prove it explicitly here, for the sake of completeness.

**Lemma 3.3.** *Let $(u_n)_n$ be a sequence in $H^1_r(\mathbb{R}^3)$ and $\bar u \in H^1_r(\mathbb{R}^3)$ such that $u_n \rightharpoonup \bar u$. Then*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx \to \int_{\mathbb{R}^3} \phi_{\bar u} \bar u^2\, dx.$$

*Proof.* Let $(u_n)_n$ and $\bar u$ be as in the assumptions. By [3, Remark 5.5], we infer that $\phi_n := \phi_{u_n} \rightharpoonup \phi_{\bar u} := \bar\phi$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$. Since $u_n \to \bar u$ in $L^{\frac{12}{5}}(\mathbb{R}^3)$ and $\phi_n \rightharpoonup \bar\phi$ in $L^6(\mathbb{R}^3)$,

$$\begin{aligned}\left|\int_{\mathbb{R}^3} \phi_n u_n^2\, dx - \int_{\mathbb{R}^3} \bar\phi \bar u^2\, dx\right| &\leqslant \left|\int_{\mathbb{R}^3} \phi_n (u_n^2 - \bar u^2)\, dx\right| + \left|\int_{\mathbb{R}^3} (\phi_n - \bar\phi)\bar u^2\, dx\right| \\ &\leqslant \|\phi_n\|_6 \|u_n - \bar u\|_{\frac{12}{5}} \|u_n + \bar u\|_{\frac{12}{5}} + \langle \bar u^2, \phi_n - \phi\rangle \xrightarrow[n\to+\infty]{} 0.\end{aligned}$$

□

By the previous lemma, we can prove this weaker Palais - Smale condition for $I_\lambda$.

**Proposition 3.4.** *Take $\lambda \in [1/2, 1]$. If $(u_n)_n$ is a bounded Palais - Smale sequence of $I_\lambda$, then it admits a converging subsequence.*

*Proof.* Take $(u_n)_n$ as in the assumption. We have that there exist $c \in \mathbb{R}$ and $\bar u \in H^1_r(\mathbb{R}^3)$ such that, up to a subsequence,

$$I_\lambda(u_n) = c + o_n(1), \quad I'_\lambda(u_n) = o_n(1) \text{ in } (H^1_r(\mathbb{R}^3))', \quad u_n \rightharpoonup \bar u \text{ in } H^1_r(\mathbb{R}^3).$$

By boundedness of $(u_n)_n$, we have $I'(u_n)[u_n] = o_n(1)$, that means

$$\|u_n\|^2 = \lambda \int_{\mathbb{R}^3} \phi_n u_n^2\, dx + o_n(1), \tag{8}$$

where we have set $\phi_n = \phi_{u_n}$. Moreover, using the weak convergence of $u_n$ to $\bar u$ and taking again into account that this implies $\phi_n \rightharpoonup \phi_{\bar u} := \bar\phi$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ by [3, Remark 5.5], we easily see for every $v \in H^1_r(\mathbb{R}^3)$,

$$\lim_n I'_\lambda(u_n)[v] = I'_\lambda(\bar u)[v] = 0$$

that is $\bar u$ is a critical point of $I_\lambda$.
In particular we deduce $I'_\lambda(\bar u)[\bar u] = 0$, that is $\|\bar u\|^2 = \lambda \int_{\mathbb{R}^3} \bar\phi \bar u^2\, dx$.
By Lemma 3.3, passing to the limit in (8), we deduce that $\lim_n \|u_n\| = \|\bar u\|$ which, together with the weak convergence, implies $u_n \to \bar u$ in $H^1_r(\mathbb{R}^3)$. □

Next two results are devoted to the verification of the mountain pass geometry for every $I_\lambda$

**Proposition 3.5.** *For all $\lambda \in [1/2, 1]$,*

$$\Gamma_\lambda := \{\gamma \in C([0,1], H^1_r(\mathbb{R}^3)) \mid \gamma(0) = 0,\ I_\lambda(\gamma(1)) < 0\} \neq \emptyset.$$

*Proof.* Since inequality (1) implies

$$I_\lambda(u_{\alpha,t}) \leqslant \|u_{\alpha,t}\|^2 - \frac{\lambda}{2}\int_{\mathbb{R}^3} \phi_{u_{\alpha,t}} u_{\alpha,t}^2\, dx,$$

proceeding as in Proposition 3.2, we find $\alpha_\lambda \in (0,1)$ and $t_\lambda > 0$ such that $I_\lambda(u_{\alpha_\lambda, t_\lambda}) < 0$. Defining $\gamma_\lambda$ as follows

$$\gamma_\lambda(s) = \begin{cases} 0 & \text{if } s = 0 \\ u_{\alpha_\lambda, s t_\lambda} & \text{if } s \in (0,1] \end{cases},$$

we have $\gamma_\lambda \in \Gamma_\lambda$. □

**Proposition 3.6.** *For any $\lambda \in [1/2, 1]$ we have $m_\lambda := \inf_{\gamma\in\Gamma_\lambda} \max_{s\in[0,1]} I_\lambda(\gamma(s)) > 0$.*

*Proof.* Set $\lambda \in [1/2, 1]$. By Proposition 3.5, certainly $m_\lambda \neq +\infty$. Now, observe that, applying [1, Lemma 2.7] for $q = 2$,

$$\int_{\mathbb{R}^3} \phi_u u^2 \, dx \leqslant C\|\phi_u\|_6 \|u\|^2_{\frac{12}{5}} \leqslant C\|\nabla \phi_u\|_2 \|u\|^2_{\frac{12}{5}} \leqslant C\|u\|^4 \tag{9}$$

and then

$$I_\lambda(u) \geqslant \|u\|^2 - \lambda \int_{\mathbb{R}^3} \phi_u u^2 \, dx \geqslant \|u\|^2(1 - \lambda C\|u\|_2^2).$$

We deduce that, for a sufficiently small $\eta > 0$, there exists $\rho > 0$ such that

$$I_\lambda(u) > 0, \ \forall u \in B_\rho \setminus \{0\}, \qquad I_\lambda(u) \geqslant \eta, \ \forall u \in \partial B_\rho.$$

So, if we take any $\gamma \in \Gamma_\lambda$, since $I_\lambda(\gamma(1)) < 0$, we deduce that $\|\gamma(1)\| > \rho$. This implies that necessarily there exists $s_\gamma \in (0,1)$ such that $\|\gamma(s_\gamma)\| = \rho$ and then

$$\max_{s\in[0,1]} I_\lambda(\gamma(s)) \geqslant I_\lambda(\gamma(s_\gamma)) \geqslant \eta.$$

As a consequence $m_\lambda \geqslant \eta > 0$. ☐

Now we are ready to prove the existence of at least a nontrivial critical point at the mountain pass level for almost every $I_\lambda$.

**Proposition 3.7.** *For almost every $\lambda \in [1/2, 1]$, there exists $u_\lambda \in H^1_r(\mathbb{R}^3)$, $u_\lambda \neq 0$, such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = m_\lambda$.*

*Proof.* By Propositions 3.5 and 3.6, we can apply the monotonicity trick and use, for example, [1, Proposition 3.1]. So, for almost any $\lambda \in [1/2, 1]$ we have a bounded Palais - Smale sequence for $I_\lambda$ at the level $m_\lambda$. Then we conclude by Proposition 3.4. ☐

Next result can be proved putting together [1, Proposition 2.8] and [1, Proposition 2.9].

**Proposition 3.8.** *For any $\lambda \in [1/2, 1]$ any critical point $u_\lambda$ of $I_\lambda$ satisfies the Pohozaev integral identity $P_\lambda(u_\lambda) = 0$, where*

$$P_\lambda(u) := \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + 3\int_{\mathbb{R}^3} u^2 \, dx - 4\lambda \int_{\mathbb{R}^3} \phi_u^2 u^2 \, dx + 3\lambda \int_{\mathbb{R}^3} \left(1 - \sqrt{1 - |\nabla \phi_u|^2}\right) dx$$

Now we are ready to prove our main result.

*Proof of Theorem 1.1.* By Propositions 3.7, there exists a sequence $(\lambda_n)_n \subset [1/2, 1]$ such that $\lambda_n \nearrow 1$ and, for all $n \in \mathbb{N}$, there exists $u_n \in H^1_r(\mathbb{R}^3) \setminus \{0\}$ such that

$$I_{\lambda_n}(u_n) = m_{\lambda_n}, \quad I'_{\lambda_n}(u_n) = 0 \text{ in } (H^1(\mathbb{R}^3))'.$$

For the sake of brevity, we will denote $\phi_n := \phi_{u_n}$. By Proposition 3.8, $P_{\lambda_n}(u_n) = 0$ and then, since

$$\frac{\|u_n\|^2}{4} \leqslant I_{\lambda_n}(u_n) - \frac{1}{4} P_{\lambda_n}(u_n) = m_{\lambda_n}$$

and $(m_{\lambda_n})_n$ is bounded since it is nonnegative and nonincreasing, we deduce that $(u_n)_n$ is bounded in $H^1_r(\mathbb{R}^3)$. It is easy to see that $(u_n)_n$ is a bounded Palais - Smale sequence for $I$ at the level $m_1 > 0$ and then, by Proposition 3.4, it possesses a subsequence converging to some $\bar{u} \in H^1_r(\mathbb{R}^3)$. Since $I'(\bar{u}) = 0$, then, arguing as in [1, Proposition 2.8], we can prove $(\bar{u}, \phi_{\bar{u}})$ is a classical solution of ($\mathcal{SBI}$). Finally, since $I(\bar{u}) = m_1 > 0$ we have that such a solution is not the trivial one.

Now we prove the existence of a radial ground state, finding a minimizer of $I$ inside

$$\mathcal{S}_r := \left\{ u \in H^1_r(\mathbb{R}^3) \setminus \{0\} \mid I'(u) = 0 \right\}.$$

Of course, $\mathcal{S}_r \neq \emptyset$ since $\bar{u} \in \mathcal{S}_r$. Moreover, by inequality (9), any $u \in \mathcal{S}_r$ satisfies

$$\|u\|^2 = \int_{\mathbb{R}^3} \phi_u u^2 \, dx \leqslant C\|u\|^4$$

and therefore $\inf_{u\in\mathcal{S}_r} \|u\| > 0$.
Since thanks to Proposition 3.8 every $u \in \mathcal{S}_r$ satisfies $P_1(u) = 0$, we have that

$$I(u) = I(u) - \frac{1}{4}P_1(u) \geqslant \frac{1}{4}\|u\|^2$$

and we conclude tha $\sigma_r := \inf_{u\in\mathcal{S}_r} I(u) > 0$.

It is easy to see that, by Proposition 2.2, if we call

$$\mathcal{S}'_r := \{(u,\phi) \in H^1_r(\mathbb{R}^3) \times \mathcal{X}_r \setminus \{(0,0)\} \mid (u,\phi) \text{ solves } (\mathcal{SBI})\},$$

then

$$\sigma_r = \inf_{u\in\mathcal{S}_r} I(u) = \inf_{(u,\phi)\in\mathcal{S}'_r} J(u,\phi)$$

and if such an infimum is achieved by some $\tilde{u} \in \mathcal{S}_r$, then $(\tilde{u}, \phi_{\tilde{u}})$ minimizes $J$ in $\mathcal{S}'_r$.
Let $(u_n)_n \subset \mathcal{S}_r$ such that $I(u_n) \to \sigma_r$. As in the initial part of the proof we show that, up to a subsequence, $(u_n)_n$ converges to $\tilde{u} \in H^1_r(\mathbb{R}^3)$ which actually turns out to be a critical point of $I$ such that $I(\tilde{u}) = \sigma_r$.
Finally we prove $\tilde{u}$ and $\phi_{\tilde{u}}$ are of class $C^2(\mathbb{R}^3)$ arguing as in [1, Proposition 2.8]. ☐

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