Improved estimates and a limit case for the electrostatic Klein-Gordon-Maxwell system*

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Abstract

We study a class of Klein-Gordon-Maxwell systems in presence of a standing wave in equilibrium with a purely electrostatic field. We improve some previous existence results in the case of an homogeneous nonlinearity. Moreover, we deal with the limit case, namely when the frequency of the standing wave is equal to the mass of the charged field.

1 Introduction

This paper is concerned with a class of Klein-Gordon-Maxwell systems written as follows

$$\begin{cases} -\Delta u + [m^2 - (e\phi - \omega)^2]u - f'(u) = 0 & \text{in } \mathbb{R}^3 \\ \Delta \phi = e(e\phi - \omega)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{1}$$

This system was introduced in the pioneering work of Benci and Fortunato [3] in 2002. It represents a standing wave $\psi = u(x)e^{i\omega t}$ (charged matter field) in equilibrium with a purely electrostatic field $\mathbf{E} = -\nabla\phi(x)$. The

*The authors are supported by M.I.U.R. - P.R.I.N. "Metodi variazionali e topologici nello studio di fenomeni non lineari"

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constant $m \geq 0$ represents the mass of the charged field and e is the coupling constant introduced in the minimal coupling rule [12].

It is immediately seen that (1) deserves some interest as system if and only if $e \neq 0$ and $\omega \neq 0$, otherwise we get $\phi = 0$. Through the paper we are looking for *nontrivial solutions*, that is solutions such that $\phi \neq 0$.

Moreover we point out that the sign of ω is not relevant for the existence of solutions. Indeed if (u, ϕ) is a solution of (1) with a certain value of ω, then $(u, -\phi)$ is a solution corresponding to $-\omega$. So, without loss of generality, we shall assume $\omega > 0$. Analogously the sign of e is not relevant, so we assume $e > 0$. Actually the results we are going to prove do not depend on the value of e.

Let us recall some previous results that led us to the present research. The first results are concerned with an homogeneous nonlinearity $f(t) = \frac{1}{p}|t|^p$. Therefore (1) becomes

$$\begin{cases} -\Delta u + [m^2 - (e\phi - \omega)^2]u - |u|^{p-2}u = 0 & \text{in } \mathbb{R}^3 \\ \Delta \phi = e(e\phi - \omega)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{2}$$

As we said before, the first result is due to Benci and Fortunato [3]. They showed the existence of infinitely many solutions whenever $p \in (4, 6)$ and $0 < \omega < m$.

In 2004 D'Aprile and Mugnai published two papers on this topic. In [8] they proved the existence of nontrivial solutions of (1) when $p \in (2, 4]$ and ω varies in a certain range depending on p:

$$0 < \omega < m\, g_0(p)$$

where

$$g_0(p) = \sqrt{\frac{p-2}{2}}.$$

Afterwards, in [9], the same authors showed that (1) has no nontrivial solutions if $p \geq 6$ and $\omega \in (0, m]$ (or $p \leq 2$).

Our first result gives a little improvement on problem (1) with $p \in (2, 4)$.

Theorem 1.1. *Let $p \in (2, 4)$. Assume that $0 < \omega < m\, g(p)$ where*

$$g(p) = \begin{cases} \sqrt{(p-2)(4-p)} & \text{if } 2 < p < 3, \\ 1 & \text{if } 3 \leq p < 4, \end{cases}$$

then (2) admits a nontrivial weak solution $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.

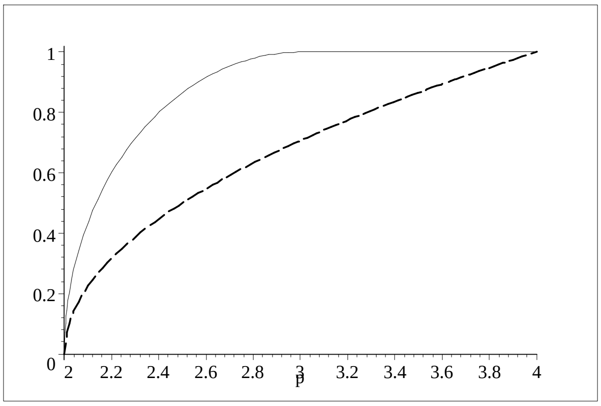

Figure 1: $g_0(p)$ dash, $g(p)$ solid

Under the above assumptions, the problem (2) is of a variational nature. Indeed its weak solutions $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ can be characterized as critical points of the functional $\mathcal{S} : H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$\mathcal{S}(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - |\nabla \phi|^2 + [m_0^2 - (\omega + e\phi)^2]u^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p.$$

The first difficulty in dealing with the functional $\mathcal{S}$ is that it is strongly indefinite, namely it is unbounded both from below and from above on infinite dimensional subspaces.

To avoid this indefiniteness, we will use a well known reduction argument, stated in Theorem 2.2. The finite energy solutions of (1) are pairs $(u, \phi_u) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$, where $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ is the unique solution of

$$\Delta \phi = e(e\phi - \omega)u^2 \quad \text{in } \mathbb{R}^3 \tag{3}$$

(see Lemma 2.1) and $u \in H^1(\mathbb{R}^3)$ is a critical point of

$$I(u) = \mathcal{S}(u, \phi_u).$$

The functional I does not present anymore the strong indefiniteness. Under the assumptions of Theorem 1.1, it will be studied by using an indirect method developed by Struwe [17] and Jeanjean [14]

In the second part of the paper we consider a more general nonlinearity $f(u)$.

Under usual assumptions, which describe behaviours analogous to $|t|^p$ (with $p \in (4, 6)$), it is easy to get a generalization of the existence result ([3]) of Benci and Fortunato; we shall state this generalization in Lemma

3.1. However we point out that all the quoted results share the assumption $\omega < m$.

We are mainly interested to study the limit case $\omega = m$, when (1) becomes

$$\begin{cases} -\Delta u + (2e\omega\phi - e^2\phi^2)u - f'(u) = 0 & \text{in } \mathbb{R}^3 \\ \Delta\phi = e(e\phi - \omega)u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

We notice that, in the first equation, besides the interaction term $(2e\omega\phi - e^2\phi^2)u$, there is no linear term in u. In this sense the situation described by (4) is analogous to the *zero-mass case* for nonlinear field equations (see e.g. [6]).

As in [6], in order to get solutions we need some stronger hypotheses on f, which force it to be inhomogeneous. More precisely we assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions.

(f1) $f \in C^1(\mathbb{R}, \mathbb{R})$;

(f2) $\forall t \in \mathbb{R} \setminus \{0\} : \ \alpha f(t) \leqslant f'(t)t$;

(f3) $\forall t \in \mathbb{R} : f(t) \geqslant C_1 \min(|t|^p, |t|^q)$;

(f4) $\forall t \in \mathbb{R} : |f'(t)| \leqslant C_2 \min(|t|^{p-1}, |t|^{q-1})$;

with $4 < \alpha \leqslant p < 6 < q$ and C_1, C_2, positive constants. We shall prove the following result

Theorem 1.2. *Assume that f satisfies the above hypotheses, then there exists a couple $(u_0, \phi_0) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ which is a weak solution of* (4).

Under the assumptions of Theorem 1.2, standard arguments (again Lemma 3.1) yield the existence of $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ weak solutions of (1) in the case $\omega < m$. The limit case $\omega = m$ is trickier.

Even if the claim of Theorem 1.2 is analogous to the quoted existence results (e.g. Theorem 1.1) and the meaning of weak solution is the same, the approach in the proof is completely different. More precisely in the zero mass case, there exists no functional $\mathcal{S}$ defined on $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that its critical points are weak solutions of (4).

As above we could consider a functional $\mathcal{S} : H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3) \to \mathbb{R}$ whose critical points are finite energy weak solutions. For every $u \in H^1(\mathbb{R}^3)$ we can find $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solution of (3), then we could consider the reduced functional $I(u) = \mathcal{S}(u, \phi_u)$. The reduced functional I has the form

$$I(u) = \mathcal{S}(u, \phi_u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + e\omega\phi_u u^2 - \int_{\mathbb{R}^3} f(u).$$

For such a functional the mountain pass geometry in $H^1(\mathbb{R}^3)$ is not immediately available.

The solution $(u_0, \phi_0) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ will be found as limit of solutions of approximating problems

$$\begin{cases} -\Delta u + (\varepsilon + 2e\omega\phi - e^2\phi^2)u - f'(u) = 0 & \text{in } \mathbb{R}^3, \\ \Delta\phi = e(e\phi - \omega)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{5}$$

For every $\varepsilon > 0$, Lemma 3.1 yields a solution $(u_\varepsilon, \phi_\varepsilon) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$. The stronger assumptions on f (subcritical at infinity, supercritical at zero) give rise to uniform estimate in $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ which allows to pass to the limit as $\varepsilon \to 0$.

Before giving the proof of Theorems 1.1 and 1.2, let us recall some other results concerning (1). In [7] there are existence and nonexistence results when f has a critical growth at infinity. In [2] it is proved the existence of a ground state for (2) (under the existence assumptions of [8]). Other recent papers (e.g. [5] and [15]) are concerned with the Klein-Gordon-Maxwell system with a completely different kind of nonlinearity, satisfying

$$\frac{1}{2}m^2t^2 - f(t) \geq 0.$$

The solutions in this case are called "nontopological solitons". In [5] it is proved the existence of a nontrivial solution if the coupling constant e is sufficiently small. There are also some results for the system (1) in a bounded spatial domain [10] and [11]. In this situation existence and nonexistence of nontrivial solutions depend on the boundary conditions, the boundary data, the kind of nonlinearity and the value of e. Lastly let us make mention of the review paper [13] which contains a large amount of references on this topic.

In the next Sections we shall prove respectively Theorems 1.1 and 1.2. Appendix A contains the proof of a certain inequality, used in Section 2, which involves only elementary Calculus arguments.

2 Proof of Theorem 1.1

We need the following:

Lemma 2.1. *For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ which satisfies*

$$\Delta\phi = e(e\phi - \omega)u^2 \quad \textit{in } \mathbb{R}^3.$$

Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ *is continuously differentiable, and on the set* $\{x \in \mathbb{R}^3 \mid u(x) \neq 0\}$,

$$0 \leqslant \phi_u \leqslant \frac{\omega}{e}. \tag{6}$$

Proof. The proof can be found in [3, 9]. □

Theorem 2.2. *The pair* $(u, \phi) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ *is a solution of* (2) *if and only if* u *is a critical point of*

$$I(u) = \mathcal{S}(u, \phi_u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + (m^2 - \omega^2) u^2 + e\omega \phi_u u^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p,$$

and $\phi = \phi_u$.

For the sake of simplicity we set $\Omega = m^2 - \omega^2 > 0$.

With our assumptions, it is a hard task to find bounded Palais-Smale sequences of functional I, therefore we use an indirect method developed by Struwe [17] and Jeanjean [14]. We look for the critical points of the functional $I_\lambda \in C^1(H^1_r(\mathbb{R}^3), \mathbb{R})$

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \Omega u^2 + e\omega \phi_u u^2 - \frac{\lambda}{p} \int_{\mathbb{R}^3} |u|^p,$$

for λ close to 1, where

$$H^1_r(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) \mid u \text{ is radially symmetric}\}.$$

Set $\delta < 1$ a positive number (which we will estimate later), $J = [\delta, 1]$ and

$$\Gamma := \{\gamma \in C([0,1], H^1_r(\mathbb{R}^3)) \mid \gamma(0) = 0, I_\lambda(\gamma(1)) < 0, \ \forall \lambda \in J\}.$$

Using a slightly modified version of [14, Theorem 1.1], it can be proved the following

Lemma 2.3. *If* $\Gamma \neq \emptyset$ *and for every* $\lambda \in J$

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0, \tag{7}$$

then for almost every $\lambda \in J$ *there is a sequence* $(v_n^\lambda)_n \subset H^1_r(\mathbb{R}^3)$ *such that*

(i) $(v_n^\lambda)_n$ *is bounded;*

(ii) $I_\lambda(v_n^\lambda) \to c_\lambda$;

(iii) $I'_\lambda(v_n^\lambda) \to 0$.

In order to apply Theorem 2.3, we have just to verify that $\Gamma \neq \emptyset$ and (7).

Lemma 2.4. *For any $\lambda \in J$, we have that $\Gamma \neq \emptyset$.*

Proof. Let $u \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ and let $\theta > 0$. Define $\gamma : [0,1] \to H_r^1(\mathbb{R}^3)$ such that $\gamma(t) = t\theta u$, for all $t \in [0,1]$. By (6), for any $\lambda \in J$, we have that

$$I_\lambda(\gamma(1)) = I_\lambda(\theta u) \leqslant \frac{\theta^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \Omega u^2 + \omega^2 u^2 - \delta \frac{\theta^p}{p} \int_{\mathbb{R}^3} |u|^p,$$

and then certainly $\gamma \in \Gamma$ for a suitable choice of θ. □

Lemma 2.5. *For any $\lambda \in J$, we have that $c_\lambda > 0$.*

Proof. Observe that for any $u \in H_r^1(\mathbb{R}^3)$ and $\lambda \in J$, by (6), we have

$$I_\lambda(u) \geqslant \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \Omega u^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p,$$

and then, by Sobolev embeddings, we conclude that there exists $\rho > 0$ such that for any $\lambda \in J$ and $u \in H_r^1(\mathbb{R}^3)$ with $u \neq 0$ and $\|u\| \leqslant \rho$, it results $I_\lambda(u) > 0$. In particular, for any $\|u\| = \rho$, we have $I_\lambda(u) \geqslant \tilde{c} > 0$. Now fix $\lambda \in J$ and $\gamma \in \Gamma$. Since $\gamma(0) = 0 \neq \gamma(1)$ and $I_\lambda(\gamma(1)) \leqslant 0$, certainly $\|\gamma(1)\| > \rho$. By continuity, we deduce that there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Therefore, for any $\lambda \in J$,

$$c_\lambda \geqslant \inf_{\gamma \in \Gamma} I_\lambda(\gamma(t_\gamma)) \geqslant \tilde{c} > 0.$$

□

Proof of Theorem 1.1. Let $\lambda \in J$ for which there exists a bounded Palais-Smale sequence $(v_\lambda^n)_n$ in $H_r^1(\mathbb{R}^3)$ for functional I_λ at level c_λ, namely

$$I_\lambda(v_\lambda^n) \to c_\lambda;$$
$$I'_\lambda(v_\lambda^n) \to 0 \text{ in } (H_r^1(\mathbb{R}^3))'.$$

Up to a subsequence, we can suppose that there exists $v_\lambda \in H_r^1(\mathbb{R}^3)$ such that

$$v_\lambda^n \rightharpoonup v_\lambda \text{ weakly in } H_r^1(\mathbb{R}^3) \tag{8}$$

and

$$v_\lambda^n(x) \to v_\lambda(x) \text{ a.e. in } \mathbb{R}^N.$$

We make the following claims:

$$I'_\lambda(v_\lambda) = 0, \tag{9}$$

$$v_\lambda \neq 0$$

$$I_\lambda(v_\lambda) \leqslant c_\lambda. \tag{10}$$

Claim (9) follows immediately by [2, Lemma 2.7].

Suppose by contradiction that $v_\lambda = 0$, then, since $v_n^\lambda \to v_\lambda(\equiv 0)$ in $L^p(\mathbb{R}^3)$ and $I'_\lambda(v_n^\lambda)[v_n^\lambda] = o_n(1)\|v_n^\lambda\|$, we have

$$\int_{\mathbb{R}^3} |\nabla v_n^\lambda|^2 + \Omega(v_n^\lambda)^2 \leqslant \int_{\mathbb{R}^3} |\nabla v_n^\lambda|^2 + \Omega(v_n^\lambda)^2 + 2e\omega\phi_{v_n^\lambda}(v_n^\lambda)^2 - e^2\phi_{v_n^\lambda}^2(v_n^\lambda)^2$$
$$= \lambda \int_{\mathbb{R}^3} |v_n^\lambda|^p + o_n(1)\|v_n^\lambda\| = o_n(1).$$

Hence $v_n^\lambda \to 0$ in $H^1(\mathbb{R}^3)$ and we get a contradiction with (7).

We pass to prove (10). Since $v_n^\lambda \to v_\lambda$ in $L^p(\mathbb{R}^3)$, by (8), by the weak lower semicontinuity of the $H^1(\mathbb{R}^3)-$norm and by Fatou Lemma, we get $I_\lambda(v_\lambda) \leqslant c_\lambda$.

Now we are allowed to consider a suitable $\lambda_n \nearrow 1$ such that, for any $n \geqslant 1$, there exists $v_n \in H_r^1(\mathbb{R}^3) \setminus \{0\}$ satisfying

$$(I_{\lambda_n})'(v_n) = 0 \text{ in } (H_r^1(\mathbb{R}^3))', \tag{11}$$

$$I_{\lambda_n}(v_n) \leqslant c_{\lambda_n}. \tag{12}$$

We want to prove that such a sequence is bounded.

By [9], v_n satisfies the Pohozaev equality

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 + 3\Omega v_n^2 + 5e\omega\phi_{v_n}v_n^2 - 2e^2\phi_{v_n}^2 v_n^2 - \frac{6\lambda_n}{p}\int_{\mathbb{R}^3} |v_n|^p = 0. \tag{13}$$

Therefore, by (11), (12) and (13) we have that the following system holds

$$\begin{cases} \int_{\mathbb{R}^3} \frac{1}{2}|\nabla v_n|^2 + \frac{1}{2}\Omega v_n^2 + \frac{e\omega}{2}\phi_{v_n}v_n^2 - \frac{\lambda_n}{p}|v_n|^p \leqslant c_{\lambda_n}, \\ \int_{\mathbb{R}^3} |\nabla v_n|^2 + 3\Omega v_n^2 + 5e\omega\phi_{v_n}v_n^2 - 2e^2\phi_{v_n}^2 v_n^2 - \frac{6\lambda_n}{p}|v_n|^p = 0, \\ \int_{\mathbb{R}^3} |\nabla v_n|^2 + \Omega v_n^2 + 2e\omega\phi_{v_n}v_n^2 - e^2\phi_{v_n}^2 v_n^2 - \lambda_n|v_n|^p = 0. \end{cases}$$

Subtracting by the first the second multiplied by α and the third multiplied by $(1-6\alpha)/p$, we get

$$\frac{p - 2\alpha p - 2 + 12\alpha}{2p}\int_{\mathbb{R}^3} |\nabla v_n|^2 + \int_{\mathbb{R}^3} \left[C_{p,\alpha}\Omega + B_{p,\alpha}e\omega\phi_{v_n} + A_{p,\alpha}e^2\phi_{v_n}^2\right] v_n^2 \leqslant c_{\lambda_n},$$

where

$$C_{p,\alpha} = \frac{(p-2)\,(1-6\alpha)}{2p},$$
$$B_{p,\alpha} = \frac{p-10\alpha p-4+24\alpha}{2p},$$
$$A_{p,\alpha} = \frac{1+2\alpha\,(p-3)}{p}.$$

It is easy to see that

$$\frac{p-2\alpha p-2+12\alpha}{2p} > 0,$$

if and only if

$$\alpha > \frac{2-p}{2(6-p)}.$$

In the Appendix (see Lemma A.1) we will prove that there exists $\alpha \in \left(\frac{2-p}{2(6-p)}, \frac{1}{6}\right)$ such that

$$C_{p,\alpha}\Omega + B_{p,\alpha} e\omega\phi_{v_n} + A_{p,\alpha} e^2\phi_{v_n}^2 \geqslant 0,$$

then we can argue that

$$\|\nabla v_n\|_2 \leqslant C \text{ for all } n \geqslant 1. \tag{14}$$

Moreover, by (11), we have

$$\Omega\int_{\mathbb{R}^3} v_n^2 \leqslant \int_{\mathbb{R}^3} |\nabla v_n|^2 + \Omega v_n^2 + 2e\omega\phi_{v_n} v_n^2 - e^2\phi_{v_n}^2 v_n^2 = \lambda_n \int_{\mathbb{R}^3} |v_n|^p. \tag{15}$$

Since for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $t^p \leqslant C_\varepsilon t^6 + \varepsilon t^2$, for all $t \geqslant 0$, taking $\varepsilon = \Omega/2$, by (15) we get

$$\frac{\Omega}{2}\int_{\mathbb{R}^3} v_n^2 \leqslant C_\varepsilon \int_{\mathbb{R}^3} v_n^6.$$

Therefore, by the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and (14) we deduce that $(v_n)_n$ is bounded in $H^1(\mathbb{R}^3)$.

Up to a subsequence, there exists $v_0 \in H_r^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup v_0 \text{ weakly in } H_r^1(\mathbb{R}^3).$$

By (11), we have that

$$I'(v_n) = (I_{\lambda_n})'(v_n) + (\lambda_n - 1)|v_n|^{p-2}v_n = (\lambda_n - 1)|v_n|^{p-2}v_n$$

so $(v_n)_n$ is a Palais-Smale sequence for the functional $I|_{H^1_r}$, since the sequence $(|v_n|^{p-2}v_n)_n$ is bounded in $\left(H^1_r(\mathbb{R}^3)\right)'$.
By [2, Lemma 2.7], we have that $I'(v_0)=0$.

To conclude the proof, it remains to check that $v_0\neq 0$.
By (11), we have

$$\int_{\mathbb{R}^3}|\nabla v_n|^2+\Omega v_n^2\leqslant\int_{\mathbb{R}^3}|\nabla v_n|^2+\Omega v_n^2+2e\omega\phi_{v_n}v_n^2-e^2\phi_{v_n}^2v_n^2\leqslant\int_{\mathbb{R}^3}|v_n|^p$$

and then, there exists $C>0$ such that $\|v_n\|_p\geqslant C$. Since $v_n\to v_0$ in $L^p(\mathbb{R}^3)$, we conclude. □

3 Proof of Theorem 1.2

The following lemma generalizes the existence result of [3].

Lemma 3.1. *Let f satisfy the following hypotheses:*

- **(f1)** $f\in C^1(\mathbb{R},\mathbb{R})$;
- **(f2)** $\exists\alpha>4$ *such that* $\forall t\in\mathbb{R}\setminus\{0\}:\ \alpha f(t)\leqslant f'(t)t$;
- **(f5)** $f'(t)=o(|t|)$ *as* $t\to 0$;
- **(f6)** $\exists C_1,C_2\geqslant 0$ *and* $p<6$ *such that* $\forall t\in\mathbb{R}:|f'(t)|\leqslant C_1+C_2|t|^{p-1}$.

Assume that $0<\omega<m$. Then (1) admits a nontrivial weak solution $(u,\phi)\in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$.

We simply give an outline of the proof.

- Using the same reduction argument (Lemma 2.1 and Theorem 2.2) applied to (1), it is immediately seen that that $(u,\phi)\in H^1(\mathbb{R}^3)\times\mathcal{D}^{1,2}(\mathbb{R}^3)$ is a solution of (1) if and only if $u\in H^1(\mathbb{R}^3)$ is a critical point of

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2+(m^2-\omega^2)u^2+e\omega\phi_u u^2-\int_{\mathbb{R}^3}f(u),$$

and $\phi=\phi_u$.

- The functional I satisfies the Palais-Smale condition in $H^1_r(\mathbb{R}^3)$.

- The functional I shows the Mountain Pass geometry.

Remark 3.2. *If f is odd, just like in [3], the $\mathbb{Z}_2$-Mountain Pass Theorem [1] yields infinitely many solutions.*

Now we can prove Theorem 1.2.

As we said in the Introduction, for every $\varepsilon > 0$, we consider the approximating problem (5). The above Lemma gives the solution $(u_\varepsilon, \phi_\varepsilon) \in H^1(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$. More precisely they are mountain pass type solutions and they are radially symmetric, in the sense that $u_\varepsilon \in H^1_r(\mathbb{R}^3)$ is a critical point of

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \varepsilon u^2 + e\omega\phi_u u^2 - \int_{\mathbb{R}^3} f(u),$$

at the level

$$c_\varepsilon = \inf_{g\in\Gamma_\varepsilon} \max_{\theta\in[0,1]} I_\varepsilon(g(\theta)),$$

where

$$\Gamma_\varepsilon = \left\{g \in C\big([0,1], H^1(\mathbb{R}^3)\big) \mid g(0) = 0,\ I_\varepsilon(g(1)) \leqslant 0,\ g(1) \neq 0\right\}.$$

Moreover, u_ε belongs to the Nehari manifold of I_ε:

$$\mathcal{N}_\varepsilon = \left\{u \in H^1(\mathbb{R}^3) \setminus \{0\} \mid \int_{\mathbb{R}^3} |\nabla u|^2 + \varepsilon u^2 + 2e\omega\phi_u u^2 - e^2\phi_u^2 u^2 = \int_{\mathbb{R}^3} f'(u)u\right\}.$$

In the sequel, we will refer to those approximating solutions as $\varepsilon-$*solutions*.

Lemma 3.3. *There exists $C > 0$ such that $c_\varepsilon < C$, for any $0 < \varepsilon < 1$.*

Proof. Let $0 < \varepsilon < 1$ and $g \in \Gamma_\varepsilon$, we have

$$c_\varepsilon \leqslant \max_{\theta\in[0,1]} I_\varepsilon(g(\theta)) = I_\varepsilon(g(\theta_0)) \leqslant I_1(g(\theta_0)).$$

□

Lemma 3.4. *There exists $C > 0$ such that $\|u_\varepsilon\|_{\mathcal{D}^{1,2}} \geqslant C$, for any $\varepsilon > 0$. Moreover, for any $\varepsilon > 0$,*

$$\int_{\mathbb{R}^3} f'(u_\varepsilon)u_\varepsilon \geqslant C. \tag{16}$$

Proof. Since u_ε is solution of (5), using (6), we have

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 &\leqslant \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2 + 2e\omega\phi_{u_\varepsilon} u_\varepsilon^2 - e^2\phi_{u_\varepsilon}^2 u_\varepsilon^2 = \int_{\mathbb{R}^3} f'(u_\varepsilon)u_\varepsilon \\ &\leqslant \int_{\mathbb{R}^3} |u_\varepsilon|^6 \leqslant C\left(\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2\right)^3\end{aligned}$$

and so we get the conclusion. □

We need a uniform boundedness estimate on the family of the $\varepsilon-$solutions, letting ε go to zero.
Actually, we have the following result

Lemma 3.5. *There exists a positive constant C which is a uniform upper bound for the family $(u_\varepsilon, \phi_{u_\varepsilon})_{\varepsilon>0}$ in the $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)-$norm.*

Proof. We have

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2 + e\omega\phi_{u_\varepsilon} u_\varepsilon^2 - \int_{\mathbb{R}^3} f(u_\varepsilon) = c_\varepsilon,$$

$$\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2 + 2e\omega\phi_{u_\varepsilon} u_\varepsilon^2 - e^2\phi_{u_\varepsilon}^2 u_\varepsilon^2 - \int_{\mathbb{R}^3} f'(u_\varepsilon)u_\varepsilon = 0.$$

By Lemma 3.3 and (**f2**) we deduce that

$$\left(\frac{\alpha}{2} - 1\right)\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2 + \left(\frac{\alpha}{2} - 2\right)\int_{\mathbb{R}^3} e\omega\phi_{u_\varepsilon} u_\varepsilon^2 \leqslant C, \tag{17}$$

while, by the second equation of (5), we have

$$\int_{\mathbb{R}^3} |\nabla \phi_{u_\varepsilon}|^2 + e^2\phi_\varepsilon^2 u_\varepsilon^2 = \int_{\mathbb{R}^3} e\omega\phi_{u_\varepsilon} {u_\varepsilon}^2. \tag{18}$$

Combining together (17) and (18), we infer that $(u_\varepsilon, \phi_{u_\varepsilon})_{\varepsilon>0}$ is bounded in the $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)-$norm. □

Now we deduce that, for any $\varepsilon_n \to 0$, there exist a subsequence of $(u_{\varepsilon_n}, \phi_{u_{\varepsilon_n}})_n$ (which we relabel in the same way), and $(u_0, \phi_0) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$\begin{aligned} u_{\varepsilon_n} &\rightharpoonup u_0, \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3), \\ \phi_{u_{\varepsilon_n}} &\rightharpoonup \phi_0, \quad \text{in } \mathcal{D}^{1,2}(\mathbb{R}^3). \end{aligned}$$

We want to show that, if the sequence $(u_{\varepsilon_n}, \phi_{u_{\varepsilon_n}})_n$ concentrates, then (u_0, ϕ_0) is a weak nontrivial solution of (4). From now on, we use u_n and ϕ_n in the place of u_{ε_n} and $\phi_{u_{\varepsilon_n}}$.

Now we can prove the existence result in the limit case.
Proof of Theorem 1.2. By [4, Lemma 13] and [16, Proposition 24], and by (16), we have that

$$\int_{\mathbb{R}^3} f'(u_0)u_0 = \lim_n \int_{\mathbb{R}^3} f'(u_n)u_n \geqslant C > 0,$$

and so $u_0 \neq 0$.

Let us show that (u_0, ϕ_0) is a weak solution of (4), namely

$$\int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \psi + 2e\omega\phi_0 u_0 \psi - e^2\phi_0^2 u_0 \psi = \int_{\mathbb{R}^3} f'(u_0)\psi,$$
$$\int_{\mathbb{R}^3} \nabla \phi_0 \cdot \nabla \psi + e^2\phi_0 u_0^2 \psi = \int_{\mathbb{R}^3} e\omega u_0^2 \psi,$$

for any ψ test function.

Since, for any $n \geqslant 1$, (u_n, ϕ_n) is a solution of (5), we have

$$\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla \psi + \varepsilon_n u_n \psi + 2e\omega\phi_n u_n \psi - e^2\phi_n^2 u_n \psi = \int_{\mathbb{R}^3} f'(u_n)\psi,$$
$$\int_{\mathbb{R}^3} \nabla \phi_n \cdot \nabla \psi + e^2\phi_n u_n^2 \psi = \int_{\mathbb{R}^3} e\omega u_n^2 \psi.$$

Let us prove that

$$\int_{\mathbb{R}^3} \phi_n u_n \psi \to \int_{\mathbb{R}^3} \phi_0 u_0 \psi. \tag{19}$$

Indeed, denoting with $K = Supp(\psi)$, we observe that

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \phi_n u_n \psi - \phi_0 u_0 \psi \right| &\leqslant \int_{\mathbb{R}^3} |\phi_n u_n \psi - \phi_n u_0 \psi| + \int_{\mathbb{R}^3} |\phi_n u_0 \psi - \phi_0 u_0 \psi| \\
&\leqslant \int_{\mathbb{R}^3} |\phi_n||u_n - u_0||\psi| + \int_{\mathbb{R}^3} |\phi_n - \phi_0||u_0||\psi| \\
&\leqslant \left(\int_{\mathbb{R}^3} |\phi_n|^6 \right)^{\frac{1}{6}} \left(\int_K |u_n - u_0|^{\frac{6}{5}} \right)^{\frac{5}{6}} \sup |\psi| \\
&\quad + \left(\int_K |\phi_n - \phi_0|^{\frac{6}{5}} \right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} |u_0|^6 \right)^{\frac{1}{6}} \sup |\psi|,
\end{aligned}$$

and so we get (19), since $u_n \rightharpoonup u_0$ and $\phi_n \rightharpoonup \phi_0$ in $H^1(K)$.
Let us prove that

$$\int_{\mathbb{R}^3} \phi_n^2 u_n \psi \to \int_{\mathbb{R}^3} \phi_0^2 u_0 \psi. \tag{20}$$

Indeed, we have

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \phi_n^2 u_n \psi - \phi_0^2 u_0 \psi \right| &\leqslant \int_{\mathbb{R}^3} \phi_n^2 |u_n - u_0||\psi| + \int_{\mathbb{R}^3} |\phi_n^2 - \phi_0^2||u_0||\psi| \\
&\leqslant \left(\int_{\mathbb{R}^3} |\phi_n|^6 \right)^{\frac{1}{6}} \left(\int_K |u_n - u_0|^{\frac{3}{2}} \right)^{\frac{2}{3}} \sup |\psi| \\
&\quad + \left(\int_K |\phi_n^2 - \phi_0^2|^{\frac{6}{5}} \right)^{\frac{5}{6}} \left(\int_{\mathbb{R}^3} |u_0|^6 \right)^{\frac{1}{6}} \sup |\psi| \\
&= o_n(1).
\end{aligned}$$

Therefore, by (19) and (20) and since ψ has compact support, we have

$$\begin{array}{ccccccccc}
\underbrace{\int_{\mathbb{R}^3} \nabla u_n \cdot \nabla \psi} & + & \underbrace{\int_{\mathbb{R}^3} \varepsilon_n u_n \psi} & + & \underbrace{\int_{\mathbb{R}^3} 2e\omega \phi_n u_n \psi} & - & \underbrace{\int_{\mathbb{R}^3} e^2 \phi_n^2 u_n \psi} & = & \underbrace{\int_{\mathbb{R}^3} f'(u_n)\psi,} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\int_{\mathbb{R}^3} \nabla u_0 \cdot \nabla \psi & + & 0 & + & \int_{\mathbb{R}^3} 2e\omega \phi_0 u_0 \psi & - & \int_{\mathbb{R}^3} e^2 \phi_0^2 u_0 \psi & = & \int_{\mathbb{R}^3} f'(u_0)\psi.
\end{array}$$

Analogously, we have

$$\begin{array}{ccccc}
\underbrace{\int_{\mathbb{R}^3} \nabla \phi_n \cdot \nabla \psi} & + & \underbrace{\int_{\mathbb{R}^3} e^2 \phi_n u_n^2 \psi} & = & \underbrace{\int_{\mathbb{R}^3} u_n^2 \psi,} \\
\downarrow & & \downarrow & & \downarrow \\
\int_{\mathbb{R}^3} \nabla \phi_0 \cdot \nabla \psi & + & \int_{\mathbb{R}^3} e^2 \phi_0 u_0^2 \psi & = & \int_{\mathbb{R}^3} u_0^2 \psi.
\end{array}$$

In particular, by this last identity, we infer that $\phi_0 \neq 0$ and we conclude. $\square$

A Appendix

Lemma A.1. *Let $p \in (2,4)$ and $\omega \in (0, g(p)\, m)$. Then there exists $\alpha \in I_p = \left(\frac{2-p}{2(6-p)}, \frac{1}{6} \right)$ such that*

$$A_{p,\alpha} e^2 \phi_{v_n}^2 + B_{p,\alpha} e\omega \phi_{v_n} + C_{p,\alpha} \Omega \geqslant 0,$$

where

$$
\begin{aligned}
A_{p,\alpha} &= \frac{1+2\alpha\,(p-3)}{p}, \\
B_{p,\alpha} &= \frac{p-10\alpha p-4+24\alpha}{2p}, \\
C_{p,\alpha} &= \frac{(p-2)\,(1-6\alpha)}{2p}.
\end{aligned}
$$

Proof. Keeping in mind (6), we have to show that

$$
f(t) = A_{p,\alpha}t^2 + B_{p,\alpha}\omega t + C_{p,\alpha}\Omega \geqslant 0, \qquad \text{for any } t \in [0,\omega]. \tag{21}
$$

First we notice that for any $\alpha \in I_p$

$$
\begin{aligned}
A_{p,\alpha} &> 0 \\
C_{p,\alpha} &> 0
\end{aligned}
$$

Now we have to distinguish two cases: $p \in (3,4)$ and $p \in (2,3]$.
In the first one, if $\alpha = \frac{4-p}{24-10p} \in I_p$, we have $B_{p,\alpha} = 0$ and so we have proved (21).
Let now consider the case $p \in (2,3]$. Since f reaches its minimum in $-\frac{B_{p,\alpha}\omega}{2A_{p,\alpha}}$ and it belongs to $[0,\omega]$, f is non-negative in $[0,\omega]$ if and only if

$$
f\left(-\frac{B_{p,\alpha}\omega}{2A_{p,\alpha}}\right) \geqslant 0,
$$

and, with straightforward calculations and using the fact that $A_{p,\alpha}+B_{p,\alpha} = C_{p,\alpha}$, this is equivalent to say that

$$
\frac{m^2}{\omega^2} \geqslant \frac{(A_{p,\alpha}+C_{p,\alpha})^2}{4A_{p,\alpha}C_{p,\alpha}}. \tag{22}
$$

We set

$$
K_p(\alpha) := \frac{(A_{p,\alpha}+C_{p,\alpha})^2}{4A_{p,\alpha}C_{p,\alpha}} = \frac{p^2}{8(p-2)} \cdot \frac{(1-2\alpha)^2}{1-6\alpha} \cdot \frac{1}{1+2\alpha(p-3)}
$$

and we shall prove that

$$
\inf_{\alpha \in I_p} K_p(\alpha) = \frac{1}{(p-2)(4-p)}, \tag{23}
$$

and then we could conclude. Indeed, if $\omega \in (0, g(p)m)$, then by (23)

$$\frac{m^2}{\omega^2} > \inf_{\alpha \in I_p} K_p(\alpha),$$

and so there exists $\alpha \in I_p$ such that

$$\frac{m^2}{\omega^2} \geqslant K_p(\alpha),$$

by which we deduce (22).
Let us now prove (23).
Let us consider the case $p = 3$: in such situation

$$K_3(\alpha) = \frac{9}{8} \cdot \frac{(1-2\alpha)^2}{1-6\alpha} \quad \text{and} \quad I_3 = \left(-\frac{1}{6}, \frac{1}{6}\right).$$

Since the function K_3 is increasing in I_3, we have

$$\inf_{\alpha \in I_3} K_3 = K_3\left(-\frac{1}{6}\right) = 1$$

and so we have proved (23).
Now, let us consider the case $p \in (2,3)$. We write $K_p(\alpha) = \frac{p^2}{8(p-2)} \cdot H_1(\alpha) \cdot H_2(\alpha)$ where

$$H_1(\alpha) := \frac{(1-2\alpha)^2}{1-6\alpha}, \qquad H_2(\alpha) := \frac{1}{1+2\alpha(p-3)}.$$

Since for $i = 1, 2$, H_i is a positive and increasing function in I_p, we have

$$\inf_{\alpha \in I_p} K_p = K_p\left(\frac{2-p}{2(6-p)}\right) = \frac{1}{(p-2)(4-p)},$$

and so we obtain (23). □

References

[1] A. Ambrosetti, P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Func. Anal. **14** (1973), 349–381.

[2] A. Azzollini, A. Pomponio, *Ground state solutions for the nonlinear Klein-Gordon-Maxwell equations*, to appear on Topol. Methods Nonlinear Anal.

[3] V. Benci, D. Fortunato, *Solitary waves of the nonlinear Klein-Gordon field equation coupled with the Maxwell equations,* Rev. Math. Phys. **14** (2002), 409–420.

[4] V. Benci, D. Fortunato, *Towards a unified theory for classical electrodynamics*, Arch. Rational Mech. Anal., **173**, (2004), 379-414.

[5] V. Benci and D. Fortunato, *Solitary waves in abelian gauge theories*, Adv. Nonlinear Stud., **8** (2008), 327–352.

[6] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations, I - Existence of a ground state*, Arch. Rational Mech. Anal., **82**, (1983), 313–345.

[7] D. Cassani, *Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell's equations*, Nonlinar Anal., **58** (7-8) (2004), 733–747.

[8] T. D'Aprile, D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A, **134**, (2004), 893–906.

[9] T. D'Aprile, D. Mugnai, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Adv. Nonlinear Stud., **4**, (2004), 307–322.

[10] P. d'Avenia, L. Pisani, G. Siciliano, *Klein-Gordon-Maxwell systems in a bounded domain*, Discrete Contin. Dyn. Syst., to appear.

[11] P. d'Avenia, L. Pisani, G. Siciliano, *Dirichlet and Neumann problems for Klein-Gordon-Maxwell systems*, Nonlinear Anal., to appear.

[12] B. Felsager, Geometry, Particle and fields, *Springer Verlag, New York*, 1998.

[13] D. Fortunato, *Solitary Waves and Electromagnetic Field*, Boll. Unione Mat. Ital. (9) **I** (2008), 767-789.

[14] L. Jeanjean, *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on* $\mathbb{R}^N$, Proc. R. Soc. Edinb., Sect. A, Math., **129**, 787–809, (1999).

[15] E. Long, *Existence and stability of solitary waves in non-linear Klein-Gordon-Maxwell equations*, Rev. Math. Phys., **18** (2006), 747–779.

[16] L. Pisani, *Remarks on the sum of Lebesgue spaces*, preprint.

[17] M. Struwe, *The existence of surfaces of constant mean curvature with free boundaries*, Acta Math. **160** (1988), no. 1-2, 19–64.