

A product integration rule on equispaced nodes for highly oscillating integrals[☆]

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ABSTRACT

This paper provides a product integration rule for highly oscillating integrands of the type

$$\int_{-a}^a e^{-i\omega(x-y)} f(x) dx, \quad a > 0, \quad \mathbf{i} = \sqrt{-1}, \quad y \in [-a, a], \quad \omega \in \mathbb{R}^+,$$

based on the approximation of f by means of the Generalized Bernstein polynomials $B_{m,\ell}f$. The rule involves the samples of f at $m+1$ equally spaced points of $[-a, a]$, and differently from the classical Bernstein polynomials, the suitable modulation of the parameter $\ell \in \mathbb{N}$ allows to increase the accuracy of the product rule, as the smoothness of f increases. Stability and error estimates are proven for f belonging to the space of continuous functions and their Sobolev-type subspaces. Finally, some numerical tests which confirm such theoretical estimates are shown.

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1. Introduction

From a numerical point of view, the main difficulty to treat integrals of the type

$$\int_{-a}^a e^{-i\omega(x-y)} f(x) dx, \quad a > 0, \quad \mathbf{i} = \sqrt{-1}, \quad y \in [-a, a], \quad \omega \in \mathbb{R}^+, \quad (1)$$

depends on the presence of the kernel $e^{-i\omega(x-y)}$, which highly oscillates for frequencies $\omega \gg 1$. For the evaluation of such integrals, many accurate formulas exist in the literature, mainly based on the zeros of

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orthogonal polynomials; see e.g. [1–6]. On the other hand, in many practical applications, the function f is known only at equispaced nodes; see e.g. [7,8].

In order to adopt the values of f at equidistant nodes, a widely used technique is based on composite quadrature rules of “lower” degree, such as Filon’s type rules, whose degree of approximation cannot be improved over the saturation class of the approximation process. Recently, in [9] starting from the knowledge of f at a finite set of equally spaced nodes, an approach has been introduced to efficiently compute weighted integrals, but not advisable in the case of oscillating integrands. For a short review on the main techniques the interested reader can consult [1].

Finding accurate quadrature rules based on equidistant points is still an open problem. Here, we introduce a product integration rule based on the approximation of the function f through the generalized Bernstein (shortly GB) polynomials $\bar{B}_{m,\ell}f$ of degree m and parameter ℓ defined in the interval $[-a, a]$. Such polynomials represent an adequate tool for our aims, since they make use of the samples of f at $m + 1$ equally spaced nodes; see e.g. [7,10–13]. Moreover, unlike the classical Bernstein polynomials $\bar{B}_m f$, they depend on a parameter ℓ that can be suitably chosen according to the smoothness degree of f . This nice property allows to avoid saturation problems [14]. However, ℓ cannot be fixed arbitrarily high. For $\ell \rightarrow \infty$, the GB polynomials tend to the Lagrange interpolation polynomial based on equidistant nodes which is a well known unstable operator. Hence, thresholds of such parameter have been studied; see e.g. [13, Remark 4.1].

The outline of the paper is as follows. In Section 2 we provide definition and main properties of GB polynomials. Section 3 contains the main results regarding the product integration rule and its error estimates. In Section 4 two numerical examples are given to corroborate the theoretical estimate. The proofs are given in the last Section 5.

2. Notation and preliminary results

In the sequel $\mathcal{C}$ denotes any positive constant having different meanings at different occurrences and the writing $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ has to be understood as $\mathcal{C}$ not depending on $a, b, \dots$

As usual, $\mathbb{P}_m$ denotes the space of the algebraic polynomials of degree less than or equal to m , and $C^0 := C^0([-a, a])$ is the space of the continuous functions on $[-a, a]$ equipped with the uniform norm $\|f\| := \max_{x \in [-a, a]} |f(x)|$.

For a fixed positive $a \in \mathbb{R}$, consider the m -th Bernstein polynomial $\bar{B}_m f$ of a given $f \in C^0$,

$$\bar{B}_m f(x) := \sum_{k=0}^m f(t_k) \bar{p}_{m,k}(x), \quad t_k := -a + k \frac{2a}{m}, \quad x \in [-a, a], \quad (2)$$

where

$$\bar{p}_{m,k}(x) := \binom{m}{k} \left(\frac{a+x}{2a} \right)^k \left(\frac{a-x}{2a} \right)^{m-k}, \quad (3)$$

with $\bar{p}_{m,k}(x) \geq 0$, and $\sum_{k=0}^m \bar{p}_{m,k}(x) = 1, \forall x \in [-a, a]$. Based on $\bar{B}_m f$, the ℓ -iterated boolean sum $\bar{B}_{m,\ell} f \in \mathbb{P}_m$, $\ell \in \mathbb{N}$, is defined for any $f \in C^0$ as

$$\bar{B}_{m,\ell} f = f - (f - \bar{B}_m f)^\ell, \quad \bar{B}_{m,1} f = \bar{B}_m f.$$

The polynomial $\bar{B}_{m,\ell} f$ can be expressed $\forall x \in [-a, a]$, as

$$\bar{B}_{m,\ell} f(x) = \sum_{j=0}^m \bar{p}_{m,j}^{(\ell)}(x) f(t_j), \quad \text{with} \quad \bar{p}_{m,j}^{(\ell)}(x) = \sum_{i=0}^m \bar{p}_{m,i}(x) c_{i,j}^{(m,\ell)}, \quad (4)$$

where $c_{i,j}^{(m,\ell)}$ are the entries of the matrix $C_{m,\ell} \in \mathbb{R}^{(m+1) \times (m+1)}$,

$$C_{m,\ell} = \mathbf{I} + (\mathbf{I} - \mathbf{A}) + \cdots + (\mathbf{I} - \mathbf{A})^{\ell-1}, \quad C_{m,1} = \mathbf{I},$$

being $\mathbf{I}$ the identity matrix of order $m+1$ and $\mathbf{A} \in \mathbb{R}^{(m+1) \times (m+1)}$ the matrix

$$(\mathbf{A})_{i,j} = p_{m,j}(t_i), \quad (i, j) \in \{0, 1, \dots, m\} \times \{0, 1, \dots, m\}.$$

By induction on $\ell = 2^p$, $p \in \mathbb{N}$, the following recurrence relation holds

$$C_{m,2^p} = C_{m,2^{p-1}} + (\mathbf{I} - \mathbf{A})^{2^{p-1}} C_{m,2^{p-1}}, \quad (5)$$

which allows for a fast construction of the subsequence $\{B_{m,2^p}\}_{p=1,2,\dots}$, since

$$B_{m,2^p} f = 2B_{m,2^{p-1}} f - B_{m,2^{p-1}}^2 f. \quad (6)$$

A survey containing properties and applications of GB polynomials is given in [15]. Moreover, specific employments of such polynomials to Fredholm and Volterra integral equations can be found in [7,13,16]. A further application for approximating Cauchy singular integrals has been proposed in [17].

As we can note by (2), the polynomials $\bar{B}_{m,\ell} f$ require the same samples of f at $m+1$ equally spaced points of $[-a, a]$. Moreover, thanks to the dependence on the parameter ℓ , they detain the special property that the speed of convergence to f increases as the smoothness of f improves, unlike the “originating” polynomial $\bar{B}_m f$.

To be more precise, let us consider the Sobolev type space of order r , $1 \leq r \in \mathbb{N}$, defined as

$$W_r = \left\{ f \in C^0 : f^{(r-1)} \in \mathcal{AC} : \|f^{(r)} \phi^r\| < \infty \right\}, \quad \|f\|_{W_r} = \|f\| + \|f^{(r)} \phi^r\|,$$

where $\phi(x) = \sqrt{a^2 - x^2}$, and $\mathcal{AC}$ denotes the space of all locally absolutely continuous functions on $[-a, a]$. Then, for each $f \in W_r$ with $1 \leq r \leq 2\ell$, by [14, Th. 2.1] we deduce

$$\|f - \bar{B}_{m,\ell} f\| \leq C \left(\frac{a^r \|f\|_{W_r}}{(\sqrt{m})^r} \right), \quad C \neq C(m, f, a). \quad (7)$$

In other words, the rate of convergence behaves like the square root of the best polynomial approximation error for this space of functions.

3. A product integration rule

By separating the kernel $e^{-i\omega(x-y)}$ into the real and imaginary parts, we are dealing with integrals of the type

$$\mathcal{I}^\omega f(y) := \int_{-a}^a \kappa(\omega(y-x)) f(x) dx, \quad \text{where} \quad \kappa(\omega(y-x)) = \begin{cases} \sin(\omega(y-x)), \\ \cos(\omega(y-x)). \end{cases} \quad (8)$$

In order to evaluate numerically the above integral, we propose to approximate f by $\bar{B}_{m,\ell} f$, getting

$$\mathcal{I}^\omega f(y) = \int_{-a}^a \kappa(\omega(y-x)) \bar{B}_{m,\ell} f(x) dx + \mathcal{R}_{m,\ell}^\omega f(y) =: \mathcal{I}_{m,\ell}^\omega f(y) + \mathcal{R}_{m,\ell}^\omega f(y). \quad (9)$$

Then, taking (4) into account, we get

$$\mathcal{I}_{m,\ell}^\omega f(y) = \sum_{j=0}^m f(t_j) \sum_{i=0}^m c_{i,j}^{(m,\ell)} \int_{-a}^a \kappa(\omega(y-x)) \bar{p}_{m,i}(x) dx =: \sum_{j=0}^m f(t_j) \sum_{i=0}^m c_{i,j}^{(m,\ell)} q_i(y). \quad (10)$$

Let us note that in the above formula the pathology of the integrand, i.e. the presence of the oscillating kernel, has been isolated in the coefficients q_i that we approximate by using a suitable technique explained in the next subsection.

Next theorem establishes conditions assuring that the rule (10) is stable and convergent for any $f \in C^0$. Moreover, it also provides an error estimate in suitable subspaces of C^0 .

Theorem 3.1. Let $f \in C^0$. For all $a, \omega \in \mathbb{R}^+$, and for each fixed ℓ , the rule (10) is stable, that is

$$\sup_m \|\mathcal{I}_{m,\ell}^\omega f\| \leq \mathcal{C} \|f\|, \quad \mathcal{C} \neq \mathcal{C}(\omega, f), \quad \mathcal{C} = \mathcal{C}(\ell). \quad (11)$$

Moreover, for any $f \in W_r$, $1 \leq r \leq 2\ell$, we have

$$\|\mathcal{R}_{m,\ell}^\omega f\| \leq \mathcal{C} \left(\frac{a^{r+1} \|f\|_{W_r}}{(\sqrt{m})^r} \right), \quad \mathcal{C} \neq \mathcal{C}(m, f, a, \omega). \quad (12)$$

3.1. The approximation of the coefficients

Now we approach to the computation of the coefficients $q_i(y)$, by proposing a suitable technique that treats the pathology of the kernel.

Let $N = \lfloor \omega \frac{a}{\pi} \rfloor + 1$, introduce the partition

$$[-a, a] = \bigcup_{h=1}^N [x_{h-1}, x_h], \quad x_0 = -a, \quad x_h = -a + h\eta, \quad \eta = \frac{2a}{N}, \quad h = 1, 2, \dots, N,$$

and consider the following decomposition in sum

$$q_i(y) = \sum_{h=1}^N \int_{x_{h-1}}^{x_h} \kappa(\omega(y-x)) \bar{p}_{m,i}(x) dx, \quad i = 0, 1, \dots, m.$$

Let us now map each interval $[x_{h-1}, x_h]$ into $[-1, 1]$ through the linear transformations $z = \gamma_h(x) := 2 \frac{x-x_{h-1}}{\eta} - 1$. In this way we get

$$q_i(y) = \frac{\eta}{2} \sum_{h=1}^N \int_{-1}^1 \bar{p}_{m,i}(\gamma_h^{-1}(z)) \kappa(\omega(y - \gamma_h^{-1}(z))) dz, \quad i = 0, 1, \dots, m,$$

and applying the m -point Gauss-Legendre quadrature rule to each integral, we have

$$q_i(y) = \frac{a}{N} \sum_{h=1}^N \left(\sum_{k=1}^m \lambda_k \bar{p}_{m,i}(\gamma_h^{-1}(z_k)) \kappa(\omega(y - \gamma_h^{-1}(z_k))) + \varepsilon_m^{i,h}(y) \right), \quad (13)$$

where $\{z_k\}_{k=1}^m$ are the zeros of the m -th Legendre polynomial and $\{\lambda_k\}_{k=1}^m$ are the related Christoffel numbers. Hence, by replacing the approximation (13) into (10), we have the following product integration rule

$$\tilde{\mathcal{I}}_{m,\ell}^\omega f(y) = \frac{a}{N} \sum_{j=0}^m f(t_j) \sum_{i=0}^m c_{i,j}^{(m,\ell)} \left(\sum_{h=1}^N \sum_{k=1}^m \lambda_k \bar{p}_{m,i}(\gamma_h^{-1}(z_k)) \kappa(\omega(y - \gamma_h^{-1}(z_k))) \right). \quad (14)$$

About the error of the product rule, next theorem holds:

Theorem 3.2. Let $f \in W_r$, with $1 \leq r \leq 2\ell$. For any $y \in [-a, a]$ and for m sufficiently large (say $m > m_0$, m_0 fixed), the following error estimate holds:

$$|\mathcal{I}^\omega f(y) - \tilde{\mathcal{I}}_{m,\ell}^\omega f(y)| \leq \mathcal{C} \left(\|f\|_{W_r} \frac{a^{r+1}}{(\sqrt{m})^r} + a \|f\| m^{\frac{3}{2}} \left(\frac{a}{N} \cdot \frac{m+\omega}{2m-1} \right)^m \right),$$

where $\mathcal{C} \neq \mathcal{C}(m, f, a, \omega)$, and $\mathcal{C} = \mathcal{C}(\ell)$.

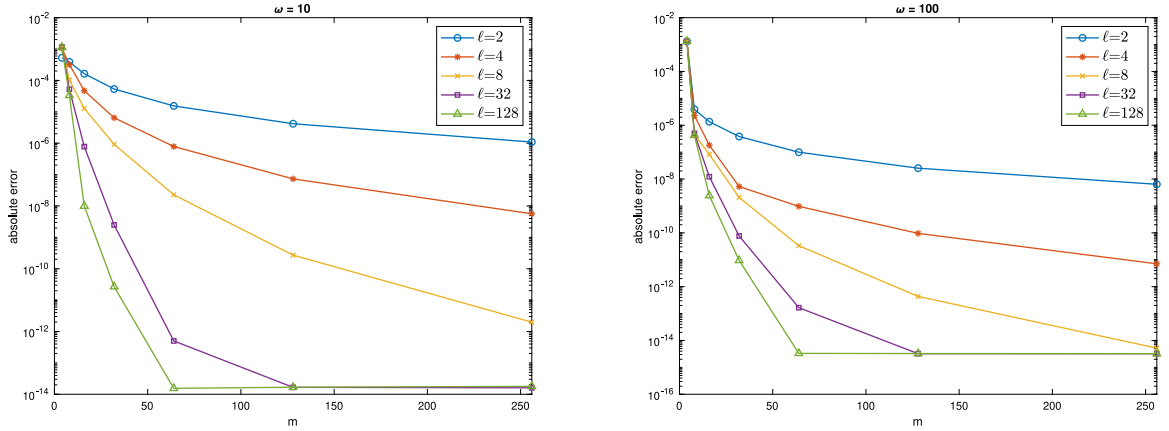


Fig. 1. Plot of the maximum absolute errors attained for integral $\mathcal{I}^\omega f_1(y)$ with different choices of ℓ .

4. Numerical tests

In this section, we show the accuracy of our product rule as well as the reliability of our theoretical estimates, to the following two integrals

$$\mathcal{I}^\omega f_1(y) = \int_{-1}^1 \sin(\omega(y-x)) f_1(x) dx, \quad \mathcal{I}^\omega f_2(y) = \int_{-2}^2 \cos(\omega(y-x)) f_2(x) dx,$$

with $f_1(x) = \tanh(x+1)$, and $f_2(x) = |x+1|^{9/2}$. In both cases, the exact value of the integrals is not known and then we consider as exact the value provided by our quadrature rule with $m = 512$. Table 1 contains, for increasing values of m , the absolute errors

$$e_{m,\ell}^\omega f_j(y) = |\tilde{\mathcal{I}}_{512,\ell}^\omega f_j(y) - \tilde{\mathcal{I}}_{m,\ell}^\omega f_j(y)|, \quad j = 1, 2,$$

obtained with $\ell = 2^8$ in two different points and for several values of ω . Moreover, in Table 2 we show the Mean Estimated Order of Convergence attained in our numerical tests. It is defined as

$$EOC_{m,\ell}^\omega f_j(y) = \frac{\log(e_{m,\ell}^\omega f_j(y)/e_{2m,\ell}^\omega f_j(y))}{\log 2}, \quad MEOC_\ell^\omega f_j(y) = \frac{1}{M-1} \sum_{k=2}^M EOC_{2^k,\ell}^\omega f_j(y), \quad j = 1, 2,$$

where M is a fixed integer. In our case we set $M = 7$.

As we can observe, for the first integral the convergence is very fast due to the analyticity of the function f_1 . For the second integral the convergence is slower since $f_2 \in W_{9/2}$. Finally, in Fig. 1 we display the maximum absolute errors attained for integral $\mathcal{I}^\omega f_1(y)$ over 256 equally spaced points of the interval $[-1, 1]$ when m is fixed and the parameter ℓ varies. It is evident that the error $e_{m,\ell}^\omega f_1(y)$ decreases for increasing values of ℓ .

5. The proofs

Proof of Theorem 3.1. First, let us prove the stability, i.e. (11). By the first identity of (10), for any $y \in [-a, a]$ we have

$$\begin{aligned} |\mathcal{I}_{m,\ell}^\omega f(y)| &\leq \|f\| \int_{-a}^a \sum_{j=0}^m \left| \sum_{i=0}^m c_{i,j}^{(m,\ell)} \bar{p}_{m,i}(x) \right| dx \leq \|f\| \int_{-a}^a \sum_{i=0}^m \bar{p}_{m,i}(x) \sum_{j=0}^m |c_{i,j}^{(m,\ell)}| dx \\ &\leq 2a \|f\| \|C_{m,\ell}\|_\infty \leq C \|f\|, \end{aligned}$$

since $\bar{p}_{m,i}(x) \geq 0$, and $\|C_{m,\ell}\|_\infty \leq 2^\ell - 1$ (see e.g. [16]).

Table 1Numerical results for integral $\mathcal{I}^\omega f_1(y)$ (top) and for integral $\mathcal{I}^\omega f_2(y)$ (bottom).

m	ω	$e_{m,2^s}^\omega f_1(-0.7)$	$e_{m,2^s}^\omega f_1(0.5)$
4	10	7.20e-04	1.30e-04
8		1.63e-05	2.25e-05
16		5.91e-09	4.15e-09
32		2.82e-12	8.81e-12
64		1.32e-15	7.42e-15
128		8.26e-15	3.23e-16
256		3.27e-15	7.28e-15
4	10^2	8.71e-04	1.17e-04
8		2.38e-07	1.06e-07
16		1.37e-10	1.17e-10
32		2.47e-12	4.09e-14
64		2.99e-15	7.88e-16
128		3.07e-15	8.41e-16
256		1.99e-15	1.90e-15
4	10^3	6.93e-04	6.64e-04
8		1.53e-09	1.44e-09
16		3.97e-12	4.25e-12
32		4.63e-14	5.18e-14
64		2.45e-15	1.53e-16
128		1.15e-15	1.07e-15
256		7.71e-16	9.89e-16

m	ω	$e_{m,2^s}^\omega f_2(-1.5)$	$e_{m,2^s}^\omega f_2(1)$
4	10	1.98e-02	6.34e-03
8		3.78e-03	3.47e-03
16		1.88e-04	1.79e-04
32		2.41e-05	2.75e-05
64		3.64e-08	1.30e-07
128		7.76e-10	3.92e-10
256		1.01e-11	4.75e-12
4	10^2	1.36e-02	7.04e-02
8		4.20e-05	2.82e-06
16		7.27e-06	1.52e-06
32		1.08e-07	1.60e-07
64		2.05e-07	1.66e-07
128		4.32e-09	4.58e-09
256		1.36e-10	1.07e-10
4	10^3	6.91e-02	2.67e-02
8		3.79e-07	4.19e-07
16		6.53e-08	7.24e-08
32		1.77e-09	1.90e-09
64		2.08e-09	2.44e-09
128		2.63e-11	2.94e-11
256		2.06e-12	2.67e-12

Table 2Mean Estimated Order of Convergence for integral $\mathcal{I}^\omega f_1(y)$ (top) and for integral $\mathcal{I}^\omega f_2(y)$ (bottom).

ω	$MEOC_{2^s}^\omega f_1(-0.7)$	$MEOC_{2^s}^\omega f_1(0.5)$
10	6.28	5.67
10^2	6.44	5.97
10^3	6.62	6.55

ω	$MEOC_{2^s}^\omega f_2(-1.5)$	$MEOC_{2^s}^\omega f_2(1)$
10	5.14	5.05
10^2	4.43	4.88
10^3	5.83	5.54

Let us now estimate the error. We have

$$|\mathcal{R}_{m,\ell}^\omega f(y)| \leq \int_{-a}^a |f(x) - \bar{B}_{m,\ell} f(x)| |\kappa(\omega(y-x))| dx \leq 2a \|f - \bar{B}_{m,\ell} f\|,$$

and by (7) the thesis follows. $\square$

Proof of Theorem 3.2. We start from

$$|\mathcal{I}^\omega f(y) - \tilde{\mathcal{I}}_{m,\ell}^\omega f(y)| \leq |\mathcal{I}^\omega f(y) - \mathcal{I}_{m,\ell}^\omega f(y)| + |\mathcal{I}_{m,\ell}^\omega f(y) - \tilde{\mathcal{I}}_{m,\ell}^\omega f(y)| =: J_1(y) + J_2(y). \quad (15)$$

By Theorem 3.1

$$J_1(y) \leq \mathcal{C} \|f\|_{W_r} \frac{a^{r+1}}{(\sqrt{m})^r}. \quad (16)$$

About J_2 , we can start from

$$J_2(y) = \left| \sum_{j=0}^m f(t_j) \sum_{i=0}^m c_{i,j}^{(m,\ell)} \frac{a}{N} \sum_{h=1}^N \varepsilon_m^{i,h}(y) \right| \leq \|f\| \frac{a}{N} \sum_{j=0}^m \sum_{i=0}^m |c_{i,j}^{(m,\ell)}| \sum_{h=1}^N |\varepsilon_m^{i,h}(y)|, \quad (17)$$

with $\varepsilon_m^{i,h}(y)$ defined in (13).

To estimate $|\varepsilon_m^{i,h}(y)|$ we recall the following Gauss–Legendre error, holding $\forall g \in W_m([-1, 1])$

$$\left| \int_{-1}^1 g(x) dx - \sum_{k=1}^m \lambda_k g(z_k) \right| \leq \mathcal{C} E_{2m-1}(f) \leq \frac{\mathcal{C}}{(2m-1)^m} (\|g\| + \|g^{(m)} \varphi^m\|), \quad \varphi(x) = \sqrt{1-x^2},$$

where $\|g\| = \sup_{z \in [-1,1]} |g(z)|$. Hence, denoting by $\kappa_y(x)$ the function $\kappa(x, y)$ regarded as function of the only variable x , we have

$$|\varepsilon_m^{i,h}(y)| \leq \frac{\mathcal{C}}{(2m-1)^m} \left\{ \|\bar{p}_{m,i}(\gamma_h^{-1}) \kappa_y(\omega(\gamma_h^{-1}))\| + \left\| [\bar{p}_{m,i}(\gamma_h^{-1}) \kappa_y(\omega(\gamma_h^{-1}))]^{(m)} \varphi^m \right\| \right\}.$$

Now, we observe that

$$\max_{z \in [-1,1]} \bar{p}_{m,i}(\gamma_h^{-1}(z)) = \max_{x \in [x_{h-1}, x_h]} \bar{p}_{m,i}(x) \leq \max_{x \in [-a,a]} \bar{p}_{m,i}(x) = \begin{cases} \binom{m}{i} \left(\frac{i}{m}\right)^i \left(\frac{m-i}{m}\right)^{m-i}, & 1 \leq i \leq m-1, \\ 1, & i=0, \text{ and } i=m. \end{cases}$$

Hence, for $1 \leq i \leq m$ by Stirling's formula it is

$$\max_{z \in [-1,1]} \bar{p}_{m,i}(z) \sim \frac{\sqrt{m}}{\sqrt{2\pi i}} \quad \text{as } m \rightarrow \infty.$$

Consequently, by Bernstein inequality for polynomials (see e.g. [18]) we deduce for $0 \leq s \leq m$, $0 \leq i \leq m$,

$$\max_{z \in [-1,1]} |\bar{p}_{m,i}^{(m-s)}(\gamma_h^{-1}(z)) \varphi^{m-s}(z)| \leq \mathcal{C} \left(\frac{a}{N}\right)^{m-s} m^{m-s} \max_{x \in [-a,a]} \bar{p}_{m,i}(x) \leq \mathcal{C} \sqrt{m} \left(\frac{a}{N}\right)^{m-s} m^{m-s}.$$

Moreover, by the definition of the kernel κ_y in (8) we have

$$\max_{z \in [-1,1]} |\kappa_y^{(s)}(\omega(\gamma_h^{-1}(z)))| \leq \left(\omega \frac{a}{N}\right)^s.$$

Thus, by applying Leibniz formula, we get

$$\begin{aligned} \max_{z \in [-1,1]} |(\bar{p}_{m,i}(\gamma_h^{-1}(z)) \kappa_y(\omega(\gamma_h^{-1}(z))))^{(m)}| &\leq \mathcal{C} \sqrt{m} \left(\frac{a}{N}\right)^m \sum_{s=0}^m \binom{m}{s} m^{m-s} \omega^s \\ &= \mathcal{C} \sqrt{m} \left(\frac{a}{N}\right)^m (m + \omega)^m, \end{aligned}$$

and then

$$|\varepsilon_m^{i,h}(y)| \leq \mathcal{C}\sqrt{m} \left(\frac{a}{N}\right)^m \left(\frac{m+\omega}{2m-1}\right)^m.$$

Now, by (17)

$$J_2(y) \leq \mathcal{C}a\|f\| \sum_{i=0}^m \|C_{m,\ell}\|_{\infty} \sqrt{m} \left(\frac{a}{N}\right)^m \left(\frac{m+\omega}{2m-1}\right)^m \leq \mathcal{C}a\|f\| m^{\frac{3}{2}} \left(\frac{a}{N}\right)^m \left(\frac{m+\omega}{2m-1}\right)^m.$$

The theorem follows by combining last inequality and (16) with (15). $\square$

Data availability

No data was used for the research described in the article.

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