Rendiconti del Circolo Matematico di Palermo Series 2
https://doi.org/10.1007/s12215-022-00775-2

On the traction problem for steady elastic oscillations equations: the double layer potential ansatz

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Received: 19 July 2021 / Accepted: 29 May 2022

Abstract
The three-dimensional traction problem for steady elastic oscillations equations is studied. Representability of its solution by means of a double layer potential is considered instead of the more usual simple layer potential.

Keywords Steady oscillations · Potential theory · Integral representations

Mathematics Subject Classification 74B05 · 31B10 · 35C15

1 Introduction

Several papers have been devoted to develop potential methods for different boundary value problems for the steady-state oscillations for isotropic elastic bodies (see, e.g., [1, 2, 15–17] and the references therein).

In particular, in [12, Chapter VII] a solution of the Dirichlet problem is sought in terms of a double layer potential. In [10] we have achieved a solution of the Dirichlet problem by a simple layer potential. In this case the boundary conditions lead to an integral system of the first kind on the boundary. The solution of this system was obtained following a method given in [3] for the Laplace equation, which can be considered as an extension to higher dimensions of Muskhelishvili method (see [4]). We observe that the method introduced in [3], which hinges on the theory of reducible operators and on the theory of differential forms, was applied to different BVPs for several PDEs in simply and multiple connected domains (see, e.g., [6–9]). We remark that our method uses neither the theory of pseudodifferential operators nor the concept of hypersingular integrals.

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Published online: 04 July 2022

In the present paper we pass to consider the traction problem with datum in $[L^p(\Sigma)]^3$ for the homogeneous system of the steady-state oscillations

$$\mu\Delta u + (\lambda + \mu)\nabla \mathrm{div}\, u + \rho\omega^2 u = 0$$

for an elastic medium with density $\rho > 0$, oscillation frequency $\omega \in \mathbb{R}$, and Lamé constants λ and μ.

We look for a solution $u : \mathbb{R}^3 \to \mathbb{C}^3$ in the form of a double layer potential with density in the Sobolev space $[W^{1,p}(\Sigma)]^3$, where Σ is the boundary of the domain Ω representing the medium. We observe that this layer potential ansatz is different from that one used in [12], where a solution is represented by means of a simple layer potential.

Our main result (Theorem 2) establishes the solvability of the problem under study whenever ω^2 is an interior traction eigenvalue or not. If it is, then the datum must satisfy certain compatibility conditions under which a singular integral system admits solutions (see Lemma 3). If not, the problem is always solvable.

The paper is organized as follows. After summarizing some notations in Sect. 2, we consider the system of elasto-static oscillations and its fundamental matrix in Sect. 3. Section 4 deals with boundary integral operators related to the system and with auxiliary results associated to certain boundary integral equations. In Sect. 5 we recall the results we have obtained in [10] for the Dirichlet problem. Section 6 is devoted to the main result of the present paper and some preliminary results useful to reach our goal.

2 Notations

In the whole paper Ω is a bounded domain (open connected set) of $\mathbb{R}^3$ such that its boundary is a Lyapunov surface Σ (i.e. Σ has a uniformly Hölder continuous normal field of some exponent $\lambda \in (0, 1]$), and such that $\mathbb{R}^3\setminus\overline{\Omega}$ is connected. The outwards unit normal vector at the point $x = (x_1, x_2, x_3) \in \Sigma$ is denoted by $n(x) = (n_1(x), n_2(x), n_3(x))$. The symbol $|\cdot|$ stands for the Euclidean norm for elements of $\mathbb{R}^3$.

For $h \in \mathbb{N}$, $C^h(\Omega)$ is the space of all complex-valued continuous functions whose derivatives are continuously differentiable up to the order h in Ω. Moreover, the symbol $C^{h,\beta}(\Omega)$ stands for the space of all functions defined in Ω having continuous derivatives up to order $h \in \mathbb{N}$ and such that the partial derivatives of order h are Hölder continuous with exponent $\beta \in (0, 1]$.

Throughout the paper we consider $p \in (1, \infty)$. $L^p(\Sigma)$ is the space of p-integrable complex-valued functions defined on Σ. By $L^p_h(\Sigma)$ we mean the space of the differential forms of degree $h \geq 1$ whose coefficients belong to $L^p(\Sigma)$.

If u is a h-form in Ω, the symbol du denotes the differential of u, while $* \, u$ denotes the dual Hodge form. Moreover, we write $\underset{\Sigma}{*}\, w = w_0$ if w is an 2-form on Σ and $w = w_0 d\sigma$.

The Sobolev space $W^{1,p}(\Sigma)$ can be defined as the space of functions in $L^p(\Sigma)$ such that their weak differential belongs to $L^p_1(\Sigma)$. If $u \in [W^{1,p}(\Sigma)]^3$, by du we denote the vector (du_1, du_2, du_3).

Finally, we distinguish by apices $+$ and $-$ the limit obtained by approaching the boundary Σ from Ω and $\mathbb{R}^3\setminus\overline{\Omega}$, respectively, that is

$$u^+(x) = \lim_{\Omega\ni y\to x} u(y) \quad \text{and} \quad u^-(x) = \lim_{\mathbb{R}^3\setminus\overline{\Omega}\ni y\to x} u(y) \quad \text{for } x \in \Sigma.$$

We shall omit the superscript $+$ when there is no ambiguity.

3 The system of elasto-static oscillations and its fundamental matrix

In this section, we recall the necessary background material related to the homogeneous system of elasto-static oscillations.

In the problem under study the domain Ω represents an elastic medium with density $\rho > 0$ and Lamé constants λ, μ satisfying conditions $\mu > 0$ and $3\lambda + 2\mu > 0$.

The homogeneous system of elasto-static oscillations is

$$\mu\Delta u + (\lambda + \mu)\nabla \mathrm{div}\, u + \rho\omega^2 u = 0 \tag{1}$$

where $u : \mathbb{R}^3 \to \mathbb{C}^3$ is the displacement vector and $\omega \in \mathbb{R}$ is the oscillation frequency [12, p. 48].

System (1) can be expressed in the matrix form

$$A(\partial_x, \omega)u = 0 \tag{2}$$

by means of the (3×3) matrix differential operator

$$A(\partial_x, \omega) = (A_{jk}(\partial_x, \omega))_{j,k=1,2,3}$$

whose entries are

$$A_{jk}(\partial_x, \omega) = \delta_{jk}(\mu\Delta + \rho\omega^2) + (\lambda + \mu)\frac{\partial^2}{\partial x_j \partial x_k}, \quad j, k = 1, 2, 3,$$

δ_{jk} being the delta Kronecker symbol. For the case $\omega = 0$ we simply write $A(\partial_x)$ and the above system turns into the Lamé one.

The fundamental matrix of (2) has the form

$$\Gamma(x, \omega) = (\Gamma_{kj}(x, \omega))_{j,k=1,2,3}$$

where

$$\Gamma_{kj}(x, \omega) = \sum_{l=1}^{2} \left(\delta_{kj}\alpha_l + \beta_l \frac{\partial^2}{\partial x_k \partial x_j} \right) \frac{e^{ik_l|x|}}{|x|}, \quad x \in \mathbb{R}^3, \omega \neq 0,$$

i is the imaginary unity, the non-negative constants k_1 and k_2 are determined by

$$k_1^2 = \rho\omega^2(\lambda + 2\mu)^{-1}, \quad k_2^2 = \rho\omega^2\mu^{-1},$$

and the costants α_l, β_l are the following ones

$$\alpha_l = \delta_{2l}(2\pi\mu)^{-1}, \quad \beta_l = (-1)^l(2\pi\rho\omega^2)^{-1}.$$

$\Gamma(x, \omega)$ is known as Kupradze's matrix and it has the property that each column and each row of it satisfy (2) for $x \neq 0$ (see [12, p. 85]). Another important property to recall is that

$$\lim_{\omega \to 0} \Gamma(x, \omega) = \Gamma(x)$$

$\Gamma(x)$ being the Somigliana (or the Kelvin) matrix, that is the fundamental matrix of $A(\partial_x)u = 0$ (see [12, p. 84 and p. 88]). In particular, the last limit relation is obtained by studying the behavior of the auxiliary matrix

$$\widetilde{\Gamma}(x,\omega)=\Gamma(x,\omega)-\Gamma(x), \tag{3}$$

which will be useful also for our scope. In particular, we mention that, for every $x \in \mathbb{R}^3$ and $\omega \in \mathbb{R}$, the following estimates hold:

$$\begin{aligned} &|\widetilde{\Gamma}_{kj}(x,\omega)| \leq |\omega|\widetilde{c}(\lambda,\mu), \quad k,j=1,2,3,\\ &\left|\frac{\partial \widetilde{\Gamma}_{kj}(x,\omega)}{\partial x_l}\right| \leq \omega^2 \overline{c}(\lambda,\mu), \quad k,j,l=1,2,3 \end{aligned} \tag{4}$$

where $\widetilde{c}(\lambda,\mu)$ and $\overline{c}(\lambda,\mu)$ are positive constants, depending on λ and μ only. Moreover,

$$\left|\frac{\partial^2 \widetilde{\Gamma}_{kj}(x,\omega)}{\partial x_l \partial x_m}\right| = \mathcal{O}\left(\frac{1}{|x|}\right), \quad k,j,l,m=1,2,3,\ x\neq 0 \tag{5}$$

For more details on the above estimates see [12, pp. 87–89].

4 Boundary integral operators

Consider now the (3×3) matrix differential operator

$$T=(T_{jk})_{j,k=1,2,3}$$

with entries

$$T_{jk}=\lambda n_j \frac{\partial}{\partial x_k}+\mu n_k \frac{\partial}{\partial x_j}+\mu\delta_{jk}\frac{\partial}{\partial n}.$$

T is known as the stress operator (see [12, p.57]).

We are interested in the kernels of the boundary integral operators $\mp I + K$ and $\pm I + K^*$, where

$$K:[L^p(\Sigma)]^3 \to [L^p(\Sigma)]^3, \quad K\varphi(x)=\int_\Sigma [T_y\Gamma(x-y,\omega)]'\varphi(y)\,d\sigma_y,$$

the prime denoting the transpose of a matrix, and

$$K^*:[L^q(\Sigma)]^3 \to [L^q(\Sigma)]^3, \quad K^*\psi(x)=\int_\Sigma [T_x\Gamma(x-y,\omega)]\psi(y)\,d\sigma_y$$

with $1<p<\infty$ and $p+q=pq$.

The operators K and K^* are adjoint ones with respect to the duality

$$\langle \psi, K\varphi\rangle = \langle K^*\psi,\varphi\rangle,$$

where $\langle f,g\rangle$ stands for the bilinear form

$$\int_\Sigma fg\,d\sigma=\int_\Sigma \sum_{j=1}^3 f_j g_j\,d\sigma.$$

Denote by $\mathcal{V}_0$ the spaces of solutions of the homogeneous Dirichlet problem

$$\begin{cases} v \in [C^{1,\lambda}(\overline{\Omega})]^3 \cap [C^2(\Omega)]^3 & \\ A(\partial_x, \omega)v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \Sigma \end{cases} \tag{6}$$

and by $\mathcal{W}_0$ the spaces of solutions of the homogeneous traction problem

$$\begin{cases} w \in [C^{1,\lambda}(\overline{\Omega})]^3 \cap [C^2(\Omega)]^3 & \\ A(\partial_x, \omega)w = 0 & \text{in } \Omega \\ Tw = 0 & \text{on } \Sigma. \end{cases} \tag{7}$$

If ω^2 is not a Dirichlet eigenvalue of (6) then $\mathcal{V}_0 = \{0\}$; analogously, if ω^2 is not a traction eigenvalue of (7) then $\mathcal{W}_0 = \{0\}$. Otherwise, such spaces are not trivial. Let us define

$$V = \left\{ Tv\Big|_{\Sigma} : v \in \mathcal{V}_0 \right\}$$

and

$$W = \left\{ w|_{\Sigma} : w \in \mathcal{W}_0 \right\}$$

In view of [12, Theorems 2.2 and 2.3, p. 413–415], we have

$$\mathcal{N}(I + K) = W \quad \text{and} \quad \mathcal{N}(I - K^*) = V. \tag{8}$$

Moreover, from (3), (4), and [12, p. 236 and p. 355] it follows

$$\dim \mathcal{N}(I + K) = \dim \mathcal{N}(I + K^*) = m_T \in \mathbb{N}$$

and

$$\dim \mathcal{N}(I - K) = \dim \mathcal{N}(I - K^*) = m_D \in \mathbb{N}.$$

If ω^2 is not an interior traction (Dirichlet) eigenvalue, then $m_T = 0$ ($m_D = 0$). Otherwise, the following results hold (see [10, Lemmas 3.1 and 3.2]).

Lemma 1

(i) *Let* $\{\phi^1, \dots, \phi^{m_T}\}$ *be a basis of* $\mathcal{N}(I + K^*)$ *and define*

$$w^j(x) = \int_\Sigma \Gamma(x - y, \omega)\phi^j(y)\, d\sigma_y, \quad x \in \mathbb{R}^3 \setminus \Sigma,\ j = 1, \dots, m_T.$$

Then

$$\phi^j = -\frac{1}{2}[Tw^j]^- \quad \text{on } \Sigma, \quad j = 1, \dots, m_T,$$

and the vector functions

$$\psi^j = -[\overline{w}^j]^- \quad \text{on } \Sigma, \quad j = 1, \dots, m_T,$$

form a basis for $\mathcal{N}(I + K)$.

(ii) *Let* $\{\eta^1, \dots, \eta^{m_D}\}$ *be a basis of* $\mathcal{N}(I - K)$ *and define*

$$v^j(x) = \int_\Sigma [T_y \Gamma(x - y, \omega)]' \eta^j(y)\, d\sigma_y, \quad x \in \mathbb{R}^3 \backslash \Sigma,\ j = 1, \dots, m_D.$$

Then

$$\eta^j = \frac{1}{2}[v^j]^- \quad \text{on } \Sigma, \quad j = 1, \dots, m_D,$$

and the vector functions

$$\chi^j = [T\overline{v}^j]^- \quad \text{on } \Sigma, \quad j = 1, \dots, m_D, \tag{9}$$

form a basis of $\mathcal{N}(I - K^*)$.

An inspection of the proof of Lemma 3.1 in [10] shows that

$$\text{the determinant of the matrix } (\langle \psi^j, \phi^l \rangle)_{j,l=1,\dots,m_T} \quad \text{does not vanish.}$$

The same happens for the part (ii) of Lemma 1, that is

$$\text{the determinant of the matrix } (\langle \eta^j, \chi^l \rangle)_{j,l=1,\dots,m_D} \quad \text{does not vanish.} \tag{10}$$

Remark 1 On account of Lyapunov–Tauber theorem [12, Theorem 8.3, p. 319], relation (9) is equivalent to

$$\chi^j = [T\overline{v}^j]^+ \quad \text{on } \Sigma, \quad j = 1, \dots, m_D. \tag{11}$$

We end this section by proving some additional results about the kernels above considered.

Lemma 2 *If* $\alpha \in \mathcal{N}(I \pm K)$ *and* $\beta \in \mathcal{N}(I \mp K^*)$, *then* $\langle \alpha, \beta \rangle = 0$.

Proof It is sufficient to note that

$$\langle \alpha, \beta \rangle = \langle \mp K\alpha, \beta \rangle = \mp \langle \alpha, K^* \beta \rangle = -\langle \alpha, \beta \rangle.$$

□

Proposition 1 *Let* $\eta \in [L^p(\Sigma)]^3$ *be a solution of* $\eta - K\eta = 0$. *Then* $\eta \in [W^{1,p}(\Sigma)]^3$.

Proof Let $\eta \in [L^p(\Sigma)]^3$ such that $\eta \in \mathcal{N}(I - K)$. It is easy to see that the potential

$$v(x) = \int_\Sigma [T_y \Gamma(x - y, \omega)]' \eta(y)\, d\sigma_y$$

satisfies the boundary condition $v^+ = 0$ on Σ. Moreover, taking (3) into account, the system $\eta - K\eta = 0$ can be rewritten as

$$-\eta(x) + \int_\Sigma [T_y\Gamma(x-y)]'\eta(y)\,d\sigma_y = H(x)$$

where

$$H(x) = -\int_\Sigma [T_y\widetilde{\Gamma}(x-y,\omega)]'\eta(y)\,d\sigma_y.$$

Note that the function H belongs to $[W^{1,p}(\Sigma)]^3$ by virtue of (4) and (5). Therefore, the double layer potential with density η

$$E(x) = \int_\Sigma [T_y\Gamma(x-y)]'\eta(y)\,d\sigma_y, \quad x \in \Omega,$$

satisfies

$$A(\partial_x)E = 0 \quad \text{in } \Omega \quad \text{and} \quad E = H \quad \text{on } \Sigma.$$

Then, following [12, Theorem 1.8 at p. 171 and pp. 313-313], there exists TE almost everywhere on Σ and it belongs to $[L^p(\Sigma)]^3$. In [5, Theorem 5] it has been proven that a solution E can be represented in terms of a simple layer potential with density $\phi \in [L^p(\Sigma)]^3$:

$$E(x) = \int_\Sigma \Gamma(x-y)\phi(y)\,d\sigma_y.$$

Accordingly, we have that the function η satisfies

$$T\int_\Sigma [T_y\Gamma(x-y)]'\eta(y)\,d\sigma_y = TE(x), \quad x \in \Sigma.$$

Now consider the following traction problem

$$A(\partial_x)w = 0 \quad \text{in } \Omega \quad \text{and} \quad Tw = TE \quad \text{on } \Sigma. \tag{12}$$

By applying [13, Theorem 1], we have that (12) admits a solution w in terms of a double layer potental with density $\eta_0 \in [W^{1,p}(\Sigma)]^3$, since the compatibility conditions

$$\int_\Sigma [TE(x)]\,(a \wedge x + b)\,d\sigma_x = 0 \quad \forall\, a, b \in \mathbb{R}^3$$

are satisfied by virtue of Gauss–Green formulas (see [12, pp. 111-114]). Then

$$T\int_\Sigma [T_y\Gamma(x-y)]'(\eta(y)-\eta_0(y))\,d\sigma_y = 0 \quad \text{on } \Sigma$$

and the double layer potential

$$\int_\Sigma [T_y\Gamma(x-y)]'(\eta(y)-\eta_0(y))\,d\sigma_y$$

has to be a rigid displacement $a \wedge x + b$ in Ω, $a, b \in \mathbb{R}^3$. Finally, $\eta = \eta_0 + a \wedge x + b$ and this concludes the proof. □

5 The Dirichlet problem associated with $A(\partial_x, \omega)u = 0$

In [10] we solved the Dirichlet problem for steady elastic oscillations in the class of potentials defined as follows.

Definition 1 We say that a function u belongs to the space $\mathcal{S}^p$ if and only if there exists $\varphi \in [L^p(\Sigma)]^3$ such that u can be represented by means of a simple layer potential with density φ, i.e.

$$u(x) = \int_\Sigma \Gamma(x-y,\omega)\varphi(y)\, d\sigma_y, \quad x \in \Omega.$$

Namely, we dealt with the solvability and representation formula of solutions of the differential problem

$$\begin{cases} u \in \mathcal{S}^p \\ A(\partial_x,\omega)u = 0 & \text{in } \Omega \\ u = f & \text{on } \Sigma,\ f \in [W^{1,p}(\Sigma)]^3 \end{cases} \tag{13}$$

where the datum f satisfies conditions

$$\int_\Sigma f\, Tv\, d\sigma = 0, \quad \forall\, v \in \mathcal{V}_0, \tag{14}$$

The method applied consists in proving that a certain singular integral operator is reducible in the following sense. We say that a continuous linear operator $S : B_1 \to B_2$ between two Banach spaces can be reduced on the left if there exists a continuous linear operator $R : B_2 \to B_1$ such that $RS = I + T$, where I stands for the identity operator on B_1 and $T : B_1 \to B_1$ is a compact operator. Analogously, one can define an operator S reducible on the right. If S is a reducible operator, its range is closed and then the equation $S\alpha = \beta$ has a solution if and only if $\langle \gamma, \beta \rangle = 0$, for any $\gamma \in B_2^*$ such that $S^*\gamma = 0$, S^* being the adjoint of S (see, e.g., [11] or [14]).

Coming back to the method, by imposing the initial condition $u|_\Sigma = f$ to a simple layer potential, we got an integral system of equations of the first kind

$$\int_\Sigma \Gamma(x-y,\omega)\varphi(y)\, d\sigma_y = f(x), \quad \text{on } \Sigma. \tag{15}$$

Following [3], we took the differential d of both sides of system (15), and we passed to the singular integral system

$$S\varphi(x) = \int_\Sigma d_x[\Gamma(x-y,\omega)]\varphi(y)\, d\sigma_y = df(x), \quad \text{a.e. } x \in \Sigma. \tag{16}$$

Note that the unknown is a vector function $\varphi \in [L^p(\Sigma)]^3$, while the data is a vector whose components are differential forms of degree 1 belonging to $L^p_1(\Sigma)$. The operator S defined by the left-hand size of (16), acting from $[L^p(\Sigma)]^3$ into $[L^p_1(\Sigma)]^3$, can be reduced on the left by the integral operator $R' : [L^p_1(\Sigma)]^3 \longrightarrow [L^p(\Sigma)]^3$ (introduced in [5]) defined as

$$R_i'[\psi](x) = \frac{(\lambda+\mu)(\lambda+2\mu)}{(\lambda+3\mu)}\mathcal{K}_{jj}[\psi](x)\, n_i(x) + \mu\mathcal{K}_{ij}[\psi](x)\, n_j(x) + \mu\frac{(\lambda+\mu)}{(\lambda+3\mu)}\mathcal{K}_{ji}[\psi](x)\, n_j(x)$$

($i = 1, 2, 3$), where

$$\mathcal{K}_{js}[\psi](x) = * \int_\Sigma d_x[s_1(x-y)] \wedge \psi_j(y) \wedge dx^s - \delta^{123}_{ihp}\int_\Sigma \frac{\partial}{\partial x_s}[K_{ij}(x-y)] \wedge \psi_h(y) \wedge dy^p,$$

$$s_1(x-y) = -\frac{1}{4\pi|y-x|}\sum_{j=1}^{3} dx^j dy^j$$

and

$$K_{ij}(x-y) = \frac{1}{4\pi}\frac{(y_i-x_i)(y_j-x_j)}{|y-x|^3}.$$

As quoted before in general, this fact implies that the range of S is closed and Eq. (16) admits a solution $\varphi \in [L^p(\Sigma)]^3$ if and only if

$$\int_\Sigma \gamma_j \wedge df_j = 0, \quad j = 1, 2, 3$$

for every $\gamma \in [W_1^{1,q}(\Sigma)]^3$ ($q = p/(p-1)$) such that $d\gamma = T(\partial_x, n)v d\sigma$, with $v \in \mathcal{V}_0$ (see [10, Theorem 5.4]).

We conclude this section by the following existence result.

Theorem 1 *Let $f \in [W^{1,p}(\Sigma)]^3$ ($1 < p < \infty$). There exists a solution of the Dirichlet problem (13) if and only if f satisfies the compatibility conditions*

$$\int_\Sigma f\chi^j \, d\sigma = 0 \quad \text{for every} \quad j = 1, \dots, m_D. \tag{17}$$

Proof In [10, Theorem 5.5] is established that there exists a solution of (13) if and only if f satisfies compatibility conditions (14) and such conditions are equivalent to (17) because of (8), Lemma 1 part (ii), and (11). □

6 The traction problem associated with $A(\partial_x, \omega)u = 0$

The aim of this section is to solve the traction problem associated with the system $A(\partial_x, \omega)u = 0$ in the following space of potentials.

Definition 2 We say that a function v belongs to the space $\mathcal{D}^p$ if and only if it can be represented by means of a double layer potential with density $\psi \in [W^{1,p}(\Sigma)]^3$, i.e.

$$v(x) = \int_\Sigma [T_y\Gamma(x-y,\omega)]'\psi(y)d\sigma_y, \quad x \in \Omega.$$

Consider the traction problem

$$\begin{cases} v \in \mathcal{D}^p \\ A(\partial_x, \omega)v = 0 & \text{in } \Omega \\ Tv = g & \text{on } \Sigma,\ g \in [L^p(\Sigma)]^3 \end{cases} \tag{18}$$

where the datum g satisfies conditions

$$\int_\Sigma gw\, d\sigma = 0, \quad \forall\, w \in \mathcal{W}_0, \tag{19}$$

or, equivalently,

$$\int_\Sigma g\psi^j\, d\sigma = 0, \quad \forall j = 1, \ldots, m_T. \tag{20}$$

We begin by showing some preliminary results.

Proposition 2 *Let $v \in \mathcal{D}^p$ with density $u \in \mathcal{S}^p$, that is*

$$v(x) = \int_\Sigma [T_y \Gamma(x-y,\omega)]' u(y) d\sigma_y,$$

where

$$u(x) = \int_\Sigma \Gamma(x-y,\omega)\varphi(y) d\sigma_y,$$

with $\varphi \in [L^p(\Sigma)]^3$. Then

$$Tv(x) = -\varphi(x) + K^{*2}\varphi(x) \tag{21}$$

for almost every $x \in \Sigma$.

Proof We start by observing that

$$2u(x) = \int_\Sigma \Gamma(x-y,\omega) Tu(y) d\sigma_y - \int_\Sigma [T_y\Gamma(x-y,\omega)]' u(y)\, d\sigma_y, \quad x \in \Omega. \tag{22}$$

Indeed, let φ_n be a sequence of polynomials such that $\varphi_n \to \varphi$ in $[L^p(\Sigma)]^3$ and define

$$u_n(x) = \int_\Sigma \Gamma(x-y,\omega)\varphi_n(y) d\sigma_y, \quad x \in \Omega.$$

Thanks to [12, formula (2.6) at p. 122], we have that

$$2u_n(x) = \int_\Sigma \Gamma(x-y,\omega) Tu_n(y) d\sigma_y - \int_\Sigma [T_y\Gamma(x-y,\omega)]' u_n(y)\, d\sigma_y, \quad x \in \Omega.$$

Letting $n \to +\infty$, we obtain (22).

On the other hand, u satisfies the following jump relation (see [12, formula (5.9) at p. 313])

$$Tu(x) = \varphi(x) + \int_\Sigma [T_x\Gamma(x-y,\omega)]\varphi(y)\,d\sigma_y \quad \text{a.e. } x \in \Sigma.$$

Then, for a.e. $x \in \Sigma$ we have

$$\begin{aligned}
Tv(x) &= T\left\{-2u(x) + \int_\Sigma \Gamma(x-y,\omega)Tu(y)\,d\sigma_y\right\} \\
&= -\left\{\varphi(x) + \int_\Sigma [T_x\Gamma(x-y,\omega)]\varphi(y)\,d\sigma_y\right\} \\
&\quad + \int_\Sigma [T_x\Gamma(x-y,\omega)]\left\{\varphi(y) + \int_\Sigma [T_y\Gamma(y-z,\omega)]\varphi(z)\,d\sigma_z\right\} d\sigma_y \\
&= -\varphi(x) + \int_\Sigma [T_x\Gamma(x-y,\omega)]\left\{\int_\Sigma [T_y\Gamma(y-z,\omega)]\varphi(z)d\sigma_z\right\}d\sigma_y
\end{aligned}$$

that is formula (21). ◻

Lemma 3 *The singular integral system*

$$-\varphi + K^{*2}\varphi = g, \tag{23}$$

where $g \in [L^p(\Sigma)]^3$, *admits a solution* $\varphi \in [L^p(\Sigma)]^3$ *if and only if conditions* (20) *and*

$$\int_\Sigma g\eta^i\,d\sigma = 0, \quad i = 1, \dots, m_D \tag{24}$$

are satisfied.

Proof Suppose that (20) and (24) are satisfied and rewrite system (23) as follows:

$$(I + K^*)(-I + K^*)\varphi = g.$$

Observe that the equation $(I + K^*)\gamma = g$ has a solution if and only if $\langle g, w\rangle = 0$ for every $w \in \mathcal{N}(I + K)$, and this is true because of (20). Let γ_0 be a solution and consider the equation

$$(-I + K^*)\varphi = \gamma_0. \tag{25}$$

This is solvable if and only if $\langle \gamma_0, \eta^i\rangle = 0$ for every $\eta^i \in \mathcal{N}(I - K)$, $i = 1, \dots, m_D$. Such compatibility conditions are satisfied since

$$\langle \gamma_0, \eta^i\rangle = \langle \gamma_0, K\eta^i\rangle = \langle K^*\gamma_0, \eta^i\rangle = -\langle \gamma_0, \eta^i\rangle + \langle g, \eta^i\rangle,$$

and then,

$$2\langle \gamma_0, \eta^i\rangle = \langle g, \eta^i\rangle = 0, \quad i = 1, \dots, m_D$$

on account of (24),

This shows that there exists a solution φ of (25). Therefore φ satisfies (23).

Conversely, if φ is a solution of (23), we have

$$(-I + K^*)(I + K^*)\varphi = g.$$

In particular, $g \in \mathcal{R}(I - K^*)$, and then $\langle g, u\rangle = 0$ for all $u \in \mathcal{N}(I - K)$. This implies that conditions (24) are fulfilled.

On the other hand, $(I + K^*)(-I + K^*)\varphi = g$. Hence $g \in \mathcal{R}(I + K^*)$ and $\langle g, v\rangle = 0$ for all $v \in \mathcal{N}(I + K)$, from which conditions (20) follow. □

Lemma 4 *Given* $\psi \in [W^{1,p}(\Sigma)]^3$ *there exist* $\varphi \in [L^p(\Sigma)]^3$ *and* $c_1, \dots, c_{m_D} \in \mathbb{C}$ *such that*

$$\psi(x) = \int_\Sigma \Gamma(x - y, \omega)\varphi(y)\, d\sigma_y + \sum_{i=1}^{m_D} c_i \eta^i(x), \quad x \in \Sigma. \tag{26}$$

The vector $(c_1, \dots, c_{m_D})$ *is the unique solution of the system*

$$\sum_{i=1}^{m_D} c_i \langle \eta^i, \chi^j\rangle = \langle \psi, \chi^j\rangle, \quad j = 1, \dots, m_D. \tag{27}$$

Proof Let $\psi \in [W^{1,p}(\Sigma)]^3$. In view of Proposition 1 the function $\psi - \sum_{i=1}^{m_D} c_i \eta^i$ belongs to $[W^{1,p}(\Sigma)]^3$ for any $c_1, \dots, c_{m_D}$. Thanks to Theorem 1, there exists $\varphi \in [L^p(\Sigma)]^3$ satisfying (26) if and only if

$$\int_\Sigma \left(\psi - \sum_{i=1}^{m_D} c_i \eta^i \right) \chi^j d\sigma = 0, \quad j = 1, \dots, m_D,$$

that is, $(c_1, \dots, c_{m_D})$ is solution of system (27).

Observe that the existence and the uniqueness of the constants $c_1, \dots, c_{m_D}$ follow from (10). □

Remark 2 The sum on the right-hand side of (26) is not present if ω^2 is not an interior Dirichlet eigenvalue.

Theorem 2 *There exists a solution of the traction problem* (18) *if and only if the datum* $g \in [L^p(\Sigma)]^3$ *satisfies compatibility conditions* (19) (*or, equivalently*, (20)). *If* ω^2 *is not an interior traction eigenvalue then the traction problem* (18) *is always solvable.*

Proof Assume that g satisfies (20). Let $(c_1, \dots, c_{m_D})$ be the solution of the system

$$\sum_{i=1}^{m_D} c_i \langle \chi^i, \eta^j\rangle = \langle g, \eta^j\rangle, \quad j = 1, \dots, m_D \tag{28}$$

and consider the double layer potential

$$\begin{aligned} v(x) = &\int_\Sigma [T_y \Gamma(x - y, \omega)]' \left(\int_\Sigma \Gamma(y - z, \omega)\varphi(z)\, d\sigma_z \right) d\sigma_y \\ &+ \sum_{i=1}^{m_D} c_i \int_\Sigma [T_y \Gamma(x - y, \omega)]' \eta^i(y)\, d\sigma_y, \quad x \in \Omega, \end{aligned}$$

where $\varphi \in [L^p(\Sigma)]^3$. By imposing the boundary condition and taking into account Proposition 2, (9), and (11), we get

$$Tv(x) = -\varphi(x) + K^{*2}\varphi(x) + \sum_{i=1}^{m_D} c_i \chi^i(x) = g(x), \quad x \in \Sigma.$$

Then v satisfies the boundary conditions if and only if

$$-\varphi + K^{*2}\varphi = g - \sum_{i=1}^{m_D} c_i \chi^i \quad \text{on } \Sigma.$$

Such a system admits a solution $\varphi \in [L^p(\Sigma)]^3$ if and only if

$$\int_\Sigma \left(g - \sum_{i=1}^{m_D} c_i \chi^i \right) \psi^j d\sigma = 0, \quad j = 1, \dots, m_T \tag{29}$$

and

$$\int_\Sigma \left(g - \sum_{i=1}^{m_D} c_i \chi^i \right) \eta^j d\sigma = 0, \quad j = 1, \dots, m_D \tag{30}$$

(see Lemma 3). Both the above set of conditions are satisfied. In fact, thanks to (20) and Lemma 2,

$$\int_\Sigma \left(g - \sum_{i=1}^{m_D} c_i \chi^i \right) \psi^j d\sigma = -\sum_{i=1}^{m_D} c_i \int_\Sigma \chi^i \psi^j d\sigma = 0,$$

and so conditions (29) are fulfilled. Conditions (30) hold in view of (28).

Conversely, let $v \in \mathcal{D}^p$ be a solution of (18) with density $\psi \in [W^{1,p}(\Sigma)]^3$. Lemma 4 infers that ψ can be written as in (26). Therefore,

$$-\varphi + K^{*2}\varphi + \sum_{i=1}^{m_D} c_i \chi^i = g \quad \text{on } \Sigma$$

because of Proposition 2 and Remark 1.

Now fix $w \in \mathcal{W}_0$. From (8), $w|_\Sigma \in \mathcal{N}(I + K)$ and, from Lemma 2, $\int_\Sigma \chi^i w \, d\sigma = 0$ ($j = 1, \dots, m_T$). On the other hand, $(-\varphi + K^{*2}\varphi) \in \mathcal{R}(I + K^*)$, and hence $\int_\Sigma (-\varphi + K^{*2}\varphi) \, w \, d\sigma = 0$ for every $w \in \mathcal{N}(I + K)$.

Accordingly,

$$\int_\Sigma g w \, d\sigma = \int_\Sigma (-\varphi + K^{*2}\varphi) w \, d\sigma + \sum_{i=1}^{m_D} c_i \int_\Sigma \chi^i w \, d\sigma = 0$$

and this concludes the proof. □

Funding Open access funding provided by Università degli Studi della Basilicata within the CRUI-CARE Agreement.

Declartions

Conflict of interest The authors declare that they have no conflict of interest.

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