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Nyström methods for approximating the solutions of an integral equation arising from a problem in mathematical biology

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ABSTRACT

The paper deals with an integral equation arising from a problem in mathematical biology. We propose approximating its solution by Nyström methods based on Gaussian rules and on product integration rules according to the smoothness of the kernel function. In particular, when the latter is weakly singular we propose two Nyström methods constructed by means of different product formulas. The first one is based on the Lagrange interpolation while the second one is based on discrete spline quasi-interpolants. The stability and the convergence of the proposed methods are proved in uniform spaces of continuous functions. Finally, some numerical tests showing the effectiveness of the methods and the sharpness of the obtained error estimates are given.

1. Introduction

We consider the following integral equations of second kind

$$f(x)\int_0^1 k(x-y)dy + \int_0^1 k(y-x)f(y)dy = g(x), \quad 0 \le x \le 1, \tag{1}$$

where k is a given convolution kernel such that

$$T(x) = \int_0^1 k(x-y)dy \neq 0, \quad \forall x \in [0,1],$$

g is a known function and f is the unknown.

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Letting

$$(Kf)(x) = \int_0^1 k(y-x)f(y)dy$$

and

$$(M_T f)(x) = f(x)T(x),$$

we can rewrite (1) as follows

$$(M_T + K)f(x) = g(x), \quad 0 \le x \le 1. \tag{2}$$

These types of integral equations are of interests because arise from a problem in mathematical biology [3,11]. In particular in [3] the authors introduced two Kantorovic methods for approximating the solution of (2), proving their convergence in L^2 spaces.

In this paper we propose to solve the equation (2) by suitable Nyström methods. We prove their stability and convergence in spaces of continuous functions, give related error estimates in the uniform norm, and prove the well conditioning of the linear systems associated to the approximating equations. If the kernel k is smooth we discretize the integral $(Kf)(x)$ by a Gauss-Legendre rule. Two Nyström methods based on product quadrature rules are proposed when k is weakly singular. The first one consists in approximating the unknown solution by a Lagrange polynomial based on Legendre zeros, while the second one approximates the function f by the discrete spline quasi-interpolant introduced in [23].

Nyström methods have been widely employed for the numerical resolution of equation (2) with $T(x) = 1$ (see, for example, [1,5–8,10,16,21,17–20] and the references therein). Moreover, very recently in [12] the authors describe a general framework for the theoretical investigation of Nyström methods based on quadrature rules of interpolatory type in spaces of continuous functions.

To our knowledge Nyström methods for approximating the solutions of integral equations of type (2) are not available in literature. We point out that, in order to solve (2) by means of one of the Nyström methods proposed for the case $T(x) = 1$, the equation (2) has to be rewritten as follows

$$f(x) + \int_0^1 \frac{k(y-x)}{\int_0^1 k(x-t)dt} f(y)dy = \frac{g(x)}{\int_0^1 k(x-t)dt}, \quad 0 \le x \le 1,$$

and the quadrature rule is applied to the operator

$$(K^* f)(x) = \int_0^1 \frac{k(y-x)}{\int_0^1 k(x-t)dt} f(y)dy.$$

However, when the exact value of the integral $\int_0^1 k(x-t)dt$ (both under the integral sign and on the right hand side) is not known, the resulting finite dimensional equations (and, then, the equivalent linear systems and the Nyström interpolants) cannot be directly used for the approximation of the solution. The introduction of an approximation of the integral $\int_0^1 k(x-t)dt$ leads to new finite dimensional equations (approximating the previous ones) and then to a new method whose stability and convergence have to be investigated.

The paper is organized as follows. In Section 2 we introduce the functional spaces in which we are going to study the integral equation and give the assumptions under which its solution is unique. Section 3 is devoted to the description of the Nyström method for the case when k is a continuous function having constant sign in the interval $[0, 1]$ and to the results dealing with its stability, convergence and well-conditioning of the associated linear systems. An error estimate in uniform norm is also given. In Section 4 we describe two Nyström methods proposed for the case of weakly singular function k, showing also in this case that both the methods are stable and convergent and lead to solving well-conditioned linear systems. For both methods we give error estimates in uniform norm and the computational details for some choices of the kernel function k. Section 5 contains some numerical tests to show the efficiency of the proposed methods. We do not make comparisons with the methods proposed in [3] because the authors in [3] did not give the necessary computational details. Finally, Section 6 is devoted to the proofs of the theoretical results.

2. Preliminaries

2.1. Notations and basic facts

In the sequel $\mathcal{C}$ will denote a positive constant which may have different values in different formulas. We will write $\mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ depends on the parameters $a, b, \ldots$ and $\mathcal{C} \neq \mathcal{C}(a, b, \ldots)$ to say that $\mathcal{C}$ is independent of the parameters $a, b, \ldots$.

By $C^0([a,b])$ we denote the space of the continuous functions on the interval $[a,b]$, $-\infty < a < b < +\infty$, equipped with the uniform norm

$$\|f\|_{[a,b]} = \sup_{x\in[a,b]} |f(x)|.$$

Moreover, we consider the Sobolev type space

$$W_r([a,b]) := \{f \in C^0([a,b]) \ : \ f^{(r-1)} \in AC(a,b), \|f^{(r)}\varphi_{a,b}^r\|_{[a,b]} < +\infty\},$$

where r is a positive integer, $\varphi_{a,b}(x) = \sqrt{(b-x)(x-a)}$ and $AC(a,b)$ denotes the set of all functions which are absolutely continuous on every closed subset of (a,b), endowed with the norm

$$\|f\|_{W_r([a,b])} = \|f\|_{[a,b]} + \|f^{(r)}\varphi_{a,b}^r\|_{[a,b]}.$$

Let us denote by $\mathbb{P}_m$ the set of all polynomials of degree at most m, and by

$$E_m(f)_{[a,b]} = \inf_{P\in\mathbb{P}_m} \|f-P\|_{[a,b]}$$

the error of the best polynomial approximation of a function $f \in C^0([a,b])$. We recall that

$$\lim_m E_m(f)_{[a,b]} = 0, \quad \forall f \in C^0([a,b]).$$

When $r > 0$ is not integer, we consider the Zygmund space

$$Z_r([a,b]) = \left\{ f \in C^0([a,b]) \ : \ \sup_{i\geq 0}(1+i)^r E_i(f)_{[a,b]} < +\infty \right\},$$

equipped with the following norm

$$\|f\|_{Z_r([a,b])} = \|f\|_{[a,b]} + \sup_{i\geq 0}(1+i)^r E_i(f)_{[a,b]}.$$

The following estimates of $E_m(f)_{[a,b]}$

$$E_m(f)_{[a,b]} \leq \frac{\mathcal{C}}{m^r}\|f\|_{W_r([a,b])}, \quad \forall f \in W_r([a,b]), \tag{3}$$

and

$$E_m(f)_{[a,b]} \leq \frac{\mathcal{C}}{m^r}\|f\|_{Z_r([a,b])}, \quad \forall f \in Z_r([a,b]) \tag{4}$$

holds (see, for example, [15, p. 172]).

In the sequel we will write $C^0 := C^0([0,1])$, $W_r := W_r([0,1])$, $E_m(f) := E_m(f)_{[0,1]}$, $Z_r := Z_r([0,1])$, $\|f\|_\infty = \|f\|_{[0,1]}$ and $\varphi = \varphi_{0,1}$.

We use the Gauss-Legendre quadrature rule on the interval $[0,1]$

$$\int_0^1 f(y)dy = \sum_{k=1}^m f(y_k)\lambda_k + R_m(f), \tag{5}$$

where $y_k, k = 1,\dots,m$, are the zeros of the Legendre polynomial p_m which is orthornormal w.r.t. the Legendre weight in $[0,1]$, λ_k, $k = 1,\dots,m$, are the corresponding Christoffel numbers and $R_m(f)$ denotes the remainder therm. We recall that (see, for example, [15, p. 337])

$$|R_m(f)| \leq E_{2m-1}(f), \quad \forall f \in C^0. \tag{6}$$

We will also consider the Lagrange polynomial interpolating the function $f \in C^0$ at the Legendre zeros in $[0,1]$, i.e.,

$$L_m(f,y) = \sum_{k=1}^m l_k(y)f(y_k), \quad l_k(y) = \lambda_k \sum_{i=0}^{m-1} p_i(y_k)p_i(y). \tag{7}$$

Denoting by $\mathbf{L}\log^+\mathbf{L}$ the set of all measurable functions $f : (0,1) \to \mathbb{R}$ for which the integral

$$\rho(f) := \int_0^1 |f(y)|(1+\log^+|f(y)|)dy, \quad \log^+ z = \begin{cases} \log z, & z > 1 \\ 0, & z \leq 1 \end{cases},$$

is finite, the following lemma can be easily deduced by [12, Lemma 7].

Lemma 2.1. *For all functions $f \in C^0$ and for all g such that $\frac{g}{\sqrt{\varphi}} \in \mathbf{L}\log^+\mathbf{L}$, there exists a constant $\mathcal{C} \neq \mathcal{C}(m, f, g)$ such that*

$$\int_0^1 |g(y)||L_m(f, y)|dy \leq \mathcal{C}\rho\left(\frac{g}{\sqrt{\varphi}}\right)\|f\|_\infty.$$

2.2. Solvability of the integral equation (2)

In this section we study the mapping properties of the operators M_T and K in order to give the assumptions under which the equation (2) admits a unique solution. We assume that

$$\lim_{h\to 0} \max_{|x-y|\leq h} \int_{-1}^1 |k(x) - k(y)|dy = 0, \tag{8}$$

$$\int_{-1}^1 |k(y)|dy < +\infty, \tag{9}$$

and

$$\int_0^1 k(x - y)dy \neq 0, \quad \forall x \in [0, 1]. \tag{10}$$

The following lemma holds.

Lemma 2.2. *Under the assumptions (8)-(10), the operator $M_T : C^0 \to C^0$ is invertible and the operator $K : C^0 \to C^0$ is compact.*

Using well-known results of functional analysis (see, for example, [13, Corollary 3.8]), the following theorem is a consequence of the previous lemma.

Theorem 2.1. *Under the assumptions (8)-(10) and $\mathrm{Ker}(M_T + K) = \{0\}$, the integral equation (1) admits a unique solution in C^0 for every choice of the right-hand side $g \in C^0$.*

Remark 2.1. (See, for example, [2, p. 8].) If k belongs to $C^0([-1, 1])$ or is weakly singular, then the hyphoteses (8), (9) and (10) are fulfilled.

In particular, the following lemma shows that the smoothness of k and g implies the smoothness of f^*.

Lemma 2.3. *If $k \in W_r([-1, 1])$ and $g \in W_r$, $r \geq 1$, then the unique solution f^* of the equation (2) belongs to W_r.*

Finally, for weakly singular kernels, the following lemma can be easily deduced by [26, Theorem 1].

Lemma 2.4. *Let k be a symmetric convolution kernel, i.e., $k(y - x) := k(|x - y|)$ with $k : (0, 1] \to \mathbb{R}$. If*

$$g \in C^s((0, 1]), \quad s \geq 1,$$
$$k(y) \in C^{s-1}((0, 1]), \quad s \geq 1,$$

and

$$|k^{(j)}(y)| \leq \mathcal{C}y^{\alpha-j}, \quad 0 < y \leq 1, \quad j = 0, 1, \ldots, s - 1, \quad -1 < \alpha < 0, \quad \mathcal{C} \neq \mathcal{C}(k, y), \tag{11}$$

then the unique solution f^* of the equation (2) belongs to $Z_{2(1+\alpha)}$.

3. Case of continuous functions *k* having constant sign

The Nyström method we propose consists in solving, instead of (2), the following approximating equations

$$(M_{T_m} + K_m) f_m(x) = g(x), \quad 0 \le x \le 1, \quad m \ge 1, \tag{12}$$

where

$$(K_m f)(x) = \sum_{k=1}^{m} k(y_k - x) f(y_k) \lambda_k$$

and

$$T_m(x) = \sum_{k=1}^{m} k(x - y_k) \lambda_k$$

are approximations of the integral $(Kf)(x)$ and the function $T(x)$, respectively, by means of the Gauss-Legendre quadrature rule (5) and

$$(M_{T_m} f)(x) := f(x) T_m(x).$$

In order to compute the approximating solutions f_m, we collocate the equations (12) at the quadrature knots $y_i, i = 1, \dots, m$, obtaining the following linear systems

$$\sum_{k=1}^{m} \left[\delta_{i,k} k(y_i - y_k)\lambda_k + k(y_k - y_i)\lambda_k\right] a_k = g(y_i), \quad i = 1, \dots, m, \tag{13}$$

whose unknowns are $a_k = f_m(y_k), k = 1, \dots, m$. The linear systems (13) are equivalent to (12) in the following sense: the array $\mathbf{a} = (a_1, \dots, a_m)^T$ is the solution of (13) if and only if the Nyström interpolant

$$f_m(x) = \frac{1}{T_m(x)} \left[g(x) - \sum_{k=1}^{m} k(y_k - x) \lambda_k a_k \right]$$

is the solution of (12).

The following theorem gives the assumptions under which the proposed method is stable and convergent. Moreover, it shows that the computation of the approximating solutions f_m requires solving well conditioned linear systems.

Theorem 3.1. *Assume that* $\operatorname{Ker}(M_T + K) = \{0\}$ *in* C^0. *If the kernel* k *belongs to* $C^0([-1, 1])$ *and has constant sign and* $g \in C^0$, *then, for a sufficiently large* m *(say* $m \ge m_0$*),*

- *the operators* $M_{T_m} + K_m : C^0 \to C^0$ *are invertible and their inverses are uniformly bounded;*
- *the unique solution* f_m^* *of (12) converges to the exact solution* f^* *of (2);*
- *the condition numbers in uniform norm of the matrices* A_m *of the linear systems (13) satisfy*

$$\sup_{m \ge m_0} \operatorname{cond}(A_m) < +\infty. \tag{14}$$

In particular, if $k \in W_r([-1, 1])$ *and* $g \in W_r$ *then the following estimate*

$$\|f^* - f_m^*\|_\infty \le \frac{\mathcal{C}}{m^r} \|f^*\|_{W_r} \tag{15}$$

holds, where $\mathcal{C} \ne \mathcal{C}(m, f^*)$.

4. Case of weakly singular functions *k*

In this section we consider the case where the kernel $k(x - y)$ is weakly singular and we propose two Nyström methods which consist in discretizing the integral operator K by product quadrature rules.

4.1. Nyström method based on a product integration formula using Lagrange interpolation

The first method we propose consists in approximating $(Kf)(x)$ by

$$(\widetilde{K}_m f)(x) = \int_0^1 k(y - x) L_m(f, y) dy$$

$$= \sum_{k=1}^{m} f(y_k) \int_0^1 k(y-x) l_k(y) dy$$

$$= \sum_{k=1}^{m} f(y_k) \lambda_k \sum_{i=0}^{m-1} p_i(y_k) c_i(x),$$

where $L_m(f)$ is the Lagrange polynomial defined in (7) and

$$c_i(x) := \int_0^1 k(y-x) p_i(y) dy. \tag{16}$$

Letting

$$\sigma_k(x) := \lambda_k \sum_{i=0}^{m-1} p_i(y_k) c_i(x)$$

we can write

$$(\widetilde{K}_m f)(x) = \sum_{k=1}^{m} \sigma_k(x) f(y_k).$$

Then, the numerical method reduces to solving the approximating equations

$$(M_T + \widetilde{K}_m) \widetilde{f}_m(x) = g(x), \quad 0 \le x \le 1, \quad m \ge 1, \tag{17}$$

with the unknowns $\widetilde{f}_m$. Collocating the above equations at the Legendre zeros $y_i, i = 1, \dots, m$, we deduce the following equivalent linear systems

$$\sum_{k=1}^{m} \left[\delta_{i,k} T(y_i) + \sigma_k(y_i) \right] \widetilde{a}_k = g(y_i), \quad i = 1, \dots, m, \tag{18}$$

whose unknowns are $\widetilde{a}_i = \widetilde{f}_m(y_i), \ i = 1, \dots, m$.

After solving the above linear systems we compute the unknowns $\widetilde{f}_m$ of the equations (17) as follows

$$\widetilde{f}_m(x) = \frac{1}{T(x)} \left[g(x) - \sum_{k=1}^{m} \sigma_k(x) \widetilde{a}_k \right]. \tag{19}$$

Each solution $\widetilde{f}_m$ of (17) furnishes a solution to (18), merely evaluating $\widetilde{f}_m$ at the nodes. The converse is also true, in fact for each solution $\widetilde{\mathbf{a}} = (\widetilde{a}_1, \dots, \widetilde{a}_m)^T$ of (18), there is a unique solution of (17) that agrees with $\widetilde{\mathbf{a}}$ at the nodes.

Concerning the stability and the convergence of the method and the conditioning of the related linear systems, the following result holds.

Theorem 4.1. *Assume that* $\mathrm{Ker}(M_T + K) = \{0\}$ *in* C^0. *If the kernel k satisfies* (10),

$$\rho\left(\frac{k(\cdot - x)}{\sqrt{\varphi}}\right) < +\infty, \tag{20}$$

and

$$\lim_{h \to 0} \rho\left(\frac{k(\cdot - x - h) - k(\cdot - x)}{\sqrt{\varphi}}\right) = 0 \tag{21}$$

uniformly w.r.t. x and $g \in C^0$, *then for a sufficiently large m (say* $m \ge m_0$*),*

- *the operators* $M_T + \widetilde{K}_m : C^0 \to C^0$ *are invertible and their inverses are uniformly bounded;*
- *the unique solution* $\widetilde{f}_m^*$ *of* (17) *converges to the exact solution* f^* *of* (2) *and the following estimate*

$$\|f^* - \widetilde{f}_m^*\|_\infty \le \mathcal{C} E_{m-1}(f^*), \quad \mathcal{C} \ne \mathcal{C}(m, f^*) \tag{22}$$

holds;

- *the condition numbers in uniform norm of the matrices $\widetilde{A}_m$ of the linear systems (18) satisfy*

$$\sup_m \operatorname{cond}(\widetilde{A}_m) < +\infty.$$

Remark 4.1. If f^* belongs to W_r, with $r \geq 1$ integer, or belongs to Z_r, with $r > 0$, by (3)-(4), the estimate of the error (22) becomes

$$\|f^* - f_m^*\|_\infty = \mathcal{O}\left(m^{-r}\right).$$

Remark 4.2. Examples of kernels satisfying (20), (21) and the assumptions of Lemma 2.4 are the following:

- $|x-y|^a$, $-1 < a < 0$;
- $\log|x-y|$;
- $|x-y|^a \log|x-y|$, $-1 < a < 0$.

In particular they satisfy the assumption (11) with $\alpha = a, -\varepsilon, a-\varepsilon$, respectively, for some $\varepsilon > 0$. Consequently, if g satisfies the assumptions of Lemma 2.4 too, (22) becomes

$$\|f^* - \widetilde{f}_m^*\|_\infty \leq \mathcal{C} \begin{cases} m^{-2(1+a)}, & k(y-x) = |x-y|^a, -1 < a < 0 \\ m^{-2(1-\varepsilon)}, & k(y-x) = \log|x-y| \\ m^{-2(1+a-\varepsilon)}, & k(y-x) = |x-y|^a \log|x-y|, -1 < a < 0 \end{cases},$$

where $\mathcal{C} \neq \mathcal{C}(m)$.

4.1.1. Computational aspects

Note that the construction of the matrices $\widetilde{A}_m$ of systems (18) and of the Nyström interpolants (19) requires the computation of the integrals $c_i(x), i = 1, \ldots, m$, given in (16). Since there is no universal method for computing such integrals, in this subsection we present the procedure we used in the cases where $k(y-x) = \log|y-x|$, $0 \leq x \leq 1$, and $k(y-x) = |y-x|^a$, $a > -1$, $0 \leq x \leq 1$, which are the most common weakly singular kernels. We note also that, for the above functions k (and in general if k is symmetric) we have $T(x) = c_0(x)$.

First, we will give some notations and properties of orthonormal Legendre polynomials on the interval $[0, 1]$. Let P_i and $\hat{P}_i$ be the Legendre polynomials which are orthogonal and orthonormal on the interval $[-1, 1]$, respectively. It is easy to see that from the orthogonality conditions we have

$$p_i(y) = \sqrt{2}\hat{P}_i(2y-1), \quad i = 0, 1, \ldots,$$

where p_i are orthonormal Legendre polynomials shifted on the interval $[0, 1]$. Also, by using the recurrence relation for the Legendre polynomials P_i and $\|P_i\|_\infty = \sqrt{\frac{2}{2i+1}}$, $i = 0, 1, \ldots$, we obtain the recurrence relation for the orthonormal Legendre polynomials shifted on the interval $[0, 1]$, i.e.,

$$\frac{i+1}{\sqrt{(2i+1)(2i+3)}} p_{i+1}(y) = (2y-1)p_i(y) - \frac{i}{\sqrt{4i^2-1}} p_{i-1}(y), \quad i = 0, 1, \ldots,$$

where $p_{-1}(y) = 0$ and $p_0(y) = 1$.

Now, using the results given in [4, p. 90] for $x \in [-1, 1]$, we deduce the recurrence relations for integrals (16), with respect to the weakly singular kernels $k(y-x) = \log|y-x|$, $0 < x < 1$, and $k(y-x) = |y-x|^a$, $a > -1$, $0 < x < 1$.

- If $k(y-x) = \log|y-x|$, $-1 < x < 1$, from [4, p. 90] we have

  $$M_{k+1}(x) = \frac{(2k+1)x}{k+2} M_k(x) - \frac{k-1}{k+2} M_{k-1}(x), \quad k = 2, 3, \ldots, \tag{23}$$

  where $M_k(x) = \int_{-1}^1 P_k(y) \log|x-y| dy$, $k = 0, 1, \ldots$, are the modified moments of the orthogonal Legendre polynomials P_k.
  Then we obtain

  $$M_k(x) = \|P_k\| \int_{-1}^{1} \log|x-y| \hat{P}_k(y) dy,$$

  and by the change of variables $y := 2y-1$ and $x := 2x-1$ we have

$$M_k(x) = \sqrt{2}\|P_k\| \int_0^1 \log|2x-2y| p_k(y)dy$$
$$= \sqrt{2}\|P_k\| \left(\log 2 \int_0^1 p_k(y)dy + c_k(x) \right).$$

Now, substituting the above relation in (23), we obtain the following recurrence relation for the integrals (16)

$$c_{k+1}(x) = \frac{(2k+1)(2x-1)}{k+2}\sqrt{\frac{2k+3}{2k+1}} \left(\log 2 \int_0^1 p_k(y)dy + c_k(x) \right)$$
$$- \frac{k-1}{k+2}\sqrt{\frac{2k+3}{2k-1}} \left(\log 2 \int_0^1 p_{k-1}(y)dy + c_{k-1}(x) \right)$$
$$- \log 2 \int_0^1 p_{k+1}(y)dy, \quad k = 2, 3, \ldots,$$

where $0 < x < 1$, and the starting integrals $c_k(x)$, $k = 0, 1, 2$, can be evaluated directly from (16). In particular we computed them using symbolic calculation in Wolfram Mathematica.

- If $k(y-x) = |y-x|^a$, $a > -1$, $-1 < x < 1$, from [4, p. 90] we have

$$M_{k+1}(x) = \frac{(2k+1)x}{k+a+2} M_k(x) - \frac{k-a-1}{k+a+2} M_{k-1}(x), \quad k = 1, 2, \ldots, \tag{24}$$

where $M_k(x) = \int_{-1}^1 P_k(y)|x-y|^a dy$, $k = 0, 1, \ldots$, are the modified moments of the orthogonal Legendre polynomials P_k. Then, proceeding as it was done in the previous case, we obtain the following recurrence relation for the integrals (16)

$$c_{k+1}(x) = \frac{(2k+1)(2x-1)}{k+a+2}\sqrt{\frac{2k+3}{2k+1}} c_k(x) - \frac{k-a-1}{k+a+2}\sqrt{\frac{2k+3}{2k-1}} c_{k-1}(x), \quad k = 1, 2, \ldots,$$

where $0 < x < 1$, and $c_k(x)$, $k = 0, 1$, can be evaluated directly from (16).

4.2. Nyström method based on a product integration formula using discrete spline quasi-interpolants

Before introducing the second Nyström method we recall the definition of discrete spline quasi-interpolants (see [24,22,23]).

Let $X_m = \{0 = x_0 < x_1 < \cdots < x_m = 1\}$ be a uniform partition of interval $I = [0, 1]$, i.e., the length of the subinterval $I_i = [x_{i-1}, x_i]$ is $h = \frac{1}{m}$. We denote by $\mathcal{S}_d(I, X_m)$ the space of splines of order d (degree $d-1$) and of class $\mathcal{C}^{d-2}$ on the uniform partition X_m and we let $J = \{1, 2, \ldots, m+d-1\}$. A basis of such space is formed by the family of B-splines $\mathcal{B} = \{B_j \mid j \in J\}$, with multiple nodes $0 = x_0 = x_{-1} = \cdots = x_{-d+1}$ and $1 = x_m = x_{m+1} = \cdots = x_{m+d-1}$. The support of B_j is $[x_{j-d}, x_j]$ for all $j \in J$. Let us define the following set

$$\mathcal{T}_m = \left\{ t_j \;\middle|\; t_j = \frac{x_{j-1} + x_{j-2}}{2}, \; j \in J \right\}.$$

A discrete quasi-interpolant spline (abbr. dQI) of order d (degree $d-1$) is a spline operator of the form:

$$Q_d(f) = \sum_{j \in J'} \mu_j(f) B_j,$$

whose coefficients $\mu_j(f)$ are the linear combinations of values of f on evaluation points of spline $\{\tau_j \mid j \in J'\}$, i.e., either on the set $\mathcal{T}_m$ for d odd or on the set X_m for d even. Also, J' is equal to J or $\{1, 2, \ldots, m\}$ for d odd or even, respectively. Moreover, Q_d is exact on $\mathbb{P}_{d-1}$, i.e.,

$$Q_d(p) = p, \quad \text{for all } p \in \mathbb{P}_{d-1}.$$

By using expansion of monomials $e_k(x) = x^k$, $k = 0, 1, \ldots$, as linear combination of B-splines (see [9,14,25]) and the exactness of Q_d on $\mathbb{P}_{d-1}$, explicit form of the functionals $\mu_j(f)$, $j \in J'$, is obtained (see [23,24,22]). Also, $Q_d(f)$ can be expressed in the quasi-Lagrange form:

$$Q_d(f, y) = \sum_{j\in J'} \widetilde{L}_j(y) f(\tau_j),$$

where the fundamental functions $\widetilde{L}_j$ are linear combination of B-splines.

Setting

$$\bar{f}_m(y) = Q_d(f, y)$$

and approximating the integral operator in (2) by

$$\begin{aligned}(\bar{K}_m f)(x) &= \sum_{j\in J'} \left(\int_0^1 \widetilde{L}_j(y) k(y-x) dy \right) f(\tau_j) \\ &= \sum_{j\in J'} \sigma_j(x) f(\tau_j),\end{aligned}$$

where $\sigma_j(x) = \int_0^1 \widetilde{L}_j(y) k(y-x) dy$, the Nyström method we propose in this subsection consists in solving the following approximating equations

$$\bar{f}_m(x) T(x) + \sum_{j\in J'} \sigma_j(x) \bar{f}_m(\tau_j) = g(x), \quad 0 \le x \le 1, \quad m = 1, 2, \ldots,$$

which can also be written in the following more compact form

$$(M_T + \bar{K}_m) \bar{f}_m(x) = g(x). \tag{25}$$

Collocating the above equation at the points τ_j, $j \in J'$, we obtain the following linear system

$$\sum_{j\in J'} \left[\delta_{i,j} T(\tau_i) + \sigma_j(\tau_i) \right] \bar{a}_j = g(\tau_i), \quad i \in J', \tag{26}$$

having $\bar{a}_j = \bar{f}_m(\tau_j)$, $j \in J'$, as unknowns. Solving the previous system we can construct the following approximating solution

$$\bar{f}_m(x) = \frac{1}{T(x)} \left(g(x) - \sum_{j\in J'} \sigma_j(x) \bar{a}_j \right). \tag{27}$$

The stability and the convergence of the method as well as the well-conditioning of the linear systems (26) are stated by the following theorem.

Theorem 4.2. *Assume that* $\mathrm{Ker}(M_T + K) = \{0\}$ *in* C^0. *If the kernel* k *satisfies* (8)-(10) *and* $g \in C^0$, *then for a sufficiently large* m *(say* $m \ge m_0$*)*

- *the operators* $M_T + \bar{K}_m : C^0 \to C^0$ *are invertible and their inverses are uniformly bounded;*
- *the unique solution* $\bar{f}_m^*$ *of* (25) *converges to the exact solution* f^* *of* (2) *and the following estimate holds true*

$$\|f^* - \bar{f}_m^*\|_\infty \le \mathcal{C} \omega \left(f^*, \frac{1}{m} \right), \quad \mathcal{C} \ne \mathcal{C}(m, d, f^*), \tag{28}$$

where ω *is the ordinary modulus of smoothness;*
- *the condition numbers in uniform norm of the matrices* $\bar{A}_m$ *of the linear systems* (26) *satisfy*

$$\sup_{m \ge m_0} \mathrm{cond}(\bar{A}_m) < +\infty.$$

Remark 4.3. In particular, if $f^* \in C^{d+1}$, with d even, by [1, Theorem 6], (28) becomes

$$\|f^* - \bar{f}_m^*\|_\infty \le \mathcal{C} \begin{cases} m^{-d-a-1}, & k(y-x) = |x-y|^a, -1 < a < 0 \\ m^{-d-1} \log m^{-1}, & k(y-x) = \log|x-y| \end{cases},$$

where $\mathcal{C} \ne \mathcal{C}(m)$.

4.2.1. Univariate quadratic splines

In this subsection we will give the computational details of the above described method in the case of discrete spline quasi-interpolant $Q_3 f$ and with respect to the kernels $k(y-x) = \log|y-x|$ and $k(y-x) = |y-x|^a$, $-1 < a < 0$, $0 \le x \le 1$.

Note that $d = 3$ and $J' = J = \{1, 2, \dots, m+2\}$. For the computation of the approximate solution (27) one has to calculate the integrals $\sigma_j(x)$, $j \in J$. As each $\widetilde{L}_j(y)$ can be represented as a linear combination of a finite number of B-splines B_j (see [24]),

$$
\begin{aligned}
\widetilde{L}_1 =& B_1 - \frac{1}{3}B_2, \\
\widetilde{L}_2 =& \frac{3}{2}B_2 - \frac{1}{8}B_3, \\
\widetilde{L}_3 =& -\frac{1}{6}B_2 + \frac{5}{4}B_3 - \frac{1}{8}B_4, \\
\widetilde{L}_j =& \frac{1}{8}(-B_{j-1} + 10B_j - B_{j+1}), \quad 4 \le j \le m-1, \\
\widetilde{L}_m =& -\frac{1}{8}B_{m-1} + \frac{5}{4}B_m - \frac{1}{6}B_{m+1}, \\
\widetilde{L}_{m+1} =& -\frac{1}{8}B_m + \frac{3}{2}B_{m+1}, \\
\widetilde{L}_{m+2} =& B_{m+2} - \frac{1}{3}B_{m+1}
\end{aligned}
$$

and $k(y-x) = k(x-y)$, $x, y \in [0,1]$, in our cases, it is sufficient to calculate the integrals $b_j(x) = \int_0^1 B_j(y)k(y-x)dy$, $j \in J$. Since each B-spline can be expressed in the Bernstein basis

$$
\left\{\beta_r^{d-1}(y) = \binom{d-1}{r}(1-y)^r y^{d-1-r} \, : \, r = 0, 1, \dots, d-1\right\}
$$

of $\mathbb{P}_{d-1}$, by using results given in [1], we have that b_j, $j \in J$, are given in the following way. For $k(y-x) = \log|y-x|$ we have

$$
\begin{aligned}
b_1^{(1)}(x) =& \frac{1}{3}h\log h + h\phi_2\left(\frac{x}{h}\right), \\
b_2^{(1)}(x) =& \frac{2}{3}h\log h + \frac{h}{2}\left(\phi_0\left(\frac{x}{h}\right) + 2\phi_1\left(\frac{x}{h}\right) + \phi_2\left(\frac{x}{h}\right) + \phi_2\left(\frac{x}{h}-1\right)\right), \\
b_j^{(1)}(x) =& h\log h + \frac{1}{2}\left(\phi_0\left(\frac{x}{h}+3-j\right) + \phi_0\left(\frac{x}{h}+2-j\right) + 2\phi_1\left(\frac{x}{h}+2-j\right)\right. \\
&\left. + \phi_2\left(\frac{x}{h}+2-j\right) + \phi_0\left(\frac{x}{h}+1-j\right)\right), \quad 3 \le j \le m, \\
b_{m+1}^{(1)}(x) =& \frac{2}{3}h\log h + \frac{h}{2}\left(\phi_0\left(\frac{x}{h}+2-m\right) + 2\phi_1\left(\frac{x}{h}+1-m\right) + \phi_0\left(\frac{x}{h}+1-m\right) + \phi_2\left(\frac{x}{h}+1-m\right)\right), \\
b_{m+2}^{(1)}(x) =& \frac{1}{3}h\log h + h\phi_0\left(\frac{x}{h}+1-m\right).
\end{aligned}
$$

In the case $k(y-x) = |y-x|^a$, $-1 < a < 0$, $x \in [0,1]$ we have

$$
\begin{aligned}
b_1^{(2)}(x) =& h^{a+1}\psi_2\left(\frac{x}{h}\right), \\
b_2^{(2)}(x) =& \frac{1}{2}h^{a+1}\left(\psi_0\left(\frac{x}{h}\right) + 2\psi_1\left(\frac{x}{h}\right) + \psi_2\left(\frac{x}{h}\right) + \psi_2\left(\frac{x}{h}-1\right)\right), \\
b_j^{(2)}(x) =& \frac{1}{2}h^{a+1}\left(\psi_0\left(\frac{x}{h}+3-j\right) + \psi_0\left(\frac{x}{h}+2-j\right) + 2\psi_1\left(\frac{x}{h}+2-j\right)\right. \\
&\left. + \psi_2\left(\frac{x}{h}+2-j\right) + \psi_0\left(\frac{x}{h}+1-j\right)\right), \quad 3 \le j \le m, \\
b_{m+1}^{(2)}(x) =& \frac{1}{2}h^{a+1}\left(\psi_0\left(\frac{x}{h}+2-m\right) + 2\psi_1\left(\frac{x}{h}+1-m\right) + \psi_0\left(\frac{x}{h}+1-m\right) + \psi_2\left(\frac{x}{h}+1-m\right)\right), \\
b_{m+2}^{(2)}(x) =& h^{a+1}\phi_0\left(\frac{x}{h}+1-m\right).
\end{aligned}
$$

The integrals $\phi_r(x) = \int_0^1 \log|y-x|\beta_r^2(y)dy$, $r = 0, 1, 2$, and $\psi_r(x) = \int_0^1 |y-x|^a \beta_r^2(y)dy$, $r = 0, 1, 2$, can be calculated with the aid of Wolfram Mathematica.

Table 1
Example 5.1.

m	$\text{cond}(A_m)$	$err_m^{(3)}$
3	4.1652	$7.205e-05$
4	4.9856	$2.286e-07$
5	5.4760	$6.712e-10$
6	5.7846	$6.370e-13$
7	5.9888	$9.301e-16$
8	6.1301	*eps*

5. Numerical tests

In this section we will show by some numerical tests the performances of the proposed methods.

For each example, we will compute the absolute errors

$$err_m = \max_{t \in X} |f(t) - f_m(t)|,$$

where X is a sufficiently large uniform mesh of $[0, 1]$ and the condition numbers in uniform norm of the matrices associated with the solved linear systems. In particular, the superscripts (3), (4.1) and (4.2) have been used for err_m and f_m in order to distinguish the implemented numerical method ((3), (4.1) and (4.2) denote Section 3, Subsection 4.1 and Subsection 4.2, respectively). When the exact solution of the integral equation is not known, the absolute errors err_m have been computed replacing f by f_m with $m = 2100$. The method given in Subsection 4.2 has been implemented with $d = 3$.

Note that, even if the method in Subsection 4.1 converges always faster than the method in Subsection 4.2 (see Examples 5.5-5.8), the latter method has the advantage of requiring a less computational cost.

Example 5.1. Consider the equation

$$f(x)\int_0^1 e^{(x-y)}dy + \int_0^1 e^{(y-x)} f(y)dy = \frac{e^{-x}}{2} + (e-1)e^{x-1}x^2\cos(x),$$

with

$$k(z) = e^z, \quad g(z) = \frac{e^{-z}}{2} + (e-1)e^{z-1}z^2\cos(z),$$

whose exact solution is $f(x) = x^2\cos(x) \in W_r,\ r \geq 1$. Since the function k is very smooth, we compute the approximate solution using the method described in Section 3. Looking at the numerical results presented in Table 1, one can see that, according to the theoretical expectations, the convergence is very fast and the condition numbers of the solved matrices are less than 6.5.

Example 5.2. Consider

$$f(x)\int_0^1 \frac{1}{((x-y)^2+4)}dy + \int_0^1 \frac{1}{((y-x)^2+4)} f(y)dy = \sin(x^2+1),$$

with

$$k(z) = \frac{1}{(z^2+4)}, \quad g(z) = \sin(z^2+1).$$

The exact solution is unknown, but, according to Theorem 3.1, $f \in W_r,\ r \geq 1$. The results reported in Table 2 confirm that also in this case the convergence is very fast and the condition numbers are very small.

Example 5.3. We take the equation

$$f(x)\int_0^1 e^{|x-y|^{\frac{11}{2}}} dy + \int_0^1 e^{|y-x|^{\frac{11}{2}}} f(y)dy = x^3\log(x+4),$$

with

$$k(z) = e^{|z|^{\frac{11}{2}}}, \quad g(z) = z^3\log(z+4),$$

Table 2
Example 5.2.

m	cond(A_m)	$err_m^{(3)}$
3	2.4431	$2.2731e-04$
4	2.6709	$5.7163e-06$
5	2.8251	$1.4087e-07$
6	2.8987	$1.7861e-10$
7	2.9617	$1.8290e-11$
8	2.9927	$1.2210e-13$
9	3.0244	$1.1507e-15$
10	3.0398	*eps*

Table 3
Example 5.3.

m	cond(A_m)	$err_m^{(3)}$
10	3.1624	$1.0991e-08$
20	3.3553	$1.2787e-10$
50	3.4819	$3.4684e-13$
100	3.5225	$3.9804e-15$
150	3.5358	*eps*

Table 4
Example 5.4.

m	cond(A_m)	$err_m^{(3)}$
50	3.1230	$2.1419e-09$
150	3.1265	$1.5680e-11$
250	3.1268	$1.5756e-12$
450	3.1269	$1.0441e-13$
650	3.1269	$1.3115e-14$
750	3.1269	$2.8589e-15$

whose exact solution is unknown. Applying Lemma 2.3 and Theorem 3.1 we deduce that $f \in W_5$ and that the error behaves as m^{-5}. This slower convergence is confirmed by the results in Table 3. In fact, in this case it is necessary to take $m = 150$ to attain an approximation of the solution with machine precision. Also in this case the condition numbers of the matrices are very small.

Example 5.4. Taking the equation

$$f(x)\int_0^1 \frac{1}{((x-y)^4+3)}dy + \int_0^1 \frac{1}{((y-x)^4+3)} f(y)dy = \left|x-\frac{1}{2}\right|^{\frac{7}{2}},$$

with

$$k(z) = \frac{1}{(z^4+3)}, \quad g(z) = \left|z-\frac{1}{2}\right|^{\frac{7}{2}},$$

by Theorem 3.1 we deduce that the unknown belongs to W_3. As one can see in Table 4, we need to take $m = 750$ to obtain an approximation of the solution with 15 exact decimal digits. This agrees with the theoretical expectations, in fact in this case the rate of converge is m^{-3}. According to (14) the condition numbers of the solved linear systems do not increase for m going to infinity.

Example 5.5. Consider the equation

$$f(x)\int_0^1 |x-y|^{-1/2}dy + \int_0^1 |y-x|^{-1/2} f(y)dy = g(x),$$

with

$$k(z) = |z|^{-1/2}, \quad g(z) = e^x(2(\sqrt{1-z}+\sqrt{z}) + \sqrt{\pi}(Erf(\sqrt{z}) + Erfi(\sqrt{1-z}))),$$

Table 5
Example 5.5.

m	$\mathrm{cond}(\widetilde{A}_m)$	$err_m^{(4.1)}$	$\mathrm{cond}(\bar{A}_m)$	$err_m^{(4.2)}$
3	1.7302	$2.925e-03$	3.0629	$1.584e-02$
4	1.9505	$1.614e-04$	3.1125	$1.163e-04$
5	2.1802	$7.275e-06$	3.1995	$4.646e-05$
6	2.3272	$2.783e-07$	3.2437	$2.149e-05$
7	2.4804	$9.186e-09$	3.2995	$1.341e-05$
8	2.5846	$2.639e-10$	3.3295	$1.004e-05$
9	2.6943	$6.693e-12$	3.3669	$8.447e-06$
10	2.7719	$1.517e-13$	3.3891	$5.604e-06$
11	2.8543	$3.236e-15$	3.4165	$3.267e-06$
12	2.9143	$6.540e-17$	3.4339	$2.479e-06$

Table 6
Example 5.6.

m	$\mathrm{cond}(\widetilde{A}_m)$	$err_m^{(4.1)}$	$\mathrm{cond}(\bar{A}_m)$	$err_m^{(4.2)}$
4	2.1977	$3.703e-04$	3.7580	$2.701e-04$
8	3.3555	$6.809e-07$	4.1433	$2.713e-05$
16	4.1437	$1.392e-12$	4.3772	$1.976e-06$
18	4.2299	$3.170e-14$	4.4052	$1.269e-06$
20	4.2953	$2.800e-15$	4.4282	$8.343e-07$
21	4.3302	$2.330e-16$	4.4159	$6.814e-07$

whose exact solution is $f(x)=e^x$, where

$$Erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt, \quad Erfi(z)=Erf(iz)/i, \quad i^2=-1,$$

are the error functions. Since the function k is weakly singular, we solve it using the methods proposed in Subsections 4.1 and 4.2. Since the solution f belongs to W_r for any $r\geq 1$, according to Remarks 4.1 and 4.3 the method proposed in Subsection 4.1 has faster convergence. This behaviour is verified by the numerical results presented in Table 5. As one can see, the condition numbers of the linear systems solved in both methods are comparable.

Example 5.6. Consider the equation

$$f(x)\int_0^1 \log|x-y|dy+\int_0^1 \log|y-x|f(y)dy=g(x),$$

with $k(z)=\log|z|$, whose exact solution is $f(x)=\frac{x}{1+x^2}$, and $g(z)$ can be explicitly obtained using Wolfram Mathematica. Also in this case the solution is very smooth and, as shown in Table 6, the convergence rate of the method in Subsection 4.1 is higher. The condition numbers of both the methods are very small.

Example 5.7. Consider the equation

$$f(x)\int_0^1 |x-y|^{-1/2}dy+\int_0^1 |x-y|^{-1/2}f(y)dy=g(x),$$

with $k(z)=|z|^{-1/2}$, whose exact solution is $f(x)=\left|x-\frac{1}{4}\right|^{9/2}$ and $g(z)$ can be explicitly obtained using Wolfram Mathematica. In this case the solution f belongs to $Z_{\frac{9}{2}}$ $(\supset C^4)$. According to Remark 4.1 the convergence order of the method given in Subsection 4.1 is $m^{-\frac{9}{2}}$, while taking into account Remark 4.3, the method given in Subsection 4.2 has convergence order $m^{-\frac{7}{2}}$. The results given in Table 7 show that the method given in Subsection 4.1 has faster convergence.

Example 5.8. Consider the equation

$$f(x)\int_0^1 |x-y|^{-1/5}dy+\int_0^1 |x-y|^{-1/5}f(y)dy=g(x),$$

Table 7
Example 5.7.

m	$\text{cond}(\widetilde{A}_m)$	$err_m^{(4.1)}$	$\text{cond}(\bar{A}_m)$	$err_m^{(4.2)}$
4	1.9505	$2.512e-03$	3.1125	$1.049e-03$
8	2.5846	$6.406e-06$	3.3295	$1.378e-04$
16	3.1165	$6.917e-08$	3.4981	$1.609e-05$
32	3.4856	$3.007e-09$	3.6245	$1.184e-06$
64	3.7094	$6.752e-11$	3.7189	$1.248e-07$
128	3.8342	$1.288e-12$	3.7886	$8.909e-09$
256	3.9004	$6.651e-14$	3.8395	$8.585e-10$
512	3.9346	$1.035e-15$	3.8765	$8.084e-11$
515	3.9348	$9.406e-16$	3.8765	$7.924e-11$

Table 8
Example 5.8.

m	$\text{cond}(\widetilde{A}_m)$	$err_m^{(4.1)}$	$\text{cond}(\bar{A}_m)$	$err_m^{(4.2)}$
100	3.3311	$2.439e-09$	3.3208	$2.585e-06$
200	3.3379	$2.573e-10$	3.3293	$8.384e-07$
300	3.3395	$5.357e-11$	3.3326	$4.287e-07$
400	3.3402	$2.049e-11$	3.3344	$2.632e-07$
500	3.3405	$8.672e-12$	3.3355	$1.782e-07$
600	3.3407	$4.673e-12$	3.3362	$1.281e-07$
800	3.3409	$1.617e-12$	3.3373	$7.351e-08$
1000	3.3411	$6.936e-13$	3.3379	$4.546e-08$
1400	3.3412	$1.697e-13$	3.3387	$1.823e-08$
1700	3.3412	$5.800e-14$	3.3391	$8.034e-09$
1900	3.3412	$2.210e-14$	3.3393	$3.469e-09$
2090	3.3412	$9.000e-16$	3.3394	$1.531e-10$

with $k(z) = |z|^{-1/5}$ and $g(z) = |\sin z|^{5/2}$ whose exact solution is unknown. According to Lemma 2.4 the unknown solution belongs to $Z_{\frac{8}{5}}$. The results given in Table 8 show that the convergence rate of the method given in Subsection 4.1 is higher than the one of the method given in Subsection 4.2. This agrees with the theoretical expectations (see Remarks 4.2 and 4.3).

6. Proofs

We first prove Lemma 2.2.

Proof of Lemma 2.2. The compactness of the operator $K : C^0 \to C^0$ under the assumptions (8) and (9) is well-known [2, p. 7].

Now we prove the invertibility of $M_T : C^0 \to C^0$. Letting

$$(\bar{K} f)(x) = \int_0^1 k(x-y) f(y) dy, \tag{29}$$

under the assumptions (8) and (10), it is easy to see that the function $T(x) = (\bar{K} f)(x)$ with $f \equiv 1$ belongs to C^0 as well as the function $\frac{1}{T(x)}$. Moreover, taking into account that

$$\|M_T f\|_\infty = \|T f\|_\infty \le \|T\|_\infty \|f\|_\infty \le \mathcal{C}\|f\|_\infty,$$
$$\left\|M_{\frac{1}{T}} f\right\|_\infty = \left\|\frac{1}{T} f\right\|_\infty \le \left\|\frac{1}{T}\right\|_\infty \|f\|_\infty \le \mathcal{C}\|f\|_\infty,$$

and

$$M_T M_{\frac{1}{T}} = M_{\frac{1}{T}} M_T = I,$$

we easily deduce that $M_T : C^0 \to C^0$ is a bounded invertible operator and its inverse is the bounded operator $M_{\frac{1}{T}} : C^0 \to C^0$. $\square$

In order to prove Lemma 2.3 we need the following result.

Lemma 6.1. *If $k \in W_r([-1, 1])$ then $k(y - \cdot) \in W_r([0, 1])$ uniformly w.r.t. y and $k(\cdot - x) \in W_r([0, 1])$ uniformly w.r.t. x.*

Proof. Taking into account that

$$\frac{\partial^r}{\partial x^r}k(y-x)=(-1)^r k^{(r)}(z)|_{z=y-x}$$

and

$$\left(\frac{x(1-x)}{(1-(y-x))(1+(y-x))}\right)^{\frac{r}{2}}\leq 1,$$

we get

$$\begin{aligned}\left|\frac{\partial^r}{\partial x^r}k(y-x)\varphi^r(x)\right| &= \left|\left[k^{(r)}(z)\varphi^r_{-1,1}(z)\right]|_{z=y-x}\right|\left(\frac{x(1-x)}{(1-(y-x))(1+(y-x))}\right)^{\frac{r}{2}}\\ &\leq \max_{0\leq x,y\leq 1}\left|\left[k^{(r)}(z)\varphi^r_{-1,1}(z)\right]|_{z=y-x}\right|\\ &\leq \max_{|z|\leq 1}\left|k^{(r)}(z)\varphi^r_{-1,1}(z)\right|. \end{aligned} \tag{30}$$

Consequently, if $k\in W_r([-1,1])$ then $k(y-\cdot)\in W_r([0,1])$ uniformly w.r.t. y. Analogously, taking into account that

$$\frac{\partial^r}{\partial y^r}k(y-x)=k^{(r)}(z)|_{z=y-x}$$

and

$$\left(\frac{y(1-y)}{(1-(y-x))(1+(y-x))}\right)^{\frac{r}{2}}\leq 1,$$

we deduce that $k(\cdot-x)\in W_r([0,1])$ uniformly w.r.t. x. □

Now we can prove Lemma 2.3.

Proof of Lemma 2.3. If we prove that $\frac{1}{T}\in W_r$ and $Kf\in W_r$ for any $f\in C^0$ then, under the assumption $g\in W_r$, we get

$$f=\frac{g}{T}-\frac{Kf}{T}\in W_r.$$

On the other hand, since, under the assumption (10), $T(x)\neq 0$ for any $x\in[0,1]$, if $T\in W_r$ then $\frac{1}{T}\in W_r$, too. Thus, we need to prove that $T\in W_r$ and $Kf\in W_r$ for any $f\in C^0$. We prove only that K is a continuous map from C^0 into W_r since, taking into account that $T(x)=(\bar{K}f)(x)$ with $f\equiv 1$ ($\bar{K}f$ defined in (29)), the proof of $T\in W_r$ is similar.

Using (30) we have

$$\begin{aligned}|(Kf)^{(r)}(x)\varphi^r(x)| &\leq \int_0^1\left|\frac{\partial^r}{\partial x^r}k(y-x)\varphi^r(x)\right||f(y)|dy\\ &\leq \|f\|_\infty\int_0^1\left|\left[k^{(r)}(z)\varphi^r_{-1,1}(z)\right]|_{z=y-x}\right|dy\\ &\leq \|f\|_\infty\|k\|_{W_r}\end{aligned}$$

and then, for $k\in W_r([-1,1])$, $K:C^0\to W_r$ is a continuous map. □

In order to prove Theorem 3.1 we need the following lemmas.

Lemma 6.2. *If $k\in C^0([-1,1])$ then the sequence $\{K_m\}_m$ is pointwise convergent to K, uniformly bounded and collectively compact.*

Proof. We first prove the pointwise convergence of $\{K_m\}_m$ to K, i.e.,

$$\lim_m\|(K-K_m)f\|_\infty=0,\quad \forall f\in C^0. \tag{31}$$

Note that $|(Kf)(x)-(K_mf)(x)|$ is the error of the Gauss-Legendre rule applied to the function $F_x(y)=k(y-x)f(y)$, for every fixed $x\in[0,1]$. Then, by (6),

$$|(Kf)(x) - (K_m f)(x)| \le E_{2m-1}(F_x). \tag{32}$$

If we prove that the function F_x is continuous, uniformly with respect to x, we deduce (31). For any $y \in [0,1]$, we have

$$\begin{aligned}\max_{x\in[0,1]} |F_x(y+h) - F_x(y)| &\le |f(y+h)| \max_{x\in[0,1]} |k(y-x+h) - k(y-x)| \\ &+ |f(y+h) - f(y)| \max_{x\in[0,1]} |k(y-x)| \\ &\le |f(y+h)| \max_{z\in[y-1,y]} |k(z+h) - k(z)| \\ &+ |f(y+h) - f(y)| \max_{z\in[y-1,y]} |k(z)| \\ &\le \|f\|_\infty \max_{z\in[-1,1]} |k(z+h) - k(z)| \\ &+ \max_{y\in[0,1]} |f(y+h) - f(y)| \max_{z\in[-1,1]} |k(z)|.\end{aligned}$$

Then, if $f \in C^0$ and $k \in C^0([-1,1])$ (i.e., f and k are uniformly continuous)

$$\lim_{h\to 0} \max_{x\in[0,1]} |F_x(y+h) - F_x(y)| = 0, \quad y \in [0,1],$$

i.e., $F_x \in C^0$ uniformly with respect to x.

Now we prove the uniformly boundedness of the sequence $\{K_m\}_m$. We have

$$\begin{aligned}|(K_m f)(x)| &\le \|f\|_\infty \sum_{k=1}^m |k(y_k - x)|\lambda_k \le \|f\|_\infty \max_{x,y\in[0,1]} |k(y-x)| \sum_{k=1}^m \lambda_k \\ &\le \|f\|_\infty \max_{z\in[-1,1]} |k(z)|\end{aligned}$$

and, then, under the assumption on k,

$$\sup_m \|K_m\|_{C^0\to C^0} \le \mathcal{C} < +\infty.$$

It remains to prove the collectively compactness of the sequence $\{K_m\}_m$, i.e., the relatively compactness in C^0 of the set

$$\{K_m f \in C^0 \,:\, m \ge 1 \text{ and } \|f\|_\infty \le 1\}.$$

Using the Ascoli-Arzelà Theorem, since the sequence $\{K_m\}_m$ is uniformly bounded, it is sufficient to prove that

$$\lim_{h\to 0} \sup_m \sup_{\|f\|_\infty = 1} \|(K_m f)(\cdot + h) - (K_m f)(\cdot)\|_\infty = 0. \tag{33}$$

The above condition can be easily proved. In fact, for any $f \in C^0$, we have

$$\begin{aligned}|(K_m f)(x+h) - (K_m f)(x)| &\le \sum_{k=1}^m |k(y_k - x - h) - k(y_k - x)||f(y_k)|\lambda_k \\ &\le \|f\|_\infty \max_{x,y\in[0,1]} |k(y-x-h) - k(y-x)| \\ &\le \|f\|_\infty \max_{z\in[-1,1]} |k(z-h) - k(z)|.\end{aligned}$$

Then, under the assumption $k \in C^0([-1,1])$, (33) follows. □

Lemma 6.3. *Assuming that the function k belongs to $C^0([-1,1])$ and has constant sign, the operators M_{T_m} are bounded, admit as uniformly bounded inverses the operators $M_{\frac{1}{T_m}}$ and pointwise converge to M_T.*

Proof. Letting

$$(K^* f)(x) = \int_0^1 k(x-y) f(y) dy \quad \text{and} \quad (K_m^* f)(x) = \sum_{k=1}^m k(x - y_k) f(y_k)\lambda_k,$$

proceeding as it was done in Lemma 6.2 for the sequence $\{K_m\}_m$, it is easy to see that, under the assumptions $k \in C^0([-1,1])$ and (10), the functions $T_m(x) = (K_m^* f)(x)$ with $f \equiv 1$ belong to C^0 as well as the functions $\frac{1}{T_m(x)}$. Moreover, the functions $T_m(x)$ uniformly converge to the function $T(x) = (K^* f)(x)$ with $f \equiv 1$. Furthermore, with $f \equiv 1$, we have

$$
\begin{aligned}
|T_m(x)| = |(\bar{K}_m f)(x)| &\le \sum_{k=1}^{m} |k(x-y_k)|\lambda_k \le \max_{x,y\in[0,1]} |k(x-y)| \sum_{k=1}^{m} \lambda_k \\
&\le \max_{z\in[-1,1]} |k(z)|,
\end{aligned}
$$

and, denoting by x_0 the point belonging to $[0,1]$ s.t. $|T_m(x_0)| = \|T_m\|_\infty$ and recalling that we are assuming k of constant sign, we get (we are assuming that k has positive sign, but the proof in case of negative sign is similar)

$$
\begin{aligned}
\|T_m\|_\infty = |T_m(x_0)| = |(\bar{K}_m f)(x_0)| = \left|\sum_{k=1}^{m} k(x_0-y_k)\lambda_k\right| &= \sum_{k=1}^{m} k(x_0-y_k)\lambda_k \\
&\ge \min_{z\in[-1,1]} k(z) \sum_{k=1}^{m} \lambda_k \\
&= \min_{z\in[-1,1]} |k(z)|,
\end{aligned}
$$

i.e.

$$
\|T_m\|_\infty \ge \min_{z\in[-1,1]} |k(z)|.
$$

Consequently,

$$
\begin{aligned}
&\|M_{T_m} f\|_\infty = \|T_m f\|_\infty \le \|T_m\|_\infty \|f\|_\infty \le \mathcal{C}\|f\|_\infty, \quad \mathcal{C} \ne \mathcal{C}(m), \\
&\left\|M_{\frac{1}{T_m}} f\right\|_\infty = \left\|\frac{1}{T_m} f\right\|_\infty \le \frac{1}{\|T_m\|_\infty}\|f\|_\infty \le \mathcal{C}\|f\|_\infty, \quad \mathcal{C} \ne \mathcal{C}(m), \\
&M_{T_m} M_{\frac{1}{T_m}} = M_{\frac{1}{T_m}} M_{T_m} = I,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_m \max_{x\in[0,1]} |(M_{T_m} f)(x) - (M_T f)(x)| &= \lim_m \max_{x\in[0,1]} |T_m(x) f(x) - T(x) f(x)| \\
&\le \|f\|_\infty \lim_m \|T_m - T\|_\infty = 0. \qquad (34)
\end{aligned}
$$

The proof of the lemma is complete. □

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. Using [13, Problem 10.3, p. 153] and Lemmas 6.2 and 6.3, we deduce that, for all sufficiently large m, the inverses $(M_{T_m} + K_m)^{-1}$ exist and are uniformly bounded and

$$
\|f_m^* - f^*\|_\infty \le \mathcal{C}\left[\|(K_m - K)f^*\|_\infty + \|(M_{T_m} - M_T)f^*\|_\infty\right]. \qquad (35)
$$

Moreover, proceeding as in [2, pp. 112-113] we get

$$
\operatorname{cond}(A_m) \le \operatorname{cond}(M_{T_m} + K_m) \le \mathcal{C} < +\infty, \quad \mathcal{C} \ne \mathcal{C}(m).
$$

It remains to prove only (15). To do this, we use (35).

Since by Lemma 6.1 and Lemma 2.3 under the assumptions $k \in W_r([-1,1])$ and $g \in W_r$ we get $k(\cdot - x) \in W_r([0,1])$ uniformly w.r.t. x and $f \in W_r$, applying (3), with $F_x(y) = k(y-x)f^*(y)$ we deduce that

$$
\begin{aligned}
E_{2m-1}(F_x) &\le 2\|f^*\|_\infty E_{m-1}(k(\cdot - x)) + \|k(\cdot - x)\|_\infty E_{m-1}(f^*) \\
&\le \frac{\mathcal{C}}{m^r}\|f^*\|_{W_r},
\end{aligned}
$$

and, taking into account (32),

$$
\|(K_m - K)f^*\|_\infty \le \frac{\mathcal{C}}{m^r}\|f^*\|_{W_r} \qquad (36)
$$

follows. Moreover, by (34),

$$
\|(M_{T_m} - M_T)f\|_\infty = \max_{x\in[0,1]} |T_m(x)f(x) - T(x)f(x)| \le \|f\|_\infty \max_{x\in[0,1]} |T_m(x) - T(x)|,
$$

where $|T_m(x) - T_m(x)|$ is the error of the Gauss-Legendre rule applied to the function $k(y-x)$ with respect to the variable y, for every fixed $x \in [0,1]$. Then, under the assumption $k \in W_r([-1,1])$, by Lemma 6.1 and (3),

$$|T(x) - T_m(x)| \le 2E_{2m-1}(k(\cdot - x)) \le \frac{\mathcal{C}}{m^r}\|k\|_{W_r[-1,1]},$$

i.e.,

$$\|(M_{T_m} - M_T)f\|_\infty \le \frac{\mathcal{C}}{m^r}\|f\|_{W_r}. \tag{37}$$

Consequently, substituting (36) and (37) into (35), (15) follows.
The proof is complete. □

In order to prove Theorem 4.1 we need the following lemma.

Lemma 6.4. *If the kernel k satisfies (20) and (21), then the sequence $\{\widetilde{K}_m\}_m$ is pointwise convergent to K, uniformly bounded and collectively compact.*

Proof. Denoting by $P \in \mathbb{P}_{m-1}$ the polynomial of best approximation of $f \in C^0$, we have

$$\begin{aligned}|(Kf)(x) - (\widetilde{K}_m f)(x)| &\le \int_0^1 |k(y-x)||f(y) - P(y)|dy \\ &+ \int_0^1 |k(y-x)||L_m(f-P,y)|dy.\end{aligned}$$

Using Lemma 2.1, we get

$$|(Kf)(x) - (\widetilde{K}_m f)(x)| \le E_{m-1}(f)\left[\int_0^1 |k(y-x)|dy + \rho\left(\frac{k(\cdot - x)}{\sqrt{\varphi}}\right)\right]$$

and by (9) and (20) we get

$$\|(K - \widetilde{K}_m)f\| \le \mathcal{C}E_{m-1}(f), \tag{38}$$

i.e., pointwise convergence of $\{\widetilde{K}_m\}_m$ to K.

Since the uniform boundedness of the sequence $\{\widetilde{K}_m\}_m$ trivially follows by Lemma 2.1, it remains to prove its collective compactness. As it was done in the proof of Lemma 6.2, we prove (33) with $\widetilde{K}_m$ in place of K_m. Applying Lemma 2.1 again, we get

$$\begin{aligned}|(\widetilde{K}_m f)(x+h) - (\widetilde{K}_m f)(x)| &\le \int_0^1 |k(y-x-h) - k(y-x)||L_m(f,y)|dy \\ &\le \mathcal{C}\rho\left(\frac{k(\cdot - x - h) - k(\cdot - x)}{\sqrt{\varphi}}\right)\|f\|_\infty.\end{aligned}$$

Then, using (21) the collective compactness of $\{\widetilde{K}_m\}_m$ follows. □

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. Applying [2, Theorem 4.1.1, p. 106] and Lemma 6.4, we deduce that, for all sufficiently large m, the inverses $(M_T + \widetilde{K}_m)^{-1}$ exist and are uniformly bounded and

$$\|f^* - \widetilde{f}_m^*\|_\infty \le \mathcal{C}\|(\widetilde{K}_m - K)f^*\|_\infty.$$

Recalling (38), (22) follows.

Moreover, proceeding as in [2, pp. 112-113] we get

$$\operatorname{cond}(\widetilde{A}_m) \le \operatorname{cond}(M_T + \widetilde{K}_m) \le \mathcal{C} < +\infty, \quad \mathcal{C} \ne \mathcal{C}(m).$$

The proof of the theorem is now complete. □

Taking into account the following lemma, the proof of Theorem 4.2 is similar to the one of Theorem 4.1.

Lemma 6.5. *If the kernel k satisfies (8) and (9), then the sequence $\{\bar{K}_m\}_m$ is pointwise convergent to K, uniformly bounded and collectively compact.*

Proof. The proof of the lemma can be found in the proof of [1, Theorem 5]. □

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