Intriguing sets of quadrics in $\mathrm{PG}(5, q)$

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Abstract

In the geometric setting of quadrics commuting with a Hermitian surface of $\mathrm{PG}(3, q^2)$, q odd, [24] a hemisystem on the Hermitian surface $\mathcal{H}(3, q^2)$, $q \geq 7$, admitting a subgroup K of $P\Omega^-(4, q)$ of order $q^2(q+1)$ is constructed. Also, a new family of Cameron–Liebler line classes of $\mathrm{PG}(3, q)$, $q \geq 5$ odd, with parameter $(q^2 + 1)/2$ is provided.

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1 Introduction and Basics

Let $\mathcal{P}$ be a finite classical polar space of rank $r \geq 2$ over a finite field of order q. We say that a set of points $\mathcal{I}$ of $\mathcal{P}$ is *intriguing* if

$$|P^\perp \cap \mathcal{I}| = \begin{cases} h_1 & if \quad P \in \mathcal{I} \\ h_2 & if \quad P \notin \mathcal{I} \end{cases},$$

for some constants h_1 and h_2 (where P ranges over the points of $\mathcal{P}$). The integers h_1 and h_2 are called the *intersection numbers* of $\mathcal{I}$. Here $\perp$ is the polarity of $\mathcal{P}$.
It turns out that an intriguing set of $\mathcal{P}$ is either an m–ovoid or an i–tight set of $\mathcal{P}$ [2]. An m–ovoid $\mathcal{O}$ of $\mathcal{P}$ is a subset of points of $\mathcal{P}$ such that every maximal of $\mathcal{P}$ meets $\mathcal{O}$ in m points. If $\mathcal{T}$ is a subset of points of $\mathcal{P}$, then the average number of points of $\mathcal{T}$ collinear with a given point is bounded

above by $i(q^{r-1}-1)/(q-1)+q^{r-1}-1$, where i is determined by the size of $\mathcal{T}$. If equality occurs, $\mathcal{T}$ is said to be i–tight. A set of points is tight if it is i–tight for some i. For more results on this topic see [2].
There exist two very nice and important objects in finite geometry admitting the same automorphism group and apparently not related each other. They are a hemisystem of the Hermitian generalized quadrangle $\mathcal{H}(3,q^2)$, q odd [11], and a Cameron–Liebler line class in $\mathrm{PG}(3,q)$, q odd, of parameter $(q^2+1)/2$ [5], both admitting the classical group $G := P\Omega^-(4,q)$ stabilizing an elliptic quadric $\mathcal{Q}^-(3,q)$ of $\mathrm{PG}(3,q)$. Under duality [19],[22], they are both examples of intriguing sets of quadrics of $\mathrm{PG}(5,q)$.
Although we were not able to establish a possible connection between the above mentioned objects, we pose the following question. There does exist a suitable chosen subgroup of G producing new hemisystems of the Hermitian generalized quadrangle and new Cameron–Liebler line classes of $\mathrm{PG}(3,q)$? The answer is affirmative. By considering a suitable subgroup of the stabilizer of a point of $\mathcal{Q}^-(3,q)$ in G, we are able to construct for $q\geq 7$ a new infinite family of hemisystems of $\mathcal{H}(3,q^2)$ and a new infinite family of Cameron–Liebler line classes with parameter $(q^2+1)/2$ for $q\geq 5$.
The paper is divided into two parts.
The first part of the paper deals with the generalized quadrangle $\mathcal{H}(3,q^2)$, the incidence structure of all points and lines (generators) of a non–singular Hermitian surface in $\mathrm{PG}(3,q^2)$, a generalized quadrangle of order (q^2,q), with automorphism group $P\Gamma U(4,q^2)$. The dual of $\mathcal{H}(3,q^2)$ is $\mathcal{Q}^-(5,q)$ [22, Theorem 3.2.3], the elliptic quadric of $\mathrm{PG}(5,q)$, a generalized quadrangle of order (q,q^2) with automorphism group $P\Gamma O^-(6,q)$.
In [24], regular systems of $\mathcal{H}(3,q^2)$ were introduced. A *regular system of order* m of $\mathcal{H}(3,q^2)$ is a set $\mathcal{R}$ of lines of $\mathcal{H}(3,q^2)$ with the property that every point lies on exactly m lines of $\mathcal{R}$, $0<m<q+1$. Segre proved that, if q is odd, such a system must have $m=(q+1)/2$, and called a regular system of $\mathcal{H}(3,q^2)$ of order $(q+1)/2$ a *hemisystem* of $\mathcal{H}(3,q^2)$. He also constructed a hemisystem of $\mathcal{H}(3,9)$ admtting the linear group $PSL(3,4)$. In [6], the nonexistence of regular systems of $\mathcal{H}(3,q^2)$ for q even was established.
Thas conjectured that there are no hemisystems of $\mathcal{H}(3,q^2)$ for $q>3$ [26], but in [11] the conjecture was disproved by constructing hemisystems of $\mathcal{H}(3,q^2)$, for all odd prime powers $q\geq 3$, and giving Segre's example for $q=3$. For other results, see [3]. Other sporadic examples can be found in [1],[4],[12]. By duality, a hemisystem of $\mathcal{H}(3,q^2)$ is a $(q+1)/2$–ovoid

of the quadric $\mathcal{Q}^-(5,q)$. Here, we will construct a new infinite family of hemisystems of $\mathcal{H}(3,q^2)$, $q \geq 7$ odd, admitting a subgroup of the stabilizer of a point of $\mathcal{Q}^-(3,q)$ in the group G.
In the second part of the paper we deal with Cameron–Liebler line classes of $\mathrm{PG}(3,q)$. A Cameron–Liebler line class $\mathcal{L}$ with parameter x in $\mathrm{PG}(3,q)$ is a set of lines of $\mathrm{PG}(3,q)$ such that every spread of $\mathrm{PG}(3,q)$ contains exactly x lines of $\mathcal{L}$ [9]. There exist classical examples of Cameron–Liebler line classes $\mathcal{L}$ with parameters $x = 1, 2$ and $x = q^2, q^2 - 1$. Namely, the set of lines through a point P and, analogously, the set of lines in a plane π form a Cameron–Liebler line class with parameter $x = 1$. The union of these two sets for $P \notin \pi$ forms a Cameron–Liebler line class with parameter $x = 2$. In general, the complement of a Cameron–Liebler line class with parameter x is a Cameron–Liebler line class with parameter $q^2 + 1 - x$. It was conjectured that no other examples of Cameron–Liebler line classes exist [9], but Bruen and Drudge [5] provided an example of Cameron–Liebler line classes in $\mathrm{PG}(3,q)$, q odd, with $x = (q^2+1)/2$, and Govaerts and Penttila [17] found a Cameron–Liebler line class in $\mathrm{PG}(3,4)$ with parameter $x = 7$, and so also with parameter $x = 10$. Very recently, a new infinite family of Cameron–Liebler line class was found for $q \equiv 9 \pmod{12}$ in [15],[20]. For further results on this topic see also [21],[16]. A Cameron–Liebler line class of $\mathrm{PG}(3,q)$ with parameter x corresponds to an x–tight set of $\mathcal{Q}^+(5,q)$, the hyperbolic quadric of $\mathrm{PG}(5,q)$. Here, we will construct a new infinite family of Cameron–Liebler line classes of $\mathrm{PG}(3,q)$, with parameter $(q^2+1)/2$, $q \geq 5$ odd, admitting the stabilizer of a point of $\mathcal{Q}^-(3,q)$ in the group G.
From [2, Theorems 11,12] the intriguing sets considered here are also instances of two–character sets of $\mathrm{PG}(5,q)$: they have two intersection numbers with respect to hyperplanes and so they give rise to strongly regular graphs (via linear representation) and two–weight codes [7].
From now on, we assume that q is a power of an odd prime.

2 A new family of hemisystems of the Hermitian surface

Let $\mathcal{H}(3,q^2)$ be a Hermitian surface of $\mathrm{PG}(3,q^2)$ and let $\mathcal{U}$ be the Hermitian polarity of $\mathrm{PG}(3,q^2)$ associated with $\mathcal{H}(3,q^2)$. Let $\mathcal{B}$ be an orthogonal polarity commuting with the Hermitian polarity $\mathcal{U}$. Set $\mathcal{V} = \mathcal{B}\mathcal{U} = \mathcal{U}\mathcal{B}$. Then $\mathcal{V}$ is a non–linear collineation and from [24], the fixed points of $\mathcal{V}$ on $\mathcal{H}(3,q^2)$

form a non–degenerate quadric $\mathcal{Q}$ which is either elliptic or hyperbolic. In particular, $\mathcal{Q} = \mathcal{H}(3, q^2) \cap \Sigma$, where Σ is a suitable subgeometry of $\mathrm{PG}(3, q^2)$ isomorphic to $\mathrm{PG}(3, q)$ [24]. Assume that $\mathcal{Q}$ is elliptic. Let $\bar{G}$ denote the stabilizer of $\mathcal{Q}$ in $PSU(4, q^2)$. From [11], $\bar{G} = PGO^-(4, q) \cap PSU(4, q^2) = PSO^-(4, q) \cdot 2$. The following results have been proved in [11].

Proposition 2.1. *The group $\bar{G}$ has three orbits on points of $\mathcal{H}(3, q^2)$.*

The $\bar{G}$–orbits have sizes: $q^2 + 1$, $q^2(q^2 + 1)(q + 1)/2$, $q^2(q^2 + 1)(q - 1)/2$: they are points on $\mathcal{Q}$, points on generators tangent to $\mathcal{Q}$ but not on $\mathcal{Q}$, and the complements of these in $\mathcal{H}(3, q^2)$, respectively.

Proposition 2.2. *$\bar{G}$ has two orbits on lines of $\mathcal{H}(3, q^2)$.*

The quadric $\mathcal{Q}$ is a partial ovoid of $\mathcal{H}(3, q^2)$ and so each generator of $\mathcal{H}(3, q^2)$ is either disjoint from $\mathcal{Q}$ or meets $\mathcal{Q}$ in exactly one point. The quadric $\mathcal{Q}$ is a *special set* of $\mathcal{H}(3, q^2)$, i.e., it is a subset of q^2+1 points of $\mathcal{H}(3, q^2)$ such that each point of $\mathcal{H}(3, q^2) \setminus \mathcal{Q}$ is conjugate to 0 or 2 points of $\mathcal{Q}$, or equivalently, any three points of $\mathcal{Q}$ generate a non–tangent plane to $\mathcal{H}(3, q^2)$, see [11] and references therein. Through each point P on $\mathcal{Q}$ there are $q+1$ generators of $\mathcal{H}(3, q^2)$, which are permuted by the stabilizer of P in $\bar{G}$. Since $\mathcal{Q}$ has q^2+1 points and $\bar{G}$ acts transitively on $\mathcal{Q}$, there are $(q + 1)(q^2 + 1)$ generators of $\mathcal{H}(3, q^2)$ permuted in a single orbit under the action of $\bar{G}$. The second orbit consists of all generators of $\mathcal{H}(3, q^2)$ disjoint from $\mathcal{Q}$.
Now, we recall the construction given in [11] of a class of hemisystems of $\mathcal{H}(3, q^2)$, admitting the group $G = P\Omega^-(4, q) \leq \bar{G}$.
The group G has the same orbits on points of $\mathcal{H}(3, q^2)$ as $\bar{G}$. Under the action of G the two $\bar{G}$–line orbits given in Proposition 2.2 split into four orbits, two of size $(q^2+1)(q+1)/2$, say $\mathcal{O}_1$ and $\mathcal{O}_2$, and two of size $q^2(q^2-1)/2$, say $\mathcal{O}_3$ and $\mathcal{O}_4$. Since G acts transitively on $\mathcal{Q}$, each orbit of size $(q^2+1)(q+1)/2$ represents a partial hemisystem. The block–tactical decomposition matrix for this orbit decomposition is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{q^2+1}{2} & \frac{q^2+1}{2} \\ q^2 & q^2 & \frac{q^2+1}{2} & \frac{q^2+1}{2} \end{bmatrix},$$

and hence the point–tactical decomposition matrix is

$$\begin{bmatrix} \frac{q+1}{2} & \frac{q+1}{2} & 0 & 0 \\ 0 & 0 & \frac{q+1}{2} & \frac{q+1}{2} \\ 1 & 1 & \frac{q-1}{2} & \frac{q-1}{2} \end{bmatrix}.$$

It follows that amalgamation of an orbit of size $(q^2+1)(q+1)/2$ and an orbit of size $q^2(q^2-1)/2$ yields a G–invariant hemisystem of $\mathcal{H}(3,q^2)$.
From the point–tactical decomposition it also follows that through any point on a generator tangent to $\mathcal{Q}$ there pass a unique line of $\mathcal{O}_1$ and a unique line of $\mathcal{O}_2$.
Let $\mathcal{R}$ be the G–invariant hemisystem of $\mathcal{H}(3,q^2)$ obtained by glueing together the orbits $\mathcal{O}_1$ and $\mathcal{O}_3$. Let us fix a point $P \in \mathcal{Q}$. The $q+1$ generators on P lie on the tangent plane π to $\mathcal{H}(3,q^2)$ at P and are such that $(q+1)/2$ of them are in $\mathcal{O}_1$ (and so belong to $\mathcal{R}$) and $(q+1)/2$ are in $\mathcal{O}_2$. Let us denote by π_i, the points of π on the generators of $\mathcal{O}_i$ on P, $i=1,2$. The stabilizer of P in G contains a subgroup K of order $q^2(q+1)$ fixing P and acting transitively on $\mathcal{Q}\setminus\{P\}$. Also, it fixes other $q-1$ permutable Baer elliptic quadrics, say $\mathcal{Q}_1,\ldots,\mathcal{Q}_{q-1}$, pairwise intersecting just in P, all embedded in $\mathcal{H}(3,q^2)$ and obtained by intersecting $\mathcal{H}(3,q^2)$ with suitable Baer subgeometries. Moreover, the union $\mathcal{O}=\mathcal{Q}\cup\mathcal{Q}_1\cup\ldots\cup\mathcal{Q}_{q-1}$ is an ovoid of $\mathcal{H}(3,q^2)$, the so called *semiclassical ovoid*, [10], [11]. Since K fixes $\mathcal{R}$, the set of generators tangent to an elliptic quadric $\mathcal{Q}_i$ not on P, splits into two orbits, say $\mathcal{L}_{i_1}$ and $\mathcal{L}_{i_2}$ of size $q^2(q+1)/2$ according as a generator meets π_1 or π_2, respectively, $i=1,\ldots,q-1$. Analogously, the generators tangent to $\mathcal{Q}$ not at P split into two orbits, say $\mathcal{L}_1$ and $\mathcal{L}_2$, of size $q^2(q+1)/2$, according as a generator meets π_1 or π_2. In particular, $\mathcal{L}_2$ belongs to $\mathcal{R}$. A generator of $\mathcal{R}$ not tangent to $\mathcal{Q}$ is necessarily tangent to a unique quadric $\mathcal{Q}_i$ due to the fact that $\mathcal{O}$ is an ovoid. Moreover, we can always assume that for $i=1,\ldots,(q-1)/2$, $\mathcal{L}_{i_1}$ belongs to $\mathcal{R}$ and hence $\mathcal{L}_{i_2}$ belongs to $\mathcal{R}$, for $i=(q+1)/2,\ldots,q-1$. Let $X:=\{\mathcal{Q}_{(q+1)/2},\ldots,\mathcal{Q}_{(q-1)},\mathcal{Q}\}$ and let $Y:=\{\mathcal{Q}_1,\ldots,\mathcal{Q}_{(q-1)/2}\}$. Let us consider a set Y' consisting of $(q-1)/2$ quadrics arbitrarily chosen among those in $\mathcal{O}$. It follows that if $|X\cap Y'|=x$ then $|Y\setminus Y'|=x$. Let us denote by $\mathcal{Q}_j$, $j=1,\ldots,x$ the quadrics in $X\cap Y'$ and by $\mathcal{Q}_k$, $k=1,\ldots,x$ the quadrics in $Y\setminus Y'$. We claim that the set

$$\mathcal{R}' := \bigcup_{j=1}^{x}\left((\mathcal{R}\setminus\mathcal{L}_{j_2})\cup\mathcal{L}_{j_1}\right)\cup\bigcup_{k=1}^{x}\left((\mathcal{R}\setminus\mathcal{L}_{k_1})\cup\mathcal{L}_{k_2}\right),$$

is a hemisystem of $\mathcal{H}(3,q^2)$.
Assume that $x=1$. Let us start from the hemisystem $\mathcal{R}$. Then $\mathcal{R}' := (\mathcal{R}\setminus\mathcal{L}_{j_2})\cup\mathcal{L}_{j_1}\cup(\mathcal{R}\setminus\mathcal{L}_{k_1})\cup\mathcal{L}_{k_2}$. Let T be a point of $\mathcal{H}(3,q^2)$. Assume that $T\in\pi_1$. Through T there pass a unique generator of $\mathcal{O}_1$, $(q-1)/2$ generators of $\mathcal{R}\setminus\{\mathcal{O}_1\}$ of which exactly one belongs to $\mathcal{L}_{k_1}$. This is due to

the fact that $\mathcal{Q}_k$ is a special set of $\mathcal{H}(3,q^2)$. Substituting the unique line of $\mathcal{R}$ in $\mathcal{L}_{k_1}$ with the unique line of $\mathcal{L}_{j_1}$ passing through T it follows that there exist $(q+1)/2$ generators of $\mathcal{R}'$ on T. Assume $T \in \pi_2$. Through T there exists a unique line ℓ of $\mathcal{L}_{j_2}$ contained in $\mathcal{R}$ and a unique line m of $\mathcal{L}_{k_1}$. Substituting ℓ with the unique line of $\mathcal{L}_{j_1}$ on T and m with the unique line of $\mathcal{L}_{k_2}$ on T, it follows that there are $(q+1)/2$ generators of $\mathcal{R}'$ on T. Assume that T is a point on a generator tangent to $\mathcal{Q}_j$ not at P. Through T there pass a unique generator of $\mathcal{L}_{j_2}$ belonging to $\mathcal{R}$. Substituting such a generator with the unique generator of $\mathcal{L}_{j_1}$ on T we have again $(q+1)/2$ generators of $\mathcal{R}'$ on T. Analogously if T is a point on a generator tangent to $\mathcal{Q}_k$ not at P. We call such a procedure *derivation* of $\mathcal{R}$ with respect to the quadrics $\mathcal{Q}_j$,$\mathcal{Q}_k$. This procedure can be iterated and at each step a hemisystem arises. In the general case $\mathcal{R}$ is obtained from $\mathcal{R}$ by applying x times the derivation procedure (*multiple derivation*). It turns out that the multiple procedure described above produces $\binom{q}{(q-1)/2}$ hemisystems of $\mathcal{H}(3,q^2)$. We have proved the following result.

Theorem 2.3. *There exists a hemisystem of* $\mathcal{H}(3,q^2)$ *admitting a group* K *of order* $q^2(q+1)$ *for all odd prime powers* $q \geq 7$. *Dually, there exists a* K*–invariant* $(q+1)/2$*–ovoid of* $\mathcal{Q}^-(5,q)$, $q \geq 7$.

Proof. From the discussion above the group G produces exactly four G–invariant hemisystems. Since in our construction a G–orbit of generators tangent to $\mathcal{Q}$ is fixed, the number of G–invariant hemisystems is $2q$. It follows that if $q \geq 7$ then there exist hemisystems of $\mathcal{H}(3,q^2)$ that are K–invariant but not G–invariant. □

Remark 2.4. It is plausible that for $q = 7, 9, 11$ our hemisystems coincide with certain examples in [4] found by computer.

Remark 2.5. A *partial quadrangle* $PQ(s,t,\mu)$ [8] is an incidence structure of points and lines with the properties that any two points are incident with at most one line, every point is incident with $t+1$ lines, every line is incident with $s+1$ points, any two non-collinear points are jointly collinear with exactly μ points, and for any point P and line l which are not incident, there is at most one point Q on l collinear with P. There are not many constructions of partial quadrangles known, see [11] and references therein. A hemisystem of a generalized quadrangle of order (s,s^2) gives a partial quadrangle $PQ((s-1)/2, s^2, (s-1)^2/2)$ (the points of the partial quadrangle being the points of the hemisystem and the lines of the partial quadrangle being the lines of the generalized quadrangle) [25]. From our construction a new $PQ((q-1)/2, q^2, (q-1)^2/2)$ arises.

Remark 2.6. From [13, Theorem 3.5] the strongly regular graphs arising via the linear representation of the two–character set obtained from non–equivalent hemisystems are not isomorphic. Also, each hemisystem of $\mathcal{H}(3, q^2)$ yields a strongly regular decomposition of the collinearity graph of $\mathcal{Q}^-(5, q)$, in the sense of [18].

Remark 2.7. Hemisystems give rise to 4–class imprimitive cometric Q–antipodal association schemes that are not metric, see [14], which are indeed rare in the literature.

3 A new family of Cameron–Liebler line classes

A *Cameron–Liebler line class* $\mathcal{L}$ is a set of lines in $\mathrm{PG}(3, q)$ such that for any line ℓ of $\mathrm{PG}(3, q)$,

$$|\{m \in \mathcal{L} \,:\, |m \cap \ell| = 1\}| = \begin{cases} (q+1)x + (q^2-1) & if \quad \ell \in \mathcal{L} \\ (q+1)x & if \quad \ell \notin \mathcal{L} \end{cases}.$$

for some fixed integer x, called the *parameter* of $\mathcal{L}$. There are many equivalent characterizations of Cameron–Liebler line classes, see [23]. Under the Klein correspondence between the lines of $\mathrm{PG}(3, q)$ and points of a Klein quadric $\mathcal{Q}^+(5, q)$, a Cameron–Liebler line class of parameter x produces an *x–tight set* of $\mathcal{Q}^+(5, q)$.

Definition 3.1. *A set of points $\mathcal{T}$ in $\mathcal{Q}^+(5, q)$ is said to be i-tight if*

$$|P^\perp \cap \mathcal{T}| = \begin{cases} i(q+1) + q^2 & if \quad P \in \mathcal{T} \\ i(q+1) & if \quad P \notin \mathcal{T} \end{cases},$$

where $\perp$ denotes the polarity of $\mathrm{PG}(5, q)$ associated with $\mathcal{Q}^+(5, q)$.

Let q be odd. Let $\mathcal{Q}^-(3, q)$ be an elliptic quadric of $\mathrm{PG}(3, q)$. Each point of $\mathcal{Q}^-(3, q)$ lies on q^2 secants to $\mathcal{Q}^-(3, q)$, and so lies on $q+1$ tangent lines. Let $G = P\Omega^-(4, q)$ be the commutator subgroup of the full stabilizer of $\mathcal{Q}^-(3, q)$ in $PGL(4, q)$. The group G has three orbits on points of $\mathrm{PG}(3, q)$, i.e., the points of $\mathcal{Q}^-(3,q)$ and other two orbits $\mathcal{O}_s$ and $\mathcal{O}_n$ of size $q^2(q^2+1)/2$. The two orbits $\mathcal{O}_s$, $\mathcal{O}_n$ correspond to points of $\mathrm{PG}(3,q)$ such that the evaluation of the quadratic form associated to $\mathcal{Q}^-(3,q)$ is a square or a non–square in $\mathrm{GF}(q)$, respectively. In its action on lines of $\mathrm{PG}(3,q)$, the group G has four orbits: two orbits, say $\mathcal{L}_1$ and $\mathcal{L}_2$, both of size $(q+1)(q^2+1)/2$, consisting of lines tangent to $\mathcal{Q}^-(3,q)$ and two orbits, say $\mathcal{L}_3$ and $\mathcal{L}_4$, both of size $q^2(q^2+1)/2$ consisting of lines secant and external to $\mathcal{Q}^-(3,q)$, respectively. The block–tactical decomposition matrix for this orbit decomposition is

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ q & 0 & \frac{q-1}{2} & \frac{q+1}{2} \\ 0 & q & \frac{q-1}{2} & \frac{q+1}{2} \end{bmatrix},$$

and hence the point–tactical decomposition matrix is

$$\begin{bmatrix} \frac{q+1}{2} & \frac{q+1}{2} & q^2 & 0 \\ q+1 & 0 & \frac{q(q-1)}{2} & \frac{q(q+1)}{2} \\ 0 & q+1 & \frac{q(q-1)}{2} & \frac{q(q+1)}{2} \end{bmatrix}.$$

Indeed, simple group–theoretic arguments show that a line of $\mathcal{L}_1$ ($\mathcal{L}_2$) contains q points of $\mathcal{O}_s$ ($\mathcal{O}_n$), that a secant line to $\mathcal{Q}^-(3,q)$ always contains $(q-1)/2$ points of $\mathcal{O}_s$ and $(q-1)/2$ points of $\mathcal{O}_n$ and that a line external to $\mathcal{Q}^-(3,q)$ contains $(q+1)/2$ points of $\mathcal{O}_s$ and $(q+1)/2$ points of $\mathcal{O}_n$.
Let $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_3$. From the orbit–decompositions above it is an easy matter to prove that $\mathcal{L}$ is the Cameron–Liebler line class constructed in [5]. In particular, $\mathcal{L}$ has the following characters with respect to line–stars of $\mathrm{PG}(3,q)$:

$$q^2+(q+1)/2, (q^2-q)/2, (q^2+q+2)/2,$$

and with respect to the line sets of planes of $\mathrm{PG}(3,q)$:

$$(q+1)/2, (q^2+q)/2, (q^2+3q+2)/2.$$

Let $P \in \mathcal{Q}^-(3,q)$ and let π be the tangent plane to $\mathcal{Q}^-(3,q)$ at P. We can distinguish the following sets of lines of $\mathrm{PG}(3,q)$:

- $t_1 : (q+1)/2$ lines in $\mathcal{L}_1$ through P;
- $t_2 : (q+1)/2$ lines in $\mathcal{L}_2$ through P;
- $s_1 : q^2$ secants on P;

- $s_2 : q^2(q^2-1)/2$ lines in $\mathcal{L}_3 \setminus s_1$;
- $u_1 : q^2(q+1)/2$ lines in $\mathcal{L}_1 \setminus t_1$;
- $u_2 : q^2(q+1)/2$ lines in $\mathcal{L}_2 \setminus t_2$;
- $e_1 : q^2$ external line lying in π;
- $e_2 : q^2(q^2-1)/2$ external lines not in π.

Let $\mathcal{L}' := t_1 \cup s_1 \cup u_1 \cup e_2$. We claim that $\mathcal{L}'$ is a Cameron–Liebler line class with parameter $(q^2+1)/2$.
First of all notice that $\mathcal{L}'$ has size $(q+1)/2+q^2+q^2(q+1)/2+q^2(q^2-1)/2 = (q^2+1)(q^2+q+1)/2$. Set $a := (q+1)(q^2+1)/2$. We distinguish several cases.
Assume that $\ell \in t_1$. Then, through a point of ℓ distinct from P, there pass $q(q+1)/2-q$ lines of e_2 and q lines of u_1. Through P there pass q^2 lines of s_1 and $(q-1)/2$ lines of t_1 distinct from ℓ. Summing up there are $a+q^2$ lines of $\mathcal{L}'$ meeting ℓ.
Assume that $\ell \in s_1$ and $\ell \cap \mathcal{Q}^-(3,q) = \{P, P'\}$. Through P there pass q^2-1 lines of s_1 distinct from ℓ and $(q+1)/2$ lines of t_1. Through P' there pass $(q+1)/2$ lines of u_1. Through a point $R \in \ell \setminus \{P, P'\}$ there pass $q(q+1)/2$ lines of e_2, if $R \in \mathcal{O}_n$, and $q(q+1)/2$ lines of e_2 and $q+1$ lines of u_1, if $R \in \mathcal{O}_s$. Summing up there are $a+q^2$ lines of $\mathcal{L}'$ meeting ℓ.
Assume that $\ell \in u_1$ and $T = \ell \cap \mathcal{Q}^-(3,q)$. Through T there pass $(q-1)/2$ lines of u_1 distinct from ℓ and a line of s_1. Through a point $R \in \ell \setminus \pi$ with $R \neq T$ there pass q lines of u_1 distinct from ℓ, a line of s_1 and $q(q+1)/2$ lines of e_2. Through the point $\ell \cap \pi$ there pass one line of t_1, $q-1$ lines of u_1 distinct from ℓ and $q(q-1)/2-q$ lines of e_2. Summing up there are $a+q^2$ lines of $\mathcal{L}'$ meeting ℓ.
Assume that $\ell \in e_2$ and $\ell \cap \pi \in \mathcal{O}_s$. Through a point R of ℓ there pass $q+1$ lines of u_1, a line of s_1 and $q(q+1)/2-1$ lines of e_2 distinct from ℓ, if $R \in \mathcal{O}_s$ and $R \notin \pi$, whereas there pass a line of s_1 and $q(q+1)/2-1$ lines of e_2 distinct from ℓ, if $R \in \mathcal{O}_n$. Through $\ell \cap \pi$ there pass a line of t_1, $q(q+1)/2-q-1$ lines of e_2 distinct from ℓ and q lines of u_1. Summing up there are $a+q^2$ lines of $\mathcal{L}'$ meeting ℓ.
Assume that $\ell \in e_2$ and $\ell \cap \pi \in \mathcal{O}_n$. Through a point R of ℓ there pass $q+1$ lines of u_1, a line of s_1 and $q(q+1)/2-1$ lines of e_2 distinct from ℓ, if $R \in \mathcal{O}_s$, whereas there pass a line of s_1 and $q(q+1)/2-1$ lines of e_2 distinct

from ℓ, if $R \in \mathcal{O}_n$ and $R \notin \pi$. Through $\ell \cap \pi$ there pass $q(q+1)/2 - q - 1$ lines of e_2 distinct from ℓ. Summing up there are $a + q^2$ lines of $\mathcal{L}'$ meeting ℓ.

Assume that $\ell \in t_2$. Through P there pass q^2 lines of s_1 and $(q+1)/2$ lines of t_1. Through a point on ℓ distinct from P there pass $q(q+1)/2 - q$ lines of e_2. Summing up there are a lines of $\mathcal{L}'$ meeting ℓ.

Assume that $\ell \in s_2$ and $\ell \cap \pi \in \mathcal{O}_n$. Through a point of $\ell \cap \mathcal{Q}^-(3,q)$ there pass one line of s_1 and $(q+1)/2$ lines of u_1. Through a point of $\ell \cap \mathcal{O}_s$ there pass $q+1$ lines of u_1, a line of s_1 and $q(q+1)/2$ lines of e_2. Through a point of $\ell \cap \mathcal{O}_n$ that is not in π there pass one line of s_1 and $q(q+1)/2$ lines of e_2. Through $\ell \cap \pi$ there pass $q(q+1)/2 - q$ lines of e_2. Summing up there are a lines of $\mathcal{L}'$ meeting ℓ.

Assume that $\ell \in s_2$ and $\ell \cap \pi \in \mathcal{O}_s$. Through a point of $\ell \cap \mathcal{Q}^-(3,q)$ there pass one line of s_1 and $(q+1)/2$ lines of u_1. Through a point of $\ell \cap \mathcal{O}_s$ that is not in π there pass $q+1$ lines of u_1, a line of s_1 and $q(q+1)/2$ lines of e_2. Through a point of $\ell \cap \mathcal{O}_n$ there pass one line of s_1 and $q(q+1)/2$ lines of e_2. Through $\ell \cap \pi$ there pass a line of t_1, q lines of u_1 and $q(q+1)/2 - q$ lines of e_2. Summing up there are a lines of $\mathcal{L}'$ meeting ℓ.

Assume that $\ell \in u_2$. Through $\ell \cap \mathcal{Q}^-(3,q)$ there pass a line of s_1 and $(q+1)/2$ lines of u_1. Through a point $R \in \ell \setminus \mathcal{Q}^-(3,q)$, $R \notin \pi$ there pass a line of s_1 and $q(q+1)/2$ lines of e_2. Through $\ell \cap \pi$ there pass $q(q+1)/2 - q$ lines of e_2. Summing up there are a lines of $\mathcal{L}'$ meeting ℓ.

Finally, assume that $\ell \in e_1$. Through a point $R \in \ell$ there pass a line of t_1, q lines of u_1 and $q(q+1)/2 - q$ lines of e_2, if $R \in \mathcal{O}_s$, and $q(q+1)/2 - q$ lines of e_2, if $R \in \mathcal{O}_n$. Summing up there are a lines of $\mathcal{L}'$ meeting ℓ.

We have proved that $\mathcal{L}'$ is a Cameron–Liebler line class with parameter $(q^2+1)/2$. In particular, $\mathcal{L}'$ has the following characters with respect to line–stars of $\mathrm{PG}(3,q)$:

$$(q+3)/2, (q^2-q)/2, (q^2+q+2)/2, (q^2+3q+4)/2, q^2+(q+1)/2,$$

and with respect to the line sets of planes of $\mathrm{PG}(3,q)$:

$$(q+1)/2, (q^2-q-2)/2, (q^2+q)/2, (q^2+3q+2)/2, q^2+(q-1)/2.$$

It turns out that, if $q > 3$, these characters are distinct from those of a Bruen–Drudge Cameron–Liebler line class.

Theorem 3.2. *There exists a Cameron–Liebler line class with parameter $(q^2+1)/2$ not equivalent to the Bruen–Drudge's example.*

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