A new quadrature scheme based on an Extended Lagrange Interpolation process

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Abstract

Let $w(x) = e^{-x^\beta} x^\alpha$, $\bar{w}(x) = xw(x)$ and let $\{p_m(w)\}_m$, $\{p_m(\bar{w})\}_m$ be the corresponding sequences of orthonormal polynomials. Since the zeros of $p_{m+1}(w)$ interlace those of $p_m(\bar{w})$, it makes sense to construct an interpolation process essentially based on the zeros of $Q_{2m+1} := p_{m+1}(w)p_m(\bar{w})$, which is called "Extended Lagrange Interpolation". In this paper the convergence of this interpolation process is studied in suitable weighted L^1 spaces, in a general framework which completes the results given by the same authors in weighted $L^p_u((0,+\infty))$, $1 \le p \le \infty$ (see [30], [27]). As an application of the theoretical results, an *extended product integration rule*, based on the aforesaid Lagrange process, is proposed in order to compute integrals of the type

$$\int_0^{+\infty} f(x)k(x,y)u(x)dx, \quad u(x) = e^{-x^\beta} x^\gamma (1+x)^\lambda, \gamma > -1, \lambda \in \mathbb{R}^+,$$

where the kernel $k(x,y)$ can be of different kinds. The rule, which is stable and fast convergent, is used in order to construct a computational scheme involving the single product integration rule studied in [22]. It is shown that the "compound quadrature sequence" represents an efficient proposal for saving 1/3 of the evaluations of the function f, under unchanged speed of convergence.

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1. Introduction

Let $w(x) = e^{-x^\beta} x^\alpha$, $\bar{w}(x) = xw(x)$ be two Generalized Freud-Laguerre weights and let $\{p_m(w)\}_m$, $\{p_m(\bar{w})\}_m$ be the corresponding sequences of orthonormal polynomials. Denoting by a_m the Mhaskar-Rakhmanov-Saff number related to w, the polynomial $\mathcal{Q}_{2m+2}(x) = p_{m+1}(w)p_m(\bar{w})(a_m - x)$ has distinct zeros [27] and therefore it makes sense to consider the Lagrange polynomial interpolating a given continuous function f at the zeros of $\mathcal{Q}_{2m+2}$. Such interpolation process belongs to the so-called *Extended interpolation processes.* Extended interpolation is of interest in the field of the polynomial approximation since it

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introduces new systems of "good" interpolation knots (see for instance [4], [3], [2], [5], [27], [29], [16] and the references therein) and the research of "well far apart" zeros of orthogonal polynomials is still an open problem [9]. On the other hand, following a procedure proposed in [26] for the automatic estimate of the numerical error by extended Gaussian rules, extended interpolation can be used in the numerical evaluation of the convergence order in interpolation processes (see for instance [4]). Moreover extended matrices of orthogonal polynomials can be employed, for instance, in the approximation of singular and hypersingular integral transforms [28], [8].

Here, in particular we consider the "truncated Lagrange polynomial sequence" $\{L^*_{2m+2}(w,\bar{w},f)\}_m$, where $L^*_{2m+2}(w,\bar{w},f) \in \mathbb{P}_{2m+1}$ interpolates f at the first $j = j(m)$ zeros of $\mathcal{Q}_{2m+2}$ and vanishes at the remaining ones, being the choice of j depending on a fixed real parameter $\theta \in (0,1)$ ([20],[17]). The same interpolation process has been already studied in weighted $L^p_u((0,+\infty))$, where

$$u(x) = e^{-x^\beta} x^\gamma (1+x)^\lambda, \quad \gamma > -\frac{1}{p}, \quad \lambda \geq 0,$$

for $1 \leq p \leq \infty$ (see [30], [27]). In the present paper we want to complete, in some sense, the aforesaid results in the case $p = 1$, by adding and proving some results partially announced in [31]. We will prove that the sequence $\{L^*_{2m+2}(w,\bar{w},f)\}_m$ behaves like the best approximation polynomial sequence in suitable weighted subspaces of L^1, for continuous functions f on $(0,+\infty)$, having an exponential growth at infinity and a possible algebraic singularity in 0, i.e. $f \in C_u$ (see Section 2.1), under more general assumption w.r.t. those stated in [30, Theorem 3.4]. Moreover, as an application of the proposed interpolation process, we determine sufficient and necessary conditions under which the integral operator

$$I_{2m+1} : f \in C_u \to \int_0^{+\infty} L^*_{2m+2}(w,\bar{w},f;x) k(x,y) u(x) dx$$

is bounded in a suitable subspace of L^1_u. This assures the stability and the convergence of the corresponding *extended product integration rule* obtained in order to approximate integrals of the types

$$I(f,y) := \int_0^{+\infty} f(x) k(x,y) u(x) dx \tag{1}$$

where the kernel $k(x,y)$ can be of different types and $u(x)$ is a given Generalized Laguerre weight. The kernel's class we can manage may contain for instance weakly singular, "nearly singular" and oscillating functions. Then we introduce an efficient product quadrature scheme obtained by "mixing" the extended rule with an analogous "one-weight" product rule involving a truncated Lagrange polynomial interpolating f just at the zeros of $p_{m+1}(w,x)(a_{m+1} - x)$ [22] (see also [20] for $\beta = 1$). As we will show, the mixed quadrature sequence allows to reduce of one third the number of samples of f needed by the one-weight product sequence.

We point out that the efficient computation of integrals (1) is needed in many contexts, as well in numerical methods for approximating the solution of integral (systems of integral) equations [18], [20], [17], [11], [7]. For instance, the Marchenko system in [7] is connected to inverse and direct scattering problems extensively treated in [32], [10]. The Wiener-Hopf integral equations, in connection with problems in radiative transfer, and related to the solution of boundary integral equations for planar problems (see [1] and the references therein, [18], [12]), are another remarkable class of integral equation for which the quadrature rules we treat here can be useful.

The outline of this paper is as follows. Section 2 contains the notations and some auxiliary results. In Section 3 we give the main results about the extended Lagrange interpolation process, while in Section 4 we introduce the extended quadrature rule, together with the study of the stability and the convergence, and then we describe the compound quadrature sequence. In Section 5 we test the proposed quadrature scheme by means of some examples, which confirm the theoretical estimates. Finally, Section 6 is devoted to the proofs of the main results.

2. Preliminary results and notations

In the sequel $\mathcal{C}$ will denote any positive constant which can be different in different formulas. Moreover $\mathcal{C} \neq \mathcal{C}(a, b, ..)$ will be used meaning that the constant $\mathcal{C}$ is independent of $a, b, ...$ The notation $A \sim B$, where A and B are positive quantities depending on some parameters, will be used if and only if $(A/B)^{\pm 1} \leq \mathcal{C}$, with $\mathcal{C}$ a positive constant independent of the above parameters.

Denote by $\mathbb{P}_m$ the space of all algebraic polynomials of degree at most m. For any bivariate function $g(x, y)$, we will denote by g_y (g_x) the function of the only variable x (y).

2.1. Spaces of functions

For $1 \leq p < \infty$, and $u(x) = e^{-x^\beta} x^\gamma (1+x)^\lambda$, $\gamma > -\dfrac{1}{p}$, $\beta > \dfrac{1}{2}$, $\lambda \geq 0$, let $L^p_u(\mathbb{R}^+)$ be the space of measurable functions f s.t. $fu \in L^p(\mathbb{R}^+)$, equipped with the norm

$$\|f\|_{L^p_u(\mathbb{R}^+)} = \left(\int_0^{+\infty} |f(x)u(x)|^p dx \right)^{\frac{1}{p}}.$$

Moreover with $\gamma \geq 0$, let $L^\infty_u(\mathbb{R}^+) =: C_u$ be the space of functions

$$C_u = \left\{ f : fu \in C^0(\mathbb{R}^+), \lim_{x \to 0^+} f(x)u(x) = 0 = \lim_{x \to +\infty} f(x)u(x) \right\},$$

equipped with the norm $\|f\|_{C_u} = \sup_{x \geq 0} |f(x)|u(x)$.

Strictly related to the weight u there is the so called Mhaskar-Rachmanov-Saff number (shortly MRS number) [25] which is explicitly defined, for any integer m, as follows [15] (see also [21])

$$a_m(u) = \left[\frac{2^{2\beta}(\Gamma(\beta))^2}{\Gamma(2\beta)} \right]^{\frac{1}{\beta}} \left(1 + \frac{2\gamma + 1}{8m} \right)^{\frac{1}{\beta}} m^{\frac{1}{\beta}} \sim m^{\frac{1}{\beta}}.$$

Since different weight functions containing the same exponential factor e^{-x^β} have equivalent MRS numbers, from now on a_m will denote any of them. Furthermore, since $a_m \sim a_{m+1}$, for the sake of brevity we will employ only a_m to denote one of them, without affecting the global discussion.

In the sequel it will be useful the following modulus of continuity [23]

$$\Omega^r_\varphi(f, t)_{u,p} = \sup_{0 < t \leq h} \|u \Delta^r_{h\varphi} f\|_{L^p(I_{rh})}, \quad r \geq 1, \quad \varphi(x) = \sqrt{x}$$

$$I_{rh} = [8r^2h^2, Ah^*], \quad h^* = \frac{1}{h^{\frac{2}{2\beta - 1}}}$$

$$\Delta^r_{h\varphi} f(x) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} f(x + (r-k)h\varphi(x)).$$

In order to evaluate $\Omega^r_\varphi(f)_{u,p}$ we recall that [23]

$$\Omega^r_\varphi(f, t)_{u,p} \leq \sup_{0 < t \leq h} h^r \|f^{(r)} \varphi^r u\|_{L^p(I_{r,h})}, \quad \mathcal{C} \neq \mathcal{C}(f, t),$$

if the sup at the right hand side is bounded.

With $r \in \mathbb{N}$ and setting $\varphi(x) = \sqrt{x}$, consider the Sobolev-type space

$$W^p_r(u) = \left\{ f \in L^p_u([0, +\infty)) : f^{(r-1)} \in AC(0, +\infty), \quad \|f^{(r)} \varphi^r u\|_p < \infty \right\},$$

equipped with the norm

$$\|f\|_{W^p_r(u)} = \|fu\|_p + \|f^{(r)} \varphi^r u\|_p.$$

Denoting by

$$E_m(f)_{u,p} = \inf_{P \in \mathbb{P}_m} \|(f-P)u\|_p, \quad 1 \le p \le \infty,$$

the error of the best approximation of $f \in L^p_u$, we recall the following weaker version of the Jackson Theorem [23]

$$E_m(f)_{u,p} \le \mathcal{C} \int_0^{\frac{\sqrt{a_m}}{m}} \frac{\Omega^r_\varphi(f,t)_{u,p}}{t} dt, \quad \mathcal{C} \ne \mathcal{C}(m,f).$$

Finally, setting $\log^+ x = \log(\max(1,x))$, by $L\log^+ L$ we denote the space of functions f defined in $(0,+\infty)$ s.t. $\|f \log^+ f\|_1 < +\infty$.

2.2. Orthogonal Polynomials

Consider the weight

$$w(x) = e^{-x^\beta} x^\alpha, \quad \alpha > -1, \quad \beta > \frac{1}{2},$$

and let $\{p_m(w)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients, i.e.

$$p_m(w,x) = \gamma_m(w)x^m + \quad \text{terms of lower degree}, \quad \gamma_m(w) > 0.$$

Denoting by $\{x_k\}_{k=1}^{m+1}$ the zeros of $p_{m+1}(w)$ in increasing order, i.e.

$$x_k < x_{k+1}, \quad k = 1, \ldots, m,$$

it is known that they lie in the interval $\left(\mathcal{C}\frac{a_m}{m^2}, a_m\right)$.

Moreover, setting $\bar{w}(x) = xw(x)$, let $\{y_k\}_{k=1}^m$ be the zeros of the corresponding m-th orthonormal polynomial $p_m(\bar{w})$. In what follows it is crucial that the zeros of $p_{m+1}(w)$ interlace with those of $p_m(\bar{w})$ [27], i.e.

$$x_k < y_k < x_{k+1}, \qquad k = 1, 2, \ldots, m.$$

Thus we set

$$z_{2i-1} := x_i, i = 1, 2, \ldots, m+1, \quad z_{2i} := y_i, \ i = 1, 2, \ldots, m.$$

From now on, for any fixed $0 < \theta < 1$, the integer j will denote the index of the zero of $Q_{2m+1} := p_{m+1}(w)p_m(\bar{w})$ s. t.

$$z_j = z_{j(m)} = \min\{z_k : z_k \ge \theta a_m, \quad k = 1, 2, .., 2m+1\}. \tag{2}$$

Finally, uniformly in $m \in \mathbb{N}$ [27]

$$\Delta z_k = z_{k+1} - z_k \sim \frac{\sqrt{a_m}}{m}\sqrt{z_{k+1}}, \quad k = 1, 2, \ldots, j.$$

2.3. A Lagrange interpolation process and a product integration rule

Here we collect some results about the truncated Lagrange process based on Freud-Laguerre zeros introduced in [14] and the related product-quadrature rule proposed in [22].

For a fixed $0 < \theta < 1$, set

$$x_{j^*} = x_{j^*(m)} = \min\{x_k : x_k \ge \theta a_m, \quad k = 1, 2, .., m+1\}$$

where x_i, $i = 1, 2, \ldots, m+1$ are the zeros of the polynomial $p_{m+1}(w)$. Denoting by $\chi_{m,\theta}$ the characteristic function of the segment $(0, x_{j^*})$, let $\mathcal{L}^*_{m+2}(w,f)$ be the "truncated Lagrange polynomial" interpolating a function f at the zeros of $p_{m+1}(w,x)(a_m - x)$ [14]

$$\mathcal{L}^*_{m+2}(w,f,x) = (a_m - x)\sum_{k=1}^{j^*} \ell_{m+1,k}(w,x)\frac{f(x_k)}{a_m - x_k}, \quad \ell_{m+1,k}(w,x) = \frac{p_{m+1}(w,x)}{p'_{m+1}(w,x_k)(x-x_k)}.$$

Setting
$$\int_0^{+\infty} f(x)\widetilde{k}(x,y)U(x)dx, \quad U(x)=e^{-\frac{x^\beta}{2}}x^\gamma(1+x)^\lambda, \gamma\geq 0, \quad \lambda\geq 0,$$
$\widetilde{k}(x,y)$ defined in $\mathbb{R}^+\times \mathrm{S}$, $\mathrm{S}\subseteq\mathbb{R}$, the following product integration rule was introduced in [22]

$$\begin{aligned} I(f,y) &= \mathcal{I}_{m+2}(f,y)+R_{m+2}(y), \quad \mathcal{I}_{m+2}(f,y)=\sum_{i=1}^{j^*}\mathcal{A}_i(y)f(x_i), \\ \mathcal{A}_i(y) &= \int_0^{+\infty}\ell_{m+1,i}(w,x)\widetilde{k}(x,y)U(x)dx. \end{aligned} \tag{3}$$

Following [22], under the assumptions

$$\sup_{y\in \mathrm{S}}\left(\frac{U}{\sqrt{w\varphi}}\widetilde{k}_y\right)\in L\log^+ L, \quad \frac{\sqrt{w\varphi}}{U}\in L^\infty, \tag{4}$$

rule (3) is stable, since for any $f\in C_U(\mathbb{R}^+)$ s.t. $\|fU\|_\infty<+\infty$, it results

$$\sup_{y\in \mathrm{S}}|\mathcal{I}_{m+2}(f,y)|\leq \mathcal{C}\|fU\|_\infty, \ \mathcal{C}\neq\mathcal{C}(m,f). \tag{5}$$

Moreover

$$\sup_{y\in \mathrm{S}}R_{m+2}(y)\leq \mathcal{C}\left\{E_{M^*}(f)_U+e^{-Am}\|fU\|_\infty\right\}, \tag{6}$$

where M^* is a proper fraction of m and $0<A\neq A(m,f)$, $0<\mathcal{C}\neq\mathcal{C}(m,f)$.

Finally we recall for the particular case $k(x,y)\equiv 1$ and $U=w$, the Truncated Gauss-Laguerre rule [19], [20]. Indeed, denoting by $\widetilde{x}_i$, $i=1,2,\ldots,m$ the zeros of the polynomial $p_m(w)$, for a fixed $0<\theta<1$ let be h the index defined as

$$\widetilde{x}_h=\widetilde{x}_{h(m)}=\min\left\{\widetilde{x}_k:\widetilde{x}_k\geq\theta a_m, \quad k=1,2,..,m\right\}. \tag{7}$$

Then the truncated Gauss-Laguerre rule is

$$\int_0^{+\infty}f(x)w(x)dx=\sum_{i=1}^{h}\lambda_{m,i}(w)f(\widetilde{x}_i)+r_m(f), \tag{8}$$

where $\{\lambda_{m,i}(w)\}_{i=1}^m$ are the Christoffel numbers w.r.t.w and $r_m(f)$ is the remainder term.

For our aims it will be useful the following error estimate [17, Prop. 2.3, p. 1050] which holds for any $f\in C_u$ under the assumption $\dfrac{w}{u}\in L^1(0,+\infty)$

$$|r_m(f)|\leq\mathcal{C}\left[E_{\widetilde{M}}(f)_u+e^{-Am}\|fu\|_\infty\right], \tag{9}$$

where $A,\mathcal{C}$ are positive constants independent of m,f and $\widetilde{M}=\left\lfloor m\left(\dfrac{\theta}{1+\theta}\right)^\beta\right\rfloor\sim m.$

3. Extended Lagrange interpolation

Denote by $L_{2m+2}(w,\bar{w},f)$ the Lagrange polynomial interpolating a given function f at the zeros of $\mathcal{Q}_{2m+2}(x)=Q_{2m+1}(x)(a_m-x)$ i.e.

$$L_{2m+2}(w,\bar{w},f;z_i)=f(z_i), \quad i=1,2,\ldots,2m+1, \quad L_{2m+2}(w,\bar{w},f;a_m)=f(a_m).$$

$L_{2m+2}(w,\bar{w},f;x)$ can be expressed in the following form

$$L_{2m+2}(w,\bar{w},f;x)=\sum_{k=1}^{2m+1} l_k(x)f(z_k)+l_{2m+2}(x)f(a_m),$$

where

$$\begin{aligned} l_k(x) &= \frac{Q_{2m+1}(x)}{Q'_{m+1}(z_k)(x-z_k)}\frac{(a_m-x)}{(a_m-z_k)}, \quad k=1,2,\dots,2m+1,\\ l_{2m+2}(x) &= \frac{Q_{2m+1}(x)}{Q_{2m+1}(a_m)}. \end{aligned}$$

Denoted by $\tilde{\chi}_{m,\theta}$ the characteristic function of the segment $(0,z_j)$ (z_j defined in (2)), let us introduce the Lagrange polynomial

$$L^*_{2m+2}(w,\bar{w},f):=L_{2m+2}(w,\bar{w},f\tilde{\chi}_{m,\theta}).$$

The operator $L_{2m+2}(w,\bar{w})$ projects C_u onto $\mathbb{P}_{2m+1}$, while $L^*_{2m+2}(w,\bar{w})$ does not. However, letting

$$\mathcal{P}^*_{2m+1}=\{q\in\mathbb{P}_{2m+1}:q(z_i)=q(a_m)=0,\quad z_i>z_j\}\subset\mathbb{P}_{2m+1},$$

with z_j defined in (2), we have that $L^*_{2m+2}(w,\bar{w})$ is a projector of C_u onto $\mathcal{P}^*_{2m+1}$. Moreover, $\bigcup_m \mathcal{P}^*_{2m+1}$ is dense in C_u. In fact, setting

$$\tilde{E}_{2m+1}(f)_u:=\inf_{P\in\mathcal{P}^*_{2m+1}}\|(f-P)u\|_\infty,$$

the following result was proved in [22]

Lemma 3.1. *For any function $f\in C_u$,*

$$\tilde{E}_{2m+1}(f)_u\le\mathcal{C}\left\{E_M(f)_u+e^{-Am}\|fu\|_\infty\right\}, \tag{10}$$

where $M=\left\lfloor 2m\left(\frac{\theta}{1+\theta}\right)^\beta\right\rfloor$ and the constants $0<A\neq A(m,f)$, $0<\mathcal{C}\neq\mathcal{C}(m,f)$.

In view of (10), $\tilde{E}_{2m+1}(f)_u$ can be estimated by the best approximation error $E_M(f)_u$, where M is a proper fraction of $2m$.

We point out that the "truncation" in the Lagrange processes based on the zeros of the Generalized Laguerre sequence $\{p_m(w)\}_m$ was introduced and studied in [14] (see also [24]) and successively applied also to the extended Lagrange interpolation in [27].

Now we are able to state our main results about the stability and the convergence of the extended Lagrange process.

Theorem 3.1. *Consider the weights $u(x)=x^\gamma(1+x)^\lambda e^{-x^\beta}$, $w(x)=x^\alpha e^{-x^\beta}$ and $\rho(x)=\frac{x^\gamma}{(1+x)^\mu}e^{-\delta x^\beta}$, with $\gamma,\mu\ge0$, $\lambda>1$, $\alpha>-1$, $\delta\ge1$, $\beta>\frac{1}{2}$. Under the assumptions*

$$\frac{\rho}{w\varphi^2}\in L\log^+L,\qquad \frac{w\varphi^2}{u}\in L^\infty(\mathbb{R}^+), \tag{11}$$

we have for any $f\in C_u$

$$\|L^*_{2m+2}(w,\bar{w},f)\rho\|_1\le\mathcal{C}\|fu\|_\infty, \tag{12}$$

with $0<\mathcal{C}\neq\mathcal{C}(m,f)$.

Remark. Conditions (11) are satisfied, if

$$\alpha + 1 - \lambda < \gamma \leq \alpha + 1,$$

and only one of the following cases holds true:

1. $\delta = 1, \qquad \mu > \gamma - \alpha,$
2. $\delta > 1, \qquad \mu \geq 0.$

That means the case $\delta = 1$ and $\mu = 0$ is not included. Nevertheless this case was already treated in [30, Th. 3.4].

Next Theorem is the key in proving our successive results on the quadrature rules.

Theorem 3.2. *Let $k(x, y)$ be a function defined in $\mathbb{R}^+ \times \mathrm{S}$. Moreover let $u(x) = e^{-x^\beta} x^\gamma (1+x)^\lambda$, $w(x) = x^\alpha e^{-x^\beta}$, with $\gamma, \lambda \geq 0$, $\alpha > -1$, $\beta > \frac{1}{2}$. If k_y, u and w satisfy the following assumptions*

$$\sup_{y\in \mathrm{S}} \left(\frac{k_y u}{w\varphi^2} \right) \in L \log^+ L, \qquad \frac{w\varphi^2}{u} \in L^\infty(\mathbb{R}^+), \tag{13}$$

then for any $f \in C_u$ it is

$$\sup_{y\in \mathrm{S}} \| L^*_{2m+2}(w, \bar{w}, f) k_y u \|_1 \leq \mathcal{C} \|fu\|_\infty, \tag{14}$$

with $0 < \mathcal{C} \neq \mathcal{C}(m, f)$. Moreover, assuming $k_y(x) \in L^1(I\!R^+)$ for any $y \in \mathrm{S}$ and holding (14), it follows

$$\sup_{y\in \mathrm{S}} \left(\frac{k_y u}{w\varphi^2} \right) \in L^1(I\!R^+). \tag{15}$$

Remark. We outline that (14) is a homogeneous inequality since the same weight appears on both sides of it. On the other hand, if k_y is a constant function, Theorem 3.2 doesn't hold true and this means that it is not possible to have an inequality of the type (12) with the same weight on the left and the right hand sides.

We conclude this Section stating an expression of the polynomial $L^*_{2m+2}(w, \bar{w}, f)$, which will be useful in what follows:

$$\begin{aligned} L^*_{2m+2}(w, \bar{w}, f, x) &= p_m(\bar{w}, x)(a_m - x) \sum_{k=1}^{j} \ell_{m+1,k}(w, x) \frac{f(x_k)}{p_m(\bar{w}, x_k)(a_m - x_k)} && (16)\\ &+ p_{m+1}(w, x)(a_m - x) \sum_{k=1}^{j} \ell_{m,k}(\bar{w}, x) \frac{f(y_k)}{p_{m+1}(w, y_k)(a_m - y_k)} && (17)\\ &+ \delta_{m,j} \frac{p_{m+1}(w, x) p_m(\bar{w}, x)}{p_{m+1}(w, a_m) p_m(\bar{w}, a_m)} f(a_m), && (18) \end{aligned}$$

where $\ell_{m+1,k}(w)$ and $\ell_{m,k}(\bar{w})$ are the fundamental Lagrange polynomials related to the zeros of $p_{m+1}(w)$ and $p_m(\bar{w})$ respectively, and $\delta m, j$ denotes the Kronecker delta.

4. Product integration rules

4.1. The extended quadrature rule

Now we propose a product integration rule based on the introduced extended interpolation process. Let us consider integrals of the type

$$I(f, y) := \int_0^{+\infty} f(x) k(x, y) u(x) dx \tag{19}$$

with $k(x,y)$ and $u(x)$ defined as in Theorem 3.2.

Approximating f by the Lagrange polynomial $L^*_{2m+2}(w,\bar{w},f)$, the following "*extended product integration rule*" can be deduced

$$\begin{aligned} I(f,y) &= \Sigma_{2m+2}(f,y) + e_{2m+2}(f,y), \quad \Sigma_{2m+2}(f,y) = \sum_{i=1}^{j} A_i(y) f(z_i), \\ A_i(y) &= \int_0^{+\infty} l_i(x) k(x,y) u(x) dx. \end{aligned} \tag{20}$$

The rule is exact for any polynomial $P \in \mathcal{P}^*_{2m+1}$. Now we prove that by (20) we are able to approximate integrals with "problematic" kernels (for instance highly oscillating, weakly singular, etc), with a rate of convergence only depending on f, since the error behaves like the best polynomial approximation of $f \in C_u$. Thus, by Theorem 3.2, the following result can be deduced

Theorem 4.1. *Assume that the functions $k(x,y)$, defined in $\mathbb{R}^+ \times \mathrm{S}$, is not a constant. Moreover let $u(x) = e^{-x^\beta} x^\gamma (1+x)^\lambda$, $w(x) = x^\alpha e^{-x^\beta}$, with $\gamma, \lambda \geq 0$, $\alpha > -1$, $\beta > \frac{1}{2}$. If k_y, u and w satisfy (13), then for any $f \in C_u$*

$$\sup_{y \in \mathrm{S}} |\Sigma_{2m+2}(f,y)| \leq \mathcal{C} \|fu\|_\infty, \tag{21}$$

and

$$\sup_{y \in \mathrm{S}} |e_{2m+2}(f,y)| \leq \mathcal{C} \left[E_M(f)_u + e^{-Am} \|fu\|_\infty \right], \tag{22}$$

where in both the estimates $\mathcal{C} \neq \mathcal{C}(m,f)$ and M is a proper fraction of $2m$.

We point out that by (21) it is no hard to deduce the stability of the rule (20), i.e.

$$\sup_{y \in \mathrm{S}} \sum_{i=1}^{j} \frac{|A_i(y)|}{u(z_i)} < +\infty.$$

Finally, we state the following equivalent expression of $\Sigma_{2m+2}(f,y)$, easily deducible by (16),

$$\Sigma_{2m+2}(f,y) = \sum_{i=1}^{j} \mathcal{B}_i(y) f(x_i) + \sum_{i=1}^{j} \mathcal{C}_i(y) f(y_i) + \delta_{m,j} \mathcal{D}_{2m+2}(y) f(a_m), \tag{23}$$

where

$$\begin{aligned} \mathcal{B}_i(y) &= \frac{1}{p_m(\bar{w},x_k)(a_m - x_k)} \int_0^{+\infty} p_m(\bar{w},x)(a_m - x) \ell_{m+1,i}(w,x) k(x,y) u(x) dx, \\ \mathcal{C}_i(y) &= \frac{1}{p_{m+1}(w,y_k)(a_m - y_k)} \int_0^{+\infty} p_{m+1}(w,x)(a_m - x) \ell_{m,i}(\bar{w},x) k(x,y) u(x) dx, \\ \mathcal{D}_{2m+2}(y) &= \frac{1}{p_{m+1}(w,a_m) p_m(\bar{w},a_m)} \int_0^{+\infty} p_m(\bar{w},x) p_{m+1}(w,x) k(x,y) u(x) dx. \end{aligned}$$

4.2. A mixed quadrature scheme

Now we observe that setting

$$\widetilde{k}(x,y) = k(x,y) e^{-x^\beta/2}, \qquad U(x) = e^{-x^\beta/2} x^\gamma (1+x)^\lambda, \tag{24}$$

integrals in (19) can be approximated also by the sequence of product integration rules $\{\mathcal{I}_m(f)\}_m$ defined in (3).

As we will show, under suitable "mixed" assumptions for the involved weights it is possible to obtain that both the rules (3) and (20) are stable and convergent with a comparable rate of convergence. For this reason we propose to construct a new sequence of quadrature rules, by "composing" the sequences $\{\mathcal{I}_{m+2}(f,y)\}_m$ and $\{\Sigma_{2m+2}(f,y)\}_m$, in order to approximate integrals of type (19) with a reduced computation of function samples, but an unchanged speed of convergence. To be more precise, for a fixed m, we consider the sequence $\mathcal{I}_{m+2}(f)$, $\Sigma_{2m+2}, \mathcal{I}_{4m+2}(f), \Sigma_{8m+2}, \mathcal{I}_{16m+2}(f), \Sigma_{32m+2}, \ldots$ in which, in the computation of the "extended" quadrature rule (20), are reused the samples of the function, used for constructing the product rule (3) of lower degree. In details, for any given $q \in \mathbb{N}$, consider the sequences

$$\{\mathcal{I}_{2^k m+2}(f)\}_{k=0}^{2q-1}, \qquad \{\mathcal{I}_{2^{2k} m+2}(f), \Sigma_{2^{2k+1} m+2}(f)\}_{k=0}^{q-1}.$$

Assuming that no truncation is performed, i.e. j is equal to the order of the quadrature rule and the same holds for $j*$ (that is the worst case), then the first sequence requires $4q + m(2^{2q} - 1)$ evaluations of the function f, while the second one only $2q + \frac{2}{3}m(2^{2q} - 1)$. This means that, using the mixed sequence, more then a third of the function evaluations is spared.

Setting

$$T_{2^n m+2}(f) = \begin{cases} \mathcal{I}_{2^n m+2}(f), & n = 0, 2, \ldots \\ \Sigma_{2^{n+1} m+2}(f), & n \text{ odd} \end{cases},$$

the following results about the stability and the convergence of the mixed sequence holds true

Theorem 4.2. *Under the assumptions*

$$\sup_{y \in \mathrm{S}} \left(\frac{u}{w\varphi^2} k_y \right) \in L \log^+ L, \quad \frac{\sqrt{w\varphi}}{U} \in L^\infty(\mathbb{R}^+) \tag{25}$$

we have for any $f \in C_u$ and any fixed $n \in \mathbb{N}$,

$$\sup_{y \in \mathrm{S}} |T_{2^n m+2}(f,y)| \le \mathcal{C} \|fu\|_\infty$$

and

$$\sup_{y \in \mathrm{S}} |I(f,y) - T_{2^n m+2}(f,y)| \le \left[E_{\widehat{M}}(f)_u + e^{-Am} \|fu\|_\infty \right],$$

where $\widehat{M}$ is a proper fraction of $2^n m + 2$ and $\mathcal{C} \neq \mathcal{C}(m,f)$.

Remark As the numerical tests will show, the extended product rule converges a little bit faster than the one-weight product rule. Since the rate of convergence of both the rules is the same, we conjecture that the better performance of the extended rule is due to the greater number of quadrature nodes belonging to the truncated interval $(0, a_m\theta)$.

5. Numerical Tests

In this section we exhibit some numerical tests showing the effectiveness of the mixed scheme obtained by combining the extended product rule (20) with the one-weight product rule (3), according to the settings in (24). Moreover, in Examples 2 and 3 we compare our results with those achieved by means of the Truncated Gaussian rule (8), reporting the values obtained for increasing values of m and the corresponding function evaluation numbers h, as defined in (7). We don't use the Gauss-Laguerre rule in Example 1, since the integrand function fk_y doesn't belong to C_u, which is the assumption for holding estimate (9).

Since $w(x) = e^{-x^\beta} x^\alpha$, is not a classical weight, the coefficients of the three term recurrence relation for $\{p_m(w)\}_m$ are generally unknown, except some special cases. To compute them, here we used the software package `OrthogonalPolynomials` by Cvetkovic and G.V. Milovanovic (see [6]) implemented in *Wolfram Mathematica Language*. Since the involved algorithm suffers of ill conditioning, the use of the extended arithmetic is required.

All the other computations have been performed in double-machine precision ($eps \approx 2.22044e-16$).

Moreover we point out that the "truncation" indices j and j^* in (20) and (3), respectively, have been empirically detected by means of the tests

$$j = \min_{1\le k\le 2m+1} |f(z_k)A_k(y)| \ge eps$$

and

$$j^* = \min_{1\le k\le m+1} |f(x_k)\mathcal{A}_k(y)| \ge eps.$$

In each example we approximate the given integral for different values of the free parameter y, producing Tables in which the first column contains the number $\#\ feval.$ of the function evaluations needed in the corresponding quadrature sum which is indicated in the second column, and the corresponding obtained values in the following columns.

Example 5.1. *For any fixed $y \in \mathrm{S} = \mathbb{R} - \{0\}$, consider the following integral with a weakly singular kernel*

$$I_1(f,y) = \int_0^{+\infty} \frac{\sin(x)}{|x-y|^{0.5}(x^2+y^2)} e^{-x}\sqrt{x}dx.$$

Therefore

$$f(x) = \sin(x), \quad k(x,y) = \frac{1}{|x-y|^{0.5}(x^2+y^2)},$$

and the parameter of the weights are:

$$\gamma = 0.5, \quad \alpha = -0.5, \quad \lambda = 0.$$

Since the function f is very smooth, the convergence is very fast, as shown in Table 1, for different values of y. We point out that the machine precision is attained for each value of y with only 57 function's evaluations.

Table 1: Evaluation of $I_1(f,y)$ by the mixed product integration scheme for $y = 1/4,\ 1/2,\ 2$.

$\#\ feval.$	Rule	$y = 1/4$	$y = 1/2$	$y = 2$
6	$\mathcal{I}_6$	1.8	1.	$1.3e-1$
9	Σ_{10}	1.81	1.12	$1.31e-1$
16	$\mathcal{I}_{18}$	1.81616	1.122	$1.318e-1$
27	Σ_{34}	1.816167	1.122013	$1.3188556383e-1$
38	$\mathcal{I}_{66}$	1.81616744	1.12201318362	$1.31885563832e-1$
57	Σ_{130}	1.8161674475678	1.12201318362232	$1.31885563832776e-1$

Example 5.2. *For any fixed $y \in \mathrm{S} = \mathbb{R} - \{0\}$, consider the integral*

$$I_2(f,y) = \int_0^{+\infty} \cos(1+x)\frac{\sin(yx)}{x^2+y^2} e^{-x}(1+x)^{\frac{1}{4}}\sqrt[4]{x}dx.$$

Therefore

$$f(x) = \cos(1+x), \quad k(x,y) = \frac{\sin(yx)}{x^2+y^2},$$

and the considered parameters of the weights are:

$$\gamma = 0.25, \quad \alpha = 0, \quad \lambda = 0.25.$$

Also in this case the function f is very smooth and consequently the convergence is very fast, as shown in Table 2. We observe that for "large" m the Gauss-Laguerre rule provides satisfactory results till y is "small", while it gives completely wrong results for $y = 20$.

Table 2: Evaluation of the integral $I_2(f, y)$ by the mixed product integration scheme for $y = 1/2,\ 6,\ 20$

# $feval.$	Rule	$y = 1/2$	$y = 6$	$y = 20$
8	$\mathcal{I}_{10}$	$-7.85e-2$	$1.52e-3$	$2e-5$
17	Σ_{20}	$-7.85e-2$	$1.512e-3$	$2.81e-5$
25	$\mathcal{I}_{34}$	$-7.8570e-2$	$1.5120e-3$	$2.8189e-5$
39	Σ_{66}	$-7.85706e-2$	$1.51207123e-3$	$2.8189743e-5$
56	$\mathcal{I}_{130}$	$-7.8570663e-2$	$1.51207123e-3$	$2.8189743e-5$
80	Σ_{258}	$-7.85706635680e-2$	$1.512071235722e-3$	$2.81897434420e-5$

Table 3: Evaluation of $I_2(f, y)$ by the Gauss-Laguerre rule $y = 1/2,\ 6,\ 20$

h	m	$y = 1/2$	$y = 6$	$y = 20$
19	32	$-7.85e-2$	$4.38e-3$	$4.96e-4$
27	64	$-7.8570e-2$	$1.e-3$	$3.77e-4$
39	128	$-7.8570663e-2$	$1.5e-3$	$3.49e-4$
56	256	$-7.85706635680e-2$	$1.5120712e-3$	$3.62e-5$
79	512	$-7.857066356804e-2$	$1.512071235722e-3$	$-2.57e-4$
79	1024	$-7.857066356804e-2$	$1.512071235722e-3$	$2.87e-5$

Example 5.3. *For any fixed $y \in \mathrm{S} = \mathbb{R} - \{0\}$ consider the integral*

$$I_3(f, y) = \int_0^{+\infty} \arctan(x)^{\frac{7}{2}} \frac{\cos(xy)}{x^2 + y^2} e^{-x}(1 + x)^{1.001} dx.$$

Therefore

$$f(x) = \arctan(x)^{\frac{7}{2}}, \quad k(x, y) = \frac{\cos(xy)}{x^2 + y^2},$$

and the parameters of the weights are

$$\gamma = 0, \quad \alpha = 0, \quad \lambda = 1.001.$$

In this case the function $f \in W_7^{\infty}(u)$ and, according to the theoretical estimate, we should obtain 7 exact digits for $m = 256$. However, inspecting Table 4 where the results obtained for different values of $y \in \mathrm{S}$ are shown, at most 10 exact digits are achieved.

Also in this case the performance of the Gaussian rule is worst then the product rule for larger value of y.

Example 5.4. *We conclude the section by proposing a test in which we compare the behavior of the two sequences $\{\mathcal{I}_m(f)\}_m$ and $\{\Sigma_m(f)\}_m$, for the same choices of the polynomial degrees. For any fixed $y \in \mathrm{S} = \mathbb{R} - \{0\}$ consider the integral*

$$I_4(f, y) = \int_0^{+\infty} \log(1 + x) \frac{\cos(xy)}{(x^2 + y^2)^2} e^{-x} \sqrt{x} dx.$$

Table 4: Evaluation of $I_3(f, y)$ by the mixed product integration scheme for $y = 1/4,\ 1/2,\ 2$

# $feval.$	Rule	$y = 1/4$	$y = 1/2$	$y = 10$
8	$\mathcal{I}_{10}$	$4.7e-1$	$2.8e-1$	$1.9e-4$
17	Σ_{20}	$4.74e-1$	$2.83e-1$	$1.94e-4$
26	$\mathcal{I}_{34}$	4.747	$2.838e-1$	$1.943e-4$
40	Σ_{66}	$4.7477e-1$	$0.28381e-1$	$1.9436e-4$
53	$\mathcal{I}_{130}$	$4.747794e-1$	$2.8381962e-1$	$1.9436514e-4$
82	Σ_{258}	$4.747794014e-1$	$2.838196282-1$	$1.94365142e-4$

Table 5: Evaluation of $I_3(f,y)$ by the Gauss-Laguerre rule for $y = 1/4,\ 1/2,\ 6$

h	m	$y = 1/4$	$y = 1/2$	$y = 6$
8	8	$4.7e-1$	$2.e-1$	$2.6e-2$
14	16	$4.747e-1$	$2.83e-1$	$2.2e-2$
20	32	4.7477	$2.838e-1$	$-1.85e-4$
29	64	$4.7477e-1$	$0.283819e-1$	$1.9e-4$
41	128	$4.74779e-1$	$2.8381962e-1$	$1.9436514e-4$
57	256	$4.74779401e-1$	$2.83819628e-1$	$1.9436514e-4$
81	512	$4.747794014e-1$	$2.838196282e-1$	$1.9436514e-2$

Here

$$f(x) = \log(1+x), \quad k(x,y) = \frac{\cos(xy)}{(x^2+y^2)^2},$$

and the parameters of the weights are:

$$\gamma = 0.5, \quad \alpha = -0.5, \quad \lambda = 0.$$

As the tests included in Table 6 show, the extended product rule converges a little bit faster than the one-weight product rule and as said before, being the rates of convergence comparable, the conjecture is that the better performance of the extended rule depend on the greater number of quadrature nodes belonging to the truncated interval $(0, a_m\theta)$*.*

Table 6: Evaluation of the integral $I_4(f,y)$ for $y = 6,\ 16,\ 20.$

	m	# $feval.$	$\mathcal{I}_m$	# $feval.$	$\Sigma_m(f)$
$y = 6$	8	8	$-1.0e-5$	8	$-1.0e-5$
	16	12	$-1.04e-5$	15	$-1.040e-5$
	32	17	$-1.0404e-5$	23	$-1.04049e-5$
	64	25	$-1.040495e-5$	34	$-1.0404952e-5$
	128	35	$-1.04049521e-5$	49	$-1.0404952100e-5$
	m	# $feval.$	$\mathcal{I}_m$	# $feval.$	$\Sigma_m(f)$
$y = 16$	8	8	$-1.6e-8$	8	$-1.6e-8$
	16	12	$-1.6e-8$	15	$-1.65e-8$
	32	17	$-1.656e-8$	23	$-1.6561e-8$
	64	25	$-1.6561e-8$	34	$-1.6561777e-8$
	128	35	$-1.6561e-8$	49	$-1.656177724e-8$
	m	# $feval.$	$\mathcal{I}_m$	# $feval.$	$\Sigma_m(f)$
$y = 20$	8	8	$-3.7e-9$	8	$-3.7e-9$
	16	12	$-3.7e-9$	15	$-3.79e-9$
	32	17	$-3.79e-9$	23	$-3.7922e-9$
	64	25	$-3.7922e-9$	34	$-3.7922782e-9$
	128	35	$-3.7922e-9$	49	$-3.79227826e-9$

6. The Proofs

We recall some polynomial inequalities that we need in the proofs.

Lemma 6.1 (Mastroianni-Szabados). *Let $A > 0$ $A_m = \left[A\frac{a_m}{m^2}, a_m\left(1 - \frac{A}{m^{\frac{2}{3}}}\right)\right]$ and $1 \le p \le \infty$. Then, for any polynomial $P_m \in I\!P_m$ there exists a constant $0 < \mathcal{C} = \mathcal{C}(A)$, $\mathcal{C} \ne \mathcal{C}(m, P_m, p)$ such that*

$$\left(\int_0^{+\infty} |P_m(x)u(x)|^p dx\right)^{\frac{1}{p}} \le \mathcal{C}\left(\int_{A_m} |P_m(x)u(x)|^p dx\right)^{\frac{1}{p}}$$

Lemma 6.2. *Let $x \in [z_1, z_{2m+1}]$ and $d = d(x) \in \{1, \ldots, 2m+1\}$ be an index of a zero of Q_{2m+1} closest to x. Then, for some positive constant $\mathcal{C} \ne \mathcal{C}(m, x, d)$, we have*

$$\frac{1}{\mathcal{C}}\left(\frac{x - z_d}{z_d - z_{d\pm1}}\right)^2 \le |Q_{2m+1}(x)|e^{-x^\beta}\left(x+\frac{a_m}{m^2}\right)^{\alpha+1}\sqrt{|a_m - x| + a_m m^{-\frac{2}{3}}} \le \mathcal{C}\left(\frac{x - z_d}{z_d - z_{d\pm1}}\right)^2. \tag{26}$$

Proof. The lemma easily follows by estimate [22, (30), p. 608] (see also [13])

$$\frac{1}{\mathcal{C}}\left(\frac{x - x_{\tilde{d}}}{x_{\tilde{d}} - x_{\tilde{d}\pm1}}\right)^2 \le p_m^2(x)e^{-x^\beta}\left(x+\frac{a_m}{m^2}\right)^{\alpha+\frac{1}{2}}\sqrt{|a_m - x| + a_m m^{-\frac{2}{3}}} \le \mathcal{C}\left(\frac{x - x_{\tilde{d}}}{x_{\tilde{d}} - x_{\tilde{d}\pm1}}\right)^2. \tag{27}$$

being $x \in [x_1, x_m]$ and $\tilde{d} = \tilde{d}(x) \in \{1, \ldots, m\}$ be an index of a zero of $p_m(w)$ closest to x, and the positive constant $\mathcal{C} \ne \mathcal{C}(m, x, \tilde{d})$. □

In particular by (27) it follows

$$|Q_{2m+1}(x)|u(x) \le \frac{\mathcal{C}}{\sqrt{a_m}}\frac{u(x)}{w(x)\varphi^2(x)}, \quad \frac{a_m}{m^2} \le x \le \theta a_m. \tag{28}$$

Lemma 6.3. *Let be $Q_{2m+1} = p_{m+1}(w)p_m(\bar{w})$. With $\{z_k\}_{k=1}^{2m+1}$ the zeros of Q_{2m+1}, it is*

$$\frac{1}{|Q'_{2m+1}(z_k)|u(z_k)} \sim \sqrt{a_m}\frac{w(z_k)\varphi^2(z_k)}{u(z_k)}\Delta z_k, \quad z_k \le z_j. \tag{29}$$

Proof. Since

$$|Q'_{2m+1}(z_k)|u(z_k) \ge \mathcal{C}\frac{z_k^{\gamma-\alpha-1}}{\sqrt{a_m}\Delta z_k}(1+z_k)^\lambda, \quad z_k \le z_j, \quad \mathcal{C} \ne \mathcal{C}(m)$$

was proved in [27], we have to prove the converse inequality. Assume $z_k = x_{\frac{k+1}{2}}$. By (27) we deduce

$$|p_m(\bar{w}; z_k)|\sqrt{\bar{w}(z_k)} \le \frac{\mathcal{C}}{\Delta z_k \sqrt[4]{a_m z_k}}, \quad z_k \le z_j$$

and taking into account

$$|p'_{m+1}(w; z_k)|\sqrt{w(z_k)} \le \frac{\mathcal{C}}{\Delta z_k \sqrt[4]{a_m z_k}}, \quad z_k \le z_j$$

we conclude

$$|Q'_{2m+1}(z_k)|u(z_k) \le \mathcal{C}\frac{z_k^{\gamma-\alpha-1}}{\sqrt{a_m}\Delta z_k}(1+z_k)^\lambda, \quad z_k \le z_j, \quad \mathcal{C} \ne \mathcal{C}(m)$$

and the Lemma follows. □

Lemma 6.4. *Let $1 \le p < +\infty$, $v^\sigma(x) = x^\sigma \in L^p$,$\tau > 0$ and let be z_j defined in (2), with $\theta \in (0,1)$ fixed. For any polynomial $P \in \mathbb{P}_{2m+1+ml}$, $l \in \mathbb{N}$ fixed, there exist $\theta_1 \in (\theta, 1)$ such that*

$$\left(\sum_{k=1}^{j} \left(\frac{|P(z_k) v^\sigma(z_k)|}{(1+z_k)^\tau} \right)^p \Delta z_k \right)^{1/p} \le \left(\int_{z_1}^{\theta_1 a_m} \left(\frac{|P(x) v^\sigma(x)|}{(1+x)^\tau} \right)^p dx \right)^{1/p}$$

being $0 < \mathcal{C} \neq \mathcal{C}(m)$.

We omit the proof since it is very similar to that of Lemma 4.3 in [14].

Finally let us state a formula for the inversion of the integration order for the Hilbert transform and a related estimate in L^1 . Denote by $H_B(g,t)$ the Hilbert transform of the function g on the compact set B, i.e.

$$H_B(g,t) = \int_B \frac{g(x)}{x-t} dx.$$

If $G \in L^\infty(B), F \log^+ F \in L^1(B)$, then

$$\int_B G H_B(F) = -\int_B F H_B(G), \tag{30}$$

and

$$\|G H_B(F)\|_1 \le \mathcal{C} + \|F \log^+ F\|_1, \quad \mathcal{C} \neq \mathcal{C}(F). \tag{31}$$

Proof of Theorem 3.1. Denote by $g_m = sgn(L^*_{2m+2}(w, \bar{w}, f))$ and set

$$A_{\widehat{\theta}m} := \left[\mathcal{C} \frac{a_m}{m^2}, \widehat{\theta} a_m \right],$$

where $\theta < \widehat{\theta}$ and $\widehat{\theta} > z_j$, by Lemma 6.1 we get

$$\begin{aligned} \|L^*_{2m+2}(w, \bar{w}, f)\rho\|_1 &\le \mathcal{C} \int_{A_m} L^*_{2m+2}(w, \bar{w}, f, x)\rho(x) g_m(x) dx \\ &= \mathcal{C} \left\{ \int_{A_{\widehat{\theta}m}} + \int_{(A_m \setminus A_{\widehat{\theta}m})} \right\} L^*_{2m+2}(w, \bar{w}, f, x)\rho(x) g_m(x) dx =: D_1 + D_2. \end{aligned} \tag{32}$$

Setting

$$\Pi(t) = \int_{A_{\widehat{\theta}m}} \frac{Q_{2m+1}(x)(a_m - x)q(x) - Q_{2m+1}(t)(a_m - t)q(t)}{(x-t)} \frac{g_m(x)\rho(x)}{q(x)} dx$$

where q is an arbitrary polynomial of degree ml, with l a fixed integer, D_1 can be expressed as follows

$$D_1 = \mathcal{C} \left| \sum_{k=1}^{j} \frac{f(z_k)}{Q'_{2m+1}(z_k)(a_m - z_k)} \Pi(z_k) \right|$$

and by (29) we have

$$D_1 \le \mathcal{C} \frac{\|fu\|_\infty}{\sqrt{a_m}} \sum_{k=1}^{j} \Delta z_k \frac{w(z_k)\varphi^2(z_k)}{u(z_k)} |\Pi(z_k)|.$$

Since $\Pi \in \mathbb{P}_{2m+1+ml}$, by Lemma 6.4 with $p = 1$ there exists $\theta_1 < \widehat{\theta}$ with $\theta < \theta_1$, such that

$$D_1 \le \mathcal{C} \frac{\|fu\|_\infty}{\sqrt{a_m}} \int_{z_1}^{\theta_1 a_m} \frac{w(t)\varphi^2(t)}{u(t)} |\Pi(t)| dt$$

and setting

$$F_m(x) := Q_{2m+1}(x)(a_m - x)g_m(x)\rho(x), \qquad G_m(x) := \frac{g_m(x)\rho(x)}{q(x)},$$

we have

$$\begin{aligned} D_1 &\le \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{a_m}}\left\{\int_{A_{\widehat{\theta}m}} \frac{w(t)\varphi^2(t)}{u(t)} \left|H_{A_{\theta m}}(F_m, t)\right| dt \right. \\ &+ \left. \int_{A_{\widehat{\theta}m}} \frac{w(t)\varphi^2(t)}{u(t)} \left|Q_{2m+1}(t)(a_m - t)q(t)H_{A_{\widehat{\theta}m}}(G_m, t)\right| dt \right\} \\ &=: \mathcal{C}\|fu\|_\infty \left\{J_1 + J_2\right\}. \end{aligned} \tag{33}$$

To evaluate J_1, by (30)

$$J_1 = \frac{\mathcal{C}}{\sqrt{a_m}}\left|\int_{A_{\widehat{\theta}m}} F_m(t)H_{A_{\widehat{\theta}m}}\left(\sigma_1\frac{w\varphi^2}{u}, t\right) dt\right|,$$

where $\sigma_1 = \mathrm{sgn}\left(H_{A_{\widehat{\theta}m}}(F_m)\right)$. Since by (28)

$$|F_m(x)| \le \mathcal{C}\sqrt{a_m}\ \frac{\rho(t)}{w(t)\varphi^2(t)},$$

setting $\sigma_2 = \mathrm{sgn}\left(H_{A_{\theta m}}\left(\sigma_1\frac{w\varphi^2}{u}\right)\right)$, it follows

$$\begin{aligned} J_1 &\le \mathcal{C}\int_{A_{\theta m}} \sigma_1(t)\frac{w(t)\varphi^2(t)}{u(t)}\left|H_{A_{\widehat{\theta}m}}\left(\sigma_2\frac{\rho}{w\varphi^2}, t\right)\right| dt \\ &\le \mathcal{C}\left\|\frac{\rho}{w\varphi^2}\log^+\frac{\rho}{w\varphi^2}\right\|_1 \le \mathcal{C}, \end{aligned} \tag{34}$$

having used (31) under the assumptions (11).

Now we estimate J_2. Recalling that there exists a polynomial $q \in \mathbb{P}_{ml}$, s. t. $q(x) \sim e^{-x^\beta}x^\gamma$ [23], and by (28) we get

$$|Q_{2m+1}(t)(a_m - t)q(t)| \le \mathcal{C}\sqrt{a_m}\frac{u(t)}{w(t)\varphi^2(t)(1+t)^\lambda}$$

and therefore

$$J_2 \le \mathcal{C}\int_{A_{\widehat{\theta m}}} \frac{1}{(1+t)^\lambda}\left|H_{A_{\widehat{\theta m}}}(G_m, t)\right| dt = \mathcal{C}\left|\int_{A_{\widehat{\theta}m}} G_m(t)H_{A_{\widehat{\theta}m}}\left(\frac{\sigma_3}{(1+\cdot)^\lambda}, t\right) dt\right|,$$

where $\sigma_3 = \mathrm{sgn}\left(H_{A_{\widehat{\theta}m}}(G_m)\right)$. Since under the assumptions

$$|G_m(t)| \le \mathcal{C}\frac{e^{-(\delta-1)t^\beta}}{(1+t)^\mu} \le \mathcal{C},$$

and $\dfrac{1}{(1+x)^\lambda} \in L\log^+ L$, by (31)

$$J_2 \le \mathcal{C} \tag{35}$$

Combining (34),(35) with (33) it follows

$$D_1 \le \mathcal{C}\|fu\|_\infty. \tag{36}$$

In order to estimate D_2 we start from

$$\begin{aligned} D_2 &= \int_{\widehat{\theta}a_m}^{a_m} L_{2m+2}^*(w,\bar{w},f,x)\rho(x)g_m(x)dx \\ &= \left| \sum_{k=1}^{j} \frac{f(z_k)}{(a_m-z_k)Q'_{2m+1}(z_k)} \int_{\widehat{\theta}a_m}^{a_m} \frac{Q_{2m+1}(x)(a_m-x)}{x-z_k}\rho(x)g_m(x)dx \right| \end{aligned}$$

and by (29) we have

$$D_2 \le \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{a_m}} \sum_{k=1}^{j} \Delta z_k \frac{w(z_k)\varphi^2(z_k)}{u(z_k)} \int_{\widehat{\theta}a_m}^{a_m} \frac{|Q_{2m+1}(x)|(a_m-x)}{|x-z_k|}\rho(x)dx.$$

Since $|x-z_k| \ge (\widehat{\theta}-\theta)a_m$ and using

$$(a_m-x)|Q_{2m+1}(x)|u(x) \le \mathcal{C}\sqrt{a_m}\frac{u(x)}{w(x)\varphi^2(x)},$$

we have

$$D_2 \le \mathcal{C}\frac{\|fu\|_\infty}{a_m} \sum_{k=1}^{j} \Delta z_k \frac{w(z_k)\varphi^2(z_k)}{u(z_k)} \int_{\widehat{\theta}a_m}^{a_m} \frac{u(x)\rho(x)}{w(x)\varphi^2(x)}dx$$

and by Lemma 6.4 with $p=1$, and with $\theta < \theta_1 < \widehat{\theta}$, we have

$$\begin{aligned} D_2 &\le \mathcal{C}\frac{\|fu\|_\infty}{a_m} \int_{z_1}^{\theta_1 a_m} \frac{w(t)\varphi^2(t)}{u(t)}dt \int_{\widehat{\theta}a_m}^{a_m} x^{2\gamma}e^{-\delta x^\beta}(1+x)^{\lambda-\mu}dx \\ &\le \mathcal{C}\|fu\|_\infty \left\| \frac{w\varphi^2}{u} \right\|_\infty \le \mathcal{C}\|fu\|_\infty \qquad (37) \end{aligned}$$

where last bound holds taking into account the second assumption in (11).
The thesis follows by combining (36), (37) with (32). □

Proof of Theorem 3.2. First we prove that the conditions in (13) imply (14).
Denoting by $g_m = \mathrm{sgn}(L_{2m+2}^*(w,\bar{w},f)k_y)$, we get

$$\begin{aligned} &\|L_{2m+2}^*(w,\bar{w},f)k_y u\|_1 = \\ &\left(\int_0^{a_m} + \int_{a_m}^{a_{2m}} + \int_{a_{2m}}^{+\infty} \right) L_{2m+2}^*(w,\bar{w},f,x)k_y(x)g_m(x)u(x)dx \\ =: &\ I_1(y)+I_2(y)+I_3(y) \qquad (38) \end{aligned}$$

where a_{2m} is the M-R-S number w.r.t. the degree $2m$.
Evaluate $I_1(y)$. Setting

$$\Pi^*(t) = \int_0^{a_m} \frac{Q_{2m+1}(x)(a_m-x)q(x) - Q_{2m+1}(t)(a_m-t)q(t)}{(x-t)} \frac{g_m(x)u(x)k_y(x)}{q(x)}dx,$$

being q an arbitrary polynomial of degree ml, l a fixed integer, by (29) we have

$$I_1(y) = \left| \sum_{k=1}^{j} \frac{f(z_k)}{Q'_{2m+1}(z_k)(a_m-z_k)} \Pi^*(z_k) \right| \le \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{a_m}} \sum_{k=1}^{j} \Delta z_k \frac{w(z_k)\varphi^2(z_k)}{u(z_k)} |\Pi^*(z_k)|.$$

Taking into account $\Pi^* \in \mathbb{P}_{2m+1+ml}$, we use Lemma 6.4 with $p=1$, and with $\theta_1 > \theta$, s.t. $\theta a_m < z_j < \theta_1 a_m$, obtaining

$$I_1(y) \quad \leq \quad \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{a_m}} \int_{z_1}^{\theta_1 a_m} \frac{w(t)\varphi^2(t)}{u(t)} |\Pi^*(t)| dt.$$

Setting

$$\begin{aligned} F_{m,y}(t) &= Q_{2m+1}(t) e^{-t^\beta} \left(t + \frac{a_m}{m^2}\right)^{\alpha+1} (a_m - t) \frac{g_m(t)k_y(t)u(t)}{e^{-t^\beta}\left(t+\frac{a_m}{m^2}\right)^{\alpha+1}}, \\ G_{m,y}(t) &= \frac{g_m(t)k_y(t)u(t)}{q(t)}, \end{aligned}$$

we have

$$\begin{aligned} I_1(y) \quad &\leq \quad \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{a_m}} \left\{ \int_{z_1}^{\theta_1 a_m} \frac{w(t)\varphi^2(t)}{u(t)} \left| H_{[0,a_m]}(F_{m,y}, t) \right| dt \right. \\ &+ \left. \int_{z_1}^{\theta_1 a_m} \frac{w(t)\varphi^2(t)}{u(t)} \left| Q_{2m+1}(t)(a_m - t) q(t) H_{[0,a_m]}(G_{m,y}, t) \right| dt \right\} \\ &=: \quad \mathcal{C}\|fu\|_\infty \left\{ I_{1,1}(y) + I_{1,2}(y) \right\}. \end{aligned} \tag{39}$$

By (30) under the second assumption in (13)

$$I_{1,1}(y) = \frac{\mathcal{C}}{\sqrt{a_m}} \left| \int_0^{a_m} F_{m,y}(t) H_{[0,a_m]}\left(\sigma_1 \frac{w\varphi^2}{u}, t\right) dt \right|,$$

where $\sigma_1 = \operatorname{sgn}\left(H_{[0,a_m]}\left(F_{m,y}\right)\right)$. Since by (27) we deduce

$$|F_{m,y}(t)| \leq \mathcal{C}\sqrt{a_m} \frac{u(t)}{w(t)\varphi^2(t)} |k_y(t)|, \quad 0 \leq t \leq a_m,$$

setting $\sigma_2 = \operatorname{sgn}\left(H_{[0,a_m]}\left(\sigma_1 \frac{w\varphi^2}{u}\right)\right)$, it follows

$$\begin{aligned} I_{1,1}(y) \quad &\leq \quad \mathcal{C} \int_0^{a_m} \frac{u(t)}{w(t)\varphi^2(t)} |k_y(t)| \left| H_{[0,a_m]}\left(\sigma_1 \frac{w\varphi^2}{u}, t\right) \right| dt \\ &= \quad \left| \int_0^{a_m} \frac{w(t)\varphi^2(t)}{u(t)} H_{[0,a_m]}\left(\sigma_2 \frac{k_y u}{w\varphi^2}, t\right) dt \right| \leq \mathcal{C} \left\| \frac{k_y u}{w\varphi^2} \log^+\left(\frac{k_y u}{w\varphi^2}\right) \right\|_1 \end{aligned}$$

and therefore by (31) under the assumptions (13)

$$\sup_{y \in \mathrm{S}} I_{1,1}(y) \leq \mathcal{C}. \tag{40}$$

In order to estimate $I_{1,2}$ by a result in [23], we choose the polynomial $q \in \mathbb{P}_{ml}$, such that $q(x) \sim e^{-x^\beta} x^{\alpha+1}$ and by (28) again, we have for any $x \in [z_1, \theta_1 a_m]$

$$|Q_{2m+1}(t)(a_m - t)q(t)| \leq \mathcal{C}\sqrt{a_m}.$$

Therefore

$$
\begin{aligned}
I_{1,2}(y) &\leq \mathcal{C}\int_{z_1}^{\theta_1 a_m} \frac{w(t)\varphi^2(t)}{u(t)} \left|H_{[0,a_m]}(G_{m,y},t)\right| dt \\
&= \mathcal{C}\left|\int_{z_1}^{\theta_1 a_m} G_{m,y}(t) H_{[0,a_m]}\left(\frac{\sigma_3 w\varphi^2}{u},t\right) dt\right|,
\end{aligned}
$$

where $\sigma_3 = \operatorname{sgn}\left(H_{[0,a_m]}(G_{m,y})\right)$.

Since

$$
|G_{m,y}(t)| \leq \mathcal{C}\frac{k_y(t)u(t)}{w(t)\varphi^2(t)},
$$

using (31) once again, under the assumptions (13), we get

$$
\sup_{y\in S} I_{1,2}(y) \leq \mathcal{C}\sup_{y\in S}\left\|\frac{k_y u}{w\varphi^2}\log^+\left(\frac{k_y u}{w\varphi^2}\right)\right\|_1 \leq \mathcal{C}. \tag{41}
$$

Combining (40),(41) with (39) it follows

$$
\sup_{y\in S} I_1(y) \leq \mathcal{C}\|fu\|_\infty. \tag{42}
$$

Now we estimate $I_2(y)$. By (29),

$$
\begin{aligned}
I_2(y) &\leq \int_{a_m}^{a_{2m}} |L^*_{2m+2}(w,\bar{w},f,x)k_y(x)|u(x)dx \\
&\leq \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{a_m}}\sum_{k=1}^{j}\Delta z_k \frac{w(z_k)\varphi^2(z_k)}{u(z_k)}\int_{a_m}^{a_{2m}} \frac{|Q_{2m+1}(x)k(x,y)|(x-a_m)u(x)}{x-z_k}dx
\end{aligned}
$$

and being $x - z_k > (1-\theta)a_m$, $\left\|\frac{w\varphi^2}{u}\right\|_\infty \leq \mathcal{C}$ and $\sum_{k=1}^{j}\Delta z_k \leq a_m$, we get

$$
I_2(y) \leq \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{a_m}}\int_{a_m}^{a_{2m}} |Q_{2m+1}(x)k(x,y)|(x-a_m)u(x)dx.
$$

Now, by (27)

$$
|Q_{2m+1}(x)|(x-a_m)u(x) \leq \mathcal{C}\frac{(x-a_m)}{\sqrt{x-a_m+a_m m^{-\frac{2}{3}}}}\frac{u(x)}{w(x)\varphi^2(x)}
$$

it follows

$$
I_2(y) \leq \mathcal{C}\|fu\|_\infty \int_0^{+\infty} |k(x,y)|\frac{u(x)}{w(x)\varphi^2(x)}dx \leq \mathcal{C}\|fu\|_\infty, \tag{43}
$$

under the first assumption in (13).

At least we estimate $I_3(y)$. By (29) we have

$$
\begin{aligned}
I_3(y) &\leq \int_{a_{2m}}^{+\infty} |L^*_{2m+2}(w,\bar{w},f,x)k_y(x)|u(x)dx \\
&\leq \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{a_m}}\sum_{k=1}^{j}\Delta z_k\frac{w(z_k)\varphi^2(z_k)}{u(z_k)}\left|\int_{a_{2m}}^{+\infty}\frac{Q_{2m+1}(x)(a_m-x)u(x)k_y(x)}{x-z_k}dx\right| \\
&\leq \mathcal{C}\frac{\|fu\|_\infty}{\sqrt{a_m}}\max_{x\geq a_{2m}}|Q_{2m+1}(x)\sqrt{x-a_m}w(x)\varphi^2(x)| \\
&\times \sum_{k=1}^{j}\Delta z_k\frac{w(z_k)\varphi^2(z_k)}{u(z_k)}\int_{a_{2m}}^{+\infty}|k_y(x)|\frac{u(x)}{w(x)\varphi^2(x)}\frac{\sqrt{x-a_m}}{x-z_k}dx.
\end{aligned}
$$

Since $x - z_k \geq a_{2m} - a_m \geq \mathcal{C}a_m$, and by using [23]

$$\max_{x \geq a_{2m}} |Q_{2m+1}(x)(a_m - x)^{\frac{1}{2}} w(x)\varphi^2(x)| \leq \mathcal{C}e^{-Am} \max_{x \leq \theta a_m} |Q_{2m+1}(x)(a_m - x)^{\frac{1}{2}} w(x)\varphi^2(x)| \leq \mathcal{C},$$

we get

$$I_3(y) \quad \leq \quad \mathcal{C}\frac{\|fu\|_\infty}{a_m} \sum_{k=1}^{j} \Delta z_k \frac{w(z_k)\varphi^2(z_k)}{u(z_k)} \int_{a_{2m}}^{+\infty} \frac{|k_y(x)|u(x)}{w\varphi^2(x)} dx,$$

and taking into account the assumptions (13), we can conclude

$$\sup_{y \in \mathrm{S}} I_3(y) \leq \mathcal{C}\|fu\|_\infty \int_{a_{2m}}^{+\infty} \frac{w(t)\varphi^2(t)}{u(t)} dt \leq \mathcal{C}\|fu\|_\infty. \tag{44}$$

Estimate (14) follows by combining (42), (43), (44) with (38).

Now we prove that (14) implies (15). Consider a function $f_0(x)$ s.t.

$$f_0(z_k) = \left[\begin{array}{ll} \mathrm{sgn}(Q'_{2m+1}(z_k)(x - z_k)), & 0 < z_k \leq 1 \\ 0, & z_k > 1 \end{array} \right.$$

and $\|f_0\| \leq 1$. Therefore we have for $x \in (0, 1)$,

$$|L^*_{2m+2}(w, \bar{w}, f_0, x)k_y(x)|u(x) = \sum_{z_1 \leq z_k \leq 1} \frac{a_m - x}{a_m - z_k} \frac{|Q_{2m+1}(x)|}{|Q'_{2m+1}(z_k)(x - z_k)|} |k_y(x)|u(x)$$

and by (29)

$$\begin{aligned} |L^*_{2m+2}(w, \bar{w}, f_0, x)k_y(x)|u(x) \quad &\geq \quad \mathcal{C}\sqrt{a_m - x}|Q_{2m+1}(x)||k_y(x)|u(x) \\ &\sum_{z_1 \leq z_k \leq 1} \frac{\Delta z_k}{|x - z_k|} \frac{\sqrt{a_m - x}}{\sqrt{a_m - z_k}} w(z_k)\varphi^2(z_k). \end{aligned}$$

Since $\sqrt{a_m - x} \sim \sqrt{a_m - z_k}$ and $|x - z_k| \leq 1$ we get

$$\begin{aligned} |L^*_{2m+2}(w, \bar{w}, f_0, x)k_y(x)|u(x) \quad &\geq \quad \mathcal{C}\sqrt{a_m - x}|Q_{2m+1}(x)||k_y(x)|u(x) \sum_{\frac{1}{2} \leq z_k \leq 1} \Delta z_k w(z_k)\varphi^2(z_k) \\ &\geq \quad \mathcal{C}\sqrt{a_m - x}|Q_{2m+1}(x)||k_y(x)|u(x) \int_{\frac{1}{2}}^{1} w(t)\varphi^2(t) dt. \end{aligned}$$

Hence by (14),

$$\begin{aligned} \|u\|_\infty \quad &\geq \quad \mathcal{C} \int_0^\infty |L^*_{2m+2}(w, \bar{w}, f_0, x)k_y(x)|u(x) dx \\ &\geq \quad \mathcal{C} \int_0^1 \sqrt{a_m - x}|Q_{2m+1}(x)||k_y(x)|u(x) dx \\ &\geq \quad \mathcal{C} \int_0^1 \frac{u(x)}{w(x)\varphi^2(x)} |k_y(x)| dx \end{aligned}$$

where last inequality follows by (27). Since $k_y \in L^1(\mathbb{R}^+)$, then

$$\int_1^{+\infty} \frac{u(x)}{w(x)\varphi^2(x)} |k_y(x)| dx < \infty, \quad \forall y \in \mathrm{S},$$

and therefore (15) follows.

□

Proof of Theorem 4.1. We omit the proof of (21) since it can be easily deduced by (14). In order to prove (22), let $P \in \mathcal{P}^*_{2m+1}$, s.t.

$$\|(f-P)u\|_\infty = \inf_{Q\in\mathcal{P}^*_{2m+1}} \|(f-Q)u\|_\infty.$$

Then, taking into account Theorem 3.2 and that under its assumption it follows $k_y \in L_1(0,+\infty)$, it is

$$\begin{aligned} |e_{2m+2}(f)| &\leq \int_0^{+\infty} |(f(x)-P(x))k(x,y)|u(x)dx \\ &+ \int_0^{+\infty} |L^*_{2m+2}(w,\bar{w},f-P,x)k(x,y)|u(x)dx \\ &\leq \|(f-P)u\|_\infty \int_0^{+\infty} |k(x,y)|dx + \mathcal{C}\|(f-P)u\|_\infty \\ &\leq \mathcal{C}\tilde{E}_{2m+1}(f,u) \end{aligned}$$

and by (10), estimate (22) follows. □

Proof of Theorem 4.2. The stability and the convergence of the mixed quadrature scheme follows by (5), (21) and (6), (22) respectively, since if the weights U, w, u and the function k satisfy conditions (25) then also (4) and (13) are fulfilled. □

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