Numerical computation of hypersingular integrals on the real semiaxis

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Abstract

In this paper we propose some different strategies to approximate hypersingular integrals

$$\oint_0^{+\infty} \frac{\mathcal{G}(x)}{(x-t)^{p+1}} dx,$$

where p is a positive integer, $t > 0$ and the integral is understood in the Hadamard finite part sense. Hadamard Finite Part integrals (shortly FP integrals), regarded as *pth* derivative of Cauchy principal value integrals, are of interest in the solution of hypersingular BIE, which model many different kind of Physical and Engineering problems (see [1] and the references therein, [2], [3], [4]).

The procedure we employ here is based on a simple tool like the "truncated" Gaussian rule (see [5]), conveniently modified to remove numerical cancellation. We will consider functions $\mathcal{G}$ having different decays at infinity. The method is shown to be numerically stable and convergent and some error estimates in suitable Zygmund-type spaces are proved. Finally, some numerical tests which confirm the efficiency of the proposed procedures are presented.

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1. Introduction

Hypersingular integrals, defined in [6], are of interest, for instance, in the numerical solution of hypersingular integral equations. As it is known, such kind of equations are model for many physics and engineering problems (see [7] and the references therein, [8], [3], [7], [2],[9]).

There is a wide literature devoted to the computation of the Finite Part (FP) of divergent integrals

$$\oint_a^{+b} \frac{g(x)}{(x-t)^{p+1}} dx, \quad p \in \{1, 2, \dots\}, \quad a < t < b,$$

for bounded intervals $[a, b]$. Limiting ourselves to global approximation methods for $p > 0$, we mention among them [7], [1], [10], [11], [8], [12], [13], [14], [15], [16], [17], [18]. An historical overview on the numerical methods for FP integrals and many properties holding in the case of bounded domains can be found in [19], [7], [1], [8]. About the papers which employ Gauss-type rules, these are nearly all devoted to the interval $[-1, 1]$. A more general approach introduced in [9] looks for determining a Gauss quadrature rule w.r.t the weight $\frac{w(x)}{(x-t)^2}$, where $w(x)$ is any Gauss classical weight on finite or infinite ranges. However,

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the Authors discuss computational details only in the interval $[-1,1]$ and for some choices of Jacobi weights w. So, FP integrals over unbounded domains received less attention in the past.

On the other hand hypersingular integrals

$$\oint_0^{+\infty} \frac{\mathcal{G}(x)}{(x-t)^{p+1}}dx, \qquad p \in \{1,2,\dots\}, t>0 \tag{1}$$

are employed in the solution of hypersingular integral equations coming from Neumann 2D elliptic problems on semiplanes by a Petrov-Galerkin infinite BEM approach [4]. In [4] FP integrals on $[a,+\infty),\ a>0$ are reduced to the interval $[0,1]$ and approximated by means of product integration rules. Nevertheless, non linear transformations can get worse the density function $\mathcal{G}$ (see [20]), while the straightforward computation on unbounded ranges can add computational and also theoretical difficulties.

Thus, we propose here some global strategies to approximate integrals of the type (1). The proposed framework allows to consider functions $\mathcal{G}$ having different decays at infinity and uses different approaches, according to the position of $t>0$.

At first we consider the case $\mathcal{G}(x) = f(x)w_\alpha(x)$, where $w_\alpha(x) = e^{-x}x^\alpha,\ \alpha \geq 0$, is a Laguerre weight. Following a very standard way, we start from the decomposition

$$\begin{aligned}
\mathcal{H}_p(fw_\alpha,t) &:= \oint_0^{+\infty} \frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx \qquad (2)\\
&= \int_0^{+\infty} \frac{f(x)-\sum_{k=0}^p \frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}}w_\alpha(x)dx + \sum_{k=0}^p \frac{f^{(k)}(t)}{k!}\oint_0^{+\infty}\frac{w_\alpha(x)}{(x-t)^{p+1-k}}dx,\\
&=: \mathcal{F}_p(fw_\alpha,t) + \sum_{k=0}^p \frac{f^{(k)}(t)}{k!}\mathcal{H}_{p-k}(w_\alpha,t)
\end{aligned}$$

focusing the attention on the first right-hand integral, since the remaining FP integrals are computable with high accuracy by standard routines (see Section 6). We use a simple tool like the Gauss-Laguerre rule properly modified in order to get a stable, convergent and efficient procedure to approximate the integral $\mathcal{F}_p(fw_\alpha,t)$. In particular, we use the "truncated" version of the Gauss-Laguerre rule [5] (see also [21]) in order to reduce the number of function computations and possible overflow ranges. Furthermore, for any fixed t, we select a suitable subsequence of "truncated" Gauss-Laguerre rule for avoiding the severe numerical cancellation arising when t is "close" to a Gaussian node. The approach for t "large" can be treated in a cheaper way, with a shrewd application of the Gaussian rule directly to $\mathcal{H}_p(fw_\alpha,t)$.

As second case we will consider density functions of the type $\mathcal{G}(x) = g(x)/(1+x)^\beta,\ \beta>1$. Indeed, by applying the aforesaid procedure to

$$\oint_0^{+\infty} \frac{\tilde{g}(x)}{(x-t)^{p+1}}e^{-x}dx, \quad \tilde{g}(x) = \frac{g(x)}{(1+x)^\beta}e^x, \tag{3}$$

the results may be rather poor, especially when $\mathcal{G}(x)$ "slowly" decays to zero as $x\to+\infty$ (see [22] about Gaussian rule deficiencies). For this reason, in some cases presented below, we show how to gain better results by making a preliminary change of variable and by applying then the above procedure. We complete this argument determining conditions on g under which the global scheme is stable and fast convergent. Also in this case, when t is "large" we suggest a different strategy.

Since the computation of the derivatives required for implementing the method can bring difficulties to the algorithm, we complete the description showing how to approximate $\{f^{(k)}\}_{k=0}^p$ by means of the derivatives of a suitable Lagrange polynomial interpolating f. In view of the behavior of the Lagrange polynomial sequence, under appropriate assumptions, the rate of convergence of the method remains unchanged, except the extra factor $\log m$.

The paper is organized as follows. In Section 2 some basic results about orthogonal polynomials and function spaces, needed to introduce the main results, are collected. Section 3 contains the definition of

Hadamard finite part integrals over $(0,+\infty)$ for functions f belonging to suitable Zygmund-type spaces, and some their properties. In Section 4 the numerical method to approximate $\mathcal{H}_p(fw_\alpha,t)$ is described and some results about the stability and the rate of convergence are stated. In the successive Section 5 we show how it is possible to speed up the convergence of the method for integrals of the type (3). In Section 6 we show how to avoid the computation of the derivatives of the function f. Section 7 contains some computational details useful in the implementation process. In Section 8 some numerical experiments are given to confirm the efficiency of the procedure. Moreover, comparisons with the method in [4] are shown. Finally, in Section 9 the proofs of the main results are stated.

2. Basic definitions and properties

Along all the paper the constant $\mathcal{C}$ will be used several times, having different meaning in different formulas. Moreover from now on we will write $\mathcal{C} \neq \mathcal{C}(a,b,\ldots)$ in order to say that $\mathcal{C}$ is a positive constant independent of the parameters $a,b,\ldots$, and $\mathcal{C} = \mathcal{C}(a,b,\ldots)$ to say that $\mathcal{C}$ depends on $a,b,\ldots$. Moreover, if $A,B \geq 0$ are quantities depending on some parameters, we will write $A \sim B$, if there exists a constant $0 < \mathcal{C} \neq (A,B)$ such that

$$\frac{B}{\mathcal{C}} \leq A \leq \mathcal{C}B.$$

Finally, $\mathbb{P}_m$ will denote the space of the algebraic polynomials of degree at most m.

2.1. Function spaces

We will denote by $Lip_\lambda((0,+\infty))$ the space of all functions f that are Lipschitz continuous of parameter $0 < \lambda \leq 1$ in $(0,+\infty)$, i.e.,

$$Lip_\lambda((0,+\infty)) = \left\{ f \in C^0((0,+\infty)) : |f(x)-f(y)| \leq \mathcal{C}|x-y|^\lambda, \quad \forall x,y \in \mathbb{R}^+ \right\}.$$

With $w_\alpha(x) = x^\alpha e^{-x}, \alpha > -1$, we will say $f \in L^1_{w_\alpha}$ if and only if $fw_\alpha \in L^1$ and we will set

$$\|f\|_{L^1_{w_\alpha}} := \|fw_\alpha\|_1 = \int_0^{+\infty} |f(x)| w_\alpha(x) dx.$$

Moreover, with $\alpha \geq 0$ we denote by C_{w_α} the following set of functions

$$C_{w_\alpha} = \begin{cases} \left\{ f \in C^0((0,+\infty)) : \lim\limits_{\substack{x\to+\infty \\ x\to 0^+}} (fw_\alpha)(x) = 0 \right\}, & \alpha > 0, \\ \left\{ f \in C^0([0,+\infty)) : \lim\limits_{x\to+\infty} (fw_\alpha)(x) = 0 \right\}, & \alpha = 0, \end{cases}$$

equipped with the norm

$$\|f\|_{C_{w_\alpha}} := \|fw_\alpha\|_\infty = \sup_{x\geq 0} |(fw_\alpha)(x)|,$$

where $C^0(E)$ is the space of the continuous functions on the set E. In the next we will use $\|f\|_E := \sup_{x\in E} |f(x)|$.

For smoother functions, we introduce the Sobolev-type spaces of order $r \in \mathbb{N}$

$$W^1_r(w_\alpha) = \left\{ f \in L^1_{w_\alpha}((0,+\infty)) \ : \ f^{(r-1)} \in AC((0,+\infty)) \text{ and } \|f^{(r)}\varphi^r w_\alpha\|_1 < +\infty \right\}$$

and

$$W^\infty_r(w_\alpha) = \left\{ f \in C_{w_\alpha} \ : \ f^{(r-1)} \in AC((0,+\infty)) \text{ and } \|f^{(r)}\varphi^r w_\alpha\|_\infty < +\infty \right\},$$

where $AC((0,+\infty))$ denotes the set of all the functions which are absolutely continuous on every closed subset of $(0,+\infty)$ and $\varphi(x)=\sqrt{x}$. We equip these spaces with the norms

$$\|f\|_{W_r^1(w_\alpha)} := \|fw_\alpha\|_1 + \|f^{(r)}\varphi^r w_\alpha\|_1$$

and

$$\|f\|_{W_r^\infty(w_\alpha)} := \|fw_\alpha\|_\infty + \|f^{(r)}\varphi^r w_\alpha\|_\infty,$$

respectively.

For any $f \in C_{w_\alpha}$, let [23]

$$\Omega_\varphi^k(f,u)_{w_\alpha} = \sup_{0<h\le u} \|w_\alpha \Delta_{h\varphi}^k f\|_{I_{kh}},$$

where $I_{kh} = \left[4k^2h^2, \frac{\mathcal{C}}{h^2}\right]$, $\mathcal{C}$ is a fixed positive constant, and

$$\Delta_{h\varphi}^k f(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f\left(x + h\varphi(x)(k-i)\right).$$

By means of the main part of the modulus of smoothness $\Omega_\varphi^k(f)$ we can define the Zygmund-type spaces

$$Z_\lambda(w_\alpha) := \left\{ f \in C_{w_\alpha} \; : \; \sup_{u>0} \frac{\Omega_\varphi^k(f,u)_{w_\alpha}}{u^\lambda} < +\infty, \quad k > \lambda \right\}, \quad \lambda \in \mathbb{R}^+,$$

equipped with the norm

$$\|f\|_{Z_\lambda(w_\alpha)} = \|fw_\alpha\|_\infty + \sup_{u>0} \frac{\Omega_\varphi^k(f,u)_{w_\alpha}}{u^\lambda}.$$

Note that for $r \le \lambda \le r+1$, it is $W_{r+1}^\infty(w_\alpha) \subseteq Z_\lambda(w_\alpha) \subseteq W_r^\infty(w_\alpha)$.

2.2. Orthogonal polynomials and Truncated Gauss-Laguerre rule

Let $w_\alpha(x) = e^{-x}x^\alpha$ be the Laguerre weight of parameter $\alpha \ge 0$ and let $\{p_m(w_\alpha)\}_m$ be the corresponding sequence of orthonormal polynomials with positive leading coefficients. Denoting by $x_{m,k}, k=1,\ldots,m$, the zeros of $p_m(w_\alpha)$ in increasing order, we recall that (see [24])

$$\frac{\mathcal{C}}{m} < x_{m,1} < x_{m,2} < \ldots < x_{m,m} < 4m + 2\alpha - \mathcal{C}m^{\frac{1}{3}}.$$

From now on, for any fixed $0<\theta<1$, the integer $j := j(m)$ will denote the index of the zero of $p_m(w_\alpha)$ s. t.

$$j = \min_{k=1,2,..,m} \{k : x_{m,k} \ge 4m\theta\}. \tag{4}$$

Inside the segment $(0, x_{m,j})$ the distance between two consecutive zeros of $p_m(w_\alpha)$ can be estimated as follows [25]

$$\Delta x_{m,k} \sim \Delta x_{m,k-1} \sim \sqrt{\frac{x_{m,k}}{m}}, \qquad \Delta x_{m,k} = x_{m,k+1} - x_{m,k}, \qquad k = 1,2,\ldots,j.$$

We recall the so called “truncated” Gauss-Laguerre rule introduced in [5] and based on the first j zeros of $p_m(w_\alpha)$, with j defined in (4), i.e.

$$\int_0^{+\infty} f(x)w_\alpha(x)dx = \sum_{k=1}^{j} f(x_{m,k})\lambda_{m,k} + R_m(f), \tag{5}$$

where $\{\lambda_{m,k}\}_{k=1}^m$ are the Christoffel numbers w.r.t. w_α and $R_m(f)$ is the remainder term.

For all $f \in W_r^1(w_\alpha)$ the following error estimate holds

$$|R_m(f)| \le \mathcal{C}\left(\frac{\|f^{(r)}\varphi^r w_\alpha\|_1}{(\sqrt{m})^r} + \|fw_\alpha\|_1 e^{-Am} \right), \tag{6}$$

where $0 < \mathcal{C} \ne \mathcal{C}(m,f)$ and $0 < A \ne A(m,f)$ [5].

2.3. Truncated Lagrange interpolation

Let $\{p_m(w_\rho)\}_m$ be the sequence of orthonormal polynomials corresponding to the weight $w_\rho(x) = e^{-2x}x^\rho$, $\rho > -1$ and let $\{y_i\}_{i=1}^m$ be the zeros of $p_m(w_\rho)$, with $\frac{\mathcal{C}}{m} < y_1 < y_2 < \dots y_m < 2m - \mathcal{C}m^{\frac{1}{3}}$. For any fixed $0 < \theta < 1$, with

$$x_\ell = \min\{y_k \geq 2m\theta, \quad k = 1, 2, \dots, m\},$$

let $\mu_{m,\theta}$ be the characteristic function of the segment $(0, y_\ell)$. For a given function f, let $L_{m+1}(w_\rho, f, x)$ be the Lagrange polynomial interpolating f at the zeros of $p_m(w_\rho, x)(2m - x)$ and let

$$L^*_{m+1}(w_\rho, f, x) := L_{m+1}(w_\rho, \mu_{m,\theta} f, x). \tag{7}$$

We point out that the previous Lagrange polynomial has been obtained by a linear change of variable in that introduced in [26] (see also [27]).

We recall the following results about the simultaneous approximation of f and its derivatives (see [26] for $p = 0$, [28] otherwise)

Theorem 2.1. *If $f \in Z_{p+\lambda}(w_\alpha)$ with $0 < \lambda \leq 1$, p nonnegative integer and α such that*

$$\frac{\rho}{2} + \frac{1}{4} \leq \alpha \leq \frac{\rho}{2} + \frac{5}{4},$$

we have

$$\|(f - L^*_{m+1}(w_\rho, f))^{(p)}\varphi^p w_\alpha\| \quad \leq \quad \mathcal{C}\log m \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega^r_\varphi\left(f^{(p)}, s\right)_{w_\alpha\varphi^p}}{s} ds + \mathcal{C}e^{-Am}\|f w_\alpha\|,$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

3. Hadamard integrals over unbounded intervals

FP integrals, introduced by Hadamard in 1923, are essentially defined as the finite part of divergent integrals. Many properties fulfilled by finite part integrals over bounded intervals can be found in [7] (see also [8], [29], [30]).

Now we consider finite parts integrals on $(0, +\infty)$, under assumptions which are appropriate to the cases we will treat here.

Definition 1. *Let $p \geq 1$. For any $0 < t < b$*

$$⨎_0^{+\infty} \frac{dx}{(x-t)^{p+1}} \quad = \quad ⨎_0^{b} \frac{dx}{(x-t)^{p+1}} + \int_b^{+\infty} \frac{dx}{(x-t)^{p+1}}$$

and

$$\begin{aligned} ⨎_0^{+\infty} \frac{dx}{(x-t)^{p+1}} &= \lim_{b\to+\infty}\left\{\lim_{\varepsilon\to0^+}\left(\int_0^{t-\varepsilon} \frac{dx}{(x-t)^{p+1}}dx + \int_{t+\varepsilon}^{b} \frac{dx}{(x-t)^{p+1}}dx - \frac{1-(-1)^p}{p\varepsilon^p}\right)\right\} \\ &= \frac{1}{p}\lim_{b\to+\infty}\left(\frac{1}{(-t)^p} - \frac{1}{(b-t)^p}\right) =: \frac{(-1)^p}{pt^p}. \end{aligned} \tag{8}$$

Let us assume $g \in C^{(p)}((0, +\infty))$ with $g^{(p)} \in Lip_\lambda((0, +\infty))$, $0 < \lambda \leq 1$ and $\|g\|_\infty \leq \mathcal{C}$. Setting

$$\zeta(g, x, t) := \frac{g(x) - g(t)}{x - t},$$

we have

$$\begin{aligned}\fint_0^{+\infty}\frac{g(x)}{(x-t)^{p+1}}dx &:= \int_0^{+\infty}\frac{g(x)-\sum_{k=0}^{p-1}\frac{g^{(k)}(t)(x-t)^k}{k!}}{(x-t)^{p+1}}dx+\sum_{k=0}^{p-1}\frac{g^{(k)}(t)}{k!}\fint_0^{+\infty}\frac{dx}{(x-t)^{p+1-k}}\\ &= \int_0^{+\infty}\frac{\partial^{p-1}}{\partial t^{p-1}}\zeta(g,x,t)\frac{dx}{x-t}+\sum_{k=0}^{p-1}\frac{g^{(k)}(t)}{k!}\fint_0^{+\infty}\frac{dx}{(x-t)^{p+1-k}}. \end{aligned} \tag{9}$$

Since $g^{(p)}\in Lip_\lambda((0,+\infty))$, it follows that $\frac{\partial^{p-1}}{\partial t^{p-1}}\zeta(g,x,t)\in Lip_\lambda((0,+\infty))$ [31] (see [3, p.4]) and therefore the first integral at right hand exists in the Cauchy principal value sense. The integrals in the summation are defined in (8).

Of course, according to (9), integrals of the type

$$\mathcal{H}_p(f\sigma_\beta,t)=\fint_0^{+\infty}\frac{f(x)}{(x-t)^{p+1}}\sigma_\beta(x)dx, \qquad \sigma_\beta(x)=\frac{1}{(1+x)^\beta}, \quad \beta>1,$$

are also defined in the Hadamard sense under the assumptions $f\in C^p([0,+\infty))$, $f^{(p)}\in Lip_\lambda((0,+\infty))$, $0<\lambda\le 1$.

Now we state the following relation which holds under the assumptions $g\in Lip_\lambda((0,+\infty))$ on any closed subset of $(0,+\infty)$, $\|g\|_\infty\le\mathcal{C}$,

$$\fint_0^{+\infty}\frac{g(x)}{(x-t)^{p+1}}dx:=\frac{1}{p!}\frac{d^p}{dt^p}\int_0^{+\infty}\frac{g(x)}{x-t}dx, \quad t>0. \tag{10}$$

(10) can be deduced by using standard arguments (see for instance [13]) and, similarly to the case of bounded intervals (see [7]), allow us to regard the $p-$th Hadamard transform of f on $(0,+\infty)$ as the $p-$th derivative of the Hilbert transform of f.

Moreover, by (10), assuming $\alpha\ge 0$, we get

$$\fint_0^{+\infty}\frac{w_\alpha(x)}{(x-t)^{p+1}}dx:=\frac{1}{p!}\frac{d^p}{dt^p}\int_0^{+\infty}\frac{w_\alpha(x)}{x-t}dx. \tag{11}$$

Finally, we consider integrals of the type

$$\mathcal{H}_p(fw_\alpha,t):=\fint_0^{+\infty}\frac{f(x)}{(x-t)^{p+1}}w_\alpha(x)dx, \tag{12}$$

with $\alpha\ge 0$. The following theorem [32] assures that the integral in (12) exists in the Hadamard sense under the more general assumption $f^{(p)}\in Z_\lambda(w_\alpha\varphi^p)$.

Theorem 3.1. *Let $p\ge 1$, $\alpha\ge 0$, $0<\lambda<1$. If $f^{(p)}\in Z_\lambda(w_\alpha\varphi^p)$ then*

$$\sup_{t>0}t^p|\mathcal{H}_p(fw_\alpha,t)|\le\mathcal{C}\left(\int_0^1\frac{\Omega_\varphi(f^{(p)},u)_{w_\alpha\varphi^p}}{u}du+\|f\|_{W_p(w_\alpha)}\right), \qquad 0<\mathcal{C}\ne\mathcal{C}(f).$$

Now we state the following integration by part rule, that can be easily deduced by the corresponding rule over bounded intervals [7]:

Lemma 3.1. *For a given function g s.t. $g^{(p)}\in Lip_\lambda((0,+\infty))$ with exponent $0<\lambda\le 1$ and $\|g\|_\infty\le\mathcal{C}$, the following integration by parts rule holds true*

$$\fint_0^{+\infty}\frac{g(x)}{(x-t)^{p+1}}dx=\frac{1}{p}\left\{\fint_0^{+\infty}\frac{g'(x)}{(x-t)^p}dx+(-1)^p\frac{g(0)}{t^p}\right\}.$$

Hence, for any $k \le p$ we deduce

$$\fint_0^{+\infty} \frac{g(x)}{(x-t)^{p+1}}dx = \frac{(p-k)!}{p!}\fint_0^{+\infty} \frac{g^{(k)}(x)}{(x-t)^{p-k+1}}dx + \sum_{r=0}^{k-1} \frac{(p-r-1)!}{p!}(-1)^{p-r}\frac{g^{(r)}(0)}{t^{p-r}}. \tag{13}$$

If in addition, $g^{(j)}(0) = 0,\ j = 0, 1, \dots, k-1$, (13) reduces to

$$\fint_0^{+\infty} \frac{g(x)}{(x-t)^{p+1}}dx = \frac{(p-k)!}{p!}\fint_0^{+\infty} \frac{g^{(k)}(x)}{(x-t)^{p-k+1}}dx.$$

4. The method

Let us start from

$$\begin{aligned}\mathcal{H}_p(fw_\alpha, t) &= \int_0^{+\infty} \frac{f(x) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x-t)^k}{(x-t)^{p+1}} w_\alpha(x)dx + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \fint_0^{+\infty} \frac{w_\alpha(x)}{(x-t)^{p+1-k}}dx \\ &=: \mathcal{F}_p(fw_\alpha, t) + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}\mathcal{H}_{p-k}(w_\alpha, t). \end{aligned} \tag{14}$$

The accurate computation of $\mathcal{H}_{p-k}(w_\alpha, t),\ k = 0, 1, \dots, p$, can be performed by efficient routines (see Section 6). So we focus our attention on the term $\mathcal{F}_p(fw_\alpha, t)$.

In what follows we will assume $0 < \theta < 1$ fixed. For any integer $m > 0$ such that $t < 4m\theta$, by using (5) with j defined in (4), we get

$$\begin{aligned}\mathcal{F}_p(fw_\alpha, t) &= \sum_{i=1}^{j} \frac{f(x_{m,i}) - \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}(x_{m,i}-t)^k}{(x_{m,i}-t)^{p+1}}\lambda_{m,i} + e_{p,m}(fw_\alpha, t) \\ &:= \mathcal{F}_{p,m}(fw_\alpha, t) + e_{p,m}(fw_\alpha, t),\end{aligned}$$

where $e_{p,m}(fw_\alpha, t)$ is the remainder term.

We observe that when t is "close" to one of the Gaussian knots, say it $x_{\bar{k}},\ \bar{k} < j$, the denominator $(x_{m,\bar{k}} - t)^{p+1}$ of the $\bar{k}$−th addendum in $\mathcal{F}_{p,m}(fw_\alpha, t)$ can become too "small", producing then numerical cancellation. This last phenomenon is all the more severe as p is "larger".

Next algorithm is useful to overcome this instability. It essentially consists in selecting, for any fixed t, a proper subsequence of the Gauss-Laguerre sequence $\{\mathcal{F}_{p,m}(fw_\alpha, t)\}_m$, in such a way that the distances $|x_{m,i} - t|$ are always large enough.

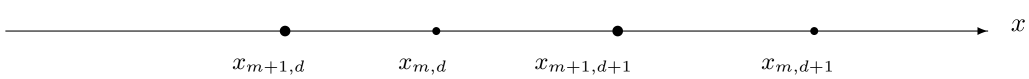
As it is well known the zeros $\{x_{m,k}\}_{k=1}^{m}$ of $p_m(w_\alpha)$ interlace the zeros $\{x_{m+1,k}\}_{k=1}^{m+1}$ of $p_{m+1}(w_\alpha)$ and the distance between two consecutive zeros of $p_m(w_\alpha)p_{m+1}(w_\alpha)$ is sufficiently large since [33, Lemma 2.1]

$$x_{m+1,k} - x_{m,k} > \mathcal{C}\sqrt{\frac{x_{m,k}}{m}}, \quad x_{m,k},\ x_{m+1,k} \in (0, 4m\theta), \quad 0 < \mathcal{C} \ne \mathcal{C}(m). \tag{15}$$

In view of this "good" distance, we propose to use the following algorithm firstly suggested in [11] in the interval $[-1, 1]$ and successively used in different contexts (see [33], [20], [16]).

Let $t > 0$ be fixed. For m sufficiently large (say $m \ge m_0 \in \mathbb{N}$), there exists an index $d \in \{1, 2, \dots, m-1\}$ s.t.

$$x_{m,d} \le t \le x_{m,d+1}.$$



Thus, two cases are possible:

$$(a) \quad x_{m,d} \le t \le x_{m+1,d+1} \qquad \text{or} \qquad (b) \quad x_{m+1,d+1} \le t \le x_{m,d+1}.$$

In the case (a) we choose

$$m^* = \begin{cases} m, & \text{if } \ t - x_{m,d} > x_{m+1,d+1} - t, \\ m+1, & \text{otherwise,} \end{cases} \tag{16}$$

and in the case (b) we choose

$$m^* = \begin{cases} m+1, & \text{if } \ t - x_{m+1,d+1} > x_{m,d+1} - t, \\ m, & \text{otherwise.} \end{cases} \tag{17}$$

In such a way, we construct the subsequence

$$\{\mathcal{F}_{p,m^*}(fw_\alpha, t)\}_{m^*} \subset \{\mathcal{F}_{p,m}(fw_\alpha, t)\}_{m \in \mathbf{N}},$$

where m^*, according to the position of $t \in (x_{m,1}, x_{m,j})$, is chosen as in (16) and (17) and we will approximate $\mathcal{F}_p(fw_\alpha, t)$ by $\mathcal{F}_{p,m^*}(fw_\alpha, t)$, i.e., coming back to (14)

$$\mathcal{H}_p(fw_\alpha, t) = \mathcal{H}_{p,m^*}(fw_\alpha, t) + e_{p,m^*}(fw_\alpha, t), \tag{18}$$

where

$$\mathcal{H}_{p,m^*}(fw_\alpha, t) = \mathcal{F}_{p,m^*}(fw_\alpha, t) + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathcal{H}_{p-k}(w_\alpha, t). \tag{19}$$

We observe that, by (15),

$$|x_{m^*,d} - t| \ge \mathcal{C} \sqrt{\frac{x_{m^*,d}}{m}}$$

which means that in any case t will be sufficiently "far" from the Gaussian knots.

Next theorem deals with the stability and the convergence of the proposed quadrature rule.

Theorem 4.1. *Let $\alpha \ge 0$, $p \ge 1$ and $0 < \lambda < 1$. If $f^{(p)} \in Z_\lambda(w_\alpha \varphi^p)$ then for any $0 < t < 4m\theta_1$, with $0 < \theta_1 < \theta < 1$,*

$$t^p |\mathcal{H}_{p,m^*}(fw_\alpha, t)| \quad \le \quad \mathcal{C} \|f\|_{Z_{p+\lambda}(w_\alpha)}$$

and

$$t^p |\mathcal{H}_p(fw_\alpha, t) - \mathcal{H}_{p,m^*}(fw_\alpha, t)| \quad \le \quad \mathcal{C} \frac{\log m}{\sqrt{m^\lambda}} \|f\|_{Z_{p+\lambda}(w_\alpha)},$$

where $0 < \mathcal{C} \ne \mathcal{C}(m, f, t)$.

Remark 4.1. *Now we observe that $f^{(p)} \in Z_\lambda(w_\alpha \varphi^p) \equiv f \in Z_{\lambda+p}(w_\alpha)$ and therefore, under more restrictive assumptions on f, for instance $f \in Z_{\lambda+p+q}(w_\alpha)$, $q \ge 0$, we get*

$$t^p |\mathcal{H}_p(fw_\alpha, t) - \mathcal{H}_{p,m^*}(fw_\alpha, t)| \le \mathcal{C} \frac{\|f\|_{Z_{\lambda+p+q}(w_\alpha)}}{\sqrt{m^{\lambda+q}}} \log m \tag{20}$$

and if $f \in W^\infty_{p+q}(w_\alpha)$, $q \ge 1$, we obtain

$$t^p |\mathcal{H}_p(fw_\alpha, t) - \mathcal{H}_{p,m^*}(fw_\alpha, t)| \le \mathcal{C} \frac{\|f\|_{W^\infty_{p+q}(w_\alpha)}}{\sqrt{m^q}} \log m.$$

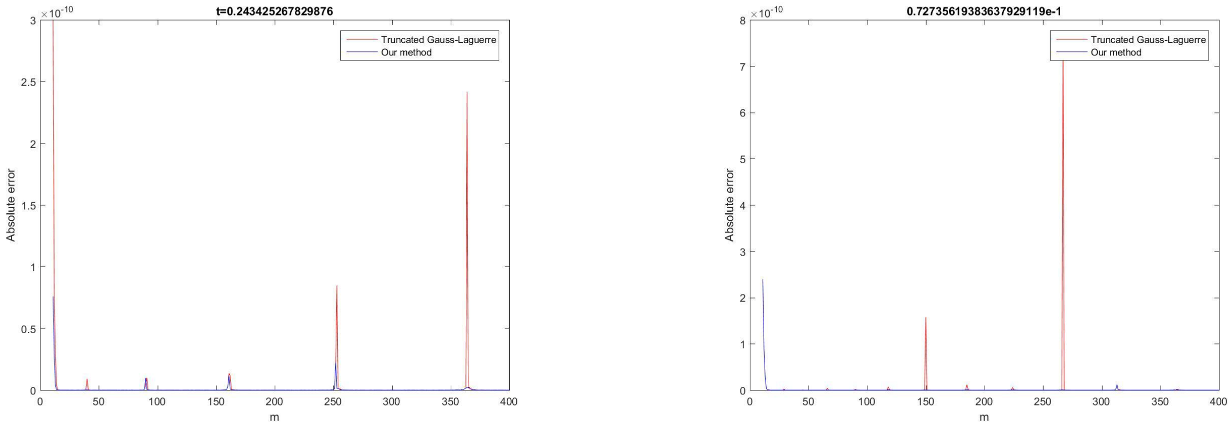


Figure 1: Comparison between the absolute errors of the rules $\mathcal{F}_{p,m}(fw_\alpha,t)$ and $\mathcal{F}_{p,m^*}(fw_\alpha,t)$

We conclude by proposing just a numerical example to highlights how the ordinary Truncated Gaussian rule suffers from instability. In the two plots in Figure 1 we compare the absolute errors obtained by implementing, for increasing values of m, the Truncated Gauss-Laguerre rule $\mathcal{F}_{p,m}(fw_\alpha,t)$ (red curve) and the modified Gauss-Laguerre rule $\mathcal{F}_{p,m^*}(fw_\alpha,t)$ (blue curve), for the integral

$$\mathcal{H}_1(fw_{\frac{1}{2}},t) = \neq\!\!\!\!\!\!\int_0^{+\infty} \frac{\sin(x+5)}{(x-t)^2}\sqrt{x}e^{-x}dx.$$

As the graph shows, the highest errors are attained by using the ordinary Truncated Gaussian rule and, for the same t, also for different values of m. Since the function f is smooth, this bad behavior is due to the closeness of some of the Gaussian abscissa to t. This shortcoming is successfully overcome by using the method in (19).

4.1. The case t "large"

The method in (19) can be used when t stays between two Gaussian nodes, i.e. for $t < 4m\theta_1$, or equivalently, $m > \frac{t}{4\theta_1}$. Thus, the "larger" t is, the "larger" m will be. For instance, for $t = 1000$ and $\theta = \frac{1}{8}$, $m > 2000$ has to be chosen. In such cases the computation of abscissas and weights in the Gauss-Laguerre rule is too much expensive, and sometimes unfeasible. For all these reasons, we propose here an alternative procedure, which is essentially a shrewd application of the Gaussian rule again, in the sense we go to precise. For any fixed t, let m be such that

$$t > x_{m,j} + 1,$$

where j is defined in (4). Setting $F_t(x) = \frac{f(x)}{(x-t)^{p+1}}$, we approximate the integral by the $m-$th truncated Gauss-Laguerre rule, i.e.

$$\mathcal{H}_p(fw_\alpha,t) = \sum_{i=1}^{j} F_t(x_{m,i})\lambda_{m,i} + R_m(F_t),. \tag{21}$$

Since f and $F_t(f)$ for $t > x_{m,j}+1$ have the same smoothness, by (6), if $f \in W_r^1(w_\alpha), r \geq 1$, then we get

$$|R_m(F_t)| \leq \mathcal{C}\left(\frac{\|F_t^{(r)}\varphi^r w_\alpha\|_1}{(\sqrt{m})^r} + \|F_t w_\alpha\|_1 e^{-Am}\right), \tag{22}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, F_t, t)$ and $0 < A \neq A(m, F_t, t)$.

We observe that this procedure can be successfully applied when F_t is a "smooth" function and for a large value of $t > x_{m,j}+1$, where j is defined in (4). Moreover, smoother is the function f and smaller is the value of m to obtain a desired precision. This means that t can be "large" but not necessarily too much. Of course, the error bound in (22) holds for a fixed m and therefore the limit on m of $R_m(F_t)$ has no meaning.

In conclusion, by using (18) or (21), we obtain an efficient procedure for the computation of $\mathcal{H}_p(fw_\alpha, t)$ for a "wide" range of t.

5. Algebraic decay functions: a particular case

In this section we go to treat integrals of the type

$$\mathcal{H}_p(f\sigma_\beta, t) = \;=\!\!\!\!\!\!\!\int_0^{+\infty} \frac{f(x)}{(x-t)^{p+1}}\sigma_\beta(x)dx,$$

with $p \geq 1$ and $\sigma_\beta(x) = \frac{1}{(1+x)^\beta}$, $\beta > 1$.

Letting $\Phi(x) := f(x)\sigma_\beta(x)e^x$ and for any function g

$$R_p(g,x,t) \;:=\; g(x) - \sum_{k=0}^{p} \frac{g^{(k)}(t)}{k!}(x-t)^k, \qquad G_p(g,x,t) := \frac{R_p(g,x,t)}{(x-t)^{p+1}}, \tag{23}$$

we can write

$$\mathcal{H}_p(f\sigma_\beta, t) \;=\; \int_0^{+\infty} G_p(\Phi,x,t)e^{-x}dx + \sum_{k=0}^{p} \frac{\Phi^{(k)}(t)}{k!} \;=\!\!\!\!\!\!\!\int_0^{+\infty} \frac{e^{-x}}{(x-t)^{p+1-k}}dx.$$

For a fixed t, apply the "truncated" Gaussian rule (5) w. r. t. w_0, i.e.,

$$\begin{aligned} \int_0^{+\infty} G_p(\Phi,x,t)e^{-x}dx \;&=\; \sum_{i=1}^{j} \lambda_{m^*,i} G_p(\Phi, x_{m^*,i}, t) + R_{m^*}(G_p(\Phi), t) \\ &=: \; \mathcal{H}_{p,m^*}(\Phi w_0, t) + R_{m^*}(G_p(\Phi), t), \end{aligned} \tag{24}$$

where m^* is defined as in (17) or (16), according to the position of $t \in (0, 4m\theta_1)$. Then, about the error estimate, we prove the following lemma:

Lemma 5.1. *Let t be fixed with $t > a > 0$ and let $f \in C^{p+1+r}([0,+\infty))$, $p \geq 1$, $r \geq 1$. Then $G_p(\Phi) \in C^r([0,+\infty))$. If, in addition,*

$$|f^{(k)}(x)| \leq \mathcal{C}, \quad k = 0,1,\ldots,p+1+r, \tag{25}$$

under the assumptions $2(\beta+p) - r > 0$, we have $G_p(\Phi) \in W_r^1(w_0)$ and then

$$|R_{m^*}(G_p(\Phi))| \leq \mathcal{C}\left(\frac{\|(G_p(\Phi))^{(r)}\varphi^r w_0\|_1}{(\sqrt{m})^r} + \|(G_p(\Phi))w_0\|_1 e^{-Am} \right), \tag{26}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m,f)$ and $0 < A \neq A(m,f)$.

In view of (26) the error of the formula (24) behaves like $\mathcal{O}(m^{-\lfloor \beta+p \rfloor})$ (where $\lfloor a \rfloor$ denotes the integer part of a). Therefore, even if the function f is very "smooth", the rate of convergence of the Gaussian rule is very poor for β "small".

Now we show that this bad behavior can be removed, by introducing a preliminary change of variable (see [22]). By using the same notation introduced in (23), we write

$$\mathcal{H}_p(f\sigma_\beta, t) \;=\; \int_0^{+\infty} \frac{G_p(f,x,t)}{(1+x)^\beta}dx + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!}\mathcal{H}_{p-k}(\sigma_\beta, t).$$

Referring the computation of $\mathcal{H}_{p-k}(\sigma_\beta, t)$ to Section 6, we introduce in the first right-hand integral the change of variable $x = \psi(y)$, where $\psi(y) := e^{qy} - 1$, with $q > 0$ fixed real parameter. Thus we have

$$\int_0^{+\infty} \frac{G_p(f, x, t)}{(1+x)^\beta} dx = q \int_0^{+\infty} G_p(f, \psi(y), t) e^{-qy(\beta-1)} dy = \int_0^{+\infty} h_t(y) e^{-y} dy,$$

where

$$h_t(y) := q G_p(f, \psi(y), t) e^{-y(q(\beta-1)-1)}.$$

Thus, by applying the truncated Gaussian rule (5) w.r.t. w_0, we get

$$\int_0^{+\infty} \frac{G_p(f, x, t)}{(1+x)^\beta} dx = \sum_{i=1}^{j} h_t(x_{m^*,i}) \lambda_{m^*,i} + R_{m^*}(h_t),$$

where m^* is chosen as in (16) and (17), according to the position of t fixed and $a < t < 4m\theta_1$, and

$$\begin{aligned} \mathcal{H}_p(f\sigma_\beta, t) &= \sum_{i=1}^{j} h_t(x_{m^*,i}) \lambda_{m^*,i} + \sum_{k=0}^{p} \frac{f^{(k)}(t)}{k!} \mathcal{H}_{p-k}(\sigma_\beta, t) + R_{m^*}(h_t) \\ &=: \hat{\mathcal{H}}_{p,m^*}(f\sigma_\beta, t) + R_{m^*}(h_t). \end{aligned} \tag{27}$$

In the following lemma we give an estimate of the error $R_{m^*}(h_t)$:

Theorem 5.1. *Let t be fixed with $0 < a < t < 4m\theta_1$ and let $f \in C^{p+1+r}([0, +\infty))$, $p \geq 1$, $r \geq 1$, such that*

$$|f^{(k)}(x)| \leq \mathcal{C}, \quad k = 0, 1, \ldots, p+1+r.$$

Under the assumption $q > \frac{1}{\beta}$ the function h_t belongs to $W_r^1(w_0)$ for any $r \geq 1$ and, then,

$$|R_{m^*}(h_t)| \leq \mathcal{C} \left(\frac{\|h_t^{(r)} \varphi^r w_0\|_1}{(\sqrt{m})^r} + \|h_t w_0\|_1 e^{-Am} \right) \tag{28}$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f)$ and $0 < A \neq A(m, f)$.

We observe that the positive parameter q can be freely chosen provided that $q > \frac{1}{\beta}$, so that the transformed integrand decays to 0 fast enough.

In conclusion, Lemma 5.1 assigns sufficient conditions on the growths at infinity of f, such that the integrand function h_t decays to zero faster than $G_p(\Phi)$ in (24). Consequently, the truncated Gaussian rule applied to h_t requires a cheaper computational effort. Moreover, in view of (28) its rate of convergence is $\mathcal{O}(m^{-\frac{r}{2}})$, for any $r \geq 1$.

We conclude observing that, following the same scheme described in Section 4 when t is "large", for a fixed m such that $t - x_{m,j} > 1$, setting $\tilde{F}_t(x) = \frac{f(x)}{(x-t)^{p+1}}$, we introduce the change of variable $x = \psi(y)$, obtaining then

$$\mathcal{H}_p(f\sigma_\beta, t) = q \int_0^{+\infty} \tilde{F}_t(\psi(y)) e^{-qy(\beta-1)} dy = \int_0^{+\infty} \tilde{h}_t(y) e^{-y} dy,$$

where $\tilde{h}_t(y) = q\tilde{F}_t(\psi(y)) e^{-y(q(\beta-1)-1)}$. Hence, by using the truncated Gaussian rule we get

$$\mathcal{H}_p(f\sigma_\beta, t) = \sum_{i=1}^{j} \tilde{F}_t(x_{m,i}) \lambda_{m,i} + \tilde{R}_m(\tilde{F}_t). \tag{29}$$

6. Approximation of the derivatives of f

Now we want to sketch how to avoid the computation of the derivatives of f in (19) or in (27), by means of a suitable Lagrange process, which results optimal in the space of functions f belongs to. Recalling the truncated Lagrange polynomial defined in (7), we use

$$f^{(k)}(t) \sim (L^*_{m+1}(w_\rho, f, t))^{(k)}, \quad 0 \le k \le p,$$

obtaining

$$\mathcal{H}_{p,m^*}(fw_\alpha, t) = \sum_{i=1}^{j} \frac{f(x_{m,i}) - \sum_{k=0}^{p} \frac{L^*_{m+1}(w_\rho,f,t))^{(k)}}{k!}(x_{m,i}-t)^k}{(x_{m,i}-t)^{p+1}} \lambda_{m,i} \tag{30}$$

$$+ \sum_{k=0}^{p} \frac{L^*_{m+1}(w_\rho, f, t))^{(k)}}{k!} \mathcal{H}_{p-k}(w_\alpha, t) + \widetilde{e}_{p,m}(fw_\alpha, t), \tag{31}$$

and, setting $\Psi(y) = f(\psi(y))$,

$$\Psi^{(k)}(y) \sim (L^*_{m+1}(w_\rho, \Psi, y))^{(k)}, \quad 0 \le k \le p,$$

obtaining,

$$\mathcal{H}_p(f\sigma_\beta, t) = q\sum_{i=1}^{j} \frac{\Psi(x_{m,i}) - \sum_{k=0}^{p} \frac{L^*_{m+1}(w_\rho,\Psi,t)^{(k)}}{k!}(x_{m,i}-t)^k}{(x_{m^*,i}-t)^{p+1}} e^{-x_{m,i}(q(\beta-1)-1)} \lambda_{m^*,i}$$

$$+ \sum_{k=0}^{p} \frac{L^*_{m+1}(w_\rho, f, t)^{(k)}}{k!} \mathcal{H}_{p-k}(\sigma_\beta, t) + \widetilde{R}_{m^*}(h_t). \tag{32}$$

Since, in view of Theorem 2.1, under the assumption

$$2\alpha - \frac{5}{2} \le \rho \le 2\alpha - \frac{1}{2}, \tag{33}$$

for functions $f \in Z_{p+\lambda}(w_\alpha)$,

$$\|(f - L^*_{m+1}(w_\rho, f))^{(k)} \varphi^k w_\alpha\| \sim \mathcal{O}\left(\frac{\log m}{(\sqrt{m})^{p+\lambda-k}}\right), \quad k \le p,$$

about the global errors $\widetilde{e}_{p,m}(fw_\alpha, t)$ and $\widetilde{R}_{m^*}(h_t)$ we can say that, except the extra factor $\log m$, they are comparable with $e_{p,m}(fw_\alpha, t)$ and $R_{m^*}(h_t)$, respectively.

We point out that a save can be reached in some cases. For instance, referring to (31), if $f \in Z_{p+\lambda}(\sqrt{w_{2\alpha}})$ with $\alpha \ge \frac{1}{2}$, since by (33) $\rho = \alpha$ is a feasible choice, the samples of the function f employed in the Gaussian rule can be reused.

7. Computational details

Now we give some details dealing with the practical computation of

$$\mathcal{H}_{p-k}(w_\alpha, t) = \int_0^{+\infty} \frac{x^\alpha e^{-x}}{(x-t)^{p+1-k}} dx, \quad \alpha \ge 0, \tag{34}$$

and

$$\mathcal{H}_{p-k}(\sigma_\beta, t) = \int_0^{+\infty} \frac{dx}{(1+x)^\beta (x-t)^{p+1-k}} dx, \quad \beta > 1. \tag{35}$$

To compute integrals of the type (34) we recall [34, p.325, n. 16]

$$\int_0^{+\infty} \frac{x^\alpha e^{-x}}{(x-t)}dx = \begin{cases} -e^{-t}E_i(t), & \alpha = 0, \\ -\pi t^\alpha e^{-t} \cot((1+\alpha)\pi) + \Gamma(\alpha)e^{-t}\,{}_1F_1(-\alpha, 1-\alpha, t), & \alpha \neq 0, \end{cases}$$

where E_i is the Exponential Integral function and ${}_1F_1$ is the Confluent Hypergeometric function. Thus, since by (11),

$$\mathcal{H}_{p-k}(w_\alpha, t) = \frac{1}{(p-k)!}\frac{d^{p-k}}{dt^{p-k}}\int_0^{+\infty} \frac{x^\alpha e^{-x}}{(x-t)}dx,$$

the computation of $\mathcal{H}_{p-k}(w_\alpha, t)$ can be performed by using the following relations [35, p. 1086, 9.213]

$$\frac{d}{dt}E_i(t) = -\frac{d}{dt}\int_{-t}^{+\infty} \frac{e^{-x}}{x}dx = \frac{e^t}{t},$$

$$\frac{d}{dt}\,{}_1F_1(a, b; t) = \frac{a}{b}\,{}_1F_1(a+1, b+1, t).$$

Now we sketch how to compute integrals in (35). By (13) with $g(x) = \frac{1}{(1+x)^\beta}$ and $k = p$, we deduce

$$\neq\!\!\!\!\!\!\int_0^{+\infty} \frac{g(x)}{(x-t)^{p+1}}dx = \frac{1}{p!}\neq\!\!\!\!\!\!\int_0^{+\infty} \frac{g^{(p)}(x)}{(x-t)}dx + \sum_{s=0}^{p-1} \frac{(p-s-1)!}{p!}(-1)^{p-s}\frac{g^{(s)}(0)}{t^{p-s}},$$

and taking into account that for $s = 0, 1, \ldots, p$,

$$\frac{d^s}{dx^s}\left[\frac{1}{(1+x)^\beta}\right] = \frac{(-1)^s}{(1+x)^{\beta+s}}\eta_s, \quad \eta_0 = 1, \quad \eta_s := \prod_{i=0}^{s-1}(\beta+i),$$

we get

$$\begin{aligned} \mathcal{H}_{p-k}(\sigma_\beta, t) &= \frac{(-1)^{p-k}\eta_{p-k}}{(p-k)!}\neq\!\!\!\!\!\!\int_0^{+\infty} \frac{dx}{(1+x)^{\beta+p-k}(x-t)} \\ &+ \sum_{s=0}^{p-k-1} \frac{(p-k-s-1)!}{(p-k)!}\frac{(-1)^{p-k}}{t^{p-k-s}}\eta_s \\ &= \frac{(-1)^{p-k}}{(p-k)!}\left[\eta_{p-k}\mathcal{H}_0(\sigma_{\beta+p-k}, t) + \sum_{s=0}^{p-k-1} \frac{(p-k-s-1)!}{t^{p-k-s}}\eta_s\right], \end{aligned}$$

where for $\delta \notin \mathbb{N}$

$$\begin{aligned} \mathcal{H}_0(\sigma_\delta, t) &= \neq\!\!\!\!\!\!\int_0^{+\infty} \frac{dx}{(1+x)^\delta(x-t)}dx \\ &= \frac{\pi}{(1+t)^\delta}\cot(\delta\pi) - \frac{\Gamma(\delta-1)}{(1+t)\Gamma(\delta)}\,{}_2F_1\left(1-\delta, 1; 2-\delta; \frac{1}{1+t}\right), \end{aligned}$$

being ${}_2F_1$ the Hypergeometric function (see [36, p. 251]).

8. Numerical experiments

In this section we propose a selection of numerical tests obtained by implementing the quadrature rules introduced in Sections 4 and 5, for integrand functions f belonging to different spaces of functions. To be more precise in Examples 8.1-8.3 we consider integrals of the type $\mathcal{H}_p(fw_\alpha,t)$ for function f with a possible exponential growth and we approximate them by

$$\bar{\mathcal{H}}_{p,m}(fw_\alpha,t)=\begin{cases}\mathcal{H}_{p,m^*}(fw_\alpha,t), & 0<t<4m\theta_1\\ \sum_{i=1}^j F_t(x_{m,i})\lambda_{m,i}, & t>x_{m,j}+1,\end{cases} \tag{36}$$

(see (18) and (21)). Since the exact values of the integrals are unknown, we will retain as exact those values computed with $m=1000$ and we will set

$$\bar{e}_{p,m}(fw_\alpha,t)=|\bar{\mathcal{H}}_{p,m}(fw_\alpha,t)-\bar{\mathcal{H}}_{p,1000}(fw_\alpha,t)|.$$

In the first two tests $p=1$ and we compare our results with those obtained by the method in [4], introduced only for this choice of p. Their procedure, after a transformation of the integral in $[0,1]$, makes use of product integration rules based on Legendre zeros and requires the computation of second kind Legendre functions, via recurrence relations.

The successive Examples 8.4 and 8.5 deal with the approximation of $\mathcal{H}_p(f\sigma_\beta,t)$ by $\hat{\mathcal{H}}_{p,m}(f\sigma_\beta,t)$ with $p>1$, where, according to the position of t,

$$\hat{\mathcal{H}}_{p,m}(fw_\alpha,t)=\begin{cases}\mathcal{H}_{p,m^*}(f\sigma_\beta,t), & 0<t<4m\theta 1\\ \sum_{i=1}^j \tilde{F}_t(x_{m,i})\lambda_{m,i}, & t>x_{m,j}+1,\end{cases} \tag{37}$$

(see (27) and (29)). Since the exact values of the integrals are unknown, we will retain as exact those values computed with $m=1000$ and we will set

$$\hat{e}_{p,m}(f\sigma_\beta,t)=|\hat{\mathcal{H}}_{p,m}(f\sigma_\beta,t)-\hat{\mathcal{H}}_{p,1000}(f\sigma_\beta,t)|.$$

In order to show how the proposed regularization in (27) really improves the procedure in (24), in Tables 7 and 9, we will set the errors

$$\bar{e}_{p,m}(f\sigma_\beta,t)=|\mathcal{H}_{p,m^*}(\Phi w_0,t)-\hat{\mathcal{H}}_{p,1000}(f\sigma_\beta,t)|.$$

We remark that, in each example the "truncation intervals" depending on a fixed θ, have been empirically detected according to the following criteria

$$j=\max_{i=1,\ldots,m^*}\lambda_{m^*,i}\left|\frac{f(x_{m^*,i})-\sum_{k=0}^p\frac{f^{(k)}(t)}{k!}(x_{m^*,i}-t)^k}{(x_{m^*,i}-t)^{p+1}}\right|\geq eps$$

in (19) and

$$j=\max_{i=1,\ldots,m^*}\lambda_{m^*,i}\,|h_t(x_{m^*,i})|\geq eps$$

in (27), where eps represents the machine precision.

We point out that all the computations have been performed in double-machine precision ($eps = 2.22044e-16$).

Finally, whereas the truncation holds, besides the number m of quadrature knots we set also the actual number $j\leq m$ of function's computations.

Example 8.1. We first consider the following Hadamard integral

$$\mathcal{H}_1(fw_{\frac{1}{2}},t)=\int_0^{+\infty}\frac{\sin(x+5)}{(x-t)^2}\sqrt{x}e^{-x}dx,\quad f(x)=\sin(x+5),\quad \alpha=\frac{1}{2},\quad p=1.$$

Since the function $f(x)$ is very smooth, the convergence of our method is very fast (see Table 1) and the machine precision is attained with only 18 quadrature knots, for different values of t. In Tables 2 are given the results obtained by the method in [4], whose convergence is much slower. The plot of $\mathcal{H}_1(fw_{\frac{1}{2}},t)$ is shown in Figure 2.

j	$\bar{e}_{1,m}(fw_{\frac{1}{2}}, 0.1)$	$\bar{e}_{1,m}(fw_{\frac{1}{2}}, 5)$	$\bar{e}_{1,m}(fw_{\frac{1}{2}}, 50)$
5 ($m = 5$)	$8.6553e-6$	$4.4598e-6$	$9.8482e-7$
10 ($m = 10$)	$4.1984e-9$	$3.7465e-9$	$8.0136e-10$
18 ($m = 20$)	*eps*	*eps*	*eps*

Table 1: Example 8.1: Errors by the present method

m	$t = 0.1$	$t = 5$	$t = 50$
100	$2.7890e-5$	$6.0310e-9$,	$3.0446e-11$
200	$2.2390e-6$	$4.2379e-10$	$3.9494e-13$
300	$1.9826e-7$	$8.8928e-11$	$5.7371e-12$
400	$1.5156e-7$	$4.1285e-11$	$3.3105e-12$
500	$2.0034e-7$	$3.1008e-11$	$9.9150e-13$
600	$1.7676e-7$	$2.7022e-11$	$1.6251e-13$
700	$1.3721e-7$	$2.3954e-11$	$1.8737e-13$
800	$9.8700e-8$	$2.0870e-11$	$3.0233e-13$
900	$6.6586e-8$	$1.7735e-11$	$2.3905e-13$

Table 2: Example 8.1: Errors by the method in [4]

Example 8.2. Consider

$$\mathcal{H}_1(fw_{\frac{3}{5}}, t) = \fint_0^{+\infty} \frac{|x-4|^{\frac{15}{2}}}{(x-t)^2} x^{\frac{3}{5}} e^{-x} dx, \quad f(x) = |x-4|^{\frac{15}{2}}, \quad \alpha = \frac{3}{5}, \quad p = 1,$$

with $f \in Z_{7.5}(w_{3/5})$. By (20) the error behaves like $m^{-\frac{13}{4}} \log m$. As shown in Table 3 the numerical results confirm such theoretical estimate. In fact, at the point 4.0001, for instance, for $m = 900$ (but only 180 computations function) 8 correct digits are achieved and $m^{-\frac{13}{4}} \log m \sim 1.7e-9$. For values of t "far" from the critical point 4, the numerical results highlight a definitive better behavior. In Table 4, we present the results obtained by the method in [4] and, as one can see, only 6 digits are attained with $m = 900$ computations of function. The plot of $\mathcal{H}_1(fw_{\frac{3}{5}}, t)$ is shown in Figure 3.

j	$\bar{e}_{1,m}(fw_{\frac{3}{5}}, 2.5)$	$\bar{e}_{1,m}(fw_{\frac{3}{5}}, 4.0001)$	$\bar{e}_{1,m}(fw_{\frac{3}{5}}, 500)$
57 (m =100)	$1.5660e-6$	$5.3871e-6$	$1.0213e-11$
82 (m =201)	$7.7518e-8$	$1.1585e-6$	$1.1538e-12$
100 (m =300)	$1.8343e-8$	$3.0130e-7$	$2.0729e-13$
115 (m =400)	$3.8055e-9$	$1.1179e-7$	$5.1859e-14$
129 (m =500)	$1.6642e-9$	$5.4807e-8$	$2.0616e-14$
141 (m =600)	$9.4205e-10$	$1.6323e-8$	$6.7410e-15$
152 (m =701)	$5.9492e-10$	$1.3827e-8$	$3.2744e-15$
153 (m =801)	$2.1716e-10$	$1.1697e-8$	$7.9002e-16$
180 (m =901)	$8.3514e-11$	$3.6227e-9$	$6.5736e-16$

Table 3: Example 8.2: Errors by the present method.

Example 8.3. Let

$$\mathcal{H}_2(fw_{\frac{1}{2}}, t) = \fint_0^{+\infty} \frac{|\sin(x-2)|^{\frac{13}{2}}}{(x-t)^3} \sqrt{x} e^{-x} dx, \quad f(x) = |\sin(x-2)|^{\frac{13}{2}}, \quad \alpha = \frac{1}{2}, \quad p = 2.$$

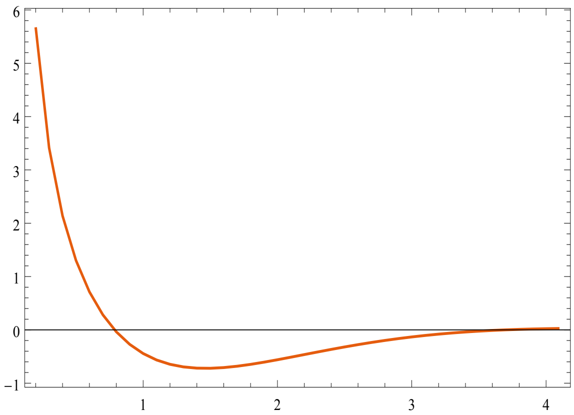


Figure 2: Example 8.1: Graph of $\mathcal{H}_1(fw_{\frac{1}{2}}, t)$

m	$t = 2.5$	$t = 4.0001$	$t = 500$
100	$6.1073e-5$	$1.3090e-4$	$4.9767e-8$
200	$1.3772e-5$	$4.6000e-5$	$9.1827e-10$
300	$3.5540e-7$	$9.1208e-6$	$2.6086e-10$
400	$3.3217e-6$	$2.6377e-7$	$1.3107e-10$
500	$3.3840e-6$	$1.7624e-6$	$1.8038e-10$
600	$2.6763e-6$	$1.0611e-6$	$7.2855e-11$
700	$1.8665e-6$	$1.5189e-7$	$5.6055e-12$
800	$1.1800e-6$	$3.1543e-7$	$4.1881e-12$
900	$6.7891e-7$	$2.9859e-7$	$1.8214e-11$

Table 4: Example 8.2: Errors by the method in [4].

Since $f \in Z_{6.5}(w_{\frac{1}{2}})$, by (20) the error behaves like $m^{-\frac{9}{4}} \log m$. By inspecting Table 5, also in this case the numerical results agree with the theoretical ones. The worst result is attained for $t = 2.0001$. The graph of $\mathcal{H}_2(fw_{\frac{1}{2}}, t)$ is shown in Figure 4.

Example 8.4. Now we consider the integral

$$\mathcal{H}_3(f\sigma_{\frac{3}{2}}, t) = \oint_0^{+\infty} \frac{\cos(\log(x+2))}{(1+x)^{\frac{3}{2}}(x-t)^4} dx, \quad f(x) = \cos(\log(x+2)), \quad \beta = \frac{3}{2}, \quad p = 3. \tag{38}$$

In this case the function $\frac{\cos(\log(x+2))}{(1+x)^{\frac{3}{2}}}$ algebraically decays and we apply (24) to

$$\mathcal{H}_3(\Phi w_0, t) = \oint_0^{+\infty} \frac{\Phi(x)}{(x-t)^4} e^{-x} dx, \quad \Phi(x) = \frac{\cos(\log(x+2))e^x}{(1+x)^{\frac{3}{2}}}.$$

Since $G_3(\Phi, t) \in W_4^1(w_0)$ for any fixed t, according to Lemma 5.1, the error behaves like m^{-2}. On the other hand, by inspecting Table 6, we observe that none truncation has been performed in view of the exponential growth of Φ and that the occurrence of overflow prevents to choose $m > 186$.

Now we show that for approximating the integral (38) the performance of the rule (37) is much better. Indeed, in Table 7 we exhibit the approximations obtained for $\mathcal{H}_3(f\sigma_{\frac{3}{2}}, t)$ with $q = 1$. Since f is very smooth, according to estimate (28), the rate of convergence is very fast and the machine precision is attained for $m = 41$ and only 22 function's computations. In Figure 5, the graph of $\mathcal{H}_3(f\sigma_{\frac{3}{2}}, t)$ is shown.

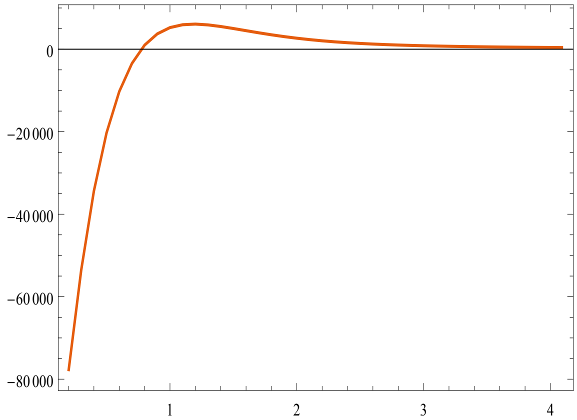


Figure 3: Example 8.2: Graph of $\mathcal{H}_1(fw_{\frac{3}{5}}, t)$

j	$\bar{e}_{2,m}(fw_{\frac{1}{2}}, \frac{1}{2})$	$\bar{e}_{2,m}(fw_{\frac{1}{2}}, 2.0001)$	$\bar{e}_{2,m}(fw_{\frac{1}{2}}, 10)$
43 $(m = 101)$	$6.9778e-7$	$2.3572e-5$	$1.6601e-6$
61 $(m = 201)$	$2.7793e-8$	$9.5002e-6$	$1.7141e-8$
75 $(m = 300)$	$1.0888e-8$	$2.2228e-6$	$3.4760e-9$
86 $(m = 401)$	$4.4911e-9$	$5.0696e-7$	$5.2015e-10$
96 $(m = 501)$	$1.6847e-9$	$1.3638e-6$	$4.0446e-10$
105 $(m = 601)$	$1.0939e-9$	$3.7675e-7$	$4.2280e-10$
114 $(m = 701)$	$3.6242e-10$	$5.1814e-7$	$3.9872e-10$

Table 5: Example 8.3: Approximation of $\mathcal{H}_2(fw_{\frac{1}{2}}, t)$.

Example 8.5. As last example we consider

$$\mathcal{H}_2(f\sigma_{\frac{5}{2}}, t) = \,=\!\!\!\!\!\!\int_0^{+\infty} \frac{(x+4)^4}{(1+x)^{\frac{5}{2}}(x^2+5)(x-t)^3} dx, \quad f(x) = \frac{(x+4)^4}{x^2+5}, \quad \beta = \frac{5}{2}, \quad p = 2.$$

Also in this case the function $\frac{(x+4)^4}{(1+x)^{\frac{5}{2}}(x^2+5)}$ has an algebraic decay at infinity. At first in Table 8 we show the numerical results obtained approximating $\mathcal{H}_2(\Phi w_0, t)$, with $\Phi(x) = f(x)\sigma_{\frac{5}{2}}(x)e^x$, by (24). In this case $G_2(\Phi, t) \in W_8^1(w_0)$ for any fixed t and, in view of Lemma 5.1, the error behaves like m^{-4}. Successively in Table 9 we present the approximations of $\mathcal{H}_2(f\sigma_{\frac{5}{2}}, t)$ obtained applying the rule (37) with $q = 1$. Also in this case, according to our expectations, without change of variable, the exponential growth of Φ produces overflow, preventing to choose $m > 190$. Otherwise, the proposed change of variable has sped up the convergence and the machine precision is attained with only 44 function's evaluations. The plot of $\mathcal{H}_2(f\sigma_{\frac{5}{2}}, t)$ is given in Figure 6.

9. The proofs

The proof of Theorem 4.1 is based on the following two results which can be found in [32]

Theorem 9.1. *Let $\alpha \geq 0$, $p \geq 1$ and $0 < \lambda < 1$. If $f^{(p)} \in Z_\lambda(w_\alpha \varphi^p)$ then for any $0 < t < 4m\theta_1$, with $0 < \theta_1 < \theta < 1$,*

$$t^p|\mathcal{H}_{p,m^*}(fw_\alpha, t)| \leq \mathcal{C}\left(\int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi(f^{(p)}, u)_{w_\alpha \varphi^p}}{u} du + \|f\|_{W_p(w_\alpha)}\right),$$

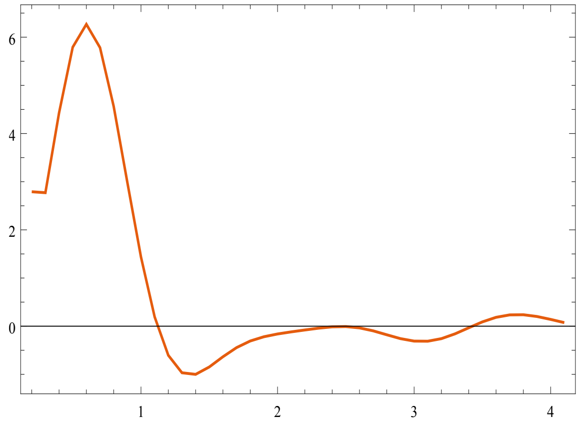


Figure 4: Example 8.3: Graph of $\mathcal{H}_2(fw_{\frac{1}{2}}, t)$.

j	$\bar{e}_{3,m}(\Phi w_0, \frac{3}{2})$	$\bar{e}_{3,m}(\Phi w_0, 8)$	$\bar{e}_{3,m}(\Phi w_0, 20)$
50 $(m = 50)$	$8.6794e-12$	$8.6794e-12$	$1.0695e-11$
100 $(m = 100)$	$4.6779e-13$	$4.9426e-13$	$5.7490e-13$
150 $(m = 150)$	$7.3022e-14$	$7.3350e-14$	$7.9083e-14$
185 $(m = 185)$	$2.5504e-14$	$2.6290e-14$	$3.0597e-14$
186 $(m = 186)$	Inf	Inf	Inf

Table 6: Example 8.4: Approximation of $\mathcal{H}_3(\Phi w_0, t)$.

where $0 < \mathcal{C} \neq \mathcal{C}(m, f, t)$.

Theorem 9.2. *Let* $\alpha \geq 0$, $p \geq 1$ *and* $0 < \lambda < 1$. *If* $f^{(p)} \in Z_\lambda(w_\alpha \varphi^p)$ *then for any* $0 < t < 4m\theta_1$, *with* $0 < \theta_1 < \theta < 1$,

$$t^p |\mathcal{H}_p(fw_\alpha, t) - \mathcal{H}_{p,m^*}(fw_\alpha, t)| \leq \mathcal{C}\left(\log m \int_0^{\frac{1}{\sqrt{m}}} \frac{\Omega_\varphi^r(f^{(p)}, u)_{w_\alpha \varphi^p}}{u} du + e^{-Am} \|f\|_{W_p^\infty(w_\alpha)}\right),$$

where $0 < \mathcal{C} \neq \mathcal{C}(m, f, t)$ *and* $0 < A \neq A(m, f, t)$.

In order to prove Lemma 5.1 and Theorem 5.1, with $R_p(f, x, t)$ and $G_p(f, x, t)$ defined in (23), we premise the following relations

$$\frac{\partial^i}{\partial x^i} R_p(f, x, t) = R_p^{(i)}(f, x, t) = \begin{cases} f^{(i)}(x) - \sum_{k=i}^p \frac{f^{(k)}(t)(x-t)^{k-i}}{(k-i)!}, & 0 \leq i \leq p, \\ f^{(i)}(x), & i \geq p+1, \end{cases}$$

and

$$\begin{aligned} \frac{\partial^r}{\partial x^r} G_p(f, x, t) &= G_p^{(r)}(f, x, t) = \sum_{i=0}^r \binom{r}{i} R_p^{(i)}(f, x, t) \left[\frac{1}{(x-t)^{p+1}}\right]^{(r-i)} = \sum_{i=0}^r c_i \frac{R_{p-i}(f^{(i)}, x, t)}{(x-t)^{p+1+r-i}} \\ &= \frac{1}{(x-t)^{p+1}} \sum_{i=0}^r c_i \left[\frac{f^{(i)}(x)}{(x-t)^{r-i}} - \sum_{k=i}^p \frac{f^{(k)}(t)}{(k-i)!(x-t)^{r-k}}\right], \end{aligned} \tag{39}$$

where

$$c_i = \binom{r}{i} (-1)^{r-i} (p+1)(p+2) \dots (p+r-i).$$

j	$\hat{e}_{3,m}(f\sigma_{\frac{3}{2}}, \frac{3}{2})$	$\hat{e}_{3,m}(f\sigma_{\frac{3}{2}}, 8)$	$\hat{e}_{3,m}(f\sigma_{\frac{3}{2}}, 20)$
12 $(m = 10)$	$1.0529e-8$	$2.8344e-11$	$2.9870e-11$
16 $(m = 21)$	$7.7561e-12$	$9.4657e-13$	$5.6634e-14$
19 $(m = 31)$	$1.7342e-13$	$3.0899e-15$	$1.1280e-16$
22 $(m = 41)$	$6.9168e-16$	eps	eps

Table 7: Example 8.4: Approximation of $\mathcal{H}_3(f\sigma_{\frac{3}{2}}, t)$.

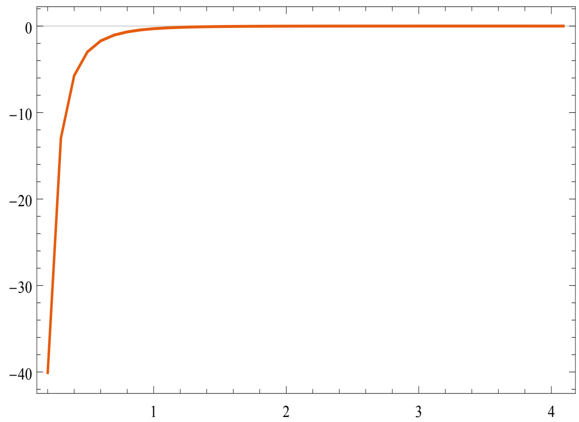


Figure 5: Example 8.4: Graph of $\mathcal{H}_3(f\sigma_{\frac{3}{2}}, t)$.

Proof of Lemma 5.1. Under the assumption $f \in C^{r+p+1}([0, +\infty))$, it follows that $\Phi \in C^{r+p+1}([0, +\infty))$ and $G_p(\Phi) \in C^r([0, +\infty))$ and, in view of (25), $\Phi^{(k)}(x)w_0(x) = \mathcal{O}\left(x^{-\beta}\right)$, $k = 1, 2, \dots r + p + 1$. Therefore, by using (39) with $f = \Phi$, we get

$$|G_p^{(r)}(\Phi, x, t)\varphi^r(x)w_0(x)| = \mathcal{O}\left(x^{\frac{r}{2}-1-p-\beta}\right),$$

from which, taking into account the hypothesis $2(\beta + p) - r > 0$, $G_p^{(r)}(\Phi) \in W_r^1(w_0)$ for any fixed $t > a > 0$ and, then, by (6), the lemma follows. □

Proof of Theorem 5.1. Setting $g(y) = G_p(f, \psi(y), t)$, by induction on i, it can be proved that

$$g^{(r)}(y) = q^r \sum_{k=1}^{r} G_p^{(k)}(f, \psi(y), t)e^{kqy}s_{k-1}^r,$$

where $\{s_{k-1}^r\}_{k=1}^r$, $r = 1, 2, \dots$, are the Stirling numbers of the second kind. Moreover, under the assumptions (25), by (39), it follows that

$$\left|G_p^{(k)}(f, \psi(y), t)\right| = \mathcal{O}\left(\psi^{-1-k}(y)\right)$$

and, therefore,

$$\left|g^{(r)}(y)\right| \le e^{-yq}, \quad \forall r \ge 1.$$

Finally, recalling that $h_t(y) = qg(y)e^{-y(q(\beta-1)-1)}$, for any fixed t and for any $r \ge 1$, we deduce

$$\left|h_t^{(r)}(y)\right| = \mathcal{O}\left(e^{-y(q\beta-1)}\right)$$

and also

$$\int_0^{+\infty} \left|h_t^{(r)}(y)\varphi^r(y)\right| w_0(y)dy \le \mathcal{C}\int_0^{+\infty} y^{\frac{r}{2}}e^{-yq\beta}dy \le \mathcal{C},$$

j	$\bar{e}_{2,m}\left(\Phi w_0, \frac{1}{3}\right)$	$\bar{e}_{2,m}\left(\Phi w_0, \frac{9}{2}\right)$	$\bar{e}_{2,m}(\Phi w_0, 25)$
10 $(m = 10)$	$1.0747e-3$	$6.5264e-5$	$6.4541e-4$
50 $(m = 50)$	$8.3845e-7$	$8.8017e-7$	$1.1296e-6$
100 $(m = 100)$	$1.3881e-7$	$1.4206e-7$	$1.5983e-7$
150 $(m = 150)$	$4.9077e-8$	$4.9835e-8$	$5.2883e-8$
190 $(m = 190)$	$2.6496e-8$	$2.7178e-8$	$2.8875e-8$
191 $(m = 191)$	Inf	Inf	Inf

Table 8: Example 8.5: Approximation of $\mathcal{H}_2(\Phi w_0, t)$.

j	$\hat{e}_{2,m}(f\sigma_{\frac{5}{2}}, \frac{1}{3})$	$\hat{e}_{2,m}\left(f\sigma_{\frac{5}{2}}, \frac{9}{2}\right)$	$\hat{e}_{2,m}(f\sigma_{\frac{5}{2}}, 25)$
10 $(m = 10$)	$6.2550e-3$	$1.0559e-3$	$6.7372-6$
21 $(m = 51$)	$1.0268e-6$	$6.5777e-8$	$1.6154e-9$
29 $(m = 100$)	$1.3905e-9$	$3.2552e-11$	$4.3751e-12$
35 $(m = 151$)	$1.5018e-11$	$2.5168e-12$	$2.1414e-14$
40 $(m = 200$)	$8.0083e-13$	$5.7405e-14$	$7.1377e-16$
44 $(m = 251$)	eps	eps	eps

Table 9: Example 8.5: Approximation of $\mathcal{H}_2(f\sigma_{\frac{5}{2}}, t)$.

being $\beta q > 1$. Thus, by (6) the lemma easily follows . $\square$

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References

[1] G. Monegato, Definitions, properties and applications of finite-part integrals, J. Comput. Appl. Math. 229 (2009) 425–439.

[2] A. Frangi, M. Guiggiani, A direct approach for boundary integral equations with high-order singularities, Int. J. Numer. Methods Eng. 49 (2000) 871–898.

[3] I. Lifanov, L. Poltavskii, G. Vainikko, Hypersingular Integral Equations and their Applications, Chapman & Hall CRC, 2003.

[4] A. Aimi, M. Diligenti, Numerical integration schemes for Petrov- Galerkin inifinite bem, Appl. Numer. Math. 58 (2008) 1084–1102.

[5] G. Mastroianni, G. Monegato, Truncated quadrature rules over $(0, \infty)$ and Nyström type methods, SIAM Jour. Num. Anal. 41 (5) (2003) 1870–1892.

[6] J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Yale Univ. Press, New Haven, 1923.

[7] G. Monegato, Numerical evaluation of hypersingular integrals, J. Comput. Appl. Math. 50 (1994) 9–31.

[8] M. Diligenti, G. Monegato, Finite-part integrals: their occurrence and computation, Rend. Circolo Mat. di Palermo 33 (1993) 39–61.

[9] G. Tsamasphyros, G. Dimou, Gauss quadrature rules for finite part integrals, Internat. J. Numer. Methods Engrg. 30 (1) (1990) 13–26.

[10] G. Criscuolo, Numerical evaluation of certain strongly singular integrals, IMA J. Numer. Anal. 34 (2) (2014) 651–674.

[11] G. Criscuolo, G. Mastroianni, Convergenza di formule Gaussiane per il calcolo delle derivate di integrali a valor principale secondo Cauchy, Calcolo 24 (2) (1987) 179–192.

[12] T. Hasegawa, T. Torii, Hilbert and hadamard transforms by generalized chebyshev expansion, J. Comput. Appl. Math. 51 (1) (1994) 71–83.

[13] N. I. Ioakimidis, On the uniform convergence of gaussian quadrature rules for cauchy principal value integrals and their derivatives, Math. Comp. 44 (1985) 191–198.

[14] P. Kolm, V. Rokhlin, Numerical quadratures for singular and hypersingular integrals, Comput. Math. Appl. 41 (3-4) (2001) 327–352.

[15] D. Occorsio, A method to evaluate the Hilbert transform on $(0, \infty)$, Appl. Math. Comput. 217 (12) (2011) 5667–5679.

[16] N. Mastronardi, D. Occorsio, Some numerical algorithms to evaluate Hadamard finite-part integrals., J. Comput. Appl. Math. 70 (1) (1996) 75–93.

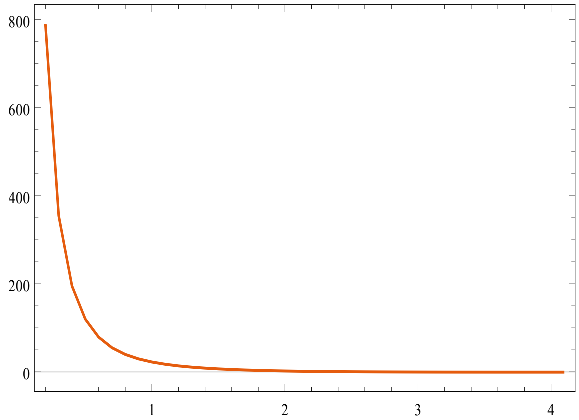

Figure 6: Example 8.5: Graph of $\mathcal{H}_2(f\sigma_{\frac{5}{2}}, t)$.

[17] G. V. Milovanovic, M. M. Spalevic, Quadrature rules with multiple nodes for evaluating integrals with strong singularities, J. Comput. Appl. Math. 189 (1-2) (2006) 689–702.
[18] S. Xiang, C. Fang, Z. Xu, On uniform approximations to hypersingular finite-part integrals, J. Math. Anal. Appl. 435 (2) (2016) 1210–1228.
[19] H. Kutt, On the numerical evaluation of finite-part integrals involving an algebraic singularity, PHD Thesis, University of Stellenbosch, 1975.
[20] B. Della Vecchia, D. Occorsio, Some algorithms for the numerical evaluation of Hadamard finite parts integrals on the semi-axis, Scientific Review 21-22 (1996) 22–35.
[21] G. Mastroianni, D. Occorsio, Numerical approximation of weakly singular integrals on the half line, J. Comput. Appl. Math. 140 (1) (2002) 587–598.
[22] G. Mastroianni, G. Monegato, Some new applications of truncated Gauss-Laguerre quadrature formulas, Numer. Algor. 49 (2008) 283–297.
[23] M. C. De Bonis, G. Mastroianni, M. Viggiano, K-functionals, moduli of smoothness and weighted best approximation on the semiaxis, in: J. S. L. Leindler, F. Schipp (Ed.), Functions, Series, Operators, Alexits Memorial Conference, Janos Bolyai Mathematical Society, Budapest, Hungary, 2002, pp. 181–211.
[24] G. Szegö, Orthogonal Polynomials, 4th Edition, Vol. 23 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 1976.
[25] C. Laurita, G. Mastroianni, l^p-convergence of Lagrange interpolation on the semiaxis, Acta Math. Hungar. 120 (2008) 249–273.
[26] G. Mastroianni, G. V. Milovanovic, Some numerical methods for second-kind Fredholm integral equations on the real semiaxis, IMA J. Numer. Anal. 29 (4) (2009) 1046–1066.
[27] G. Mastroianni, D. Occorsio, Lagrange interpolation at Laguerre zeros in some weighted uniform spaces, Acta Math. Hungar. 91 (1-2) (2001) 27–52.
[28] M. C. De Bonis, D. Occorsio, On the simultaneous approximation of a Hilbert transform and its derivatives on the real semiaxis, Appl. Numer. Math. 114 (2017) 132–153.
[29] G. Monegato, J. Lyness, The Euler-Maclaurin expansion and finite-part integrals, Numer. Math. 81 (1998) 273–291.
[30] P. Martin, F. Rizzo, Hypersingular integrals: How smooth must the density be?, Int. J. Numer. Methods Eng. 39 (1996) 687–704.
[31] N. Muskhelishvili, Singular Integral Equations. Boundary Problems of Function Theory and Their Application to Mathematical Physics, Dover Books on Mathematics, Dover Publications, 1953.
[32] M. C. De Bonis, D. Occorsio, Error bounds for a Gauss-type quadrature rule to evaluate hypersingular integrals, in progress.
[33] M. C. De Bonis, B. Della Vecchia, G. Mastroianni, Approximation of the Hilbert transform on the real semiaxis using Laguerre zeros, J. Comput. Appl. Math. 140 (1-2) (2002) 209–229.
[34] A. Prudnikov, Y. Bryčkov, O. Maričev, Elementary functions, integrals and series, Volume I, Gordon and Breach Science Publishers, NY, London, Tokyo, 1986.
[35] I. Gradshteyn, I. Ryzhik, Table of Integrals, Series, and Products, 5th Edition, Academic Press, Inc., Boston, MA, 1994.
[36] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher transcendental functions, Tables of integrals transforms, Vol. II, McGraw-Hill Book Company, Inc., New York , Toronto, London, 1954.